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Realizability: a historical essay

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Dedicated to Anne S. Troelstra on his 60th Birthday

1. Introduction

The purpose of this short paper is to sketch the development of a few basic topics in the history of realizability. The number of topics is quite limited and reflects very much my own personal taste, prejudices and area of competence.

Realizability has, over the past 60 years, developed into a subject of such dimensions that a comprehensive overview would require a fat book. Maybe someone, some day ought to write such a book. But it will not be easy. Quite apart from the huge amount of literature to cover, there is the task of creating unity where there is none. For realizability has many faces, each of them turned towards different areas of logic, mathematics and computer science, and this proliferation shows no signs of diminishing in our day. Like an aggressive carcinoma, realizability stretches out its tentacles to ever more remote fields: linear logic, complexity theory and rewrite theory have already been infected. The theory of subrecursive hierarchies too. Everything connected with the λ-calculus is heavily engaged. Proof theory is suffering. Intuitionism is dead.

This is just to name a few! Did you think that *at least* the realm of classical logic would be safe? Recently, Krivine came up with a realizability interpretation for ZF set theory!

Confronted with this mess, I have acted like the typical impostor who has walked into the hospital claiming to be a surgeon, and is now wielding the knives in the operating theatre: I took the nearest scalpel at hand and cut out everything that would not fit into either one of my two major streams: metamathematics of intuitionistic arithmetical theories, and topos-theoretic developments.

Needless to say, there is no question of even starting to list what I have omitted – sometimes to my great regret, although I realise that such hollow apologies just reverberate in the vast emptiness I have created‡.

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‡ The disappointed reader can take solace in the availability of an *excellent* proof-theoretical survey on realizability (Troelstra 1998).

Therefore, let's get physical and say something concrete about what *is* in this paper. According to me, there are three landmark publications in realizability. These are:

(1) Kleene's original 1945 paper, *On the Interpretation of Intuitionistic Number Theory* (Kleene 1945).
(2) Troelstra's *Metamathematical Investigations* from 1973 (Troelstra 1973).
(3) Hyland's *The Effective Topos* from 1981 (Hyland 1980).

Of these three, both (1) and (3) initiated a whole new strand of research. I have therefore decided that the material I wish to present, naturally divides into two *periods*: 1940–1980 and 1980–2000. This is not to say that suddenly there were, after 1980, no more purely syntactical presentations of Realizabilities (quite to the contrary, thanks to computer science, syntax is back!), but I do feel that although many of these matters still need and deserve to be investigated (and need all the elegance and expository skills we can muster), no radically new vistas have emerged from this research. Therefore, in my account of the second period I have concentrated on what I regard as more innovative research.

The second item in my list is of a different kind. This monumental work brought together all existing results, many of which were due to its author, and ordered them in such a way that the diligent student could see at once the similarities between them. It charted the territory, and in this way achieved something of conceptual value: the notion that all these systems, interpretations and axiomatisations were manifestations of a common pattern. What exactly this pattern is, we still do not know. But it is my feeling that the categorical analyses of later years owe a lot to this work.

When it appeared, it made a 'daunting' impression on some people. And it certainly did so on me when I was Troelstra's student. But now I experience a sensation of dry, austere beauty in its relentless pursuit of order. And let us not forget it set new standards of presentation and notation. For although Kleene's first paper is a gem of readability, regrettably Kleene later adopted a style of writing which was so cluttered with notation that it takes a strong man to fight through it.

I have therefore decided to dedicate this paper in gratitude to my mentor Anne Troelstra, who has contributed so much to the subject matter.

2. The first 40 years: 1940–1980

2.1. *The origin of realizability*

In his overview paper: 'Realizability: a retrospective survey' (Kleene 1973), Stephen Cole Kleene recounts how his idea for numerical realizability developed. He wished to give some precise meaning to the intuition that there should be a connection between intuitionism and the theory of recursive functions (both theories stressing the importance of extracting information *effectively*). He started to think about this in 1940[†].

[†] For some biographical details on Kleene and a personal appreciation, see the obituary by his friend Saunders Mac Lane (Mac Lane 1996).

In order to appreciate the originality of his thinking, one should recall that the formal system of intuitionistic arithmetic **HA** did not exist at the time[†].

As an example of a precise connection between Intuitionism and the theory of recursive functions, Kleene starts by conjecturing a weak form of Church's Rule: if a closed formula of the form $\forall x \exists y \varphi(x, y)$ is provable in intuitionistic number theory, then there must be a general recursive function F such that for all n, the formula $\varphi(\overline{n}, \overline{F(n)})$ is true. One arrives at this conjecture by unravelling the meaning that such a statement must have for an intuitionist.

Conjecturing this, at a time when intuitionism was still clouded by Brouwer's mysticism, the formal system in question hardly established, and the content of the conjecture blatantly false for Peano arithmetic, was imaginative indeed!

But, this was still far away from the actual development of realizability. Often, one encounters the opinion that realizability was inspired by the so-called 'Brouwer–Heyting–Kolmogorov interpretation' (an attempt to clarify the constructive meaning of the logical operations). This was not the case. Kleene starts by quoting Hilbert and Bernays (1934). They, in their 'Grundlagen der Mathematik', explain the 'finitist' position in mathematics. The relevant passage is the one about 'existential statements as incomplete communications', which, since it is philosophy, can only be appropriately understood in the original German:

> Ein *Existenzsatz* über Ziffern, also ein Satz von der Form 'es gibt eine Ziffer n von der Eigenschaft $\mathscr{A}(n)$' ist finit aufzufassen als ein 'Partialurteil', d.h. als eine unvollständige Mitteilung einer genauer bestimmten Aussage, welche entweder in der direkten Angabe einer Ziffer von der Eigenschaft $\mathscr{A}(n)$ oder der Angabe eines Verfahrens zur Gewinnung einer solchen Ziffer besteht […].[‡]

Kleene then asks: 'Can we generalise this idea to think of *all*[§] (except, trivially, the simplest) intuitionistic statements as incomplete communications?'[¶]

He outlines in what sense every logical sentence is 'incomplete' and what would constitute its 'completion'. For the implication case, Kleene interestingly says that he first tried an inductive clause inspired by 'Heyting's "proof-interpretation" ', but that it 'didn't work' and so, 'Heyting's proof-interpretation failed to help me to my goal'[‖]. Since Kleene does not reveal what this first try was, we are free to conjecture. It is just conceivable

[†] Well, … there is a system closely resembling **HA** in Gödel (1932). Kleene appears to have been at least initially unaware of this, for although his 1945 paper gives the reference, the retrospective survey stresses that 'Heyting Arithmetic […] does not occur as a subsystem readily separated out from Heyting's full system of intuitionistic mathematics', and quotes Kleene's own formalism, which later appeared in Kleene (1952), as the thing he had in mind.

[‡] An *existential statement* about numbers, *i.e.* a statement of the form 'there exists a number n with property $\mathscr{A}(n)$' is finitistically taken as a 'partial judgement', that is, as an incomplete rendering of a more precisely determined proposition, which consists in either giving directly a number n with the property $\mathscr{A}(n)$, or a procedure by which such a number can be found … .

[§] My italics.

[¶] It is, however, fair to say that Hilbert and Bernays did not limit their treatment of the finitist position to existential statements; they had a lot more to say, and also included negations and $\forall\exists$-statements in their account.

[‖] In the words of Troelstra and van Dalen (1988, volume I, page 9), the Heyting proof interpretation clause for implication is: 'A proof of $A \to B$ is a construction which transforms any hypothetical proof of A into a proof of B'.

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that he tried: a realizer for $A \to B$ is a partial recursive function that sends proofs of A to proofs of B.

