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SUBSTRUCTURAL INQUISITIVE LOGICS

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Abstract. This paper shows that any propositional logic that extends a basic substructural logic *BSL* (a weak, nondistributive, nonassociative, and noncommutative version of Full Lambek logic with a paraconsistent negation) can be enriched with questions in the style of inquisitive semantics and logic. We introduce a relational semantic framework for substructural logics that enables us to define the notion of an inquisitive extension of λ, denoted as $\lambda^?$, for any logic λ that is at least as strong as *BSL*. A general theory of these "inquisitive extensions" is worked out. In particular, it is shown how to axiomatize $\lambda^?$, given the axiomatization of λ. Furthermore, the general theory is applied to some prominent logical systems in the class: classical logic *Cl*, intuitionistic logic *Int*, and t-norm based fuzzy logics, including for example Łukasiewicz fuzzy logic *Ł*. For the inquisitive extensions of these logics, axiomatization is provided and a suitable semantics found.

§1. Introduction. Traditionally, the aim of logic has been to provide criteria that would help us to distinguish between valid and invalid arguments. Arguments are usually understood as linguistic structures that consist solely of declarative sentences (premises and a conclusion). However, it has been recognized by many logicians that also logical relations among nondeclarative sentences, such as imperatives and questions, and their contribution to deductive reasoning can be modelled using the tools of modern formal logic (see, e.g., Rescher, 1966; Vranas, 2010; Groenendijk & Stokhof, 1997; Harrah, 2002; Wiśniewski, 1995, 2013; Peliš, 2016).

One of the promising approaches to the logical analysis of questions has been developed within the framework known as inquisitive semantics (see Ciardelli & Roelofsen, 2011; Ciardelli, Groenendijk, & Roelofsen, 2013; Ciardelli, 2016b, 2018). The logic of questions determined by standard inquisitive semantics is based on classical logic, which plays the role of the background logic of declarative sentences. It was shown in (Punčochář, 2016, 2017) that it is possible to formulate a generalized inquisitive semantics, in which classical logic, as the background logic of declarative sentences, is replaced by intuitionistic logic. This paper goes much further in this direction. It provides a suitable general semantic framework for logics extending a basic propositional substructural logic that we will denote as *BSL*: a weak, nondistributive, nonassociative, and noncommutative version of Full Lambek logic with a paraconsistent negation. The semantic framework is an extended version of the groupoid semantics introduced in (Došen, 1989). It is shown that such a framework enables us to enrich every logic that is at least as strong as *BSL* with questions in the style of inquisitive semantics. A logic λ enriched with questions in this way is called the inquisitive

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extension of λ. We will study these inquisitive extensions on a general level and apply the general theory to some important special cases.

The paper is structured as follows. §2 provides an introduction to the standard propositional inquisitive semantics and shows its essential mathematical features that are the subject of generalization in later sections. Some examples from natural language are provided that motivate the attempts to generalize the standard inquisitive semantics.

§3 introduces a semantic framework based on structures that are called information models. The framework can be seen as a modified version of the semantics from (Došen, 1989). A Hilbert system for *BSL* is formulated and completeness w.r.t. information models demonstrated.

§4 forms the core of the paper. It extends the standard substructural language with inquisitive disjunction and introduces the notion of the inquisitive extension for any logic that is at least as strong as *BSL*. A general theory of these inquisitive extensions is presented. A Hilbert system *InqBSL* is constructed that determines the inquisitive extension of *BSL*. Moreover, it is shown that if $BSL \oplus A_\lambda$ is an axiomatization of a propositional logic λ, then $InqBSL \oplus A_\lambda$ is an axiomatization of the inquisitive extension of λ. What is presented in literature as the standard framework of propositional inquisitive semantics (usually denoted as *InqB*) corresponds in our framework to one particular information model. It is shown that the standard inquisitive logic, i.e., the logic determined by *InqB*, is the inquisitive extension of classical logic. The main results of §4 generalize those of (Punčochář, 2016, 2017).

§5 is concerned with an application of the general theory of inquisitive extensions to several special cases: classical logic *Cl*, intuitionistic logic *Int*, and t-norm based fuzzy logics.

§2. Basic inquisitive semantics and logic. This section provides an introduction to propositional inquisitive semantics. We explain the motivation behind this approach and the mathematical core, on which the semantic framework is based.

Inquisitive semantics can be seen as an extension or generalization of the standard view according to which the meaning of a sentence is identified with its truth conditions which are in turn represented as the set of possible worlds in which the sentence is true. This picture is obviously applicable only to declarative sentences and not to questions, for questions do not have truth conditions and truth values.

Inquisitive semantics works with a more general notion of sentential meaning that is applicable to declarative sentences as well as to questions. Sentential meaning is modeled as consisting of an informative part and an inquisitive part. The informative content of a sentence can be represented in the standard way as a set of possible worlds and the sentence provides the information that the actual world is located somewhere in the set. The inquisitive content can be understood as a request to locate the actual world more precisely. The request can be identified with the set of those nonempty subsets of the informative content that contain enough information to settle the request. A proposition, then, is a pair consisting of a set of possible worlds (informative content) and a set of its subsets (inquisitive content). Sets of possible worlds are also called information states, so a proposition is an information state and a set of its substates.

Furthermore, it is assumed that these two aspects of a given proposition are related in a particular way. The function of the informative content is to exclude some possibilities and the inquisitive content should not exclude any further possibilities. This corresponds to an additional constraint: the informative content is required to be equal to the union of the inquisitive content. Since it is always possible to retrieve the informative content from inquisitive content as its union, one can represent the sentential meaning simply by the inquisitive content, i.e., as a set of sets of possible worlds.

Propositions are also required to be downward closed, that is, closed under subsets. This is motivated in the following way: if a set of worlds is regarded as a sufficient potential localization of the actual world, then every subset of this set must also be regarded as its sufficient potential localization. The empty set represents an inconsistent body of information and is contained in every proposition.

If one defines propositions in this way, as nonempty downward closed sets of information states, one is able to distinguish declarative and inquisitive propositions. A proposition is declarative if it contains its own informative content, that is, if it contains its own union. Otherwise, it is inquisitive.

In other words, the inquisitive content of a sentence can be understood as the issue that is raised by uttering the sentence. A sentence expresses a declarative proposition if the information it provides resolves the issue it raises. Let us now reconstruct more formally the basic framework of propositional inquisitive semantics. Suppose that a set of atomic formulas is given.

DEFINITION 2.1. *A possible world is any function that assigns a truth value* (*either T or F*) *to every atomic formula. An information state is any set of possible worlds. A proposition is any nonempty set of information states that is closed under subsets. A declarative proposition is any proposition that contains its own union. An inquisitive proposition is any proposition that is not declarative.*

Note that declarative propositions are, algebraically speaking, principal ideals in the algebra of information states. As the next step, we introduce a formal language and assign a proposition (in the sense of Definition 2.1) to every sentence of this language. We will start with a language of declarative sentences, which will be later expanded with questions.

Consider a basic propositional formal language L of declarative sentences defined in the following way:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \to \alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha.$$

The formulas from L will be called L-formulas (an analogous terminology will be used for the languages defined below). We assign to every L-formula α a proposition $||\alpha||$. For any given information state a we have to determine whether $a \in ||\alpha||$. Informally, an information state a is contained in the proposition that is expressed by α if a represents a body of information that implies α. It is natural to make this more precise by the following condition:

(P) $a \in ||\alpha||$ iff α is true in every world of a.

The meaning of the phrase "true in a world" is specified by the standard semantics of classical propositional logic. One can easily observe that every L-formula expresses a declarative proposition in the sense of Definition 2.1. Note that (P) implies $\emptyset \in ||\alpha||$, for every L-formula α.

To be able to take into account also questions it is useful to proceed in a different way and assign propositions to L-formulas directly (i.e., without the reference to the standard semantics of classical logic) with the help of the so-called support relation ($\vDash$). This relation between information states and L-formulas can be determined recursively via the following Kripke-style semantic clauses:[1]

- $a \vDash p$ iff p is true in every world of a.
- $a \vDash \neg\alpha$ iff for any b, if $a \cap b \neq \emptyset$, then $b \nvDash \alpha$.
- $a \vDash \alpha \to \beta$ iff for any b, if $b \vDash \alpha$, then $a \cap b \vDash \beta$.
- $a \vDash \alpha \wedge \beta$ iff $a \vDash \alpha$ and $a \vDash \beta$.
- $a \vDash \alpha \vee \beta$ iff there are b, c such that $b \vDash \alpha$, $c \vDash \beta$ and $a \subseteq b \cup c$.

"$a \vDash \alpha$" is read as "a supports α". Then $||\alpha||$ can be defined as the set of those states that support α and this definition is equivalent to the former one, which is based on the condition (P). This is a consequence of the following fact that can be easily proved by induction:

(S) $a \vDash \alpha$ iff α is true in every world of a.

The advantage of defining the notion of a proposition in terms of the support relation, instead of the relation of truth, is that in this way it can be simply and naturally extended so that it encompasses not only declarative sentences but also interrogatives. This can be done with the help of a new binary connective $\vee\!\!\!\vee$ that is called inquisitive disjunction. Let us now introduce a new language $L^?$ (inquisitive extension of the language L) that is obtained from L by adding this new symbol:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \vee\!\!\!\vee \varphi.$$

The semantic clauses for the language $L^?$ are the same as the clauses for L, and contain, in addition, the clause for $\vee\!\!\!\vee$:

$$a \vDash \varphi \vee\!\!\!\vee \psi \text{ iff } a \vDash \varphi \text{ or } a \vDash \psi.$$

Propositions expressed by $L^?$-formulas are defined in the same way as before: $||\varphi||$ is the set of states that support φ. We have already observed that L-formulas express declarative propositions. Such formulas will be called *statements*. We will use for them the variables $\alpha, \beta, \gamma, \ldots$. Inquisitive disjunction generates formulas that express inquisitive propositions. Such formulas will be called *questions*. For instance, the formula $p \vee\!\!\!\vee q$ represents the question *whether p or q*. A polar question $?p$, read as *whether p*, is equivalent to *whether p or* $\neg p$. So, one can define:

$$?\varphi =_{def} \varphi \vee\!\!\!\vee \neg\varphi.$$

The support relation can be viewed as a unifying notion suitable for both statements and questions. If φ represents a declarative sentence, the meaning of $a \vDash \varphi$ is that a contains enough information to establish that φ, or briefly that a implies φ. If φ represents an interrogative sentence then the meaning of $a \vDash \varphi$ is that a contains enough information to resolve φ. If α, β are statements, the semantic clause for inquisitive disjunction amounts to the following: an information state resolves the question *whether* α *or* β iff it implies α or β (which differs from the claim that the state implies $\alpha \vee \beta$). Note that in this interpretation α and β represent direct answers to the question $\alpha \vee\!\!\!\vee \beta$.

[1] The semantic clauses in the basic framework of inquisitive semantics are usually formulated in an equivalent but different way (see, e.g., Ciardelli, 2016b). We use this nonstandard formulation of the conditions since it is suitable for the generalization that will be presented in the following sections.

The support relation also allows for defining such semantic notions as logical validity, entailment, and logical equivalence for the language $L^?$, thus determining the standard propositional inquisitive logic (*InqB*). Logical validity is defined as universal support and entailment as preservation of support. Logical equivalence is mutual entailment.

DEFINITION 2.2. *Let φ, ψ be $L^?$-formulas and Δ a set of $L^?$-formulas. Logical validity, entailment and equivalence are defined in the following way*:

(a) *φ is logically valid in InqB (symbolically $\vDash_{InqB} \varphi$) if it is supported by every information state.*
(b) *φ is a consequence of Δ in InqB ($\Delta \vDash_{InqB} \varphi$) if every information state that supports every formula from Δ supports also φ.*
(c) *φ and ψ are logically equivalent in InqB ($\varphi \equiv_{InqB} \psi$) if they are supported by the same states, i.e., if $\varphi \vDash_{InqB} \psi$ and $\psi \vDash_{InqB} \varphi$.*

A remarkable feature of logical validity thus defined is that it is not closed under uniform substitution. This feature is well-motivated, since atomic formulas represent declarative sentences whose logical behavior differs from the behavior of questions. So, it is not always possible to substitute questions for atoms.[2] The set of logically valid formulas is closed under the substitution of statements.

The logical notions introduced in Definition 2.2 generalize the corresponding notions of classical logic. Besides the standard cases, in which premises and conclusions are statements, one can also consider arguments consisting purely of questions, or hybrid cases in which premises contain a question or the conclusion is a question. It was shown in (Ciardelli, 2016b) that all these cases have a clear meaning. Let Δ be a set of L-formulas that represent a description of a context in which an argument is formulated. Let α and β be statements and φ, ψ questions. One might consider the following four cases:[3]

(a) $\Delta, \alpha \vDash_{InqB} \beta$,
(b) $\Delta, \varphi \vDash_{InqB} \beta$,
(c) $\Delta, \alpha \vDash_{InqB} \psi$,
(d) $\Delta, \varphi \vDash_{InqB} \psi$.

These cases are informally interpreted in the following way:

(a) In the context Δ, the statement α entails the statement β.
(b) In the context Δ, the question φ presupposes the statement β.
(c) In the context Δ, the statement α resolves the question ψ.
(d) In the context Δ, any information that resolves the question φ resolves also the question ψ.

For example, it holds that $p \vee\!\!\!\vee q \vDash_{InqB} p \vee q$. This means that the question *whether p or q* pressuposes the statement *p or q*.

A complete Hilbert style axiomatization of the consequence relation and logical validity of *InqB* for the language $L^?$ without the disjunction $\vee$ was presented for example in (Ciardelli & Roelofsen, 2011). The system is obtained by adding the following two

2 For example, $\neg\neg p \rightarrow p$ is logically valid in *InqB* but its substitutional instance $\neg\neg ?p \rightarrow ?p$ is not.

3 As usual, "$\Delta, \alpha \vDash \beta$" is an abbreviation for "$\Delta \cup \{\alpha\} \vDash \beta$".

schemata to an axiomatization of intuitionistic logic (where inquisitive disjunction plays the role of intuitionistic disjunction):

$$KP \quad (\neg\varphi \to (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\neg\varphi \to \psi) \mathbin{\vee\!\!\!\vee} (\neg\varphi \to \chi)),$$
$$DN \quad \neg\neg p \to p,$$

where p ranges over atomic formulas. The axiomatization of *InqB* over a language that contains both disjunctions, $\mathbin{\vee\!\!\!\vee}$ and $\vee$, is more complicated. We will see a Hilbert style axiomatization in §5.1. It is worth mentioning that the unrestricted presence of both disjunctions leads to a peculiar feature of the resulting language. One can form disjunctions of questions like $(p \mathbin{\vee\!\!\!\vee} q) \vee (r \mathbin{\vee\!\!\!\vee} s)$. It is not clear whether this construction has a natural language counterpart. Assume that φ and ψ are questions. Then, according to the inquisitive logic, $\varphi \vee \psi$ is a question that is resolved by disjunctions $\alpha \vee \beta$, where α resolves φ and β resolves ψ. For example, $(p \mathbin{\vee\!\!\!\vee} q) \vee (r \mathbin{\vee\!\!\!\vee} s)$ is resolved by the following statements: $p \vee r, p \vee s, q \vee r, q \vee s$. One surprising consequence is that $\vee$ is not generally idempotent in the context of the language $L^?$. For example, $(p \mathbin{\vee\!\!\!\vee} q) \vee (p \mathbin{\vee\!\!\!\vee} q)$ does not entail $p \mathbin{\vee\!\!\!\vee} q$, as is evident from the observation that $p \vee q$ resolves $(p \mathbin{\vee\!\!\!\vee} q) \vee (p \mathbin{\vee\!\!\!\vee} q)$ but not $(p \mathbin{\vee\!\!\!\vee} q)$. These observations also motivate some peculiar features of the inquisitive logic introduced in §4, namely the replacement of the rule $R4$ by $R4^*$ and the restriction related to the axiom $A8$.

