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Exponential mixing property for Hénon–Sibony maps of $\mathbb{C}^k$

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Abstract. Let f be a Hénon–Sibony map, also known as a regular polynomial automorphism of $\mathbb{C}^k$, and let μ be the equilibrium measure of f. In this paper we prove that μ is exponentially mixing for plurisubharmonic observables.

Key words: equilibrium measure, exponential mixing, positive closed current
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1. *Introduction and main results*

Let f be a polynomial automorphism of $\mathbb{C}^k$. It can be extended to a birational map of $\mathbb{P}^k$. The set I_+ (respectively I_-) where f (respectively f^{-1}) is not defined is called the *indeterminacy set* of f (respectively f^{-1}). We say that f is a *Hénon–Sibony map* or a *regular polynomial automorphism* in the sense of Sibony if I_+ and I_- are non-empty and they satisfy $I_+ \cap I_- = \varnothing$. There is a very large class of polynomial automorphisms of $\mathbb{C}^k$ satisfying these properties (see [**13**, **14**]). For example, every polynomial automorphism of $\mathbb{C}^2$ is conjugated either to a Hénon–Sibony map, or an elementary polynomial automorphism, which has the form $g(z_1, z_2) := (az_1 + p(z_2), bz_2 + c)$, where a, b, c are constants in $\mathbb{C}$ with $a, b \neq 0$ and p a polynomial. The latter map preserves the family of lines where z_2 is constant.

We first recall some basic properties of f. The indeterminacy sets $I_\pm$ are contained in the hyperplane at infinity $L_\infty := \mathbb{P}^k \backslash \mathbb{C}^k$. There exists an integer s such that $\dim I_+ = k - 1 - s$ and $\dim I_- = s - 1$. The set I_- is attractive for f and I_+ is attractive for f^{-1}. Moreover, $f(L_\infty \backslash I_+) = I_-$ and $f^{-1}(L_\infty \backslash I_-) = I_+$. Denote by d_+ and d_- the algebraic degrees of f and f^{-1}, respectively; we have $d_+^s = d_-^{k-s}$. When $k = 2s$, we have $d_+ = d_-$. In the case $k = 2$, f is called a *generalized Hénon map* of $\mathbb{C}^2$ (see [**13**]).

We define the *Green functions* by

$$G^+(z) := \lim_{n\to\infty} d_+^{-n} \log^+ \|f^n(z)\| \quad \text{and} \quad G^-(z) := \lim_{n\to\infty} d_-^{-n} \log^+ \|f^{-n}(z)\|,$$

where $\log^+ := \max\{\log, 0\}$. They are Hölder continuous and plurisubharmonic (p.s.h. for short) on $\mathbb{C}^k$ and they satisfy $G^+ \circ f = d_+ G^+$ and $G^- \circ f^{-1} = d_- G^-$. Define the *Green currents* of bidegree $(1, 1)$ by $T_+ := dd^c G^+$ and $T_- := dd^c G^-$. Sibony showed that f admits an invariant probability measure μ, called the *equilibrium measure*, and it satisfies $\mu = T_+^s \wedge T_-^{k-s}$, which turns out to be a measure of maximal entropy (unique when $k = 2$). Hence, μ plays a very important role in the study of complex dynamics. For more dynamical properties of Hénon–Sibony maps, the reader may refer to [1, 2, 10–12, 14].

The current T_+^s (respectively T_-^{k-s}) is supported in the boundary of the *filled Julia set* K_+ (respectively K_-). Recall that K_+ (respectively K_-) is the set of points $z \in \mathbb{C}^k$ such that the orbit $(f^n(z))_{n\in\mathbb{N}}$ (respectively $(f^{-n}(z))_{n\in\mathbb{N}}$) is bounded in $\mathbb{C}^k$. We have

$$K_+ = \{G^+ = 0\}, \quad K_- = \{G^- = 0\} \quad \text{and} \quad \overline{K}_\pm \cap L_\infty = I_\pm$$

in $\mathbb{P}^k$. The open set $\mathbb{P}^k \backslash \overline{K}_+$ (respectively $\mathbb{P}^k \backslash \overline{K}_-$) is the immediate basin of I_- for f (respectively I_+ for f^{-1}). Define $K := K_+ \cap K_-$. It is a compact subset of $\mathbb{C}^k$ and we have $\operatorname{supp}(\mu) \subseteq K$.