Kleene's realizability was, at least conceptually, a major advance. Its achievement is not so much a philosophical explanation of the intuitionistic connectives – Troelstra (Troelstra 1973, page 188) has said 'it cannot be said to make the intended meaning of the logical operators more precise. As a "philosophical reduction" of the interpretation of the logical operators it is also only moderately successful; e.g. negative formulae are essentially interpreted by themselves'. In fact, Kleene admits this explicitly in his 1945 paper[†]. On the other hand, by providing an interpretation which can be read and checked by the classical mathematician, he did put forward an interpretation of the intuitionistic connectives in terms of the classical ones (this, in contrast to the so-called BHK or 'proof'-interpretation, which interprets the intuitionistic connectives in terms of themselves)[‡].

More importantly, realizability, as it is designed to handle 'information' about formulas rather than proofs, already hints at the role intuitionism would come to play in theoretical computer science some forty years later: it foreshadows the view of intuitionistic formulas as *datatypes*, and intuitionistic logic as the logic of *information*.

But the scope of realizability is wider than just 'interpreting the logic'. Realizability also provides models for theories that are classically *inconsistent*, models therefore whose internal logic is strictly non-classical (important examples are: Brouwer's theory of choice sequences; parts of (suitably formalised) recursive analysis; set-theoretic interpretations of the polymorphic λ-calculus; synthetic domain theory). It is in some of these models, that the statement 'realizability is equivalent to truth' can be given a precise meaning. And for the intuitionist, (an abstract form of) realizability *does* represent the intuitionistic connectives faithfully, as can be seen from van Oosten (1991b).

2.2. *Realizability and glued realizability*

In this and the next section, I will introduce realizability and some of its most important variations, which I call 'glued realizability'. (The term 'gluing' has its mathematical origin in algebraic geometry; here, I use it loosely to mean 'welding two interpretations together'. There is a precise connection between the two meanings of the word, provided by Topos Theory; see Section 3.1.)

My treatment will not be entirely faithful to history; as often in Mathematics, the chronological order is not always the most systematic way of presenting things. However, I shall do my best to sketch the history as I go along.

As I have already said, realizability was introduced in Kleene (1945). The definition specifies, in an inductive way, what it means that a natural number *n realizes* a sentence ϕ of the language of arithmetic. The inductive clauses are:

† 'The analysis that leads to this truth definition is not to be regarded as more than a partial analysis of the intuitionistic meaning of the statements … ' (Section 2).

‡ Again quoting Troelstra and van Dalen (1988, Volume I, page 9): ' … it is not hard to show that, on a very 'classical' interpretation of construction and mapping, [Heyting's clauses] justify the principles of two-valued (classical) logic'.

(1) n realizes F, where F is an atomic sentence, if and only if $n = 0$ and F is true;
(2) n realizes a conjunction $\phi \wedge \psi$, if and only if $n = \langle m, k \rangle$, and m realizes ϕ and k realizes ψ
(here, $\langle \cdot, \cdot \rangle$ denotes a primitive recursive bijection: $\mathbb{N}^2 \to \mathbb{N}$);
(3) n realizes a disjunction $\phi \vee \psi$ if and only if *either* $n = \langle 0, m \rangle$ and m realizes ϕ, *or* $n = \langle 1, m \rangle$ and m realizes ψ;
(4) n realizes an implication $\phi \to \psi$ if and only if n is the Gödel number of a partial recursive function F such that for each m that realizes ϕ, $F(m)$ is defined and realizes ψ;
(5) n realizes an existential statement $\exists x \phi(x)$ if and only if $n = \langle m, k \rangle$ and k realizes $\phi(\overline{m})$ (here and later, $\overline{m}$ is the *numeral*, that is, a canonical term that denotes m);
(6) n realizes a universal statement $\forall x \phi(x)$ if and only if n is the Gödel number of a total recursive function F such that for all numbers m, $F(m)$ realizes $\phi(\overline{m})$.

The acronym **HA** stands for Heyting Arithmetic, the formal system of intuitionistic first-order arithmetic.

Suppose now, that $\mathscr{P}$ is a set of sentences of the language of **HA**, such that $\mathscr{P}$ contains every theorem of **HA** and, moreover, if both ϕ and $\phi \to \psi$ are elements of $\mathscr{P}$, then so is ψ. Important examples of such $\mathscr{P}$ are: the set of all theorems of **HA** (the minimal $\mathscr{P}$), the set of all arithmetical sentences (the maximal $\mathscr{P}$), and the set of all sentences true in some model $\mathscr{M}$ of **HA**.

The definition of 'n realizes-$\mathscr{P}$ ϕ' is similar in structure to that of 'n realizes ϕ'; it has the same inductive clauses except for:

(4′) n realizes-$\mathscr{P}$ $(\phi \to \psi)$ if and only if $(\phi \to \psi)$ is an element of $\mathscr{P}$ and n is the Gödel number of a partial recursive function F such that for each m that realizes-$\mathscr{P}$ ϕ, $F(m)$ is defined and realizes-$\mathscr{P}$ ψ;
(6′) n realizes-$\mathscr{P}$ a universal statement $\forall x \phi(x)$ if and only if $\forall x \phi(x)$ is an element of $\mathscr{P}$ and n is the Gödel number of a total recursive function F such that for all numbers m, $F(m)$ realizes-$\mathscr{P}$ $\phi(\overline{m})$.

I call the notion 'realizes-$\mathscr{P}$' *glued realizability* with respect to $\mathscr{P}$. Note that ordinary realizability is glued realizability with respect to the maximal choice of $\mathscr{P}$ (a trivial gluing), so it suffices to formulate results for the 'realizes-$\mathscr{P}$' notion.

The basic theorem is:

If **HA** $\vdash \phi$, then there is a number n such that n realizes-$\mathscr{P}$ ϕ. Moreover, if there is a number n such that n realizes-$\mathscr{P}$ ϕ, then $\phi \in \mathscr{P}$.

An easy consequence of this theorem is, that there are formulas $\phi(x)$ with one free variable x, such that the sentence $\neg\forall x(\phi(x) \vee \neg\phi(x))$ is consistent with **HA**. It also follows, that the rule of 'double-negation shift':

$$\forall x \neg\neg\phi(x) \to \neg\neg\forall x \phi(x)$$

is not a derived rule of **HA** (Kleene 1945).

In Kleene (1945) only two forms of gluing are considered: the minimal gluing (which is called $\vdash$-realizability) and the maximal one (ordinary realizability). From the minimal

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gluing, Kleene obtained the weak Church's rule mentioned in Section 2.1: if $\forall x \exists y \phi(x, y)$ is a theorem of **HA**, then for some total recursive function G one has that for all n, $\phi(\overline{n}, \overline{G(n)})$ is a theorem of **HA**, and hence true. This is in Kleene (1945). Incidentally, this result might also have been obtained by gluing with the set of all (classically) true arithmetical sentences.

As a corollary of this proof, one obtains the existence property for **HA**: if $\mathbf{HA} \vdash \exists x \phi(x)$, then for some n, $\mathbf{HA} \vdash \phi(\overline{n})$. Similarly, one obtains the disjunction property: if $\mathbf{HA} \vdash \phi \vee \psi$, then $\mathbf{HA} \vdash \phi$ or $\mathbf{HA} \vdash \psi$. These conclusions are *not* explicitly in Kleene (1945), contrary to what Kleene later said[†].

2.3. *Formalised realizability and* **q**-*realizability*

The definition of realizability only involves first-order properties of indices of partial recursive functions.

The predicate $T(e, x, y)$ (y codes a computation with program e on input x) and the function $U(y)$ (the output of the computation y codes) are primitive recursive and hence representable in **HA**; I shall also use T and U for the representing formulas, treating them as a relation symbol (function symbol) of **HA**.

Therefore, as was immediately seen by Kleene, realizability can be formalised in **HA** itself. This is already in Kleene (1945); the details are in Nelson (1947).

I shall abbreviate $\exists z T(x, y, z)$ by $xy\downarrow$, and also use xy to denote $U(z)$ if $T(x, y, z)$. The following presentation of formalised (glued) realizability is based on Troelstra (1973).

Suppose that for each formula A a formula $P(A)$ is specified such that $P(A)$ has at most the same free variables as A, and, moreover:

($P1$) $\mathbf{HA} \vdash A \Rightarrow \mathbf{HA} \vdash P(A)$, for sentences A.