A crucial feature of propositional inquisitive semantics and logic that makes the framework suitable for a formal representation of questions is that every $L^?$-formula can be represented by a finite set of L-formulas. It is possible to assign to every $L^?$-formula φ a finite set of L-formulas $\mathcal{R}(\varphi)$ such that φ is logically equivalent (in *InqB*) to the inquisitive disjunction of the formulas from $\mathcal{R}(\varphi)$. Intuitively, this means that $\mathcal{R}(\varphi)$ is a complete set of direct answers to φ. Formally, this amounts to a particular kind of disjunctive normal form in inquisitive logic. The formulas from $\mathcal{R}(\varphi)$ are called resolutions of φ and the set of resolutions can be defined recursively by the following equations:

- $\mathcal{R}(p) = \{p\}$,
- $\mathcal{R}(\neg\varphi) = \{\bigwedge_{\alpha\in\mathcal{R}(\varphi)} \neg\alpha\}$,
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha\in\mathcal{R}(\varphi)} \alpha \to f(\alpha); f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta; \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \vee \psi) = \{\alpha \vee \beta; \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \mathbin{\vee\!\!\!\vee} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

For example, $\mathcal{R}(?p) = \{p, \neg p\}$. Notice that for any L-formula α, $\mathcal{R}(\alpha) = \{\alpha\}$. The case of implication deserves some clarification. The equation says that for every function f that assigns formulas of $\mathcal{R}(\psi)$ to the formulas of $\mathcal{R}(\varphi)$, the set $\mathcal{R}(\varphi \to \psi)$ contains conjunction of all implications of the form $\alpha \to f(\alpha)$, where $\alpha \in \mathcal{R}(\varphi)$. For example, $\mathcal{R}(p \to ?q) = \{p \to q, p \to \neg q\}$. Now the disjunctive normal form theorem can be stated for propositional inquisitive logic (see, e.g., Ciardelli, 2016b, p. 57).

THEOREM 2.3. *For any φ from $L^?$, if $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ then it holds*:

$$\varphi \equiv_{InqB} \alpha_1 \mathbin{\vee\!\!\!\vee} \cdots \mathbin{\vee\!\!\!\vee} \alpha_n.$$

The concept of resolution allows for a specific reduction of the inquisitive logic of questions to the background logic of declarative sentences, which in the case of the basic framework of inquisitive semantics is just classical propositional logic (*Cl*). The reduction of the logical validity of *InqB* to the logical validity of *Cl* has the following form (Ciardelli, 2016b, p. 62).

THEOREM 2.4. *Let φ be an $L^?$-formula. φ is logically valid in InqB iff there is $\alpha \in \mathcal{R}(\varphi)$ that is logically valid in Cl.*

The reduction of the inquisitive consequence relation $\vDash_{InqB}$ to the consequence relation of classical logic $\vDash_{Cl}$ is more complicated (Ciardelli, 2016b, p. 63).

THEOREM 2.5. *For any $L^?$-formulas $\varphi_1, \dots, \varphi_n, \psi$,*

$$\varphi_1, \dots, \varphi_n \vDash_{InqB} \psi$$

iff for every $\alpha_1 \in \mathcal{R}(\varphi_1), \dots, \alpha_n \in \mathcal{R}(\varphi_n)$ there is $\beta \in \mathcal{R}(\psi)$ such that

$$\alpha_1, \dots, \alpha_n \vDash_{Cl} \beta.$$

So, the consequence relation of *InqB* is determined by the following three factors: 1. the equations that define resolutions, 2. the reduction of the consequence relation of the logic of questions to the consequence relation of the background logic of declarative sentences (this reduction is expressed in Theorem 2.5), and 3. the consequence relation of the background logic of declarative sentences. The main goal of this paper is to show that it is possible to formulate a general semantic framework in which the first two factors are fixed and the third can be varied. Questions can be treated in the same way as in the basic framework of inquisitive semantics and the background logic of declarative sentences can be replaced by any of a large class of logical systems that extend a weak version of Full Lambek logic. In this class, we find the most prominent logical systems studied in the literature on nonclassical logic as, for example, linear logics, superintuitionistic logics, relevant logics, and many-valued logics including fuzzy logics.

It can be argued that the task to model questions over nonclassical logics is interesting not only for mathematical reasons but also because it will have applications to the logical analysis of natural language. For example, if the background logic of declarative sentences is classical, the system is not immune to the well-known paradoxes of material implication and irrelevance. This has unwanted consequences even for arguments involving questions. Consider the following argument forms that are valid in *InqB*. The concrete examples of problematic natural language arguments that fall under these forms illustrate that the project to base a logic of questions on a nonclassical logic of declarative sentences is worth pursuing.

(a) $\neg(p \to q) \vDash_{InqB} ?p$. *Example*: The statement *it is not the case that if I die today, I will be living tomorrow* resolves the question *whether I will die today*.

(b) $\neg p \to \neg(q \to r), \neg q \vDash_{InqB} ?p$. *Example*: The statements *if god does not exist, then it is not the case that if I pray, my prayers will be answered* and *I do not pray* together resolve the question *whether god exists*.

(c) $p \wedge \neg p \vDash_{InqB} ?q$. *Example*: The statement *it is raining and it is not raining* resolves the question *whether Jane passed the exam*.

(d) $?q \vDash_{InqB} p \vee \neg p$. *Example*: The question *whether Berlin is the capital of Germany* presupposes the statement *Goldbach's conjecture is true or false*.

(e) $p \to ?q \vDash_{InqB} (p \wedge r) \to ?q$. *Example*: Every answer to the conditional question *whether Ms. X will win the presidential election if a sufficient number of people will vote for her* answers the question *whether Ms. X will win the presidential election if a sufficient number of people will vote for her but the results will be manipulated*.

It is clear that the unintuitive evaluation of these arguments is not the result of the way in which questions are modelled in inquisitive semantics but rather of the background logic of declarative sentences. The various nonclassical logics mentioned above were proposed as better models of portions of natural language. Their inquisitive extensions add questions to these models and thus show that the project of inquisitive semantics is viable even if these models are considered instead of the highly idealized classical logic.

§3. A semantic framework for substructural logics. In this section, we develop a semantics for substructural logics that turns out to be particularly suitable for a formal representation of questions in the style of inquisitive semantics. One can find several alternative semantic frameworks in the literature on substructural logics, but most of them do not meet our requirements. In particular, it should be possible to extend the framework with the standard semantic clause for inquisitive disjunction and this new connective should have its characteristic logical properties known from the standard propositional inquisitive semantics. That means that the new connective should differ from the standard noninquisitive disjunction and should interact with the other connectives in a suitable way.

First of all, we need semantic structures, in which the standard noninquisitive disjunction is equipped with a nonstandard semantic clause. Such structures are usually used for nondistributive substructural logics, as, for example, the ordered monoids in (Ono & Komori, 1985), closure frames in (Restall, 2000, chap. 12), or phase structures in (Paoli, 2002, chap. 7). However, the mentioned frameworks cannot be used directly for our purposes. If we extend them with inquisitive disjunction, it does not interact well with the other connectives. For example, Restall's closure frames (as well as Paoli's phase structures that can be viewed as special cases of closure frames) do not validate the most distinctive feature of inquisitive logic, namely distributivity of implication over inquisitive disjunction (where antecedents are restricted to formulas of the declarative language). Even though the semantics of Ono and Komori is more suitable with respect to our purposes, it also cannot be adopted without any modifications. For example, the resulting framework would not validate the distributivity of noninquisitive disjunction over the inquisitive one, which is necessary for the crucial result concerning transformation to the disjunctive normal form.

A suitable semantic framework that we are going to introduce and use is strongly inspired by (Došen, 1989). The framework also has many similarities with various other semantics that have appeared in the literature on nonclassical logics, for example in (Urquhart, 1972; Fine, 1974; Wansing, 1993; Fine, 2014; Yang, 2014).

We will be concerned with logics of questions in the later sections. The aim of this section is to describe the semantic structures and to show that over a language of declarative sentences these structures determine a basic substructural logic that we denote as *BSL*.

The semantic structures that are particularly suitable for our purposes will be called *information models*. An information model is any structure of the following type satisfying the conditions that are specified below:

$$\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V \rangle.$$

S is an arbitrary nonempty set; $+$ and $\cdot$ are binary operations on S; 0 and 1 are two distinguished elements of S; C is a binary relation on S; and V is a valuation—a function that assigns to every atomic formula a subset of S. Several conditions are required to be satisfied. $\langle S, +, 0 \rangle$ is a join-semilattice with the least element 0. This means that we assume that the following equations generally hold:

- $a + a = a$,
- $a + b = b + a$,

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- $a + (b + c) = (a + b) + c$,
- $a + 0 = a$.

The semilattice determines an ordering of S which is defined in the standard way: $a \leq b$ iff $a + b = b$. As a consequence, it holds for any a that $0 \leq a$ and for any a, b that $a \leq a + b$. We also assume that the following equations hold:

- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,
- $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$,
- $1 \cdot a = a$,
- $0 \cdot a = 0$.

The relation C satisfies the following conditions:

- there is no a such that $0Ca$,
- if aCb, then bCa,
- if aCb and $a \leq c$, then cCb,
- if $(a + b)Cc$, then aCc or bCc.

Moreover, the valuation V assigns to every atomic formula an ideal in $\mathcal{M}$. An ideal I in $\mathcal{M}$ is a subset of S satisfying:

- $0 \in I$,
- if $a \in I$ and $b \leq a$, then $b \in I$,
- if $a \in I$ and $b \in I$, then $a + b \in I$.

The elements of S will be called *information states*, or just *states*. Unlike in the basic framework of inquisitive semantics, information states are not defined as sets of possible worlds in this general setting. Instead, they are regarded as primitive entities.

The operation $+$ is viewed as a specific algebraic operation on information states. The state $a + b$ is interpreted as the state containing the information that is common to the states a and b. It contains the common content of these states. Officially, information states are primitive entities, but we will see that if information states are viewed as consisting of sentences, the $+$ operation corresponds to intersection, while if information states are viewed from the dual perspective as consisting of open possibilities (possible worlds), the $+$ operation behaves like union.

The operation $\cdot$ is interpreted as *fusion* of information states. In $a \cdot b$ the information from a is fused with the information from b. This operation generalizes intersection of information states (viewed as sets of possible worlds). However, the basic properties of intersection, such as idempotence, commutativity and associativity, are not generally required. So, for example, we admit cases, where the order, in which information is combined, can have an effect on the result of the combination. In general, it is also not the case that $a \cdot b$ is under a. This corresponds to the situation when some information from a is lost when a is updated with b.

The state 1 is called the *logical state*. Validity in a model will be defined with respect to this state. The state 0 will be called the *trivially inconsistent state*. The semantics will also allow for nontrivially inconsistent states, i.e., states in which a contradiction is supported though not every formula is supported. However, it will be possible to prove that the state 0 supports every formula.

The relation C is called a *compatibility relation*. We read the claim that aCb as "the state a is compatible with the state b". The compatibility relation will be used in the semantic clause for negation. It is similar to the compatibility relation of (Restall, 2000, p. 238), and closely related to the treatment of negation based on an incompatibility relation known for example from (Dunn, 1993). It might be of some interest to observe that the same conditions which we require for the compatibility relation, are also used in definitions of the so called contact algebras studied in the region-based theory of space (see, e.g., Düntsch & Winter, 2005).

The fourth condition that we require for the compatibility relation (if $(a + b)Cc$, then aCc or bCc) deserves some comment. One can easily observe that this condition holds in the special case where information states are sets of possible worlds and compatibility of two states means that they have nonempty intersection. Since $+$ corresponds to union in this case, the condition says that the union of a and b has nonempty intersection with c only if a or b has nonempty intersection with c. The condition is well motivated even in a more general setting as is shown by the following informal argument: Assume that two abstract states a and b are both incompatible with a third state c. Intuitively, that means that there are two pieces of information such that the first one is supported by a, the second one by b and the negation of both is supported by c. It follows that the disjunction of the two pieces of information is in the common content of a and b, and, as a consequence, their common content $a + b$ is incompatible with c.

The requirement that the valuation V assigns ideals to atomic formulas stems from the interpretation of $+$ and 0. The following conditions are consequences of this interpretation:

(a) The trivially inconsistent state 0 must support p.

The state $a + b$ is the state supporting the information that is common to a and b:

(b) $a + b$ supports p iff a supports p and b supports p.

(a) and (b) together are equivalent to the requirement that the set of states that support p is an ideal in $\mathcal{M}$. Later on, we will need the following monotonicity property of information models.

LEMMA 3.1. *For any states a, b, c, d of any information model, it holds that*:

if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$ and $a \cdot c \leq b \cdot d$.

Proof. Assume $a \leq b$ and $c \leq d$, i.e., $b = a + b$ and $d = c + d$. Then $b + d = (a + b) + (c + d) = (a + c) + (b + d)$, i.e., $a + c \leq b + d$. Moreover, $a \cdot c \leq (a \cdot c) + (b \cdot c) + (a \cdot d) + (b \cdot d) = (a + b) \cdot (c + d) = b \cdot d$. □

Note that it holds as a special case of Lemma 3.1 that

if $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Now we introduce a propositional language L_s that is standardly used in substructural logic. It is constructed in the following way:[4]

$$\alpha ::= p \mid \bot \mid t \mid \neg\alpha \mid \alpha \to \alpha \mid \alpha \wedge \alpha \mid \alpha \otimes \alpha \mid \alpha \vee \alpha.$$

This will be our language of declarative sentences. From now on, the variables $\alpha, \beta, \gamma, \delta$ will range over the formulas of this language. Besides the connectives $\neg, \wedge, \vee, \to$ of the language L, the language L_s contains also the constant $\bot$ representing a (strong) contradiction, the constant t for logical truth and fusion or intensional conjunction $\otimes$, which is

[4] For the sake of simplicity, the language contains only one implication, even though we will not assume, on the general level, that $\otimes$ is commutative.

residuated by implication in substructual logics. In the next section, we will enrich this language with inquisitive disjunction.

The next step is to introduce the semantic clauses for the language L_s. With respect to a given information model $\mathcal{M}$, we define recursively a support relation $\vDash$ between information states from S and L_s-formulas.

- $a \vDash p$ iff $a \in V(p)$.
- $a \vDash \bot$ iff $a = 0$.
- $a \vDash t$ iff $a \leq 1$.
- $a \vDash \neg\alpha$ iff for any b, if bCa then $b \nvDash \alpha$.
- $a \vDash \alpha \to \beta$ iff for any b, if $b \vDash \alpha$, then $a \cdot b \vDash \beta$.
- $a \vDash \alpha \wedge \beta$ iff $a \vDash \alpha$ and $a \vDash \beta$.
- $a \vDash \alpha \otimes \beta$ iff there are b, c such that $b \vDash \alpha$, $c \vDash \beta$, and $a \leq b \cdot c$.
- $a \vDash \alpha \vee \beta$ iff there are b, c such that $b \vDash \alpha$, $c \vDash \beta$, and $a \leq b + c$.

Notice that these semantic clauses resemble those that we used for the basic framework of inquisitive semantics. We will return to this point in the next section when we introduce inquisitive disjunction in the general setting.

If $a \vDash \alpha$, we say that a supports α. We say that an L_s-formula α is valid in an information model $\mathcal{M}$ if the logical state 1 supports α in $\mathcal{M}$. An L_s-formula is valid in a class of information models if it is valid in every model of this class. An information model is called a model of a given set of L_s-formulas if every formula from the set is valid in the model. Let Δ be a set of L_s-formulas and $\mathcal{C}$ a class of information models. The class of information models of Δ will be denoted as $Mod(\Delta)$. The set of L_s-formulas that are valid in $\mathcal{C}$ will be denoted as $Log(\mathcal{C})$. Note that if $\mathcal{C}$ is empty, then $Log(\mathcal{C})$ is the set of all L_s-formulas.

As already mentioned, the semantics is very similar to the semantics introduced in (Došen, 1989). However, there are some differences. Most importantly, the semantic structures of Došen's framework do not involve the compatibility relation. Došen defines $\neg\varphi$ as $\varphi \to \bot$. The compatibility relation enables us to have a paraconsistent negation of the kind that is used for example in relevant logics. Such negation is not reducible to implication of an explosive contradiction. Another difference is that the trivially inconsistent state 0 is not required to be a part of Došen's semantic structures. Besides the fact that the presence of 0 allows for a simplification of the semantic clause for disjunction (Došen works with a more complicated condition), this state will also play an important role in the later sections, where its presence allows us to prove the crucial results on the products of information models (in particular, Lemma 4.12 used in the proof of Lemma 4.13 and Theorem 4.14).

The proposition $||\alpha||_{\mathcal{M}}$ expressed by α in a given information model $\mathcal{M}$ is the set of states of $\mathcal{M}$ that support α. A crucial feature of the semantics is that every formula of the language L_s expresses an ideal. That means that all L_s-formulas express propositions of the same kind as propositions expressed by atomic formulas.