It was proved in [7] that μ is mixing. For $0 < \alpha \leq 2$, Dinh [4] showed that the speed of mixing is exponential when $k = 2s$ for real-valued $\mathscr{C}^\alpha$ functions. In [15], exponential mixing is also achieved for generic birational maps of $\mathbb{P}^k$ for $\mathscr{C}^\alpha$ observables with $0 < \alpha \leq 2$. However, $\mathscr{C}^\alpha$ functions do not have good invariance properties. For example, the pull-back of a $\mathscr{C}^\alpha$ function by a birational map may not even be continuous any more. So, in this case, it is natural to ask whether the exponential mixing property holds for other spaces of test functions. In this paper, we will extend the result of [4] to a class of plurisubharmonic test functions. It is known that the space spanned by those functions is an important space of test functions in complex dynamics, as it is invariant under the action of holomorphic or meromorphic maps. Moreover, p.s.h. functions may have singularities along analytic sets and this allows one to study the action of the dynamical system on analytic sets using p.s.h. functions (see e.g. [9]).

When f is an endomorphism of $\mathbb{P}^k$ with algebraic degree $d \geq 2$, one can also construct the Green current T and the equilibrium measure $\mu := T^k$ by using a similar way as above. Moreover, μ is mixing for all d.s.h. (see the definition below) observablesand the speed is exponential (see [5, 9]). The advantage here is that f has no singularities on $\mathbb{P}^k$, that is, it is holomorphic everywhere. Therefore, some invariant properties and good estimates of d.s.h. functions can be obtained under the action of f.

In the rest of this paper, we fix a Hénon–Sibony map f of $\mathbb{C}^k$. For simplicity, we assume that $k = 2s$ (see also Remark 3.3 and [15]). Denote $d := d_+ = d_-$. The case $k = 2$ and $s = 1$ is already interesting (see [10]). Here is the first main result of this paper.

THEOREM 1.1. *Let f be a Hénon–Sibony map of $\mathbb{C}^k$ as above and assume that $k = 2s$. Let d be the algebraic degree and let μ be the equilibrium measure of f. Then, for any open neighborhood D of K, there exists a constant $c > 0$ depending on D such that*

$$\left| \int (\varphi \circ f^n) \psi \, d\mu - \left(\int \varphi \, d\mu \right) \left(\int \psi \, d\mu \right) \right| \leq c d^{-n/2} \|\varphi\|_{L^\infty(D)} \|\psi\|_{L^\infty(D)}$$

for all $n \geq 0$ and all functions φ and ψ on $\mathbb{C}^k$ which are bounded and p.s.h. on D.

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Remark 1.2. For the inequality above, note that the values of the integrals on the left-hand side do not depend on the values of φ and ψ outside K since $\mathrm{supp}(\mu) \subseteq K$. A main novelty here is that observables are not even globally defined and that requires a good extension lemma (see Lemma 2.5 below). It is easy to see that this result still holds when φ and ψ are differences of bounded p.s.h. functions on D, e.g. when φ and ψ are of class $\mathscr{C}^2$ on $\mathbb{C}^k$.

Another version of Theorem 1.1 has been proved in [4] for $\varphi, \psi \in \mathscr{C}^2$ and it can be extended to the $\mathscr{C}^\alpha$ case, $0 < \alpha \leq 2$, using interpolation theory between Banach spaces. In this case, one can assume that φ and ψ are of class $\mathscr{C}^2$ and p.s.h. because we can write $\mathscr{C}^2$ functions as differences of $\mathscr{C}^2$ functions which are p.s.h. near K. A key step in the proof of [4] is to consider the functions

$$(\varphi(z) + A)(\psi(w) + A) \quad \text{and} \quad (-\varphi(z) + A)(\psi(w) - A)$$

as test functions on $\mathbb{C}^k \times \mathbb{C}^k$ for the system $(z, w) \mapsto (f(z), f^{-1}(w))$. These two functions are p.s.h. when A is large enough and they play a 'linear' role in the setting of the system $(z, w) \mapsto (f(z), f^{-1}(w))$. Some general estimates for the latter system imply the desired result.