($P2$) $\mathbf{HA} \vdash (P(A) \wedge P(A \to B)) \to P(B)$ for all A, B.

($P3$) $\mathbf{HA} \vdash F \to P(F)$ for all atomic formulas F.

Then define for each formula ϕ a formula 'x realizes-P ϕ' that has one extra free variable x, as follows:

(1) x realizes-P F is $x = x \wedge F$ if F is an atomic formula.

(2) x realizes-P $\phi \wedge \psi$ is $((x)_0$ realizes-P $\phi) \wedge ((x)_1$ realizes-P $\psi)$,
where $(\cdot)_0$ and $(\cdot)_1$ are the projection functions corresponding to $\langle \cdot, \cdot \rangle$.

(3) x realizes-P $(\phi \to \psi)$ is $P(\phi \to \psi) \wedge \forall y(y$ realizes-P $\phi \to xy\downarrow \wedge\, xy$ realizes-P $\psi)$.

(4) x realizes-P $\exists y \phi(y)$ is $(x)_1$ realizes-P $\phi((x)_0)$.

(5) x realizes-P $\forall y \phi(y)$ is $P(\forall y \phi(y)) \wedge \forall y(xy\downarrow \wedge\, xy$ realizes-P $\phi(y))$.

[†] The first proof of the existence and disjunction properties for **HA** was given in Harrop (1956). In Kleene (1962), Kleene says Harrop 'rediscovered' these results, and in a footnote he details: '[the Existence property] appears explicitly in Kleene (1945, page 115, lines 8–7 from below) or Kleene (1952, page 509, lines 15–11 from below), taking $n = 0$. [the Disjunction property] is included in [this]' (reference numbers changed). These references are also given in Troelstra and van Dalen (1988, Volume I, pages 175–6). However, it is simply not there. Kleene was not above drawing obvious inferences, so one can safely assume that the existence property had not occurred to him at the time.

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As is well known, disjunction is definable in arithmetic: $\phi \vee \psi$ is provably equivalent to $\exists x((x = 0 \to \phi) \wedge (x \neq 0 \to \psi))$. Therefore, a realizability clause for disjunction is not needed.

One has the theorem:

$$\text{If } \mathbf{HA} \vdash \phi, \text{ then } \mathbf{HA} \vdash \exists x(x \text{ realizes-}P\,\phi);$$

moreover,

$$\mathbf{HA} \vdash \exists x(x \text{ realizes-}P\,\phi) \to P(\phi).$$

Important examples of P satisfying $(P1)$–$(P3)$ above, are: $P(A) \equiv 0 = 0$ (we have *ordinary formalised realizability*); $P(A) \equiv A$ (this formalised glued realizability is called **q**-*realizability*); $P(A) \equiv \text{Prov}(A)$ or more generally, P any interpretation of **HA** into itself. Note, that if P satisfies $(P1)$–$(P3)$, then $P'(A) \equiv \exists x(x \text{ realizes-}P\ A)$ does so also.

q-realizability gives Church's Rule for **HA**:

$$\text{If } \mathbf{HA} \vdash \forall x \exists y \phi(x, y), \text{ then } \mathbf{HA} \vdash \exists z \forall x(zx\downarrow \wedge\, \phi(x, zx)).$$

In particular, the total recursive function from weak Church's rule is actually *provably recursive*. But this version (and the even stronger 'Extended Church's Rule') appears first in Troelstra (1971) (also in Troelstra (1973)), although there is a **q**-version for 'analysis' in Kleene (1969). The reader will find that **q**-realizability looks different from the presentation above in these sources; the form presented here is equivalent, but has nicer proof-theoretic properties and was first given in Grayson (1981a).

2.4. *The logic of realizability*

Kleene's original conjecture that realizability might mirror intuitionistic reasoning faithfully, was disproved: Rose (Rose 1953) and later Ceitin, gave examples of propositional formulas that are realizable (even 'absolutely': there is a number n that realizes every substitution instance of the formula, where one substitutes **HA**-sentences for the propositional variables), but not provable in the intuitionistic calculus†. The 'predicate logic of realizability' is quite complicated, and was investigated by the Russian Plisko in a series of papers. Of course, there are several ways to define what it means for a formula in predicate logic to be 'realizable'. An interesting theorem (Plisko 1983) of his concerns what he calls 'absolutely realizable predicate formulas'. Consider a purely relational formula $\varphi = \varphi[P_1, \ldots, P_k]$ with all predicate symbols shown, P_i being n_i-ary. Let $F_i : \mathbb{N}^{n_i} \to \mathscr{P}(\mathbb{N})$ be a k-tuple of functions. We can now define the notion n realizes φ, relative to $(F_1, \ldots, F_k)$, by letting the variables run over $\mathbb{N}$, and putting

$$n \text{ realizes } P_i(m_1, \ldots, m_{n_i}) \text{ if and only if } n \in F_i(m_1, \ldots, m_{n_i}).$$

We say that a sentence φ of purely relational predicate logic is *absolutely realizable* if there is a number n such that for all k-tuples $(F_1, \ldots, F_k)$, n realizes φ relative to $(F_1, \ldots, F_k)$. The theorem is that the logic of absolutely realizable predicate formulas is Π^1_1-complete.

† Ceitin's example is $[\neg(p_1 \wedge p_2) \wedge (\neg p_1 \to q_1 \vee q_2) \wedge (\neg p_2 \to q_1 \vee q_2)] \to [(\neg p_1 \to q_1) \vee (\neg p_1 \to q_2) \vee (\neg p_2 \to q_1) \vee (\neg p_2 \to q_2)]$.

However, the logic of realizability can be viewed in a different light. Making use of formalised realizability, one can consider the collection of (say, propositional) formulas φ such that every arithmetical substitution instance (again, by substituting **HA**-sentences for the propositional variables) is provably realized in **HA**. This notion can be formalised in second-order intuitionistic arithmetic **HAS**†. Gavrilenko (1981) has the interesting theorem: suppose φ is a propositional formula with the property that **HAS** proves that every arithmetical substitution instance of it is realizable. Then φ is a theorem of intuitionistic propositional logic‡. Anticipating further developments, I mention here the following theorem of my own (van Oosten 1991b): let $\mathbf{HA}^+$ be an expansion of **HA** by new constants **k** and **s**, a partial binary function (or ternary relation which is single-valued) and axioms saying that this structure is a partial combinatory algebra (see Section 2.6 for a definition). One can define realizability with respect to this. Suppose that φ is a purely relational predicate formula all of whose arithmetical substitution instances are realizable in this abstract sense, provably in $\mathbf{HA}^+$. Then φ is provable in the intuitionistic predicate calculus.

2.5. *Axiomatisation of realizability*

As we have seen, the logic of realizability is too complicated to axiomatise. However, the situation is quite different for formalised realizability. The formulas x realizing A all have a syntactic property: they are (up to equivalence) *almost negative*, that is: built from Σ^0_1-formulas using only $\wedge$, $\rightarrow$ and $\forall$. Conversely, if A is an almost negative formula, there is a 'partial term' t_A (an expression of arithmetic expressing a (possibly non-terminating) computation; see Troelstra (1998) for details), containing the same free variables as A, such that the equivalence

$$A \leftrightarrow t_A\!\downarrow \wedge\, t_A \text{ realizes } A$$

is provable in **HA** ('$t_A\!\downarrow$' means that the computation represented by t_A terminates). This was observed by Kleene in Kleene (1960).

Exploiting the idempotency of the formalised realizability translation, one can then prove that formalised realizability is axiomatised by the scheme

$$\forall x(A(x) \rightarrow \exists y B(x,y)) \rightarrow \exists e \forall x(A(x) \rightarrow \exists y(T(e,x,y) \wedge B(x,U(y))))$$

where $A(x)$ must be an almost negative formula. This scheme is called the *extended Church's thesis* (ECT_0)§. The exact formulation of the axiomatisation is:

(i) $\mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi \leftrightarrow \exists x(x \text{ realizes } \varphi)$

† One needs second-order since it involves a truth definition for Gödel numbers of formulas.