LEMMA 3.2. *For any information model $\mathcal{M}$, any states a, b from $\mathcal{M}$ and any L_s-formula α, the following conditions hold*:

(a) $0 \vDash \alpha$,

(b) *if* $a \vDash \alpha$ *and* $b \leq a$, *then* $b \vDash \alpha$,

(c) *if* $a \vDash \alpha$ *and* $b \vDash \alpha$, *then* $a + b \vDash \alpha$.

Proof. By straightforward induction on the complexity of α. □

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THEOREM 3.3. *For any information model $\mathcal{M}$ and any L_s-formula α, $||\alpha||_{\mathcal{M}}$ is an ideal in $\mathcal{M}$.*

Proof. This follows immediately from Lemma 3.2. □

We will denote the logic determined by our semantics as *BSL* (basic substructural logic).

DEFINITION 3.4. *Let α be an L_s-formula and Δ a nonempty set of L_s-formulas. We say that α is semantically BSL-valid (symbolically $\vDash_{BSL} \alpha$) if α is valid in every information model. α is a semantic BSL-consequence of Δ (symbolically $\Delta \vDash_{BSL} \alpha$) if for any state a of any information model, if a supports everything from Δ, then a supports α.*

While the *BSL*-consequence relation is defined with respect to all states, *BSL*-validity is defined only with respect to logical states. However, there is a close connection between these two semantic notions that is expressed in the following lemma.

LEMMA 3.5. *$\alpha_1, \dots, \alpha_n \vDash_{BSL} \beta$ iff $\vDash_{BSL} (\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta$.*

Proof. First, assume $\alpha_1, \dots, \alpha_n \vDash_{BSL} \beta$ and take an arbitrary information model $\mathcal{M}$. If a state a of $\mathcal{M}$ supports $\alpha_1 \wedge \cdots \wedge \alpha_n$, then, according to our assumption, a, i.e., $1 \cdot a$, supports also β. If follows that the state 1 supports $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta$.

Now assume that $\vDash_{BSL} (\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta$ and that a is a state of an information model $\mathcal{M}$ which supports the formulas $\alpha_1, \dots, \alpha_n$. Since the state 1 of $\mathcal{M}$ supports $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta$, it follows that $1 \cdot a$, i.e., a, supports β. □

The logic of all information models can be axiomatized by a Hilbert style axiomatic system *BSL* that contains the following six axioms

$$\begin{array}{ll} A1\ \alpha \to \alpha & A2\ \bot \to \alpha \\ A3\ (\alpha \wedge \beta) \to \alpha & A4\ (\alpha \wedge \beta) \to \beta \\ A5\ \alpha \to (\alpha \vee \beta) & A6\ \beta \to (\alpha \vee \beta) \end{array}$$

plus one distributive axiom

$$D1\ (\alpha \otimes (\beta \vee \gamma)) \to ((\alpha \otimes \beta) \vee (\alpha \otimes \gamma))$$

and nine rules of inference

$$\begin{array}{ll} R1\ \alpha, \alpha \to \beta / \beta & R2\ \alpha \to \beta / (\beta \to \gamma) \to (\alpha \to \gamma) \\ R3\ \gamma \to \alpha, \gamma \to \beta / \gamma \to (\alpha \wedge \beta) & R4\ \alpha \to \gamma, \beta \to \gamma / (\alpha \vee \beta) \to \gamma \\ R5\ \alpha \to (\beta \to \gamma) / (\alpha \otimes \beta) \to \gamma & R6\ (\alpha \otimes \beta) \to \gamma / \alpha \to (\beta \to \gamma) \\ R7\ t \to \alpha / \alpha & R8\ \alpha / t \to \alpha \\ R9\ \alpha \to \neg\beta / \beta \to \neg\alpha & \end{array}$$

This logic is equivalent to Došen's system E^+ from (Došen, 1988) enriched with the axiom $A2$ for $\bot$ and the rules $R7$, $R8$, $R9$ for t and $\neg$. The system can be viewed also as a nondistributive and noncommutative version of the Hilbert system for Full Lambek logic with a paraconsistent negation used in (Bílkova, Majer, & Peliš, 2016) and (Sedlár, 2015). The original Lambek logic was introduced in (Lambek, 1958) as a logic of syntactic types. It concerned a language containing an associative but noncommutative binary connective (our intensional conjunction) residuated by two symmetric implications. A nonassociative version of Lambek logic was formulated in (Lambek, 1961). We extend the language with conjunction, disjunction, negation and the two constants $\bot$ and t, respectively representing explosive contradiction and logical truth. However, for the sake of simplicity, we consider only one implication in the language.

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The rules $R3$, $R4$ and axiom $D1$ could be replaced with the rule $\alpha, \beta/\alpha \wedge \beta$ and axioms $((\gamma \rightarrow \alpha) \wedge (\gamma \rightarrow \beta)) \rightarrow (\gamma \rightarrow (\alpha \wedge \beta))$, $((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)$, and the resulting system would generate the same logic. The other rules cannot be replaced by corresponding axioms in a similar way. This is obvious from the fact that such axioms would not be sound with respect to our semantics while the system *BSL* is sound with respect to the semantics, as is observed in the following Lemma.

LEMMA 3.6. *Every axiom of the system BSL is semantically BSL-valid and all the rules of BSL preserve semantic BSL-validity in all information models.*

Proof. We will consider just two cases for illustration. The semantic *BSL*-validity of an axiom schema means that all instances of the schema are supported by the state 1 of every information model. For example, let us consider the schema $A5$. Assume that an arbitrary information model is given. We will show that in the model $1 \vDash \alpha \rightarrow (\alpha \vee \beta)$. So, we have to show that for any state a, if a supports α, then $1 \cdot a$, i.e., a, supports $\alpha \vee \beta$. Suppose that $a \vDash \alpha$. According to Lemma 3.2(a), the state 0 supports every formula. In particular, $0 \vDash \beta$. Moreover, $a + 0 = a$. So, there are states b and c ($b = a$ and $c = 0$) such that b supports α, c supports β and $a \leq b + c$. So, $a \vDash \alpha \vee \beta$.

An inference rule preserves semantic validity in an information model if for any instance of the rule such that the premises of the rule are supported by the state 1 of the model, the conclusion of the rule is also supported by the state 1 of the model. For example, let us consider the rule $R6$. Assume that in a given information model $1 \vDash (\alpha \otimes \beta) \rightarrow \gamma$. We will show that then $1 \vDash \alpha \rightarrow (\beta \rightarrow \gamma)$. Suppose that a supports α. We have to prove that $1 \cdot a$, i.e., a, supports $\beta \rightarrow \gamma$. Assume b supports β. It follows that $a \cdot b$ supports $\alpha \otimes \beta$. Then according to our first assumption $1 \cdot (a \cdot b)$, i.e., $a \cdot b$, supports γ, which is what we needed. □

A proof in the system *BSL* is defined in the standard way as a finite sequence of L_s-formulas such that every formula in the sequence is either an instance of an axiom schema, or a formula that is derived according to an inference rule from some formulas that occur earlier in the sequence. Some examples of proofs in *BSL* can be found in Appendix (Lemma 7.1). We say that α is *BSL*-provable (symbolically, $\vdash_{BSL} \alpha$), if there is a proof $\beta_1, \ldots, \beta_n$ such that $\alpha = \beta_n$. The expression $\alpha_1, \ldots, \alpha_n \vdash_{BSL} \beta$ is an abbreviation for $\vdash_{BSL} (\alpha_1 \wedge \cdots \wedge \alpha_n) \rightarrow \beta$, and if Δ is a set of L_s-formulas, the expression $\Delta \vdash_{BSL} \beta$ means that there are $\alpha_1, \ldots, \alpha_n \in \Delta$ such that $\alpha_1, \ldots, \alpha_n \vdash_{BSL} \beta$. We say that two L_s-formulas α, β are provably equivalent ($\alpha \dashv\vdash_{BSL} \beta$) if $\alpha \vdash_{BSL} \beta$ and $\beta \vdash_{BSL} \alpha$.

An important feature of *BSL* that we state without proof is that provably equivalent L_s-formulas are replaceable in the following sense. Assume that α, β, γ are any L_s-formulas such that α is a subformula of γ and $\gamma[\alpha/\beta]$ is an L_s-formula that is obtained from γ by the replacement of an occurrence of the subformula α with β. Then $\alpha \dashv\vdash_{BSL} \beta$ implies $\gamma[\alpha/\beta] \dashv\vdash_{BSL} \gamma$.

Completeness of the Hilbert system *BSL* with respect to the class of all information models can be proved using a canonical model construction. We will be concerned not only with the logic *BSL* but also with its extensions. These extensions will be called *logics of declarative sentences*.

DEFINITION 3.7. *A set of L_s-formulas λ is called a logic of declarative sentences if the following three conditions are satisfied*:

- *λ contains all the axioms of BSL,*
- *λ is closed under the rules of BSL,*

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- *λ is closed under substitutions of L_s-formulas.*[5]

The class of logics of declarative sentences contains many interesting logical systems that were studied separately in the literature. We have already mentioned relevant logics, linear logics, multi-valued logics, intuitionistic logic and its extensions (called intermediate or superintuitionistic logics) and so on (for more examples, see Galatos, Jipsen, Kowalski, & Ono, 2007).

Instead of $\alpha \to \beta \in \lambda$, we write just $\alpha \vdash_\lambda \beta$ to indicate that we will be interested in logics of declarative sentences that are determined by an axiomatic system.

For the rest of this section, we will suppose that a logic of declarative sentences λ is fixed. In the case of our semantics the canonical model for λ can be constructed from arbitrary theories instead of from some special (for instance *prime*) theories as is usual.

DEFINITION 3.8. *A nonempty set of L_s-formulas Δ is an λ-theory if it satisfies the following two conditions*:

(a) *if $\alpha \in \Delta$ and $\beta \in \Delta$, then $\alpha \wedge \beta \in \Delta$,*
(b) *if $\alpha \in \Delta$ and $\alpha \vdash_\lambda \beta$, then $\beta \in \Delta$.*

Now we can define the canonical model for λ.[6]

DEFINITION 3.9. *The canonical model for the logic of declarative sentences λ is the structure*

$$\mathcal{M}^\lambda = \langle S^\lambda, +^\lambda, \cdot^\lambda, 0^\lambda, 1^\lambda, C^\lambda, V^\lambda \rangle,$$

where

- *S^λ is the set of all λ-theories,*
- *$\Gamma +^\lambda \Delta = \Gamma \cap \Delta$,*
- *$\Gamma \cdot^\lambda \Delta = \{\alpha;$ for some $\gamma \in \Gamma$ and $\delta \in \Delta, \gamma \otimes \delta \vdash_\lambda \alpha\}$,*
- *0^λ is the set of all L_s-formulas,*
- *$1^\lambda = \lambda$,*
- *$\Gamma C^\lambda \Delta$ iff for all α, if $\neg\alpha \in \Gamma$, $\alpha \notin \Delta$.*
- *$\Gamma \in V^\lambda(p)$ iff $p \in \Gamma$.*

Using this construction, completeness can be proved in a routine way. The details of the proofs of the following two Theorems can be found in Appendix.

THEOREM 3.10. *$\mathcal{M}^\lambda$ is an information model.*

THEOREM 3.11. *For any L_s-formula α and any λ-theory Γ, it holds in $\mathcal{M}^\lambda$ that $\Gamma \vDash \alpha$ iff $\alpha \in \Gamma$.*

COROLLARY 3.12. *$\alpha \in \lambda$ iff α is valid in $\mathcal{M}^\lambda$.*

THEOREM 3.13. *$Log(Mod(\lambda)) = \lambda$.*

---

[5] A substitution of L_s-formulas is a function s that assigns to every atomic formula p an L_s-formula $s(p)$. If s is a substitution and α an L_s-formula, then $s(\alpha)$ is the formula that is obtained from α by simultaneous replacement of every occurrence of the atomic formula p in α with $s(p)$. λ is closed under substitutions of L_s-formulas if for every $\alpha \in \lambda$ and every substitution of L_s-formulas s, $s(\alpha) \in \lambda$.

[6] (Došen, 1989) uses a different canonical model construction which was formulated for a sequent system of a logic (Došen's L) that does not validate modus ponens and adjunction.

Proof. It is obvious that if $\alpha \in \lambda$, then α is valid in all models of λ. It follows that $\lambda \subseteq Log(Mod(\lambda))$. Now assume that $\alpha \in Log(Mod(\lambda))$. It follows that α is supported also by the logical state 1 in the canonical model $\mathcal{M}^{\lambda}$. As a consequence, $\alpha \in \lambda$. □

THEOREM 3.14. *$\Delta \vDash_{BSL} \beta$ iff $\Delta \vdash_{BSL} \beta$, for any set of L_s-formulas Δ.*

Proof. The right-to-left direction is soundness of the system *BSL* w.r.t. information models, which follows from Lemma 3.6. We will prove the left-to-right direction. Assume $\Delta \nvdash_{BSL} \beta$. Let $Th(\Delta) = \{\alpha; \Delta \vdash_{BSL} \alpha\}$. $Th(\Delta)$ is a *BSL*-theory. For any $\alpha \in \Delta$, the state $Th(\Delta)$ supports α in the canonical model. But $Th(\Delta)$ does not support β. So, $\Delta \nvDash_{BSL} \beta$. □

COROLLARY 3.15. *If $\Delta \vDash_{BSL} \beta$, there is a finite $\Gamma \subseteq \Delta$ such that $\Gamma \vDash_{BSL} \beta$.*

§4. Inquisitive extensions of substructural logics.

This section forms the core of this paper. It shows that our framework for substructural logics allows for the introduction of inquisitive disjunction, and, consequently, that every logic of declarative sentences can be extended with questions. A general theory of these "inquisitive extensions" is provided.

We will work with the language $L_s^?$ that is defined as the language L_s, enriched with the additional binary connective $\mathbin{\vee\!\!\!\vee}$:

$$\varphi ::= p \mid \bot \mid t \mid \neg\varphi \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi \mid \varphi \vee \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi.$$

This will be our language of declarative sentences and questions. From now on, the variables $\varphi, \psi, \chi, \vartheta$ will range over the formulas of this language.

The semantic clauses defining the support relation simply extend those introduced in the previous section. The difference is that now the clauses are defined generally, for the whole language $L_s^?$, and the clause for inquisitive disjunction is added:

$$a \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi \text{ iff } a \vDash \varphi \text{ or } a \vDash \psi.$$

The concepts of validity in a model and in a class of models are defined as in the previous section. The set of $L_s^?$-formulas that are valid in a class of information models $\mathcal{C}$ will be denoted as $Log^?(\mathcal{C})$. Note that if $\mathcal{C}$ is empty, $Log^?(\mathcal{C})$ is the set of all $L_s^?$-formulas. If an $L_s^?$-formula is valid in all information models, we say that it is *InqBSL*-valid. If we define *InqBSL*-consequence relation in the same way as in the previous section, it is again reducible to *InqBSL*-validity in the sense of Lemma 3.5.

A simple observation shows that (a) and (b) of Lemma 3.2 hold even with respect to the language $L_s^?$.

LEMMA 4.1. *For any information model $\mathcal{M}$, any states a, b from $\mathcal{M}$ and any $L_s^?$-formula φ, the following conditions hold*:

(a) *$0 \vDash \varphi$,*
(b) *if $a \vDash \varphi$ and $b \leq a$, then $b \vDash \varphi$.*

However, (c) of Lemma 3.2, saying that the set of states supporting a given formula is closed under $+$, cannot be extended to $L_s^?$. For example, if we have two states, a and b, of an information model, such that neither $a \leq b$, nor $b \leq a$, and if $V(p) = \{c; c \leq a\}$ and $V(q) = \{c; c \leq b\}$, then both a and b support $p \mathbin{\vee\!\!\!\vee} q$, but $a + b$ does not support $p \mathbin{\vee\!\!\!\vee} q$. So, propositions expressed by L_s-formulas are always ideals, and propositions expressed by $L_s^?$-formulas express nonempty downward closed sets but not necessarily ideals.

It is obvious that within this semantic framework, the whole propositional inquisitive semantics can be viewed as semantics consisting of one particular information model $\mathcal{M}_{InqB} = \langle S, +, \cdot, 0, 1, C, V\rangle$, where

- the set of states S is the set of all sets of possible worlds,[7]
- the operation $+$ is union,
- the operation $\cdot$ is intersection,
- the inconsistent state 0 is the empty set $\emptyset$,
- the logical state 1 is the set of all possible worlds,
- two states a and b are compatible (aCb) iff $a \cap b \neq \emptyset$,
- $a \in V(p)$ iff p is true in every world of a.