We will use the method of [4]. However, the same idea as above cannot be directly applied because the two test functions above may not be p.s.h. when φ and ψ are not of class $\mathscr{C}^2$. We need to introduce several new test functions in $\mathbb{C}^k \times \mathbb{C}^k$ and prove that they satisfy good properties required in this approach (see Lemma 3.1 below).

Recall that a function is *quasi-plurisubharmonic* (*quasi-p.s.h.* for short) if locally it is the difference of a p.s.h. function and a smooth one. A function u on $\mathbb{P}^k$ with values in $\mathbb{R} \cup \{\pm\infty\}$ is said to be *d.s.h.* if outside a pluripolar set it is equal to a difference of two quasi-p.s.h. functions. Two d.s.h. functions are identified when they are equal out of a pluripolar set. Denote the set of d.s.h. functions by $\mathrm{DSH}(\mathbb{P}^k)$. Clearly it is a vector space and it is equipped with a norm

$$\|u\|_{\mathrm{DSH}} := \left| \int_{\mathbb{P}^k} u \, \omega_{\mathrm{FS}}^k \right| + \min \|T^\pm\|,$$

where ω_{FS} is the standard Fubini–Study form on $\mathbb{P}^k$ and the minimum is taken on all positive closed (1, 1)-currents $T^\pm$ such that $dd^c u = T^+ - T^-$.

A positive measure ν on $\mathbb{P}^k$ is said to be *moderate* if for any bounded family $\mathscr{F}$ of d.s.h. functions on $\mathbb{P}^k$, there are constants $\alpha > 0$ and $c > 0$ such that

$$\nu\{z \in \mathbb{P}^k : |\psi(z)| > M\} \leq c e^{-\alpha M} \tag{1.1}$$

for $M \geq 0$ and $\psi \in \mathscr{F}$ (see [5, 6, 9]). The papers [5, 7] show that if f is a Hénon–Sibony map of $\mathbb{C}^k$ or, more generally, a regular birational map of $\mathbb{P}^k$, then the equilibrium measure μ of f is moderate. Using the moderate property of μ, we can remove the boundedness conditions of φ and ψ, but the estimate on the mixing will be a little bit weaker.

THEOREM 1.3. *Let f be a Hénon–Sibony map of $\mathbb{C}^k$ and assume that $k = 2s$. Let d be the algebraic degree and let μ be the equilibrium measure of f. Then for any two d.s.h.*

functions φ *and* ψ *on* $\mathbb{P}^k$, *there exists a constant* $c > 0$ *depending on* φ, ψ *such that*

$$\left| \int (\varphi \circ f^n)\psi \, d\mu - \left(\int \varphi \, d\mu \right)\left(\int \psi \, d\mu \right) \right| \leq cn^2 d^{-n/2}$$

for all $n \geq 0$.

It is not hard to see that one can choose a common constant c for every compact family of d.s.h. observables. However, we do not know if the factor n^2 is removable but its presence seems to be natural as it somehow represents the role of the singularities of φ and ψ. More precisely, those functions satisfy exponential estimates (see e.g. [9]), which suggests that their singularities may contribute to some factors exponentially less important than the main factor $d^{-n/2}$ in our estimate.

2. *Estimates on iterations of positive closed currents*

In this section, we recall some known results and get a slightly more general version (see Proposition 2.4 below), which will be used for proving our main theorems.

Recall that $K = \{G^+ = G^- = 0\}$ and D is an open neighborhood of K. Define $G := \max\{G^+, G^-\}$, which is continuous and p.s.h. on $\mathbb{C}^k$ since it is equal to the maximum of two p.s.h. functions. Observe that $K = \{G = 0\}$. Fix a small positive constant δ such that $\delta < \min_{\partial D} G$. Since $\mathbb{P}^k \backslash \overline{K}_+$ (respectively $\mathbb{P}^k \backslash \overline{K}_-$) is the immediate basin of I_- for f (respectively I_+ for f^{-1}), we can find U_i, V_i, $i = 1, 2$, which are open subsets of $\mathbb{P}^k$, such that

$$\overline{K}_+ \Subset U_i, \overline{K}_- \Subset V_i, U_1 \Subset U_2, V_1 \Subset V_2, f^{-1}(U_i) \Subset U_i, f(V_i) \Subset V_i$$

and $U_2 \cap V_2 \Subset \{G < \delta\}$. Then observe that $K \Subset U_1 \cap V_1 \Subset U_2 \cap V_2 \Subset \{G < \delta\} \Subset D$.