‡ Regrettably, recently Albert Visser and the author discovered that Gavrilenko's proof contains a gap. Nevertheless, we remain convinced that his theorem is true, and that the proof can be patched.

§ This is a very debatable choice of name. It has nothing to do with Church's informal Thesis, which says that every intuitively computable function is recursive. In the metamathematics of intuitionistic arithmetic, 'Church's Thesis' stands for the formal statement that says that *all* functions from numbers to numbers are recursive. However, from the perspective of higher order arithmetic, the scheme ECT_0 not only strengthens this but also incorporates a *choice principle*.

(ii) $\mathbf{HA} \vdash \exists x(x \text{ realizes } \varphi) \Leftrightarrow \mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi$

The same axiomatisation holds true if **HA** is augmented by Markov's Principle MP: $\forall x(A(x) \vee \neg A(x)) \rightarrow (\neg\neg\exists x A(x) \rightarrow \exists x A(x))$. These axiomatisation results were given, independently, in Dragalin (1969) and Troelstra (1971); see also Troelstra (1973) for a thorough exposition.

Let us look at a minor application. Obviously, Markov's Principle is an example of a predicate logical scheme that is intuitionistically underivable. However, one can prove that the scheme

$$\forall x(A(x) \vee \neg A(x)) \wedge (\forall x A(x) \rightarrow \exists y B) \rightarrow \exists y(\forall x A(x) \rightarrow B)$$

is derivable in $\mathbf{HA} + \mathrm{MP} + \mathrm{ECT}_0$. So one sees that the introduction of realizability influences the predicate logic, at least if MP is assumed†.

Another application is, that the scheme IP of independence of premisses: $(\neg A \rightarrow \exists y B) \rightarrow \exists y(\neg A \rightarrow B)$ (y not free in B) is not derivable in **HA**, since it is easily shown to be inconsistent with ECT_0 (Troelstra 1973).

2.6. *Extensions and generalisations of realizability*

The first realizability definition based on a general notion of *combinatory algebra* appears in Staples (1973). Feferman, in Feferman (1975), sets out to code what he calls 'explicit mathematics' in a language for *partial* combinatory algebras (the system was later called **APP** by Troelstra and Van Dalen).

A partial combinatory algebra (or pca) A is a set A equipped with a partial binary operation $x, y \mapsto xy$ such that there are elements (*combinators*) **k** and **s** satisfying the postulates:

(**k**) $\mathbf{k}x$ and $(\mathbf{k}x)y$ are always defined, and $(\mathbf{k}x)y = x$;

(**s**) $\mathbf{s}x$ and $(\mathbf{s}x)y$ are always defined; and $((\mathbf{s}x)y)z$ is defined if and only if all of xz, yz and $(xz)(yz)$ are defined; in which case $((\mathbf{s}x)y)z = (xz)(yz)$.

The combinator axioms (**k**) and (**s**) mirror the two schemes that axiomatise intuitionistic purely implicational logic: $A \rightarrow (B \rightarrow A)$ and $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$.

However, as several people have observed (for example, Aczel (1980)), the (**s**)-axiom is slightly stronger than needed. It is enough to assume that if $(xz)(yz)$ is defined, then so is $((\mathbf{s}x)y)z$, and $((\mathbf{s}x)y)z = (xz)(yz)$ (this weakening also occurs in the $\leqslant$-pca's of van Oosten (1997a), and in recent work of John Longley).

The natural numbers with partial recursive application form a partial combinatory algebra. Another example is the set of functions $\mathbb{N} \rightarrow \mathbb{N}$. Every function α codes a partial continuous operation (with open domain): $\mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$‡. This partial combinatory algebra was at the basis of Kleene's *function realizability* (Kleene 1965; Kleene and Vesley 1965; Kleene 1969). This was an interpretation of 'intuitionistic analysis' (a theory that treats

† Whether the predicate logic of $\mathbf{HA} + \mathrm{ECT}_0$ properly extends intuitionistic predicate logic is still an open problem as far as I am aware.

‡ For details, see, for example, Troelstra (1973).

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numerical functions as well as natural numbers; the functions often being seen as reals). Function realizability vindicates Brouwer's opinion[†] that every well-defined function on the reals must be continuous. A **q**-variant of function realizability establishes for this system the following rule: if an existential statement $\exists\alpha A(\alpha)$ can be proved (α a variable for reals), then $A(r)$ can be established for some *recursive* real r.

At this point it is worth mentioning an older version of function realizability, which appeared in Kleene (1957). This version used relative computability with total functions as oracles. The notion is formulated as 'e realizes$^\Phi \varphi$' where Φ is a string of functions. Using a Gödel numbering for Turing machines with oracles, let φ_e^Φ be the partial function coded by e using oracles Φ. The clause for $\forall\alpha\psi$ reads: e realizes$^\Phi$ $\forall\alpha\psi$ iff for all functions α, $\varphi_e^{\alpha,\Phi}$ realizes$^{\alpha,\Phi}$ $\psi(\alpha)$. So, if $\forall\alpha\exists\beta\psi$ is realized (relative to oracles Φ), β is obtained recursively in α, Φ. One says a closed formula is realizable, if some number realizes it with respect to all oracles. Later, Kleene dismissed this version because the later notion is equivalent (Kleene and Vesley 1965).

There is a twist in Kleene's definition of function realizability: one has realizability clauses like for number realizability (using functions as realizers), but at the end one says that a formula is 'realizable', provided there is a *recursive* function realizing the formula. This is a notion that was later called 'relative realizability' in a generalised setting (see Section 3.5).

A different type of generalisation is Kreisel's *modified realizability*, which was originally conceived for the system $\mathbf{HA}^\omega$. $\mathbf{HA}^\omega$ is 'Gödel's T with predicate logic'. One builds a type structure from one basic type o and type constructors $\times$ and $\Rightarrow$; one has variables of each type, typed combinators for pairing and projections, **k** and **s** of each appropriate type, and combinators for primitive recursion. For any formula A, a formula 'x realizes A' can be defined in a completely straightforward way: the type of the variable x is determined by the logical form of A. So, if the type of realizers of A is σ, and the type of realizers of B is τ, the type of realizers of $A \to B$ is $(\sigma \Rightarrow \tau)$. This 'typed realizability', defined by Kreisel in 1959 (Kreisel 1959)[‡], predates the slogan 'formulae as types' (Howard 1980) by 10 years! Of course, it came to be used in the late seventies to interpret versions of Martin-Löf's type theory (for example, Diller (1980) and the thesis Swaen (1989)), and analogous versions for systems based on PCF have been studied by John Longley. Troelstra found an axiomatisation for modified realizability for $\mathbf{HA}^\omega$ (Troelstra 1973).

But, it is the untyped 'collapse' of this realizability that most people know as 'modified realizability'. The structure of hereditary recursive operations (Troelstra 1973) is a typed structure that models $\mathbf{HA}^\omega$ and is itself definable in **HA**. Using the fact that **HA** is a subsystem of $\mathbf{HA}^\omega$, one can construct from Kreisel's definition a new notion of realizability for **HA**. Each formula gets *two* sets of realizers, the *actual* realizers being a subset of the *potential* ones[§]. One gets two, intertwined, inductive definitions for both types of realizers: see Troelstra (1973) for the formal definition. Here I just give the most distinctive clause:

[†] He called it a 'theorem'.

[‡] In a footnote!

[§] This modified realizability is also reminiscent of Kolmogorov's interpretation of intuitionism by 'problems'; see, for example, Medvedev (1962).

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n is an actual realizer of an implication $\phi \to \psi$ if n is the Gödel number of a partial recursive function that sends every actual realizer of ϕ to an actual realizer of ψ, *and* every potential realizer of ϕ to a potential realizer of ψ.