The set of $L_s^?$-formulas that are logically valid in *InqB* can be determined as $Log^?(\mathcal{M}_{InqB})$. In this particular model, $\varphi \otimes \psi$ is equivalent to $\varphi \wedge \psi$, t can be defined as $p \vee \neg p$, and $\bot$ as $p \wedge \neg p$. Moreover, $\neg\varphi$ is in $\mathcal{M}_{InqB}$ equivalent to $\varphi \to \bot$.

Let $\mathcal{C}$ be the class of all information models. We have seen in the previous section that $Log(\mathcal{C})$ is completely axiomatized by *BSL*. A natural question arises whether there is a similar axiomatization of $Log^?(\mathcal{C})$. It turns out that $Log^?(\mathcal{C})$ can be axiomatized by a system that we will call *InqBSL*. It contains the axiomatic schemata of the system *BSL*, with the difference that now the variables range over the whole language $L_s^?$. To these we add two more axioms for $\vee$ and two for $\mathbin{\vee\!\!\!\vee}$. There is a crucial restriction concerning the axiom $A8$. The formula α is $\mathbin{\vee\!\!\!\vee}$-free, i.e., α ranges only over the L_s-formulas.

$A1\ \varphi \to \varphi$	$A2\ \bot \to \varphi$
$A3\ (\varphi \wedge \psi) \to \varphi$	$A4\ (\varphi \wedge \psi) \to \psi$
$A5\ \varphi \to (\varphi \vee \psi)$	$A6\ \psi \to (\varphi \vee \psi)$
$A7\ (\varphi \vee \psi) \to (\psi \vee \varphi)$	$A8\ (\alpha \vee \alpha) \to \alpha$
$A9\ \varphi \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)$	$A10\ \psi \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)$.

Moreover, we add four distributive laws to the distributive axiom $D1$. In the axiom $D5$, it is again assumed that α ranges only over L_s-formulas.

$$
\begin{array}{ll}
D1 & (\varphi \otimes (\psi \vee \chi)) \to ((\varphi \otimes \psi) \vee (\varphi \otimes \chi)) \\
D2 & (\varphi \otimes (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\varphi \otimes \psi) \mathbin{\vee\!\!\!\vee} (\varphi \otimes \chi)) \\
D3 & (\varphi \wedge (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\varphi \wedge \psi) \mathbin{\vee\!\!\!\vee} (\varphi \wedge \chi)) \\
D4 & (\varphi \vee (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\varphi \vee \psi) \mathbin{\vee\!\!\!\vee} (\varphi \vee \chi)) \\
D5 & (\alpha \to (\psi \mathbin{\vee\!\!\!\vee} \chi)) \to ((\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi)).
\end{array}
$$

The converse implications to axioms $D1 - D5$ are provable in the system. This will be shown in Appendix as a part of the proof of Theorem 4.3.[8] The rules of *InqBSL* are basically the rules of *BSL* with two modifications. First, the rule $R10$ for inquisitive disjunction is added. Second, the rule $R4$ for noninquisitive disjunction (i.e., $\varphi \to \chi, \beta \to \chi/(\varphi \vee \beta) \to \chi$) is not sound when it is formulated for the whole language $L_s^?$.[9] It has to be replaced with a modified rule $R4^*$.

[7] Let us recall that possible worlds are functions from the set of atomic formulas to the set of truth values $\{T, F\}$.

[8] See Lemma 7.8.

[9] Consider the following instance of the rule: $p \to (p \mathbin{\vee\!\!\!\vee} q), q \to (p \mathbin{\vee\!\!\!\vee} q)/(p \vee q) \to (p \mathbin{\vee\!\!\!\vee} q)$. The premises are valid in all models but the conclusion is not. Notice that this feature of the semantics is quite intuitive. The validity of the first premise says that the statement p resolves the question *whether p or q*. The second premise says that the statement q resolves the question *whether p or q*. Intuitively, from these it should not be possible to infer that the statement *p or q* resolves the question *whether p or q*.

$R1\ \varphi, \varphi \to \psi / \psi$	$R2\ \varphi \to \psi / (\psi \to \chi) \to (\varphi \to \chi)$
$R3\ \chi \to \varphi, \chi \to \psi / \chi \to (\varphi \wedge \psi)$	$R4^*\ \varphi \to \chi, \psi \to \vartheta / (\varphi \vee \psi) \to (\chi \vee \vartheta)$
$R5\ \varphi \to (\psi \to \chi) / (\varphi \otimes \psi) \to \chi$	$R6\ (\varphi \otimes \psi) \to \chi / \varphi \to (\psi \to \chi)$
$R7\ t \to \varphi / \varphi$	$R8\ \varphi / t \to \varphi$
$R9\ \varphi \to \neg\psi / \psi \to \neg\varphi$	$R10\ \varphi \to \chi, \psi \to \chi / (\varphi \vee\!\!\!\vee \psi) \to \chi.$

LEMMA 4.2. *Every axiom of InqBSL is InqBSL-valid and every rule of InqBSL preserves InqBSL-validity in every information model.*

Proof. The proof for the axioms $A1-A6$ and $D1$, and for the rules $R1-R3$, $R5-R9$ is the same as in the case of *BSL*. We need to verify *InqBSL*-validity of the new axioms $A7-A10$ and $D2-D5$, and to show that the new rules $R4^*$ and $R10$ preserve *InqBSL*-validity. We will discuss just a few cases.

$A8$: This schema says that the propositions expressed by $\vee\!\!\!\vee$-free formulas are closed under $+$. We know this fact from Lemma 3.2(c).

$D5$: Assume that a does not support $(\alpha \to \psi) \vee\!\!\!\vee (\alpha \to \chi)$ where α is $\vee\!\!\!\vee$-free. Then there are two states, b and c, such that b supports α but $a \cdot b$ does not support ψ, and c supports α but $a \cdot c$ does not support χ. Since α is $\vee\!\!\!\vee$-free, $b + c$ supports α but $a \cdot (b + c)$, i.e., $(a \cdot b) + (a \cdot c)$, does not support $\psi \vee\!\!\!\vee \chi$ (if it did, then, due to persistence, $a \cdot b$ would support ψ or $a \cdot c$ would support χ).

$R4^*$: Assume that in some information model, 1 supports both $\varphi \to \chi$ and $\psi \to \vartheta$. Moreover, assume that some state a supports $\varphi \vee \psi$. Then there are states b and c such that b supports φ, c supports ψ, and $a \leq b + c$. It follows that b supports χ and c supports ϑ. As a consequence, a supports $\chi \vee \vartheta$. We have proved that 1 supports $(\varphi \vee \psi) \to (\chi \vee \vartheta)$.

$R10$: Assume that a state 1 of an information model supports $\varphi \to \chi$ and $\psi \to \chi$. Assume that a state a supports $\varphi \vee\!\!\!\vee \psi$, i.e., φ or ψ. Both cases imply that a supports χ. So, 1 supports $(\varphi \vee\!\!\!\vee \psi) \to \chi$. □

If an $L_s^?$-formula φ is provable in the system *InqBSL*, we say that it is *InqBSL*-provable and write $\vdash_{InqBSL} \varphi$. Instead of $\vdash_{InqBSL} \varphi \to \psi$, we write $\varphi \vdash_{InqBSL} \psi$, and if $\varphi \vdash_{InqBSL} \psi$ and $\psi \vdash_{InqBSL} \varphi$, we write $\varphi \dashv\vdash_{InqBSL} \psi$. The system *InqBSL* has the property that provably equivalent $L_s^?$-formulas are replaceable.[10]

We can define the notion of resolution for the language $L_s^?$. The definition simply extends the analogous definition formulated in §2 for the language $L^?$.

- $\mathcal{R}(p) = \{p\}$,
- $\mathcal{R}(\bot) = \{\bot\}$,
- $\mathcal{R}(t) = \{t\}$,
- $\mathcal{R}(\neg\varphi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \neg\alpha\}$,
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \alpha \to f(\alpha); f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta; \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \otimes \psi) = \{\alpha \otimes \beta; \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \vee \psi) = \{\alpha \vee \beta; \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

We obtain the following result that corresponds to the key feature of standard propositional inquisitive logic that was semantically expressed in Theorem 2.3.

[10] See Appendix, Lemma 7.7.

THEOREM 4.3. *For any $L^?_s$-formula φ, if $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, then it holds*:

$$\varphi \dashv\vdash_{InqBSL} \alpha_1 \mathbin{\rotatebox[origin=c]{0}{$\vee$}\!\!\vee} \cdots \mathbin{\vee\!\!\vee} \alpha_n.$$

Proof. See Appendix. □

Now we will consider and relate the axiomatic extensions of *BSL* and *InqBSL*. We will restrict ourselves to the cases in which the extra axioms and rules involve only L_s-formulas. The result of extending *BSL* and *InqBSL* with an additional set of such axioms and rules A will be denoted as $BSL \oplus A$ and $InqBSL \oplus A$, respectively. Let us fix any such set A.

LEMMA 4.4. *Any L_s-formula α that is provable in $BSL \oplus A$ is provable also in $InqBSL \oplus A$.*

Proof. Any axiom of the system *BSL* is also an axiom of the system *InqBSL*. Any inference rule of the system *BSL* is also an inference rule of the system *InqBSL*, with the exception of $R4$. However, the rule $R4$ restricted to L_s-formulas can be simulated in *InqBSL* using $A8$ and $R4^*$. □

DEFINITION 4.5. *We say that a set of $L^?_s$-formulas Δ is $\mathbin{\vee\!\!\vee}$-prime if it holds for any $L^?_s$-formulas φ, ψ that if $\varphi \mathbin{\vee\!\!\vee} \psi \in \Delta$, then $\varphi \in \Delta$ or $\psi \in \Delta$.*

Suppose that Δ is a deductively closed set of $L^?_s$-formulas (w.r.t. $\vdash_{InqBSL}$). Intuitively, we can say that Δ is $\mathbin{\vee\!\!\vee}$-prime if and only if it resolves every question it contains.

THEOREM 4.6. *Let $\mathcal{C}$ be a class of information models satisfying the following two conditions*:

(a) *$Log(\mathcal{C})$ is the set of L_s-formulas provable in $BSL \oplus A$,*
(b) *$Log^?(\mathcal{C})$ is $\mathbin{\vee\!\!\vee}$-prime.*

Then $Log^?(\mathcal{C})$ is the set of $L^?_s$-formulas that are provable in $InqBSL \oplus A$.

Proof. In Lemma 4.2 we showed that the axioms of *InqBSL* are valid and the rules of *InqBSL* preserve validity in every information model. Moreover, we assume that the axioms and rules from A are sound with respect to $\mathcal{C}$. It follows that if φ is provable in $InqBSL \oplus A$, then $\varphi \in Log^?(\mathcal{C})$. We will prove the converse implication. Suppose that φ is an $L^?_s$-formula, and $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. We show that if $\varphi \in Log^?(\mathcal{C})$, then φ is provable in $InqBSL \oplus A$. We can proceed by the following steps:

1. $\varphi \in Log^?(\mathcal{C})$ (assumption),
2. $\alpha_1 \mathbin{\vee\!\!\vee} \cdots \mathbin{\vee\!\!\vee} \alpha_n \in Log^?(\mathcal{C})$ (from 1, Lemma 4.2, and Theorem 4.3),
3. for some $\alpha \in \mathcal{R}(\varphi)$, $\alpha \in Log(\mathcal{C})$ (2 and assumption (b)),
4. for some $\alpha \in \mathcal{R}(\varphi)$, α is provable in $BSL \oplus A$ (3 and assumption (a)),
5. for some $\alpha \in \mathcal{R}(\varphi)$, α is provable in $InqBSL \oplus A$ (4 and Lemma 4.4),
6. $\alpha_1 \mathbin{\vee\!\!\vee} \cdots \mathbin{\vee\!\!\vee} \alpha_n$ is provable in $InqBSL \oplus A$ (from 5, using $A9$, $A10$),
7. φ is provable in $InqBSL \oplus A$ (6 and Theorem 4.3). □

COROLLARY 4.7. *For any information model $\mathcal{M}$, if $Log(\mathcal{M})$ is the set of L_s-formulas provable in $BSL \oplus A$, then $Log^?(\mathcal{M})$ is the set of $L^?_s$-formulas provable in $InqBSL \oplus A$.*

Proof. It is obvious that $Log^?(\mathcal{M})$ is $\mathbin{\vee\!\!\vee}$-prime. So the statement follows immediately from Theorem 4.6. □

Now we will be concerned with the question when a class of information models determines a $\bigvee\!\!\!\!\bigvee$-prime set of $L_s^?$-formulas and we will show that one important sufficient condition for this is that the class is closed under products of its models.

DEFINITION 4.8. *The product of* $\mathcal{M}_1 = \langle S_1, +_1, \cdot_1, 0_1, 1_1, C_1, V_1\rangle$ *and* $\mathcal{M}_2 = \langle S_2, +_2, \cdot_2, 0_2, 1_2, C_2, V_2\rangle$ *is the structure* $\mathcal{M}_1 \times \mathcal{M}_2 = \langle S, +, \cdot, 0, 1, C, V\rangle$*, where*

- $S = S_1 \times S_2$ *(Cartesian product of* S_1 *and* S_2*),*
- $\langle a, b\rangle + \langle c, d\rangle = \langle a +_1 c, b +_2 d\rangle$*,*
- $\langle a, b\rangle \cdot \langle c, d\rangle = \langle a \cdot_1 c, b \cdot_2 d\rangle$*,*
- $0 = \langle 0_1, 0_2\rangle$*,*
- $1 = \langle 1_1, 1_2\rangle$*,*
- $\langle a, b\rangle C\langle c, d\rangle$ *iff* aC_1c *or* bC_2d*,*
- $V(p) = V_1(p) \times V_2(p)$*.*

We will state some basic facts about products of information models that will be needed in what follows. The first observation is that the orderings in the models $\mathcal{M}_1$ and $\mathcal{M}_2$ (defined by $+_1$ and $+_2$) determine the ordering in the product $\mathcal{M}_1 \times \mathcal{M}_2$ (defined by $+$) in the following way.

LEMMA 4.9. *Let* $\leq_1$ *and* $\leq_2$ *be the orderings of two information models* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*, and* $\leq$ *the ordering of the resulting product* $\mathcal{M}_1 \times \mathcal{M}_2$*. Then* $\langle a, b\rangle \leq \langle c, d\rangle$ *iff* $a \leq_1 c$ *and* $b \leq_2 d$*.*

Proof. $\langle a, b\rangle \leq \langle c, d\rangle$ iff $\langle a, b\rangle + \langle c, d\rangle = \langle c, d\rangle$ iff $\langle a +_1 c, b +_2 d\rangle = \langle c, d\rangle$ iff $a +_1 c = c$ and $b +_2 d = d$ iff $a \leq_1 c$ and $b \leq_2 d$. □

LEMMA 4.10. *If* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are information models, then* $\mathcal{M}_1 \times \mathcal{M}_2$ *is also an information model.*

Proof. It can be easily verified that the product satisfies all the conditions from the definition of information models. For illustration, we will show that the relation C in $\mathcal{M}_1 \times \mathcal{M}_2$ satisfies the fourth condition for the compatibility relation.

Assume $(\langle a, b\rangle + \langle c, d\rangle)C\langle e, f\rangle$, i.e., $\langle a +_1 c, b +_2 d\rangle C\langle e, f\rangle$. This means, according to the definition, that $(a +_1 c)C_1e$ or $(b +_2 d)C_2f$. It follows that aC_1e or cC_1e or bC_2f or dC_2f. As a consequence, $\langle a, b\rangle C\langle e, f\rangle$ or $\langle c, d\rangle C\langle e, f\rangle$. □

LEMMA 4.11. *For any information models* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*, and any* $L_s^?$*-formula* φ*, the following holds*:

(a) $\langle a, 0_2\rangle \vDash \varphi$ *in* $\mathcal{M}_1 \times \mathcal{M}_2$ *iff* $a \vDash \varphi$ *in* $\mathcal{M}_1$*.*
(b) $\langle 0_1, b\rangle \vDash \varphi$ *in* $\mathcal{M}_1 \times \mathcal{M}_2$ *iff* $b \vDash \varphi$ *in* $\mathcal{M}_2$*.*

Proof. We will proceed by induction on φ and prove (a). We will write simply 0 instead of 0_1 and 0_2, and 1 instead of 1_1 and 1_2. It will be clear from the context in which model the respective state is. Moreover, we will omit the reference to the model to which the support relation is relative, since it will also be clear from the context. So, we are going to prove that $\langle a, 0\rangle \vDash \varphi$ iff $a \vDash \varphi$, for any $L_s^?$-formula φ. We will show only the inductive steps for the less obvious cases: negation, implication, fusion, and noninquisitive disjunction. The induction hypothesis is: $\langle a, 0\rangle \vDash \varphi$ iff $a \vDash \varphi$, and $\langle a, 0\rangle \vDash \psi$ iff $a \vDash \psi$.