We define a norm on the space of real currents with support in $\overline{V}_1$. Let ω_{FS} be the standard Fubini–Study form on $\mathbb{P}^k$. Let Ω be a real $(s+1, s+1)$-current supported in $\overline{V}_1$ and assume that there exists a positive closed $(s+1, s+1)$-current Ω' supported in $\overline{V}_1$ such that $-\Omega' \leq \Omega \leq \Omega'$. Define the norm $\|\Omega\|_*$ as

$$\|\Omega\|_* := \min\{\|\Omega'\|, \Omega' \text{ as above}\},$$

where $\|\Omega'\| = \langle \Omega', \omega_{\mathrm{FS}}^{s-1} \rangle$ is the mass of Ω'. We have the following lemma.

LEMMA 2.1. *Let* Ω *be a real* dd^c*-exact* $(s+1, s+1)$*-current supported in* $\overline{V}_1$ *and assume that* $\Omega \geq -S$ *for some positive closed current* S *supported in* $\overline{V}_1$. *Then* $\|\Omega\|_* \leq 2\|S\|$.

Proof. Note that $\Omega + 2S$ is a positive closed current supported in $\overline{V}_1$ and Ω satisfies

$$-(\Omega + 2S) \leq \Omega \leq \Omega + 2S.$$

The mass of $\Omega + 2S$ is $2\|S\|$ because Ω is dd^c-exact. □

We need the following estimate [4, Proposition 2.1].

PROPOSITION 2.2. *Let* f *be a Hénon–Sibony map with* $k = 2s$. *Let* R *be a positive closed* (s, s)*-current of mass* 1 *supported in* U_1 *and smooth on* $\mathbb{C}^k$. *Let* Φ *be a real*

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smooth (s, s)-form with compact support in $V_1 \cap \mathbb{C}^k$. Assume that $dd^c\Phi \geq 0$ on U_2 and $\|dd^c\Phi\|_* < \infty$. Then there exists a constant $c > 0$ independent of R and Φ such that

$$\langle d^{-sn}(f^n)^* R - T_+^s, \Phi\rangle \leq cd^{-n}\|dd^c\Phi\|_*$$

for every $n \geq 0$.

Remark 2.3. Note that the support of $d^{-sn}(f^n)^* R - T_+^s$ is in U_1 and the support of Φ is in V_1. Therefore, the value of $\langle d^{-sn}(f^n)^* R - T_+^s, \Phi\rangle$ does not depend on the value of Φ outside $U_1 \cap V_1$. Thus, for the above proposition, the condition that Φ is smooth can be replaced by Φ being smooth on $U_1 \cap V_1$.

We will use Proposition 2.2 to prove the following estimate, which will be crucial in the proof of exponential mixing for plurisubharmonic observables. The case for $\varphi \in \mathscr{C}^2$ was shown in [4].

PROPOSITION 2.4. *Let f be a Hénon–Sibony map with $k = 2s$. Let φ be a bounded real-valued function on $\mathbb{P}^k$ that is p.s.h. on D. Let R (respectively S) be a positive closed (s, s)-current of mass 1 with support in U_1 (respectively V_1) and smooth on $\mathbb{C}^k$. Then there exists a constant $c > 0$ independent of φ, R and S such that*

$$\langle d^{-2sn}(f^n)^* R \wedge (f^n)_* S - \mu, \varphi\rangle \leq cd^{-n}\|\varphi\|_{L^\infty}$$

for every $n \geq 0$.

Before proving Proposition 2.4, we prove a 'regularization' lemma for φ first. Fix an open set D_1 such that $\{G < \delta\} \Subset D_1 \Subset D$.