Features of HRO-modified realizability for **HA** are that it validates the scheme IP (see the last paragraph of Section 2.5) and refutes Markov's Principle. By a **q**-version of this realizability one can obtain an IP-rule for **HA** (I believe this was first noticed in van Oosten (1991a)). Beeson (1975) applies modified realizability to show that although, in formalisations of elementary recursion theory, the Myhill–Shepherdson and Kreisel–Lacombe–Shoenfield theorems seem to require Markov's Principle, they do not, conversely, imply it, for these theorems hold under modified realizability.

The idea of actual and potential realizers can, of course, be applied to different partial combinatory algebras, and this was done by Kleene ('special realizability' in Kleene and Vesley (1965)) and Joan Moschovakis (Moschovakis 1971). Moschovakis shows the consistency of Kleene and Vesley's 'basic system' of intuitionistic analysis together with the scheme $(\neg A \to \exists\alpha B(\alpha)) \to \exists\alpha(\neg A \to B(\alpha))$ and the scheme $\exists\alpha A(\alpha) \to \exists\alpha(\mathrm{GR}(\alpha) \wedge A(\alpha))$ for *closed* $\exists\alpha A(\alpha)$†. She uses the partial combinatory algebra of functions together with its subalgebra of recursive functions; recent work of Awodey, Birkedal and Scott (Awodey *et al.* 1999); see also Section 3.5) is closely related to hers (the relationship is made precise in Birkedal and van Oosten (2000)). In general, as shown, for example, in van Oosten (1997b), modified realizability interpretations are also intimately connected with what I have called 'Kripke models of realizability' (van Oosten 1991a); see the next section.

Recently, modified realizability has enjoyed renewed interest, mainly by the efforts of Thomas Streicher, Martin Hyland and Luke Ong (Streicher 1994; Hyland and Ong 1993); see also van Oosten (1997b) and Birkedal and van Oosten (2000).

For an extension of formalised Kleene-realizability to second-order arithmetic **HAS**, see Troelstra (1973). Troelstra shows that the following principle of second-order arithmetic is valid under his extension:

$$\mathrm{UP} \qquad \forall X \exists n A(X,n) \to \exists n \forall X A(X,n).$$

The initials UP stand for uniformity principle. This principle received much attention in connection with the effective topos: see Sections 3.1 and 3.2. Saying that every function from sets of numbers to numbers must be constant, it is very non-classical; however, it can be shown that $\mathbf{HAS} + \mathrm{UP}$ has no non-classical first-order consequences (van Oosten 1991a).

2.7. *Kripke models of realizability*

This is really a prelude to a general topos-theoretic account of realizability. But topos theory was slow to catch up with realizability, and long after the logical significance of toposes had been grasped, it was not yet clear what toposes could do for realizability.

† The formula $\mathrm{GR}(\alpha)$ expresses the fact that α is recursive. That this is a non-trivial result is apparent from the fact that the 'basic system' contains the axiom scheme of so-called 'bar induction' (a principle of induction over definable well-founded trees), which fails badly for the recursive universe.

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A Kripke model of realizability is a Kripke model of the theory **APP**, that is, a system of partial combinatory algebras $(A_p)_{p\in P}$ indexed by some partially ordered set P, together with maps $A_p \to A_q$ for $p \leqslant q$, satisfying the usual conditions. As a simple example, take the partial order $\{0 < 1\}$, let A_1 be the pca of function realizability and A_0 be its sub-pca of recursive functions. One can also take A_1 the graph model $\mathscr{P}(\omega)$ and A_0 its subalgebra on the recursively enumerable subsets of $\mathbb{N}$. See Section 3.5 for more about this.

In general, if $(A_p)_{p\in P}$ is a Kripke model of realizability, to any formula φ a P-indexed system $([\![\varphi]\!]_p)_{p\in P}$ of sets of realizers is assigned (which is a subset of $(A_p)_{p\in P}$ in the sense of Kripke models).

The first example I know of such a Kripke model of realizability is the unpublished paper de Jongh (1969). De Jongh wished to establish the theorem that a formula A is provable in intuitionistic predicate calculus if and only if each of its arithmetical substitutions is provable in **HA**. He succeeded partially: the full theorem was first proved by Leivant in his thesis (and Leivant used proof theory). In van Oosten (1991b) I was able to revive De Jongh's original realizability method to prove the full theorem.

Another example occurs in Goodman (1978). The models of De Jongh and Goodman are strikingly similar: in both cases, A_p is the set of indices of functions that are partial recursive in some set $X_p \subseteq \mathbb{N}$, with $X_p \subseteq X_q$ for $p \leqslant q$. However, Goodman, whose aim was to interpret a version of $\mathbf{HA}^\omega$ with decidable equality at all types, also brings the $\neg\neg$-translation into the picture, so strictly speaking his model transcends the definition of a Kripke model of realizability, and might rather be called a (generalised) Beth model of realizability.

Much work on combinations of realizability with Kripke forcing was done by Jim Lipton (Lipton 1990; Lipton and O'Donnell 1996).

2.8. *Extensional realizability*

'Extensional realizability' not only defines realizers, but, simultaneously, an equivalence relation on them; the idea is that a realizer for an implication $A \to B$ should send equivalent realizers for A to equivalent realizers for B. The origin is, of course, again Kreisel's modified realizability; just as HRO is a model for $\mathbf{HA}^\omega$ that is definable in **HA**, we have the models HEO of 'hereditarily effective operations' and $\mathrm{HRO_E}$, the extensional collapse of HRO (Troelstra 1973). HEO in combination with modified realizability had already been considered in Troelstra (1973), but the first extensional realizability for $\mathbf{HA}^\omega$, in combination with Kripke forcing, was used by Beeson (Beeson 1979), who extended Goodman's theorem to the statement that $\mathbf{E}-\mathbf{HA}^\omega + \mathrm{AC}$ is conservative over **HA**.

The first appearance in print of a definition for extensional realizability that was suitable for first-order arithmetic was in Pitts' thesis (Pitts 1981)[†].

Extensional realizability was used by Beeson (Beeson 1982; Beeson 1985) in connection with Martin-Löf's Type Theory, and by Diller, Troelstra and Renardel (Diller and Troelstra 1984; Renardel de Lavalette 1990). Martin Hyland studied extensional realizability

[†] According to Pitts, the idea came from Robin Gandy.

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from a topos-theoretic point of view, and noted its salient higher-order logical properties in Hyland (1982) – see also the next chapter.

In van Oosten (1997a), two versions of extensional realizability for **HA**, analogous to HEO and HRO_E, are compared and found non-equivalent. It is shown that the HEO-version is not idempotent, but, nevertheless, an axiomatisation for this realizability is obtained over a conservative extension of **HA**. The usual Troelstra-type results are obtained: a **q**-version is defined, and an 'extensional Church's rule' for **HA** is derived.

3. The period 1980–2000

Around 1970, Lawvere and Tierney had generalised Grothendieck's notion of 'topos' to the definition of *elementary topos*; in subsequent work they (and also others, like Michael Barr and Peter Freyd) have shown that very many results in the theory of Grothendieck topoi can in fact be derived from the axioms for an elementary topos. An impressive account of elementary topos theory (I mean 'theory of elementary toposes'; the theory itself is at places far from 'elementary') of the seventies, which has served as a standard reference to this day, is Johnstone (1977).

Logicians discovered that toposes generalised semantical ideas that had developed in the sixties: Cohen forcing for ZF set theory (later, reformulated by Solovay[†] in terms of Boolean-valued models[‡]), Kripke and Beth models for intuitionistic predicate logic, and topological models. All these semantics fall, from the point of view of a topos theorist, under the heading of 'localic toposes', or, to use a more familiar term for logicians, Heyting-valued semantics.