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Negation. $\langle a, 0\rangle \nvDash \neg\varphi$ iff there are b, c such that $\langle a, 0\rangle C\langle b, c\rangle$ and $\langle b, c\rangle \vDash \varphi$ iff there are b, c such that aCb or $0Cc$ and $\langle b, c\rangle \vDash \varphi$ iff[11] there is b such that aCb and $\langle b, 0\rangle \vDash \varphi$ iff there is b such that aCb and $b \vDash \varphi$ iff $a \nvDash \neg\varphi$.

Implication. $\langle a, 0\rangle \nvDash \varphi \to \psi$ iff there are b, c such that $\langle b, c\rangle \vDash \varphi$ and $\langle a\cdot b, 0\cdot c\rangle \nvDash \psi$ iff there is b such that $\langle b, 0\rangle \vDash \varphi$ and $\langle a \cdot b, 0\rangle \nvDash \psi$ iff there is b such that $b \vDash \varphi$ and $a \cdot b \nvDash \psi$ iff $a \nvDash \varphi \to \psi$.

Fusion. $\langle a, 0\rangle \vDash \varphi \otimes \psi$ iff there are b, c, d, e such that $\langle b, c\rangle \vDash \varphi$, $\langle d, e\rangle \vDash \psi$, and $\langle a, 0\rangle \le \langle b \cdot d, c \cdot e\rangle \vDash \varphi$ iff there are b, d such that $\langle b, 0\rangle \vDash \varphi$, $\langle d, 0\rangle \vDash \psi$, and $a \le b \cdot d$ iff there are b, d such that $b \vDash \varphi$, $d \vDash \psi$, and $a \le b \cdot d$ iff $a \vDash \varphi \otimes \psi$.

Disjunction. $\langle a, 0\rangle \vDash \varphi \vee \psi$ iff there are b, c, d, e such that $\langle b, c\rangle \vDash \varphi$, $\langle d, e\rangle \vDash \psi$, and $\langle a, 0\rangle \le \langle b + d, c + e\rangle$ iff there are b, d such that $\langle b, 0\rangle \vDash \varphi$, $\langle d, 0\rangle \vDash \psi$, and $a \le b + d$ iff there are b, d such that $b \vDash \varphi$, $d \vDash \psi$, and $a \le b + d$ iff $a \vDash \varphi \vee \psi$. □

LEMMA 4.12. *For any information models $\mathcal{M}_1$ and $\mathcal{M}_2$, and any L_s-formula α, the following holds*:

$$\langle a, b\rangle \vDash \alpha \text{ in } \mathcal{M}_1 \times \mathcal{M}_2 \text{ iff } a \vDash \alpha \text{ in } \mathcal{M}_1 \text{ and } b \vDash \alpha \text{ in } \mathcal{M}_2.$$

Proof. Since $\langle a, b\rangle = \langle a, 0_2\rangle + \langle 0_1, b\rangle$ this Lemma is a straightforward consequence of Lemma 4.11. □

The previous Lemma can also be expressed in the following elegant way. It holds for any L_s-formula α that

$$||\alpha||_{\mathcal{M}_1\times\mathcal{M}_2} = ||\alpha||_{\mathcal{M}_1} \times ||\alpha||_{\mathcal{M}_2}.$$

This claim does not hold generally, for all $L_s^?$-formulas. Consider two states, a and b, such that a supports p but not q in $\mathcal{M}_1$, and b supports q but not p in $\mathcal{M}_2$. Then $a \vDash p \vee\!\!\!\vee q$ in $\mathcal{M}_1$ and $b \vDash p \vee\!\!\!\vee q$ in $\mathcal{M}_2$ but $\langle a, b\rangle \nvDash p \vee\!\!\!\vee q$ in $\mathcal{M}_1 \times \mathcal{M}_2$.

LEMMA 4.13. *If a class of information models $\mathcal{C}$ is closed under products, then $Log^?(\mathcal{C})$ is $\vee\!\!\!\vee$-prime.*

Proof. Assume that $\mathcal{C}$ is closed under products, and that $\varphi \notin Log^?(\mathcal{C})$ and $\psi \notin Log^?(\mathcal{C})$. Then there are two information models $\mathcal{M}_1, \mathcal{M}_2$ in $\mathcal{C}$ such that $1_1 \nvDash \varphi$ in $\mathcal{M}_1$ and $1_2 \nvDash \psi$ in $\mathcal{M}_2$. It follows from Lemma 4.11 that $\langle 1_1, 0_2\rangle \nvDash \varphi$ and $\langle 0_1, 1_2\rangle \nvDash \psi$ in $\mathcal{M}_1 \times \mathcal{M}_2$. As a consequence, $\langle 1_1, 1_2\rangle \nvDash \varphi \vee\!\!\!\vee \psi$ in $\mathcal{M}_1 \times \mathcal{M}_2$. So, $\varphi \vee\!\!\!\vee \psi \notin Log^?(\mathcal{C})$. □

THEOREM 4.14. *For any class of information models $\mathcal{C}$ that is closed under products, if $Log(\mathcal{C})$ is the set of L_s-formulas provable in $BSL \oplus A$, then $Log^?(\mathcal{C})$ is the set of $L_s^?$-formulas provable in $InqBSL \oplus A$.*

Proof. This is a consequence of Theorem 4.6 and Lemma 4.13. □

Now we will show that every logic of declarative sentences can be enriched with questions in the same way in which classical logic is enriched with questions in the basic inquisitive semantics.

[11] The left-to-right direction of this equivalence uses the fact that there is no c such that $0Cc$ and that $\langle b, c\rangle \vDash \varphi$ implies $\langle b, 0\rangle \vDash \varphi$, which holds because $\langle b, 0\rangle \le \langle b, c\rangle$ and support is downward persistent. In the right-to-left direction, it suffices to take $c = 0$.

DEFINITION 4.15. *Let λ be a logic of declarative sentences. The inquisitive extension of λ, denoted as $\lambda^?$, is the set of all $L_s^?$-formulas that are valid in every model of λ. In symbols, $\lambda^? = Log^?(Mod(\lambda))$.*

LEMMA 4.16. *It holds for any logic of declarative sentences λ that $Mod(\lambda)$ is closed under products.*

Proof. Take two arbitrary models $\mathcal{M}_1, \mathcal{M}_2$ from $Mod(\lambda)$. If $\alpha \in \lambda$, then $1_1 \vDash \alpha$ in $\mathcal{M}_1$ and $1_2 \vDash \alpha$ in $\mathcal{M}_2$. It follows from Lemma 4.12 that $\langle 1_1, 1_2 \rangle \vDash \alpha$ in $\mathcal{M}_1 \times \mathcal{M}_2$. We have proved that $\mathcal{M}_1 \times \mathcal{M}_2$ is also in $Mod(\lambda)$. □

THEOREM 4.17. *If $BSL \oplus A_\lambda$ is a sound and complete axiomatization of a logic of declarative sentences λ, then $InqBSL \oplus A_\lambda$ is a sound and complete axiomatization of $\lambda^?$.*

Proof. Let us assume that λ is the set of L_s-formulas provable in the system $BSL \oplus A_\lambda$. According to Lemma 4.16, $Mod(\lambda)$ is closed under products of its models. Since $\lambda = Log(Mod(\lambda))$ (Theorem 3.13), it follows from Theorem 4.14 that $\lambda^?$ is the set of $L_s^?$-formulas provable in $InqBSL \oplus A_\lambda$. □

We obtain the following corollary as a special case of the previous theorem, when we assume that A_λ is empty. In the formulation of the corollary, the expression *BSL* (*InqBSL*) does not stand (as usual) for the axiomatic system but rather for the set of L_s-formulas ($L_s^?$-formulas) provable in the system.

COROLLARY 4.18. *$BSL^? = InqBSL$.*

This result can be expressed also in the following alternative way.

COROLLARY 4.19. *The system InqBSL is a sound and complete axiomatization of the set of InqBSL-valid formulas.*

Let us recall that for any logic of declarative sentences λ, $\mathcal{M}^\lambda$ is the canonical model of λ defined in the previous section.

COROLLARY 4.20. *$\lambda^? = Log^?(\mathcal{M}^\lambda)$, for any logic of declarative sentences λ.*

Proof. This follows form Corollaries 3.12 and 4.7 and Theorem 4.17. □

DEFINITION 4.21. *We say that a set of $L_s^?$-formulas is an inquisitive logic if it is the inquisitive extension of some logic of declarative sentences.*

We provide a general characterization of inquisitive logics.

THEOREM 4.22. *A set of $L_s^?$-formulas Δ is an inquisitive logic iff the following four conditions are satisfied*:

1. *Δ contains all the axioms of InqBSL,*
2. *Δ is closed under the rules of InqBSL,*
3. *Δ is closed under substitutions of L_s-formulas.*
4. *Δ is $\mathbb{V}$-prime.*

Proof. First, suppose that Δ is an inquisitive logic and $\Delta = \lambda^?$, where λ is a logic of declarative sentences. The condition 4 follows directly from Lemmas 4.13 and 4.16. Moreover, it follows from Theorem 4.17 that Δ is a set of $L_s^?$-formulas provable in $InqBSL \oplus \lambda$ (we can take all the formulas of λ as the extra axioms of Δ). Then the conditions 1 and

2 have to be satisfied. Furthermore, the assumption that λ is closed under substitutions of L_s-formulas and simple inspection of the axioms and rules of the system *InqBSL* lead to the conclusion that Δ is closed under substitutions of L_s-formulas. So, 3 holds, too.

Second, assume that 1-4 hold. Take $\lambda = \{\alpha \in \Delta; \alpha$ is an L_s-formula$\}$. Then 1-3 guarantee that λ is a logic of declarative sentences. We will show that $\Delta = \lambda^?$. Assume that φ is an $L_s^?$-formula and $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. The following equivalences hold:[12] $\varphi \in \lambda^?$ iff $\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n \in \lambda^?$ iff $\alpha_i \in \lambda$, for some $\alpha_i \in \mathcal{R}(\varphi)$ iff $\alpha_i \in \Delta$, for some $\alpha_i \in \mathcal{R}(\varphi)$ iff $\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n \in \Delta$ iff $\varphi \in \Delta$. □

At the end of this section we will show that every inquisitive logic is related to its background logic of declarative sentences in the same way in which the basic inquisitive logic is related to classical logic. For the special case of *InqB*, this relation was expressed in Theorems 2.4 and 2.5. Now we want to express a generalized version of these results. The generalization of Theorem 2.4 has the following form.

THEOREM 4.23. *Let λ be a logic of declarative sentences and φ an $L_s^?$-formula. $\varphi \in \lambda^?$ iff for some $\alpha \in \mathcal{R}(\varphi)$, $\alpha \in \lambda$.*

Proof. This follows from Theorems 4.3 and 4.22. □

COROLLARY 4.24. *$\lambda^?$ is decidable iff λ is decidable.*

To be able to formulate the generalization of Theorem 2.5, we will have to introduce suitable notation. Let λ be any logic of declarative sentences, $\alpha_1, \dots, \alpha_n, \beta$ any L_s-formulas and $\varphi_1, \dots, \varphi_n, \psi$ any $L_s^?$-formulas. We define:

- $\varphi_1, \dots, \varphi_n \vDash_{\lambda^?} \psi$ iff $(\varphi_1 \wedge \dots \wedge \varphi_n) \to \psi \in \lambda^?$,
- $\alpha_1, \dots, \alpha_n \vDash_\lambda \beta$ iff $(\alpha_1 \wedge \dots \wedge \alpha_n) \to \beta \in \lambda$.

Now the generalization of Theorem 2.5 can be stated.

THEOREM 4.25. *For any logic of declarative sentences λ and any $L_s^?$-formulas $\varphi_1, \dots, \varphi_n, \psi$,*

$$\varphi_1, \dots, \varphi_n \vDash_{\lambda^?} \psi$$

iff for every $\alpha_1 \in \mathcal{R}(\varphi_1), \dots, \alpha_n \in \mathcal{R}(\varphi_n)$ there is $\beta \in \mathcal{R}(\psi)$ such that

$$\alpha_1, \dots, \alpha_n \vDash_\lambda \beta.$$

Proof. First, assume $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi \in \lambda^?$, $\alpha_1 \in \mathcal{R}(\varphi_1), \dots, \alpha_n \in \mathcal{R}(\varphi_n)$. Let $\mathcal{M}$ be a model of λ and a its state such that $a \vDash \alpha_1 \wedge \cdots \wedge \alpha_n$. Then $a \vDash \varphi_1 \wedge \cdots \wedge \varphi_n$, and consequently $a \vDash \psi$. So, $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \psi$ is valid in every model of λ, which means that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \psi \in \lambda^?$. Let $\mathcal{R}(\psi) = \{\beta_1, \dots, \beta_m\}$. It follows that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to (\beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m) \in \lambda^?$. Using soundness of *D*5, we can conclude that $((\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta_1) \vee\!\!\!\vee \cdots \vee\!\!\!\vee ((\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta_m) \in \lambda^?$. Since $\lambda^?$ is $\vee\!\!\!\vee$-prime, it holds for some $\beta \in \mathcal{R}(\psi)$ that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta \in \lambda^?$. Thus, for some $\beta \in \mathcal{R}(\psi)$, $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta \in \lambda$.

Second, assume that for every $\alpha_1 \in \mathcal{R}(\varphi_1), \dots, \alpha_n \in \mathcal{R}(\varphi_n)$ there is some $\beta \in \mathcal{R}(\psi)$ such that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \beta \in \lambda$. Let $\mathcal{M}$ be a model of λ and a its state such that $a \vDash$

[12] Notice that the following part of the proof resembles the proof of Theorem 4.6. However, now instead of implications, we can and must use the corresponding equivalences.

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$\varphi_1 \wedge \cdots \wedge \varphi_n$. Then for some $\alpha_1 \in \mathcal{R}(\varphi_1), \ldots, \alpha_n \in \mathcal{R}(\varphi_n)$, $a \vDash \alpha_1 \wedge \cdots \wedge \alpha_n$. Consequently, $a \vDash \beta$, for some $\beta \in \mathcal{R}(\psi)$. Thus, $a \vDash \psi$. We have proved that $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi \in \lambda^?$. □

§5. Case studies. In the previous section, we formulated a general theory of inquisitive extensions of a large class of logics of declarative sentences. In this section, we apply it to a few important special cases. By definition, the class of all information models provides an adequate semantics for the inquisitive extension of the logic axiomatized by *BSL*. Corollary 4.18 tells us how to axiomatize the extension. By selecting a suitable class of information models, we can obtain an adequate semantics for the inquisitive extension of some special logic that we are interested in. Moreover, if we have the axiomatization of the logic formulated in such a way that it enriches *BSL* with some extra axioms and rules, Theorem 4.17 tells us how to obtain an axiomatization of the inquisitive extension.

In this section, we will consider some important logics and specify semantics and an axiomatization for their inquisitive extensions. The selected logics are: classical logic *Cl*, intuitionistic logic *Int*, and the class of t-norm based fuzzy logics, in particular, Łukasiewicz fuzzy logic *Ł*. Classical logic is the underlying logic of declaratives in the standard system *InqB*. Intuitionistic logic is probably the most famous nonclassical logic and also the first nonclassical logic for which the inquisitive extension was formulated (Punčochář, 2016). Łukasiewicz logic is probably the most prominent fuzzy logic and a typical example of a substructural logic since it does not validate contraction, which corresponds to one of the structural rules.

We will be using these axiomatic schemata:

*Sch*1	$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$	contraposition
*Sch*2	$(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$	transitivity
*Sch*3	$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$	exchange
*Sch*4	$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$	contraction
*Sch*5	$\neg\neg\alpha \to \alpha$	double negation
*Sch*6	$\alpha \to (\beta \to \alpha)$	weakening
*Sch*7	$\neg\alpha \to (\alpha \to \bot)$	ex falso quodlibet
*Sch*8	$(\alpha \to \bot) \to \neg\alpha$	indirect proof
*Sch*9	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
*Sch*10	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Wajsberg axiom.