LEMMA 2.5. *Let φ be a bounded real-valued function on $\mathbb{P}^k$ that is p.s.h. on D_1. There exist a function ϕ with compact support in $\mathbb{C}^k$ and an open set D' satisfying $U_2 \cap V_2 \Subset D' \Subset D_1$ such that ϕ is p.s.h. on D_1 and smooth outside D' satisfying $\phi = \varphi$ on $U_2 \cap V_2$ and*

$$\|\phi\|_{L^\infty} \leq c\|\varphi\|_{L^\infty} \quad \text{and} \quad \|\phi\|_{\mathscr{C}^2(\mathbb{P}^k\setminus D')} \leq c\|\varphi\|_{L^\infty}$$

for some constant $c > 0$ independent of φ.

Proof. Using regularizations by convolution, one can find a family of smooth p.s.h. functions G_ϵ which decreases to G when ϵ decreases to 0. Since G is continuous, this convergence is locally uniform. Hence, there exist positive constants $\kappa_1 < \kappa_2$ and λ such that

$$\{G < \delta\} \Subset \{G_\lambda < \kappa_1\} \Subset \{G_\lambda < \kappa_2\} \Subset D_1.$$

Since φ is bounded, after adding some constant we can assume that $\varphi \geq 0$. Define

$$\tau := \|\varphi\|_{L^\infty} \cdot (G_\lambda - \kappa_1)/(\kappa_2 - \kappa_1).$$

Consider the function

$$\phi := \chi \cdot \max\{\varphi, \tau\},$$

where χ is a real cut-off function satisfying $\chi(z) = 0$ for $z \notin D$, $\chi(z) = 1$ for $z \in D_1$ and $|\chi'|, |\chi''|$ being bounded by some constant.

For $z \in D_1$, we have $\phi = \max\{\varphi, \tau\}$. Hence, ϕ is p.s.h. on D_1 because it is equal to the maximum of two p.s.h. functions on D_1. Now we let $D' := \{G_\lambda < \kappa_2\}$. When $z \in \mathbb{P}^k \backslash D'$, we have $(G_\lambda - \kappa_1)/(\kappa_2 - \kappa_1) \geq 1$. In this case, $\phi = \chi\tau$, so ϕ is smooth outside D'. When $z \in \{G_\lambda < \kappa_1\}$, we have $\tau(z) \leq 0 \leq \varphi(z)$, so $\phi = \varphi$ inside $\{G_\lambda < \kappa_1\}$. Since $U_2 \cap V_2 \Subset \{G_\lambda < \kappa_1\}$, we get $\phi = \varphi$ on $U_2 \cap V_2$.

Now we prove the two estimates. For the first inequality,

$$\|\phi\|_{L^\infty} = \sup_{z \in \mathbb{P}^k \backslash D'} \chi(z)\tau(z) = \sup_{z \in D \backslash D'} \tau(z) \leq c_1 \|\varphi\|_{L^\infty}$$

for some constant $c_1 > 0$ independent of φ. For the second one,

$$\|\phi\|_{\mathscr{C}^2(\mathbb{P}^k \backslash D')} = \|\chi\tau\|_{\mathscr{C}^2(\mathbb{P}^k \backslash D')} \leq c_2 \|\varphi\|_{L^\infty}$$

for some constant $c_2 > 0$ independent of φ. We take $c = \max\{c_1, c_2\}$ and finish the proof of this lemma. □

Now consider the canonical inclusions of $\mathbb{C}^k$ and $\mathbb{C}^k \times \mathbb{C}^k$ in $\mathbb{P}^k$ and $\mathbb{P}^{2k}$. We will use z, w and (z, w) for the canonical coordinates of $\mathbb{C}^k$ and $\mathbb{C}^k \times \mathbb{C}^k$. Write $[z : t]$, $[w : t]$ and $[z : w : t]$ for the homogeneous coordinates. Denote by L'_∞ the hyperplane at infinity of $\mathbb{P}^{2k}$.

Define an automorphism of $\mathbb{C}^k \times \mathbb{C}^k$ by $F(z, w) := (f(z), f^{-1}(w))$. Then F is also a Hénon–Sibony map (see [4, Lemma 3.2]). The algebraic degrees of F and F^{-1} are both equal to d. The Green current of bidegree $(2s, 2s)$ of F is $T^s_+ \otimes T^s_-$ satisfying $F^*(T^s_+ \otimes T^s_-) = d^{2s} T^s_+ \otimes T^s_-$.