Denis Higgs (Higgs 1973; Higgs 1984) had proved in 1973 that the category of '$\mathscr{H}$-valued sets' is equivalent to the topos of sheaves over $\mathscr{H}$, for a complete Heyting algebra $\mathscr{H}$. So Kripke semantics, topological semantics, and so on, have a *natural* extension to higher-order languages[§]. This is important for the development of intuitionistic elementary mathematics: the real numbers are constructed by Dedekind cuts, which needs second-order arithmetic (logicians had been describing models for analysis completely independent of second-order arithmetic).

It seems that no one in the traditional logicians' world of the seventies was more influential in pushing topos semantics than Dana Scott. Martin Hyland has testified[¶] that Scott's coming to Oxford in the mid-seventies meant a 'change in ways of doing logic'. Much of this can probably be attributed to a different cultural background: most of all, the model theorist Scott advocated the view of realizability (and other 'interpretations') as *models*, to be treated as syntax-free as possible.

Anyway, the reader who wishes to see a representative sample of work from the seventies on sheaf models, is referred to the 'Durham Proceedings' (Fourman *et al.* 1979). All this

[†] And, independently, by Scott and Vopěnka; see Scott's Foreword to Bell (1977).

[‡] It was Scott who first observed that Cohen's forcing over a poset was Kripke forcing combined with the ¬¬-translation.

[§] This point is emphasised in Scott's Foreword to Bell (1977), where the failure by logicians to spot this fact is attributed to 'the first-order disease'.

[¶] In his lecture at the realizability workshop in Trento.

work concerns *Grothendieck topoi*, however, and realizability was markedly absent. In fact, what did one know about non-Grothendieck topoi? Finite sets (not very entertaining); and yes; the Lawvere/Tierney axioms are sufficiently *algebraic* to ensure that a *free topos* exists; but what did one know about it? Finally, there were the toposes arising by the so-called *filter-quotient* construction, which had been used to give topos-theoretic proofs of Cohen's independence results.

3.1. *The effective topos*

A completely new type of topos was discovered around 1979 (apparently following some ideas of Scott; independently, there had been work of W. Powell along similar lines) by Martin Hyland, Peter Johnstone and Andy Pitts. The relevant publications are Hyland *et al.* (1980), Pitts (1981) and Hyland (1980).

It was well known, and amply demonstrated in Fourman and Scott's paper Fourman and Scott (1979) that Boolean-valued sets generalise to Heyting-valued sets for a complete Heyting algebra. The completeness of the algebra is used for interpretation of the quantifiers. Now, in Fourman and Scott (1979), Fourman and Scott had dissected the construction of the topos of $\mathscr{H}$-sets into two logically meaningful steps. First, one has a model of many-sorted intuitionistic predicate logic without equality. The predicates of sort X (where X is a set) are functions from X into the set of propositions $\mathscr{H}$. Since $\mathscr{H}$ itself exists as a sort, one has, in fact, second-order propositional logic too. The next step is adding equality as a general $\mathscr{H}$-valued symmetric and transitive (but not necessarily reflexive!) relation, and considering *all* possible such. One obtains a topos, and the validity of a formula φ in the internal logic of this topos is connected to the validity in the underlying model of many-sorted predicate logic of a translation of φ into the 'logic of identity and existence' (Scott 1979).

Hyland, Johnstone and Pitts discovered a useful generalisation of the first step in this construction, calling it 'tripos' for 'topos-representing indexed pre-ordered set'†. The 'Theory of triposes' is the subject matter of Andy Pitts' thesis (Pitts 1981), but a major application of the idea is the 'effective topos', discovered by Martin Hyland and described in the classic paper Hyland (1980). Let the 'domain of propositions' be the powerset of $\mathbb{N}$. For any set X, the set of predicates on X, that is, the set $\mathscr{P}(\mathbb{N})^X$ is preordered by: $\varphi \leqslant \psi$ if and only if there is a partial recursive function F such that for each $x \in X$ and each $n \in \varphi(x)$, $F(n)$ is defined and $F(n) \in \psi(x)$. Then $\mathscr{P}(\mathbb{N})^X$ is a Heyting (pre)algebra, and although it is not complete, adjoints to the map $\mathscr{P}(\mathbb{N})^f : \mathscr{P}(\mathbb{N})^Y \to \mathscr{P}(\mathbb{N})^X$ (for functions $f : X \to Y$) exist. One can mimic the the construction of the topos of $\mathscr{H}$-valued sets completely, and one gets the *Effective topos* $\mathscr{E}\!f\!f$.

In $\mathscr{E}\!f\!f$, the standard truth definition for first-order arithmetic (based on the natural numbers object) is equivalent to Kleene's 1945-realizability. But much more is true: standard second-order arithmetic in $\mathscr{E}\!f\!f$ is captured by an informal reading of Troelstra's realizability for **HAS** (as shown in van Oosten (1991a)), and standard analysis in $\mathscr{E}\!f\!f$

† 'Tripos' is also the name of the major Mathematics exam at the University of Cambridge – a typical Cambridge pun, in more than one way.

(using the Dedekind reals) turns out to be equivalent to Markov-style recursive analysis. The finite type structure over the natural numbers is the structure HEO. All these different, hitherto unrelated bits of research fell into their right place.

Even more strikingly, the *proof-theoretic* results obtained by realizability also received a wider significance in the effective topos. The role of the almost negative formulas is explained by the fact that the category of Sets is contained in $\mathcal{E}ff$ as '$\neg\neg$-sheaves' (see the section 'Basic facts from the logic of sheaves' in Hyland (1980)).

In a little series of never-published, hand-written notes, Robin Grayson (Grayson 1981a; Grayson 1981c; Grayson 1981b) gave accounts of results that he and Hyland had obtained independently. He described the construction of toposes for modified and extensional realizability. He explained the topos-theoretic counterpart of **q**-realizability. By *gluing* the toposes Sets and $\mathcal{E}ff$ along the embedding (see Wraith (1974) for this construction) one gets a topos corresponding to a sort of **q**-realizability. Replacing Sets by the free topos (with natural numbers object) $\mathcal{F}$ and constructing $\mathcal{E}ff$ over $\mathcal{F}$, one obtains versions of existence properties for higher-order intuitionistic arithmetic **HAH** and Church's Rule for **HAH**[†]. Let us sketch the argument for Church's Rule. So $\mathcal{F}$ is the free topos, $\mathcal{E}ff(\mathcal{F})$ the effective topos constructed over it, and $\mathcal{E}$ the gluing of $\mathcal{F}$ to $\mathcal{E}ff(\mathcal{F})$. The satisfaction relation $\mathcal{E} \models \varphi$ can be expressed in $\mathcal{F}$. Now suppose **HAH** $\vdash \forall x : N \exists y : N \psi(x, y)$, so $\mathcal{E} \models \forall x \exists y \psi$. By the realizability construction, we have

$$\mathcal{F} \models \exists f : N \forall x : N \exists y : N(T(f, x, y) \wedge \mathcal{E} \models \psi(x, U(y))).$$

Now there is a logical functor $\mathcal{E} \to \mathcal{F}$ (a general feature of the gluing construction), whence

$$\mathcal{F} \models \exists f : N \forall x : N \exists y : N(T(f, x, y) \wedge \psi(x, U(y)))$$

so **HAH** proves this formula, and we are done[‡].

In his beautiful recent paper Hyland (2002), Martin Hyland sketches various ideas for applications of the topos-theoretic point of view to different interpretations, in particular, Martin-Löf's type theory, and the Dialectica interpretation.

3.2. *Modest sets and internal completeness*

In his paper Hyland (1980), Hyland had singled out an interesting subcategory of $\mathcal{E}ff$: the subcategory on what he called 'effective objects'. This category generalises Eršov's 'Numerierungen' (Eršov 1973): it is equivalent to the category whose objects are pairs (X, μ) with X a set and $\mu : A \to X$ a surjective function from a subset of $\mathbb{N}$ to X; morphisms $(X, \mu) \to (Y, \nu)$ are functions $f : X \to Y$ such that for some partial recursive

[†] The existence property for **HAH** was first proved by Lambek and Phillip Scott in 1978, using Friedman-style **q**-realizability. That this was essentially a gluing construction, was realised by Peter Freyd, who appears to have been surprised by the fact that in $\mathcal{F}$ the terminal object is indecomposable and projective, but nevertheless gave an algebraic proof of it. For good or ill, Freyd's proof was again syntacticised by Lambek and Phillip Scott in Lambek and Scott (1986).