We assume that the instances of these schemata are L_s-formulas.

5.1. $\boldsymbol{Cl^?}$**.** Let *Cl* be the set of L_s-formulas valid in classical propositional logic. It can be axiomatized by adding the schemata *Sch*1 − *Sch*6 to *BSL*.[13] All the other schemata from the list are derivable in the resulting system. We have already formulated a suitable semantics for the inquisitive extension of *Cl* in our framework. It is provided by the model $\mathcal{M}_{InqB}$ that we constructed in the previous section. It is easy to prove by induction that a state supports a $\vee\!\!\!\vee$-free formula in this model iff the formula is true (according to classical logic) in every world of the state. As a consequence, $Log(\mathcal{M}_{InqB})$ is the set of classically valid formulas. It follows from Corollary 4.7 that $Log^?(\mathcal{M}_{InqB})$ is the inquisitive extension of *Cl* and it can

[13] Usually, $\otimes$ is not present in the language of classical propositional logic because it would collapse to $\wedge$. Here we are still working with the language L_s. In the mentioned axiomatization for classical logic $\alpha \otimes \beta$ is provably equivalent to $\alpha \wedge \beta$.

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be axiomatized by *InqBSL* enriched with schemata $Sch1 - Sch6$. This logic corresponds to the basic propositional inquisitive semantics *InqB*. A system of natural deduction for this logic involving both disjunctions was constructed also in (Ciardelli, 2016a).

An alternative semantics for the inquisitive extension of classical logic can be formulated as follows. Let $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V \rangle$ be an information model. We say that $\mathcal{M}$ is *classical* if (a) $\langle S, +, \cdot, 0, 1 \rangle$ is a bounded Boolean lattice, where $+$ is join, $\cdot$ is meet, 0 is the least element, and 1 is the greatest element; (b) the compatibility relation C is determined by the lattice in the following way:

$$aCb \text{ iff } a \cdot b \neq 0.$$

The valuation V assigns to every atomic formula a principal ideal in $\mathcal{M}$. The following two results can be easily verified.

THEOREM 5.1. *For any L_s-formula α, $\alpha \in Cl$ iff α is valid in every classical model.*

LEMMA 5.2. *The class of classical models is closed under products.*

We proved in Theorems 4.14 and 4.17 that if a class of information models $\mathcal{C}$ is closed under products and $Log(\mathcal{C})$ is λ then $Log^?(\mathcal{C})$ is $\lambda^?$. Lemma 5.2 shows that the class of classical models is closed under products and Theorem 5.1 that this class determines classical logic. As a consequence, we obtain the following semantic characterization of the inquisitive extension of classical logic.

THEOREM 5.3. *For any $L_s^?$-formula φ, $\varphi \in Cl^?$ iff φ is valid in every classical model.*

5.2. $Int^?$. Let *Int* be the set of intuitionistically valid L_s-formulas. An axiomatization of intuitionistic logic can be obtained by adding the schemata $Sch2 - Sch4$ and $Sch6 - Sch8$ to *BSL*.[14] So, the inquisitive extension of *Int* can be axiomatized by adding these schemata to *InqBSL*. A system of natural deduction for the inquisitive extension of intuitionistic logic was introduced in (Punčochář, 2017) and a suitable semantics for this logic was the main topic of that paper. A similar semantics for intuitionistic logic (without inquisitive disjunction) appeared also in (Došen, 1989) and (Ono & Komori, 1985).

In the current framework we can obtain an adequate semantics for intuitionistic logic as follows. Let $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V \rangle$ be an information model. We say that $\mathcal{M}$ is *intuitionistic* if (a) $\langle S, +, \cdot, 0, 1 \rangle$ is a bounded distributive lattice, where $+$ is join, $\cdot$ is meet, 0 is the least element, and 1 is the greatest element; (b) the compatibility relation C is determined again by aCb iff $a \cdot b \neq 0$. The valuation V assigns to every atomic formula an arbitrary ideal in $\mathcal{M}$. We will state the following results without proof since they can be easily reconstructed from (Punčochář, 2017).

THEOREM 5.4. *For any L_s-formula α, $\alpha \in Int$ iff α is valid in every intuitionistic model.*

LEMMA 5.5. *The class of intuitionistic models is closed under products.*

Using Theorems 4.14 and 4.17 we obtain a semantic characterization of the inquisitive extension of intuitionistic logic.

THEOREM 5.6. *For any $L_s^?$-formula φ, $\varphi \in Int^?$ iff φ is valid in every intuitionistic model.*

[14] The schema $Sch1$ is derivable in the resulting system. Again, $\alpha \otimes \beta$ is provably equivalent to $\alpha \wedge \beta$.

It is obvious that classical information models introduced in the previous subsection form a special subclass of intuitionistic information models, namely those based on Boolean lattices and equipped with a valuation assigning principal ideals. Let us state without proof that if we take these Boolean structures and allow valuations assign arbitrary (and not only principal) ideals, then we obtain an alternative semantics for intuitionistic logic and its inquisitive extension.

5.3. Inquisitive extensions of t-norm based fuzzy logics. Our aim in this subsection is to characterize syntactically and semantically the inquisitive extensions of t-norm based fuzzy logics, with a special attention paid to Łukasiewicz fuzzy logic *Ł*. Fuzzy logics were introduced as logical models of vagueness. Vague sentences are not just true or false. They can be more or less true and more or less false. For this reason, the two truth values of classical logic are replaced by a continuum of truth values represented by the closed interval [0, 1]. Following (Hájek, 1998) we can consider propositional fuzzy logics that are fully determined by special binary operations on this interval called continuous t-norms. A continuous t-norm is a continuous, commutative, associative and monotone binary function $*$ on [0, 1] such that $1 * x = x$ and $0 * x = 0$ for any x from [0, 1]. Given a continuous t-norm $*$ there is a unique binary residuated operation $\Rightarrow_*$ on [0, 1] satisfying:

$$x * y \leq z \text{ iff } x \leq y \Rightarrow_* z.$$

Every continuous t-norm $*$ determines a fuzzy logic λ_* in the following way. Any function e that assigns to atomic formulas real numbers from the closed interval [0, 1] will be called a *standard evaluation*. Any standard evaluation e can be extended so that it assigns real numbers from the interval [0, 1] to all L_s-formulas. The value of atomic formulas is given directly by e. The value of the remaining L_s-formulas can be defined recursively by the following clauses:

- $e(\bot) = 0$,
- $e(t) = 1$,
- $e(\neg\alpha) = e(\alpha) \Rightarrow_* 0$,
- $e(\alpha \to \beta) = e(\alpha) \Rightarrow_* e(\beta)$,
- $e(\alpha \wedge \beta) = min\{e(\alpha), e(\beta)\}$,
- $e(\alpha \otimes \beta) = e(\alpha) * e(\beta)$,
- $e(\alpha \vee \beta) = max\{e(\alpha), e(\beta)\}$.

Then for every L_s-formula, we define $\alpha \in \lambda_*$ iff for every standard evaluation e, $e(\alpha) = 1$. Probably the most important and most studied special case is Łukasiewicz fuzzy logic that is determined in this way by the following continuous t-norm:

$$a * b = max\{0, a + b - 1\}.$$

Let us denote this t-norm as the Łukasiewicz t-norm. Let *Ł* be the set of L_s-formulas valid in Łukasiewicz fuzzy logic. It can be axiomatized by *BSL* plus *Sch*2, *Sch*3, *Sch*6 − *Sch*10 (the schemata *Sch*1 and *Sch*5 are derivable in the resulting system). Theorem 4.17 guarantees that if we add these schemata to *InqBSL*, we obtain an axiomatization of $Ł^?$. Our last task is to provide an adequate semantics for $Ł^?$.

The semantics we are going to introduce stems from the algebraic semantics described above. We want to transform algebraic models of fuzzy logic into our information models. However, the algebraic models based on the interval [0, 1] are insufficient for two related reasons. First, inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$ and noninquisitive disjunction $\vee$ collapse into one connective in linear models. This can be easily verified by inspecting the respective

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semantic clauses. The second reason is that the set of algebraic models on the interval [0, 1], given by the set of possible standard evaluations, is not closed under products. To be able to apply Theorem 4.14, we need to close the class of models under products. This motivates the following construction in which models are built from n-tuples of real numbers instead of single numbers.

We will take a more general perspective and construct an adequate semantics for any t-norm based fuzzy logic. Let us fix an arbitrary continuous t-norm $*$. We introduce a class of information models that we will call *fuzzy models for* $*$. These are structures of the form $\mathcal{M}^n_E = \langle S, +, \cdot, 0_n, 1_n, C, V\rangle$, where $n \geq 1$ is a natural number, $E = \langle e_1, \ldots, e_n\rangle$ is an n-tuple of standard evaluations, and it holds:

- $S = [0, 1]^n$,
- $\langle x_1, \ldots, x_n\rangle + \langle y_1, \ldots, y_n\rangle = \langle max\{x_1, y_1\}, \ldots, max\{x_n, y_n\}\rangle$,
- $\langle x_1, \ldots, x_n\rangle \cdot \langle y_1, \ldots, y_n\rangle = \langle x_1 * y_1, \ldots, x_n * y_n\rangle$,
- 1_n is an n-place sequence of 1's,
- 0_n is an n-place sequence of 0's,
- $\langle x_1, \ldots, x_n\rangle C\langle y_1, \ldots, y_n\rangle$ iff for some i $(1 \leq i \leq n)$, $x_i * y_i \neq 0$,
- $\langle x_1, \ldots, x_n\rangle \in V(p)$ iff for all i $(1 \leq i \leq n)$, $x_i \leq e_i(p)$.

Note that $\langle x_1, \ldots, x_n\rangle \leq \langle y_1, \ldots, y_n\rangle$ iff for all i $(1 \leq i \leq n)$, $x_i \leq y_i$. The following lemmas can be proved in a strightforward way.

LEMMA 5.7. *Every fuzzy model for $*$ is an information model.*

We say that a fuzzy model $\mathcal{M}^n_E$ for $*$ is *simple* if $n = 1$. In this case, E is just a standard evaluation.[15]

LEMMA 5.8. *Assume that $\mathcal{M}^1_E = \langle S, +, \cdot, 0, 1, C, V\rangle$ is a simple fuzzy model, $x \in [0, 1]$, and α is an L_s-formula. Then $x \vDash \alpha$ in $\mathcal{M}^1_E$ iff $x \leq E(\alpha)$.*

Proof. A straightforward induction on the complexity of α. □

It is not problematic to assume that, in general,

(A) $\langle\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_m\rangle\rangle = \langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$.

Then we can observe that every fuzzy model is the product of a finite set of simple models:

$$\text{if } E = \langle e_1, \ldots, e_n\rangle, \text{ then } \mathcal{M}^n_E = \mathcal{M}^1_{e_1} \times \cdots \times \mathcal{M}^1_{e_n}.$$

Using Lemma 4.12, we get the following generalization of Lemma 5.8.

LEMMA 5.9. *Let $\mathcal{M}^n_E$ be a fuzzy model for $*$, where $E = \langle e_1, \ldots, e_n\rangle$, $a_1, \ldots, a_n \in [0, 1]$, and α an L_s-formula. Then $\langle a_1, \ldots, a_n\rangle \vDash \alpha$ in $\mathcal{M}^n_E$ iff for every i $(1 \leq i \leq n)$, $a_i \leq e_i(\alpha)$.*

THEOREM 5.10. *For any L_s-formula α, $\alpha \in \lambda_*$ iff α is valid in every fuzzy model for $*$.*

Proof. It holds that $\alpha \in \lambda_*$ iff $e(\alpha) = 1$, for every standard evaluation e iff (Lemma 5.9) α is valid in every fuzzy model for $*$. □

Under the assumption (A), the following Lemma is obvious.

LEMMA 5.11. *The class of fuzzy models is closed under products.*

[15] The following result can be generalized to all residuated lattices.

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THEOREM 5.12. *For any $L_s^?$-formula φ, $\varphi \in \lambda_*^?$ iff φ is valid in every fuzzy model for $*$.*

Proof. This follows from Theorems 4.14, 4.17, and 5.10 and Lemma 5.11. □

COROLLARY 5.13. *For any $L_s^?$-formula φ, $\varphi \in Ł^?$ iff φ is valid in every fuzzy model for the Łukasiewicz t-norm.*

We have shown that the inquisitive extension of Łukasiewicz fuzzy logic is sound and complete with respect to the class of models that we have constructed. We leave for future research whether this formal semantics has an intuitive informal interpretation comparable to the interpretation of the standard propositional inquisitive semantics.

§6. Conclusion. The main aim of this paper has been to show that the methods of logical modelling of questions that are used in inquisitive semantics are quite universal and that any propositional logic extending a basic substructural logic can be enriched with questions in the style of inquisitive semantics.

Let us summarize the main results of the paper. In §3, we have introduced a relational semantics based on "information models" and a Hilbert system *BSL*, a nondistributive, nonassociative, and noncommutative version of Full Lambek logic with a paraconsistent negation, and we demonstrated completeness of the system with respect to the semantics.§4 shows that the semantic framework is suitable for the introduction of questions in the style of inquisitive semantics, i.e., via the so called inquisitive disjunction. The notion of inquisitive extensions of substructural logics was introduced. For any logic λ extending *BSL*, the inquisitive extension of λ, denoted as $\lambda^?$, was defined, and it was shown how to axiomatize $BSL^?$, and any $\lambda^?$, given the axiomatization of λ. We have shown that $\lambda^?$ is complete with respect to any model with respect to which λ is complete, and the same holds for any class of information models that is closed under products. In §5, we found suitable classes of information models for the inquisitive extensions of classical logic *Cl*, intuitionistic logic *Int*, and Łukasiewicz fuzzy logic *Ł*. In future research, we would like to develop, within our general framework, semantics for other particular cases of substructural logics. In particular, we would like to focus on the inquisitive extensions of the relevant logics *R* and *E*. We expect that a systematic study of a well-behaving relevant logic of questions might be of special interest.

§7. Appendix. Proofs of three Theorems were left for this Appendix. In particular, we have to prove Theorems 3.10, 3.11, and 4.3.

Proof of Theorem 3.10. We say that an inference rule is *BSL*-admissible if it holds for any instance $\alpha_1, \ldots, \alpha_n/\beta$ of the rule that if $\alpha_1, \ldots, \alpha_n$ are *BSL*-provable, then β is also *BSL*-provable. The following Lemma shows some examples of *BSL*-provable formulas and *BSL*-admissible rules that will be used in the proof of the Theorem.

LEMMA 7.1. *The instances of the following schemata are BSL-provable*:

(a) $\alpha \to \neg\neg\alpha$,
(b) $(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$.

Moreover, the following rules are BSL-admissible:

(c) $\alpha \to \beta, \beta \to \gamma / \alpha \to \gamma$,
(d) $\alpha, \beta / \alpha \wedge \beta$,

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(e) $\alpha \to \beta, \gamma \to \delta/(\alpha \otimes \gamma) \to (\beta \otimes \delta)$,

(f) $(\delta_1 \otimes \gamma_1) \to \alpha, (\delta_2 \otimes \gamma_2) \to \beta/((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to (\alpha \wedge \beta)$,

(g) $(\beta_1 \otimes \gamma_1) \to \alpha, (\beta_2 \otimes \gamma_2) \to \alpha/((\beta_1 \wedge \beta_2) \otimes (\gamma_1 \vee \gamma_2)) \to \alpha$,

(h) $(\gamma_1 \otimes \beta_1) \to \alpha, (\gamma_2 \otimes \beta_2) \to \alpha/((\gamma_1 \vee \gamma_2) \otimes (\beta_1 \wedge \beta_2)) \to \alpha$.

Proof. (a) $\alpha \to \neg\neg\alpha$ can be derived in *BSL* by this two-step proof: $\neg\alpha \to \neg\alpha$ (*A*1), $\alpha \to \neg\neg\alpha$ (*R*9).

Before proving (b), we will prove (c): $\alpha \to \beta, \beta \to \gamma$ (assumptions), $(\beta \to \gamma) \to (\alpha \to \gamma)$ (*R*2), $\alpha \to \gamma$ (*R*1).