Denote by $I^F_\pm$ the indeterminacy sets of F. Let Δ be the diagonal of $\mathbb{C}^k \times \mathbb{C}^k$ and let $\overline{\Delta}$ be its closure in $\mathbb{P}^{2k}$. From [4, Lemma 3.2], we know that

$$I^F_+ = \{[z : w : 0] : \ [z : 0] \in I_+ \text{ and } [w : 0] \in I_-\},$$

$$I^F_- = \{[z : w : 0] : \ [z : 0] \in I_- \text{ and } [w : 0] \in I_+\}$$

and

$$I^F_\pm \cap \overline{\Delta} = \varnothing, \quad F(\overline{\Delta}) \cap L'_\infty \subset I^F_-.$$

We will use F to prove Proposition 2.4 by following the same strategy as Proposition 3.1 in [4]. We also need that every positive current on $\mathbb{P}^k$ can be regularized on every neighborhood of its support since $\mathbb{P}^k$ is homogeneous (see e.g. [8]). For the convenience of the reader, we present full details here although some parts of the proof have appeared in [4] already.

Proof of Proposition 2.4. Since the support of the measure $d^{-2sn}(f^n)^* R \wedge (f^n)_* S - \mu$ is in $U_1 \cap V_1$ and the constant c in Proposition 2.4 is independent of φ, R and S, we can assume that φ is smooth on $U_2 \cap V_2$ and p.s.h. on D_1 in order to apply Proposition 2.2. Then we obtain the general case by approximating φ by a decreasing sequence of smooth functions which are p.s.h. on D_1.

On the other hand, using Lemma 2.5, we can assume that φ is smooth on $(\mathbb{P}^k\backslash D')\cup(U_2\cap V_2)$, with compact support in $\mathbb{C}^k$ and $\|\varphi\|_{\mathscr{C}^2(\mathbb{P}^k\backslash D')}\leq c'\|\varphi\|_{L^\infty}$ for some constant $c'>0$. After multiplying φ by some constant, we can assume that $|\varphi|\leq 1$.

Replacing R and S by $d^{-s}f^*(R)$ and $d^{-s}f_*(S)$, we can also assume that

$$\operatorname{supp}(R)\cap L_\infty\subset I_+\quad\text{and}\quad\operatorname{supp}(S)\cap L_\infty\subset I_-.$$

Consider the current $R\otimes S$ in $\mathbb{C}^k\times\mathbb{C}^k$. By the above assumptions on R and S, we have

$$\overline{\operatorname{supp}(R\otimes S)}\cap L'_\infty\subset I_+^F.$$

Since $\dim I_+^F=2s-1$, by Skoda's extension theorem [3, Theorem III.2.3], the trivial extension of $R\otimes S$ (which we still denote by $R\otimes S$) to $\mathbb{P}^{2k}$ is a positive closed $(2s,2s)$-current of mass 1 and satisfies

$$\operatorname{supp}(R\otimes S)\cap L'_\infty\subset I_+^F.$$

Define $\widehat{\varphi}(z,w):=\varphi(z)$ on $\mathbb{C}^k\times\mathbb{C}^k$. Since $T_\pm$ are invariant and have continuous potentials out of $I_\pm$, we have

$$\langle d^{-2sn}(f^n)^*R\wedge(f^n)_*S-\mu,\varphi\rangle=\langle d^{-2sn}(f^n)^*R\otimes(f^n)_*S-T_+^s\otimes T_-^s,\widehat{\varphi}[\Delta]\rangle.$$

Since $\mathbb{P}^k$ is homogeneous, using a regularization of $[\Delta]$, one can find a smooth positive closed form Θ of mass 1 with support in a small neighborhood W of $\overline{\Delta}$ such that

$$\begin{aligned}&|\langle d^{-2sn}(f^n)^*R\otimes(f^n)_*S-T_+^s\otimes T_-^s,\widehat{\varphi}[\Delta]\rangle\\&\quad-\langle d^{-2sn}(f^n)^*R\otimes(f^n)_*S-T_+^s\otimes T_-^s,\widehat{\varphi}\Theta\rangle|\leq d^{-n}.\end{aligned}$$

Note that Θ may depend on n. We can choose W such that $W\cap I_+^F=\varnothing$.