[‡] By the way, existence properties for **HAS** had first been obtained by Friedman (Friedman 1977) using **q**-realizability. Note, that Friedman's 'set existence property for **HAS**' is *not* automatically subsumed by the existence property for full **HAH**.

function F, $F(n)$ is defined for all $n \in \mathrm{dom}(\mu)$ and $F(n) \in \mathrm{dom}(\nu)$ and $f(\mu(n)) = \nu(F(n))$. Abstractly, the effective objects are (in $\mathscr{E}ff$) $\neg\neg$-separated quotients of subobjects of N. The concrete representation just given was later called the *category of modest sets* by Dana Scott (Scott 1986).

Hyland noticed that the effective objects allow an interesting generalisation of Troelstra's uniformity principle (see Section 2.6). Recall that Sets is included in $\mathscr{E}ff$ as $\neg\neg$-sheaves. Now any function from a quotient of a set to an effective object is necessarily constant in $\mathscr{E}ff$; in fact, for an effective object A and a quotient B of a set, the diagonal embedding $A \to A^B$ is an isomorphism.

Around 1985, Moggi and Hyland made an important discovery. This 'uniformity principle' meant that a specific internal category in $\mathscr{E}ff$ (basically, the internal full subcategory of separated subquotients of N) was *complete* in a sense, without being a preorder[†].

This meant several things. For example, Scott used it in Scott (1986) to show that intuitionistically it may happen that a set A is in bijective correspondence with 2^{2^A}[‡]. It could also be used to obtain a set-theoretic interpretation of Girard's second-order λ-calculus F[§].

The precise meaning of 'complete' (this is not expressible in the internal language of the topos) took a while to sort out. A basic observation came from Freyd: take the property that $A \to A^B$ is an isomorphism for each set B (in fact, just the set 2 will suffice; but note the *set* 2, not the *object* 2 in $\mathscr{E}ff$!) as a *defining* property A can have; call A 'discrete' if it has this property. Eventually, Hyland, Robinson and Rosolini showed that the discrete objects, as a fibration over $\mathscr{E}ff$, are complete, and weakly equivalent to the fibration obtained by 'externalising' the aforementioned internal category in $\mathscr{E}ff$; from this, it follows that the internal category is 'weakly complete'[¶]. This is explained in Hyland *et al.* (1990) and Hyland (1988).

Of course this does *not* mean that the category of modest sets is complete, as Robinson (Robinson 1989) and Rosolini (Rosolini 1990) hastened to point out. But it may serve very well for interpretations of theories in, say, system F and related programming languages such as Quest. Such 'PER' models were constructed by Abadi, Cardelli, Longo, Freyd, Hyland, Robinson, Rosolini and many, many others; by now, PER models form a standard tool in the semantics of programming languages.

For historical reasons, quotients of sets are called 'uniform objects'. The notions 'uniform' and 'discrete' can be applied to *maps* as well, and give rise to a factorisation system on $\mathscr{E}ff$ very much in analogy with the 'monotone-light' factorisation system on the category of T_0-topological spaces (Carboni *et al.* 1997).

Important applications of the completeness of 'pers' come from synthetic domain theory (see Section 3.6).

[†] Contradicting a classical theorem of Peter Freyd.
[‡] Contradicting Cantor's theorem.
[§] Contradicting a well-known result of Reynolds.
[¶] Basically, the problem resides in the absence of choice in $\mathscr{E}ff$. Call the internal category C. For another arbitrary one, say D, we have the object C^D of diagrams in C of type D, and an object E of pairs (d, c) where d is a diagram, and c a limit for this diagram. The projection: $E \to C^D$ is an epimorphism in $\mathscr{E}ff$, but there need not be a section of it, which would assign a limit to each diagram.

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3.3. *Realizability as a universal construction*

The effective topos has intriguing, not to say mystifying, aspects. One way of attacking its mystery is to look for universal properties it may enjoy. Around 1990, two papers appeared with rather similar-looking constructions of $\mathscr{E}\!f\!f$: Carboni *et al.* (1988) and Robinson and Rosolini (1990). The key word here is *completion*.

We have seen that the effective topos is a two-step construction. But there are many ways in which to cover a distance by two steps ...

Let us consider two completion processes: given a finite-limit category C, one can add coproducts to it; or one can add stable quotients of equivalence relations to it, making it *exact*. The first construction belongs to folklore and results in $\mathrm{Fam}(C)$: objects are families $(C_i)_{i\in I}$ of objects of C indexed by a set I; a morphism $(C_i)_{i\in I} \to (D_j)_{j\in J}$ consists of a function $f : I \to J$ and an I-indexed collection of arrows $(f_i : C_i \to D_{f(i)})_{i\in I}$ of C. The second construction is detailed in Carboni and Magno (1982) and results in the category $(C)_{\mathrm{ex/lex}}$.

Performing the two in succession gives $(\mathrm{Fam}(C))_{\mathrm{ex/lex}}$, which is a topos, the topos $\mathrm{Sets}^{C^{\mathrm{op}}}$†.

Now suppose one does not add *all* coproducts, just the *recursive* ones. That is, one take $\mathrm{Fam}_R(C)$: objects are now families indexed by a subset I of $\mathbb{N}$, and morphisms $(C_i)_{i\in I} \to (D_j)_{j\in J}$ need a *partial recursive* function $I \to J$. The main result of Robinson and Rosolini (1990) is that $(\mathrm{Fam}_R(\mathrm{Sets}))_{\mathrm{ex/lex}}$ is a topos, *the effective topos*. Note the mirroring in the two cases: for a Grothendieck topos, at least for presheaf toposes, one completes a *small* category with all coproducts indexed by Sets; for $\mathscr{E}\!f\!f$, one completes Sets by coproducts indexed by a small category R!

It follows from the general theory of ex/lex completions that the category $\mathrm{Fam}_R(\mathrm{Sets})$ (into which Sets embeds) is equivalent to the full subcategory of *projective objects* of $\mathscr{E}\!f\!f$; and, moreover, that every object of $\mathscr{E}\!f\!f$ is a quotient of a projective object.

On the other hand, the construction given in Carboni *et al.* (1988) presents $\mathscr{E}\!f\!f$ as $(\mathrm{Asm})_{\mathrm{ex/reg}}$; that is, make Asm exact but preserve the regular structure, where Asm is the category of *assemblies*, the $\neg\neg$-separated objects of the effective topos‡. In a completely analogous way, the topos of sheaves over $\mathscr{H}$ ($\mathscr{H}$ a complete Heyting algebra) is $(\mathrm{Fam}(\mathscr{H}))_{\mathrm{ex/reg}}$.

It is amusing to note that $(\mathrm{Asm})_{\mathrm{ex/lex}}$ also yields a topos; now not the effective topos, but a topos for *extensional realizability* (van Oosten 1997a)§.

An interesting result in this area is due to John Longley (Longley 1995). We can construct $\mathscr{E}\!f\!f$ over any partial combinatory algebra A; call it $\mathscr{E}\!f\!f_A$. How 'functorial' is $\mathscr{E}\!f\!f_A$ in A? Longley defines a 2-category **Pca** of partial combinatory algebras, such that the category $\mathbf{Pca}(A, B)$ is equivalent to the category of exact functors $\mathscr{E}\!f\!f_A \to \mathscr{E}\!f\!f_B$ that commute with the inclusions from Sets into these toposes. At first sight, his definition

† For a recent explanation of when (if) $(C)_{\mathrm{ex/lex}}$ is a topos, see Menni (1999).

‡ The constructions ex/lex and ex/reg are well explained in Carboni and Vitale (1998) and Carboni (1995).