(b) $(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$ is *BSL*-provable:

1.	$(\alpha \wedge \beta) \to \alpha$	*A*3
2.	$\alpha \to \neg\neg\alpha$	L7.1(a)
3.	$(\alpha \wedge \beta) \to \neg\neg\alpha$	1,2, L7.1(c)
4.	$\neg\alpha \to \neg(\alpha \wedge \beta)$	3, *R*9
5.	$(\alpha \wedge \beta) \to \beta$	*A*4
6.	$\beta \to \neg\neg\beta$	L7.1(a)
7.	$(\alpha \wedge \beta) \to \neg\neg\beta$	5,6, L7.1(c)
8.	$\neg\beta \to \neg(\alpha \wedge \beta)$	7, *R*9
9.	$(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$	4,8, *R*4

(d) $\alpha, \beta/\alpha \wedge \beta$ is *BSL*-admissible: α, β (assumptions), $t \to \alpha, t \to \beta$ (*R*8), $t \to (\alpha \wedge \beta)$ (*R*3), $\alpha \wedge \beta$ (*R*7).

(e) $\alpha \to \beta, \gamma \to \delta/(\alpha \otimes \gamma) \to (\beta \otimes \delta)$ is *BSL*-admissible:

1.	$\gamma \to \delta$	assumption
2.	$(\delta \to (\beta \otimes \delta)) \to (\gamma \to (\beta \otimes \delta))$	1, *R*2
3.	$(\beta \otimes \delta) \to (\beta \otimes \delta)$	*A*1
4.	$\beta \to (\delta \to (\beta \otimes \delta))$	3, *R*6
5.	$\beta \to (\gamma \to (\beta \otimes \delta))$	2,4, L7.1(c)
6.	$(\beta \otimes \gamma) \to (\beta \otimes \delta)$	5, *R*5
7.	$(\beta \otimes \gamma) \to (\beta \otimes \gamma)$	*A*1
8.	$\beta \to (\gamma \to (\beta \otimes \gamma))$	7, *R*6
9.	$\alpha \to \beta$	assumption
10.	$\alpha \to (\gamma \to (\beta \otimes \gamma))$	8,9, L7.1(c)
11.	$(\alpha \otimes \gamma) \to (\beta \otimes \gamma)$	10, *R*5
12.	$(\alpha \otimes \gamma) \to (\beta \otimes \delta)$	6,11, L7.1(c)

(f) $(\delta_1 \otimes \gamma_1) \to \alpha, (\delta_2 \otimes \gamma_2) \to \beta/((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to (\alpha \wedge \beta)$ is *BSL*-admissible:

1.	$(\delta_1 \wedge \delta_2) \to \delta_1$	*A*3
2.	$(\gamma_1 \wedge \gamma_2) \to \gamma_1$	*A*3
3.	$((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to (\delta_1 \otimes \gamma_1)$	1,2, L7.1(e)
4.	$(\delta_1 \wedge \delta_2) \to \delta_2$	*A*4
5.	$(\gamma_1 \wedge \gamma_2) \to \gamma_2$	*A*4
6.	$((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to (\delta_2 \otimes \gamma_2)$	4,5, L7.1(e)
7.	$(\delta_1 \otimes \gamma_1) \to \alpha$	assumption
8.	$((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to \alpha$	3,7, L7.1(c)
9.	$(\delta_2 \otimes \gamma_2) \to \beta$	assumption
10.	$((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to \beta$	6,9, L7.1(c)
11.	$((\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2)) \to (\alpha \wedge \beta)$	8,10, *R*3

(g) $(\beta_1 \otimes \gamma_1) \rightarrow \alpha, (\beta_2 \otimes \gamma_2) \rightarrow \alpha / ((\beta_1 \wedge \beta_2) \otimes (\gamma_1 \vee \gamma_2)) \rightarrow \alpha$ is *BSL*-admissible:

1.	$(\beta_1 \wedge \beta_2) \rightarrow \beta_1$	*A*3
2.	$\gamma_1 \rightarrow \gamma_1$	*A*1
3.	$((\beta_1 \wedge \beta_2) \otimes \gamma_1) \rightarrow (\beta_1 \otimes \gamma_1)$	1,2, L7.1(e)
4.	$(\beta_1 \otimes \gamma_1) \rightarrow ((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2))$	*A*5
5.	$((\beta_1 \wedge \beta_2) \otimes \gamma_1) \rightarrow ((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2))$	3,4, L7.1(c)
6.	$(\beta_1 \wedge \beta_2) \rightarrow \beta_2$	*A*4
7.	$\gamma_2 \rightarrow \gamma_2$	*A*1
8.	$((\beta_1 \wedge \beta_2) \otimes \gamma_2) \rightarrow (\beta_2 \otimes \gamma_2)$	6,7, L7.1(e)
9.	$(\beta_2 \otimes \gamma_2) \rightarrow ((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2))$	*A*6
10.	$((\beta_1 \wedge \beta_2) \otimes \gamma_2) \rightarrow ((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2))$	8,9, L7.1(c)
11.	$(((\beta_1 \wedge \beta_2) \otimes \gamma_1) \vee ((\beta_1 \wedge \beta_2) \otimes \gamma_2)) \rightarrow$ $\rightarrow ((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2))$	5,10, *R*4
12.	$((\beta_1 \wedge \beta_2) \otimes (\gamma_1 \vee \gamma_2)) \rightarrow$ $\rightarrow (((\beta_1 \wedge \beta_2) \otimes \gamma_1) \vee ((\beta_1 \wedge \beta_2) \otimes \gamma_2))$	*D*1
13.	$((\beta_1 \wedge \beta_2) \otimes (\gamma_1 \vee \gamma_2)) \rightarrow ((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2))$	11,12, 7.1(c)
14.	$(\beta_1 \otimes \gamma_1) \rightarrow \alpha$	assumption
15.	$(\beta_2 \otimes \gamma_2) \rightarrow \alpha$	assumption
16.	$((\beta_1 \otimes \gamma_1) \vee (\beta_2 \otimes \gamma_2)) \rightarrow \alpha$	14,15, *R*4
17.	$((\beta_1 \wedge \beta_2) \otimes (\gamma_1 \vee \gamma_2)) \rightarrow \alpha$	13,16, 7.1(c)

(h) $(\gamma_1 \otimes \beta_1) \rightarrow \alpha, (\gamma_2 \otimes \beta_2) \rightarrow \alpha / ((\gamma_1 \vee \gamma_2) \otimes (\beta_1 \wedge \beta_2)) \rightarrow \alpha$ is *BSL*-admissible:

1.	$(\gamma_1 \otimes \beta_1) \rightarrow \alpha$	assumption
2.	$\gamma_1 \rightarrow (\beta_1 \rightarrow \alpha)$	1, *R*6
3.	$(\beta_1 \wedge \beta_2) \rightarrow \beta_1$	*A*3
4.	$(\beta_1 \rightarrow \alpha) \rightarrow ((\beta_1 \wedge \beta_2) \rightarrow \alpha)$	3, *R*2
5.	$\gamma_1 \rightarrow ((\beta_1 \wedge \beta_2) \rightarrow \alpha)$	2,4, L7.1(c)
6.	$(\gamma_2 \otimes \beta_2) \rightarrow \alpha$	assumption
7.	$\gamma_2 \rightarrow (\beta_2 \rightarrow \alpha)$	6, *R*6
8.	$(\beta_1 \wedge \beta_2) \rightarrow \beta_2$	*A*4
9.	$(\beta_2 \rightarrow \alpha) \rightarrow ((\beta_1 \wedge \beta_2) \rightarrow \alpha)$	8, *R*2
10.	$\gamma_2 \rightarrow ((\beta_1 \wedge \beta_2) \rightarrow \alpha)$	7,9, L7.1(c)
11.	$(\gamma_1 \vee \gamma_2) \rightarrow ((\beta_1 \wedge \beta_2) \rightarrow \alpha)$	5,10, *R*4
12.	$((\gamma_1 \vee \gamma_2) \otimes (\beta_1 \wedge \beta_2)) \rightarrow \alpha$	11, *R*5

□

We are proving that the canonical structure $\mathcal{M}^\lambda$ is indeed an information model. This is a direct consequence of the Lemmas 7.2–7.6.

LEMMA 7.2. *0 and 1 are λ-theories. Moreover, if Δ and Γ are λ-theories, then $\Delta + \Gamma$ and $\Delta \cdot \Gamma$ are also λ-theories.*

Proof. Obviously, the set of all L_s-formulas is a λ-theory. 1 is a λ-theory due to the rule *R*1 and Lemma 7.1(d).

It is easy to verify that λ-theories are closed under intersection, so the case of $+$ is immediate. Suppose that Δ and Γ are λ-theories. We will show that $\Delta \cdot \Gamma$ is also a λ-theory. First, assume that $\alpha, \beta \in \Delta \cdot \Gamma$, i.e., there are $\delta_1, \delta_2 \in \Delta$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $(\delta_1 \otimes \gamma_1) \vdash_\lambda \alpha$ and $(\delta_2 \otimes \gamma_2) \vdash_\lambda \beta$. Lemma 7.1(f) gives us $(\delta_1 \wedge \delta_2) \otimes (\gamma_1 \wedge \gamma_2) \vdash_\lambda \alpha \wedge \beta$. Since $\delta_1 \wedge \delta_2 \in \Delta$ and $\gamma_1 \wedge \gamma_2 \in \Gamma$, $\alpha \wedge \beta \in \Delta \cdot \Gamma$. It remains to be proved that if $\alpha \in \Delta \cdot \Gamma$ and $\alpha \vdash_\lambda \beta$, then $\beta \in \Delta \cdot \Gamma$. Assume $\alpha \in \Delta \cdot \Gamma$ and $\alpha \vdash_\lambda \beta$. So, there is $\delta \in \Delta$ and $\gamma \in \Gamma$ such that $\delta \otimes \gamma \vdash_\lambda \alpha$. Due to Lemma 7.1(c), $\delta \otimes \gamma \vdash_\lambda \beta$. So $\beta \in \Delta \cdot \Gamma$. □

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LEMMA 7.3. *In the canonical model, $\langle S, +, 0\rangle$ is a join-semilattice with the least element* 0*.*

Proof. It is obvious that $\langle S, \cap\rangle$ is a semilattice. The ordering $\leq$ is identical with the superset relation $\supseteq$. The state 0, i.e., the set of all L_s-formulas, is the least element of S with respect to this ordering. □

LEMMA 7.4. *For any λ-theories Γ, Δ, Ω the following conditions are satisfied*:

(a) $\Omega \cdot (\Gamma + \Delta) = (\Omega \cdot \Gamma) + (\Omega \cdot \Delta)$,
(b) $(\Gamma + \Delta) \cdot \Omega = (\Gamma \cdot \Omega) + (\Delta \cdot \Omega)$,
(c) $1 \cdot \Gamma = \Gamma$,
(d) $0 \cdot \Gamma = 0$.

Proof. (a) First, we will prove $\Omega \cdot (\Gamma \cap \Delta) \subseteq (\Omega \cdot \Gamma) \cap (\Omega \cdot \Delta)$. Assume that $\alpha \in \Omega \cdot (\Gamma \cap \Delta)$. So, there are $\beta \in \Omega$ and $\gamma \in \Gamma \cap \Delta$ such that $\beta \otimes \gamma \vdash_\lambda \alpha$. Since $\gamma \in \Gamma$ and $\gamma \in \Delta$, it holds that $\alpha \in (\Omega \cdot \Gamma) \cap (\Omega \cdot \Delta)$.

Now we will prove $(\Omega \cdot \Gamma) \cap (\Omega \cdot \Delta) \subseteq \Omega \cdot (\Gamma \cap \Delta)$. Assume that $\alpha \in \Omega \cdot \Gamma$ and $\alpha \in \Omega \cdot \Delta$. This means that there are $\beta_1 \in \Omega$ and $\gamma_1 \in \Gamma$ such that $\beta_1 \otimes \gamma_1 \vdash_\lambda \alpha$, and there are $\beta_2 \in \Omega$ and $\gamma_2 \in \Delta$ such that $\beta_2 \otimes \gamma_2 \vdash_\lambda \alpha$. It follows from Lemma 7.1(g) that $(\beta_1 \wedge \beta_2) \otimes (\gamma_1 \vee \gamma_2) \vdash_\lambda \alpha$. It holds that $\beta_1 \wedge \beta_2 \in \Omega$. Moreover, due to axiom schemata $A5$ and $A6$, $\gamma_1 \vee \gamma_2 \in \Gamma \cap \Delta$. As a consequence, $\alpha \in \Omega \cdot (\Gamma \cap \Delta)$.

(b) is proved similarly using Lemma 7.1(h) and axioms $A5$ and $A6$.

(c) First, assume $\alpha \in \lambda \cdot \Gamma$. There are $\beta \in \lambda$, $\gamma \in \Gamma$ such that $\beta \otimes \gamma \vdash_\lambda \alpha$. Due to $R6$ and $R1$, $\gamma \vdash_\lambda \alpha$. So, $\alpha \in \Gamma$.

Assume $\alpha \in \Gamma$. Using $A1$, $R5$ and $R8$, we obtain $t \otimes \alpha \vdash_\lambda \alpha$. Since $t \in \lambda$ (due to $A1$ and $R7$), it holds $\alpha \in \lambda \cdot \Gamma$.

(d) We show that $\alpha \in 0 \cdot \Gamma$, for any L_s-formula α. Take an arbitrary L_s-formula $\beta \in \Gamma$. It holds that $\bot \otimes \beta \vdash_\lambda \alpha$ (due to $A2$ and $R5$). Since $\bot \in 0$, it follows that $\alpha \in 0 \cdot \Gamma$. □

LEMMA 7.5. *For any λ-theories Γ, Δ, Ω the following conditions are satisfied*:

(a) *it is not the case that* $0C\Gamma$,
(b) *if* $\Gamma C \Delta$*, then* $\Delta C \Gamma$,
(c) *if* $\Gamma C \Delta$ *and* $\Gamma \leq \Omega$*, then* $\Omega C \Delta$,
(d) *if* $(\Gamma + \Delta) C \Omega$*, then* $\Gamma C \Omega$ *or* $\Delta C \Omega$.

Proof. The conditions (a) and (c) are obvious. We will prove that (b) and (d) hold. (b) Assume that $\Delta C \Gamma$ does not hold. So, there is an L_s-formula β such that $\neg\beta \in \Delta$ and $\beta \in \Gamma$. According to Lemma 7.1(a), $\beta \vdash_\lambda \neg\neg\beta$. So, $\neg\neg\beta \in \Gamma$. Since $\neg\beta \in \Delta$, it does not hold that $\Gamma C \Delta$.

(d) Assume that neither $\Gamma C \Omega$, nor $\Delta C \Omega$. Therefore, there are two L_s-formulas α and β such that $\neg\alpha \in \Gamma$, $\alpha \in \Omega$, $\neg\beta \in \Delta$, $\beta \in \Omega$. We will show that it does not hold that $(\Gamma \cap \Delta) C \Omega$. Due to $A5$ and $A6$, $\neg\alpha \vee \neg\beta \in \Gamma \cap \Delta$. It follows from Lemma 7.1(b) that $\neg(\alpha \wedge \beta) \in \Gamma \cap \Delta$. However, $\alpha \wedge \beta \in \Omega$. □

LEMMA 7.6. *For any atomic formula p, the following holds*:

(a) $0 \in V(p)$,
(b) *if* $\Gamma \in V(p)$ *and* $\Delta \leq \Gamma$*, then* $\Delta \in V(p)$,
(c) *if* $\Gamma \in V(p)$ *and* $\Delta \in V(p)$*, then* $\Gamma + \Delta \in V(p)$.

Proof. Obvious. □

This finishes the proof of Theorem 3.10.

Proof of Theorem 3.11. We are proving that for any L_s-formula α and any λ-theory Γ, it holds in $\mathcal{M}^\lambda$ that $\Gamma \vDash \alpha$ iff $\alpha \in \Gamma$.

We will proceed by induction. The cases for atomic formulas, for t and $\bot$ are straightforward and use $A1$, $A2$, $R1$, $R7$, and $R8$. The induction hypothesis is that the statement holds for some L_s-formulas α and β. For any L_s-formula γ we will define the following set:

$$Th(\gamma) = \{\delta; \gamma \vdash_\lambda \delta\}.$$

Note that the rule $R3$ and Lemma 7.1(c) guarantee that $Th(\gamma)$ is an λ-theory. Moreover, $\gamma \in Th(\gamma)$, due to $A1$.

Negation. It holds that $\Gamma \vDash \neg\alpha$ iff for every λ-theory Δ, if $\Delta C \Gamma$, then $\Delta \nvDash \alpha$ iff for every λ-theory Δ, if $\Delta C \Gamma$, then $\alpha \notin \Delta$ iff $\neg\alpha \in \Gamma$. We will prove the last equivalence. *First*, assume $\neg\alpha \notin \Gamma$. We have to show that there is a λ-theory Δ such that $\Delta C \Gamma$ and $\alpha \in \Delta$. Take $\Delta = Th(\alpha)$. So, Δ is a λ-theory such that $\alpha \in \Delta$. For the contradiction, assume that there is β such that $\neg\beta \in \Delta$ but $\beta \in \Gamma$. So, $\alpha \vdash_\lambda \neg\beta$. Due to $R9$, $\beta \vdash_\lambda \neg\alpha$, so $\neg\alpha \in \Gamma$, which contradicts our assumption. *Second*, if we assume that there is λ-theory Δ such that $\Delta C \Gamma$ and $\alpha \in \Delta$, it immediately follows that $\neg\alpha \notin \Gamma$.

Implication. It holds that $\Gamma \vDash \alpha \to \beta$ iff for every λ-theory Δ, if $\Delta \vDash \alpha$, then $\Gamma \cdot \Delta \vDash \beta$ iff for every λ-theory Δ, if $\alpha \in \Delta$, then $\beta \in \Gamma \cdot \Delta$ iff $\alpha \to \beta \in \Gamma$. We will prove the last equivalence. *First*, assume $\alpha \to \beta \in \Gamma$. Take an arbitrary λ-theory Δ such that $\alpha \in \Delta$. It holds that $(\alpha \to \beta) \otimes \alpha \vdash_\lambda \beta$ (due to $A1$ and $R5$). So, $\beta \in \Gamma \cdot \Delta$. *Second*, assume $\alpha \to \beta \notin \Gamma$. We have to prove that there is a λ-theory Δ such that $\alpha \in \Delta$ and $\beta \notin \Gamma \cdot \Delta$. Let $\Delta = Th(\alpha)$. So, Δ is a λ-theory such that $\alpha \in \Delta$. For the contradiction, suppose that $\beta \in \Gamma \cdot \Delta$. That means that there are $\gamma \in \Gamma$ and $\delta \in \Delta$ such that $\gamma \otimes \delta \vdash_\lambda \beta$. It follows that $\alpha \vdash_\lambda \delta$, and due to $A1$ and Lemma 7.1(e), $\gamma \otimes \alpha \vdash_\lambda \gamma \otimes \delta$. Using Lemma 7.1(c), we obtain $\gamma \otimes \alpha \vdash_\lambda \beta$. Then, due to $R6$, $\gamma \vdash_\lambda \alpha \to \beta$. Therefore, $\alpha \to \beta \in \Gamma$, which is in contradiction with our assumption.

Conjunction. $\Gamma \vDash \alpha \wedge \beta$ iff $\Gamma \vDash \alpha$ and $\Gamma \vDash \beta$ iff $\alpha \in \Gamma$ and $\beta \in \Gamma$ iff $\alpha \wedge \beta \in \Gamma$. The last equivalence is due to $A3$ and $A4$, and the fact that Γ is a λ-theory.

Fusion. $\Gamma \vDash \alpha \otimes \beta$ iff there are λ-theories Δ, Ω such that $\Delta \vDash \alpha$, $\Omega \vDash \beta$, and $\Delta \cdot \Omega \subseteq \Gamma$ iff there are λ-theories Δ, Ω such that $\alpha \in \Delta$, $\beta \in \Omega$, and $\Delta \cdot \Omega \subseteq \Gamma$ iff $\alpha \otimes \beta \in \Gamma$. We will prove the last equivalence. *First*, assume that there are λ-theories Δ, Ω such that $\alpha \in \Delta$, $\beta \in \Omega$, and $\Delta \cdot \Omega \subseteq \Gamma$. It holds that $\alpha \otimes \beta \in \Delta \cdot \Omega$ ($A1$) and thus $\alpha \otimes \beta \in \Gamma$. *Second*, assume $\alpha \otimes \beta \in \Gamma$. Let $\Delta = Th(\alpha)$ and $\Omega = Th(\beta)$. So, Δ, Ω are λ-theories such that $\alpha \in \Delta$, $\beta \in \Omega$. We have to show that $\Delta \cdot \Omega \subseteq \Gamma$. Let $\gamma \in \Delta \cdot \Omega$. So, for some $\delta \in \Delta$ and $\omega \in \Omega$, $\delta \otimes \omega \vdash_\lambda \gamma$. It follows that $\alpha \vdash_\lambda \delta$ and $\beta \vdash_\lambda \omega$. From Lemma 7.1(e) we obtain $\alpha \otimes \beta \vdash_\lambda \delta \otimes \omega$. Therefore, $\delta \otimes \omega \in \Gamma$ and so also $\gamma \in \Gamma$.

Disjunction. It holds that $\Gamma \vDash \alpha \vee \beta$ iff there are λ-theories Δ, Ω such that $\Delta \vDash \alpha$, $\Omega \vDash \beta$, and $\Delta \cap \Omega \subseteq \Gamma$ iff there are λ-theories Δ, Ω such that $\alpha \in \Delta$, $\beta \in \Omega$, and $\Delta \cap \Omega \subseteq \Gamma$ iff $\alpha \vee \beta \in \Gamma$. We will prove the last equivalence. *First*, assume that there are λ-theories Δ, Ω such that $\alpha \in \Delta$, $\beta \in \Omega$, and $\Delta \cap \Omega \subseteq \Gamma$. Due to $A5$ and $A6$, $\alpha \vee \beta \in \Delta \cap \Omega$. So, $\alpha \vee \beta \in \Gamma$. *Second*, assume $\alpha \vee \beta \in \Gamma$. Let $\Delta = Th(\alpha)$ and $\Omega = Th(\beta)$. So, Δ, Ω are λ-theories such that $\alpha \in \Delta$, $\beta \in \Omega$. We have to show that $\Delta \cap \Omega \subseteq \Gamma$. Let $\gamma \in \Delta \cap \Omega$. So, $\alpha \vdash_\lambda \gamma$ and $\beta \vdash_\lambda \gamma$. Using $R4$ we obtain $\alpha \vee \beta \vdash_\lambda \gamma$. It follows that $\gamma \in \Gamma$. This finishes the proof of Theorem 3.11.

Proof of Theorem 4.3. We are proving that in *InqBSL* every $L_s^?$-formula is provably equivalent to the inquisitive disjunction of its resolutions. We will need two Lemmas. The first says that provably equivalent $L_s^?$-formulas are replaceable in *InqBSL*.

LEMMA 7.7. *Let φ, ψ, χ be any $L_s^?$-formulas such that φ is a subformula of χ, and let $\chi[\psi/\varphi]$ be an $L_s^?$-formula that is obtained from χ by the replacement of an occurrence of the subformula φ with ψ. Then*

$$\text{if } \varphi \dashv\vdash_{InqBSL} \psi, \text{ then also } \chi \dashv\vdash_{InqBSL} \chi[\psi/\varphi].$$

Proof. Assume $\varphi \dashv\vdash_{InqBSL} \psi$. We need to prove $\neg\varphi \dashv\vdash_{InqBSL} \neg\psi$, and that for any $L_s^?$-formula ϑ:

(a)	$\vartheta \to \varphi \dashv\vdash_{InqBSL} \vartheta \to \psi$,	(b)	$\varphi \to \vartheta \dashv\vdash_{InqBSL} \psi \to \vartheta$,
(c)	$\vartheta \wedge \varphi \dashv\vdash_{InqBSL} \vartheta \wedge \psi$,	(d)	$\varphi \wedge \vartheta \dashv\vdash_{InqBSL} \psi \wedge \vartheta$,
(e)	$\vartheta \otimes \varphi \dashv\vdash_{InqBSL} \vartheta \otimes \psi$,	(f)	$\varphi \otimes \vartheta \dashv\vdash_{InqBSL} \psi \otimes \vartheta$,
(g)	$\vartheta \vee \varphi \dashv\vdash_{InqBSL} \vartheta \vee \psi$,	(h)	$\varphi \vee \vartheta \dashv\vdash_{InqBSL} \psi \vee \vartheta$,
(i)	$\vartheta \mathbin{\vee\!\!\!\vee} \varphi \dashv\vdash_{InqBSL} \vartheta \mathbin{\vee\!\!\!\vee} \psi$,	(j)	$\varphi \mathbin{\vee\!\!\!\vee} \vartheta \dashv\vdash_{InqBSL} \psi \mathbin{\vee\!\!\!\vee} \vartheta$.

We will not go into details but $\neg\varphi \dashv\vdash_{InqBSL} \neg\psi$ can be obtained from $A1$, $R1$, $R2$, and $R9$. (a) is given by $A1$, $R1$, $R2$, $R5$, and $R6$, (b) by $R2$. (c), (d) can be obtained from $R1$, $R2$, $R3$, $A3$, and $A4$. (e) is due to $A1$ and a *InqBSL*-variant of Lemma 7.1(e), and (f) due to $A1$, $R1$, $R2$, $R5$, and $R6$. (g) and (h) is given by $A1$ and $R4^*$, and (i), (j) can be obtained using $A9$, $A10$, $R1$, $R2$, and $R10$. □

LEMMA 7.8. *Let φ, ψ, χ be any $L_s^?$-formulas and α any L_s-formula. Then it holds*:

(a)	$\neg(\varphi \mathbin{\vee\!\!\!\vee} \psi) \dashv\vdash_{InqBSL} \neg\varphi \wedge \neg\psi$,
(b)	$\alpha \to (\psi \mathbin{\vee\!\!\!\vee} \chi) \dashv\vdash_{InqBSL} (\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi)$.
(c)	$\varphi \wedge (\psi \mathbin{\vee\!\!\!\vee} \chi) \dashv\vdash_{InqBSL} (\varphi \wedge \psi) \mathbin{\vee\!\!\!\vee} (\varphi \wedge \chi)$,
(d)	$\varphi \otimes (\psi \mathbin{\vee\!\!\!\vee} \chi) \dashv\vdash_{InqBSL} (\varphi \otimes \psi) \mathbin{\vee\!\!\!\vee} (\varphi \otimes \chi)$,
(e)	$\varphi \vee (\psi \mathbin{\vee\!\!\!\vee} \chi) \dashv\vdash_{InqBSL} (\varphi \vee \psi) \mathbin{\vee\!\!\!\vee} (\varphi \vee \chi)$,
(f)	$(\varphi \mathbin{\vee\!\!\!\vee} \psi) \to \chi \dashv\vdash_{InqBSL} (\varphi \to \chi) \wedge (\psi \to \chi)$.
(g)	$(\varphi \mathbin{\vee\!\!\!\vee} \psi) \wedge \chi \dashv\vdash_{InqBSL} (\varphi \wedge \chi) \mathbin{\vee\!\!\!\vee} (\psi \wedge \chi)$,
(h)	$(\varphi \mathbin{\vee\!\!\!\vee} \psi) \otimes \chi \dashv\vdash_{InqBSL} (\varphi \otimes \chi) \mathbin{\vee\!\!\!\vee} (\psi \otimes \chi)$,
(i)	$(\varphi \mathbin{\vee\!\!\!\vee} \psi) \vee \chi \dashv\vdash_{InqBSL} (\varphi \vee \chi) \mathbin{\vee\!\!\!\vee} (\psi \vee \chi)$.

Proof. We will show the details of just a few cases. First, let us consider the right-to-left implication of (a):

1.	$(\neg\varphi \wedge \neg\psi) \to \neg\varphi$	$A3$
2.	$\varphi \to \neg(\neg\varphi \wedge \neg\psi)$	1, $R9$
3.	$(\neg\varphi \wedge \neg\psi) \to \neg\psi$	$A4$
4.	$\psi \to \neg(\neg\varphi \wedge \neg\psi)$	3, $R9$
5.	$(\varphi \mathbin{\vee\!\!\!\vee} \psi) \to \neg(\neg\varphi \wedge \neg\psi)$	2,4, $R10$
6.	$(\neg\varphi \wedge \neg\psi) \to \neg(\varphi \mathbin{\vee\!\!\!\vee} \psi)$	5, $R9$.

The left-to-right implications of (b)–(e) are the Axioms $D2 - D5$. The converse implications can be derived in the system. For an illustration, we will prove the right-to-left implication of (e):

1.	$\varphi \to \varphi$	$A1$
2.	$\psi \to (\psi \vee\!\!\!\vee \chi)$	$A9$
3.	$(\varphi \vee \psi) \to (\varphi \vee (\psi \vee\!\!\!\vee \chi))$	1,2, $R4^*$
4.	$\chi \to (\psi \vee\!\!\!\vee \chi)$	$A10$
5.	$(\varphi \vee \chi) \to (\varphi \vee (\psi \vee\!\!\!\vee \chi))$	1,4, $R4^*$
6.	$((\varphi \vee \psi) \vee\!\!\!\vee (\varphi \vee \chi)) \to (\varphi \vee (\psi \vee\!\!\!\vee \chi))$	3,5, $R10$.

We will also show the left-to-right implication of the equivalence (h):

1.	$(\varphi \otimes \chi) \to ((\varphi \otimes \chi) \vee\!\!\!\vee (\psi \otimes \chi))$	$A9$
2.	$\varphi \to (\chi \to ((\varphi \otimes \chi) \vee\!\!\!\vee (\psi \otimes \chi)))$	1, $R6$
3.	$(\psi \otimes \chi) \to ((\varphi \otimes \chi) \vee\!\!\!\vee (\psi \otimes \chi))$	$A10$
4.	$\psi \to (\chi \to ((\varphi \otimes \chi) \vee\!\!\!\vee (\psi \otimes \chi)))$	3, $R6$
5.	$(\varphi \vee\!\!\!\vee \psi) \to (\chi \to ((\varphi \otimes \chi) \vee\!\!\!\vee (\psi \otimes \chi)))$	2,4, $R10$
6.	$((\varphi \vee\!\!\!\vee \psi) \otimes \chi) \to ((\varphi \otimes \chi) \vee\!\!\!\vee (\psi \otimes \chi))$	5, $R5$. □

The rest of the proof of Theorem 4.3 is analogous to the proof of the corresponding results from, for example, (Ciardelli, 2016b) or (Punčochář, 2017). We can proceed by induction using Lemmas 7.7 and 7.8. The case of atomic formulas and the constants $\bot$ and t is immediate, since $\mathcal{R}(p) = \{p\}$, $\mathcal{R}(\bot) = \{\bot\}$, and $\mathcal{R}(t) = \{t\}$.

The induction hypothesis says that for given $L_s^?$-formulas φ and ψ, such that $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{R}(\psi) = \{\beta_1, \ldots, \beta_m\}$, we have:

$$\varphi \dashv\vdash_{InqBSL} \alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n,$$
$$\psi \dashv\vdash_{InqBSL} \beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m.$$

Negation: Then due to the equivalence (a) of Lemma 7.8:

$$\neg\varphi \dashv\vdash_{InqBSL} \bigwedge_{\alpha \in \mathcal{R}(\varphi)} \neg\alpha.$$

Implication: Due to the equivalences (b) and (f) of Lemma 7.8, the following formulas are provably equivalent in *InqBSL*:

1. $\varphi \to \psi$,
2. $(\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n) \to (\beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m)$,
3. $(\alpha_1 \to (\beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m)) \wedge \cdots \wedge (\alpha_n \to (\beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m))$,
4. $((\alpha_1 \to \beta_1) \vee\!\!\!\vee \cdots \vee\!\!\!\vee (\alpha_1 \to \beta_m)) \wedge \cdots \wedge ((\alpha_n \to \beta_1) \vee\!\!\!\vee \cdots \vee\!\!\!\vee (\alpha_n \to \beta_m)))$,

Distributivity of $\wedge$ over $\vee\!\!\!\vee$ (equivalences (c) and (g) of Lemma 7.8) implies that 4 (and, consequently, also 1–3) is *InqBSL*-equivalent to the inquisitive disjunction of the formulas from $\{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \alpha \to f(\alpha); f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$.

Conjunction, fusion, disjunction: Let $\circ \in \{\wedge, \otimes, \vee\}$. The formula $\varphi \circ \psi$ is provably equivalent in *InqBSL* to $(\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n) \circ (\beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m)$. Using (c), (d), (e), (g), (h), (i) of Lemma 7.8, we can observe that these formulas are *InqBSL*-equivalent to the inquisitive disjunction of the formulas from $\{\alpha \circ \beta; \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$.

Inquisitive disjunction: The formula $\varphi \vee\!\!\!\vee \psi$ is provably equivalent to $\alpha_1 \vee\!\!\!\vee \cdots \alpha_n \vee\!\!\!\vee \beta_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \beta_m$. This finishes the proof of Theorem 4.3.

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