§ Menni (1999) has an independent, abstract argument that $\mathrm{Asm}_{\mathrm{ex/lex}}$ is a topos. He obtains a whole hierarchy of toposes, starting with $\mathscr{E}\!f\!f$ and $\mathrm{Asm}_{\mathrm{ex/lex}}$. See also Menni (2000).

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looks like a hack, but, a 1-cell from A to B in **Pca** is nothing but an *internal* partial combinatory algebra in Asm(B) (assemblies over B; that is, a $\neg\neg$-separated internal pca in $\mathcal{E}ff_B$ for which the domain of the application map is $\neg\neg$-closed) with global sections A; a 2-cell between such is an internal 'ordinary' pca-morphism. Viewed in this way, and combined with Pitts' *iteration results* (Pitts 1981), the construction becomes a lot more transparent, and its connection to the exact completions business should be obvious.

Recently, a lot of work has been devoted to the question of when an exact completion is (locally) cartesian closed (Rosický 1997; Carboni and Rosolini 2000; Birkedal *et al.* 1998). Much of this work was prompted by the appearance of Scott's 'new category' (Scott 1996)†. This category is 'almost' an exact completion of the category of T_0-topological spaces; in fact, it is the 'regular completion' of T_0-spaces (Rosolini 2000).

The relationships between these various completions, and when they have nice properties (being locally cartesian closed or toposes) have been systematically studied by Matías Menni in his thesis (Menni 2000); obtaining a synthesis of all previous work in this area.

3.4. *Axiomatisation revisited*

In his seminal paper, Hyland (1980), Hyland concluded with the comment:

> What we lack, above all [...] is any real information analogous to the results obtained in Troelstra (1973) axiomatising realizability [...] we have no good information in this area. We cannot properly be said to understand realizability until we do.

Was it not about time, after 1990 and all these further results on $\mathcal{E}ff$ has appeared, to use them in order to obtain more 'information in this area'?

In van Oosten (1994), I gave the construction of a series of theories of higher order arithmetic (2nd, 3rd, ... order) that are true in $\mathcal{E}ff$, and realizabilities for these theories that are also true in $\mathcal{E}ff$, and can be axiomatised over the theories. This is based on the fact that in $\mathcal{E}ff$ realizability can be defined in such a way that in $\mathcal{E}ff$ a sentence is equivalent to its own realizability. The details are worked out for 2nd and 3rd order arithmetic; the axioms characterising the 2nd order realizability are the uniformity principle, the extended Church's thesis and Shanin's principle, which says that for any subset X of N there is a $\neg\neg$-closed subset A of N such that

$$X = \{x \mid \exists y \langle x, y\rangle \in A\}.$$

(For Shanin's principle, consult Gordeev (1980).)

The construction of these theories is motivated by the fact that the relevant arithmetical objects are covered by *definable* projective objects; for example, Ω^N is covered by $(\Omega_{\neg\neg})^N$; that this is a cover is the content of Shanin's Principle.

A corollary of the treatment for 3rd order arithmetic is that from the axioms that characterise its realizability, one can prove a completeness property of the category of modest sets.

† I prefer 'new category' as a name to this category's official name, **Equ**, pronounced 'eek'. 'New category' is reminiscent of 'new foundations', 'new age' and 'new economy', making it a cool object of study.

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Yet, we are a long way from understanding realizability axiomatically. We may ask the following question. For an arbitrary topos $\mathscr{E}$ with natural numbers object, let $\mathscr{E}ff(\mathscr{E})$ be the effective topos constructed over it. The construction $\mathscr{E} \mapsto \mathscr{E}ff(\mathscr{E})$ is *not* idempotent up to equivalence, although Pitts (1981) shows it gives rise to a monad ('the effective monad') on a certain category of toposes and geometric morphisms. Is there any way of characterising the algebras for this monad? Is there any reasonable system of meaningful conditions on $\mathscr{E}$ ensuring that $\mathscr{E} \to \mathscr{E}ff(\mathscr{E})$ is an equivalence?

What does $\mathscr{E}ff(\mathscr{F})$ look like[†]? Is it an exact completion[‡]?

3.5. *Relative realizability*

From around 1997, a group of people around Dana Scott at CMU in Pittsburgh has been working on realizability: Steve Awodey, Andrej Bauer and Lars Birkedal. In recent papers, Awodey *et al.* (1999), Awodey and Birkedal (1999) and Birkedal (2000), they study what they call 'relative realizability'.

Suppose a pca A has a subset $A_\sharp$ that is closed under the application and contains a choice for **k** and **s** for A; in other words, a sub-pca. One can define a tripos on Sets in the following way: predicates on X are functions $X \to \mathscr{P}(A)$, but the order between two such functions has to be realized by an element of $A_\sharp$. Call the resulting topos $\mathscr{E}ff_{A_\sharp,A}$.

Usually, $A_\sharp$ consists of 'recursive' or 'recursively enumerable' elements of A; see the examples cited in Section 2.7. Part of the motivation for studying this situation is the 'study of computable operations and maps on data that is not necessarily computable, such as the space of all real numbers'.

$\mathscr{E}ff_{A_\sharp,A}$ compares nicely to the toposes $\mathscr{E}ff_{A_\sharp}$ and $\mathscr{E}ff_A$: there is a geometric morphism $\mathscr{E}ff_{A_\sharp} \to \mathscr{E}ff_{A_\sharp,A}$ that is *local*, and there is a *logical functor* $\mathscr{E}ff_{A_\sharp,A} \to \mathscr{E}ff_A$.

The reader can see that the *notion* of relative realizability is very old: in fact, Kleene's function realizability from Kleene and Vesley (1965) (see Section 2.6) is of this form. However, the analysis is quite nice. The relative situation can also be studied in connection with modified realizability, leading to a more complete understanding of Moschovakis' work. These relationships are made precise in Birkedal and van Oosten (2000). We see, that the 'logical functor' $\mathscr{E}ff_{A_\sharp,A} \to \mathscr{E}ff_A$ is a filter quotient situation, and we arrive at a very general definition of 'modified realizability' with respect to an internal pca in a topos $\mathscr{E}$, and an open subtopos of $\mathscr{E}$.

Also, the work of Thomas Streicher (Streicher 1997) deserves mention. He exploits relative realizability in order to obtain a topos for computable analysis.

Finally, note that the motivation of letting computable things act on non-computable data is reminiscent of Kleene's setup for higher-type recursive functionals (Kleene (1959) and later papers).

[†] Recall that $\mathscr{F}$ denotes the free topos with natural numbers object.

[‡] $\mathscr{F}$ is embedded as a full reflective subcategory in $\mathscr{E}ff(\mathscr{F})$, and the inclusion preserves epimorphisms; hence the reflection preserves projectives. Therefore, if $\mathscr{E}ff(\mathscr{F})$ is an exact completion, then $\mathscr{F}$ has enough projectives; I do not know whether this is true.

3.6. *Non-classical theories*

A useful feature of $\mathscr{E}\!f\!f$ and related topoi is that one often finds in them models for inherently non-classical theories, theories that have no classical models (sometimes not even models in Grothendieck topoi).

Here I will just point to a few interesting topics that deserve further research.

Synthetic domain theory: This aims for a suitable category of objects that carry a *natural* domain structure such that *any* map is automatically continuous between these objects. It was suggested by Dana Scott. Rosolini, at the time Scott's student, was the first who made real progress in setting up the theory (Rosolini 1986); later work was presented in, among other papers, Hyland (1992), Phoa (1990), Taylor (1991) and Reus and Streicher (1999). In van Oosten and Simpson (2000), the force of a truly axiomatic and rigorously internal approach is advocated.

Algebraic set theory: In their elegant little book (Joyal and Moerdijk 1995), Joyal and Moerdijk present a novel way of looking at set theory. They point to a model in $\mathscr{E}\!f\!f$, which needs to be further investigated.

Intuitionistic non-standard arithmetic. There are also interesting models for this in $\mathscr{E}\!f\!f$, as pointed out in Moerdijk (1995). This must definitely also be studied more closely.

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