In the following, we will estimate the term

$$\langle d^{-2sn}(f^n)^*R\otimes(f^n)_*S-T_+^s\otimes T_-^s,\widehat{\varphi}\Theta\rangle.$$

Fix an integer $m>0$ large enough. Since $\widehat{\varphi}\Theta$ has compact support in $\mathbb{C}^k\times\mathbb{C}^k$ and

$$(f^n)^*R\otimes(f^n)_*S=(F^n)^*(R\otimes S)$$

in $\mathbb{C}^k\times\mathbb{C}^k$, we have for $n>m$,

$$\begin{aligned}&\langle d^{-2sn}(f^n)^*R\otimes(f^n)_*S-T_+^s\otimes T_-^s,\widehat{\varphi}\Theta\rangle\\&\quad=\langle d^{-2sn}(F^n)^*(R\otimes S)-d^{-2sm}(F^m)^*(T_+^s\otimes T_-^s),\widehat{\varphi}\Theta\rangle\\&\quad=\langle d^{-2s(n-m)}(F^{n-m})^*(R\otimes S)-T_+^s\otimes T_-^s,d^{-2sm}(F^m)_*(\widehat{\varphi}\Theta)\rangle\\&\quad=\langle d^{-2s(n-2m)}(F^{n-2m})^*T-T_+^s\otimes T_-^s,\Phi\rangle,\end{aligned}$$

where $T:=d^{-2sm}(F^m)^*(R\otimes S)$ and $\Phi:=d^{-2sm}(F^m)_*(\widehat{\varphi}\Theta)$.

Note that for m,n big enough, T has support in a small neighborhood U of $K_+^F:=K_+\times K_-$ and Φ has support in a small neighborhood V of $K_-^F:=K_-\times K_+$. Since m is large and φ is smooth p.s.h. on $U_2\cap V_2$, there exists a neighborhood $U'\Supset U$ such that on U', $dd^c\Phi\geq 0$ and Φ is smooth.

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Define $\widehat{\omega}(z, w) := \omega(z)$. Since $\|\varphi\|_{\mathscr{C}^2(\mathbb{P}^k\setminus D')} \le c'$ and φ is p.s.h. on D_1, we have $dd^c\varphi \ge -c'\omega$. It follows that

$$dd^c\Phi \ge -d^{-2sm}(F^m)_*(c'\widehat{\omega} \wedge \Theta). \tag{2.1}$$

Using Lemma 2.1, we obtain $\|dd^c\Phi\|_* \le 2c'$ because the operator $d^{-2sm}(F^m)_*$ preserves the mass of positive closed (k, k)-currents and $c'\widehat{\omega} \wedge \Theta$ has mass c'.

Notice that the choices of W, U, V, U' and m do not depend on φ and n. Proposition 2.2 and Remark 2.3 applied to F, T and Φ implies that there exists $c > 0$ such that

$$\langle d^{-2s(n-2m)}(F^{n-2m})^*T - T^s_+ \otimes T^s_-, \Phi\rangle \le cd^{-n}$$

for all n. The proof of the proposition is complete. □

Remark 2.6. The main difference between the proof of Proposition 2.4 and [**4**, Proposition 3.1] is the term $dd^c\Phi$ in (2.1). Here we only obtain a lower bound for it, while, in [**4**, Proposition 3.1], there is also an upper bound due to the assumption that $\varphi \in \mathscr{C}^2$. So, there is an extra lower bound in the conclusion of [**4**, Proposition 3.1].

3. *Proofs of the main theorems*

Proof of Theorem 1.1. By Remark 1.2, we can assume that φ and ψ are bounded on $\mathbb{C}^k$ and $\|\varphi\|_{L^\infty} = \|\varphi\|_{L^\infty(D)}$, $\|\psi\|_{L^\infty} = \|\psi\|_{L^\infty(D)}$. After multiplying them by some constant, one can assume that $\|\varphi\|_{L^\infty} \le 1/2$ and $\|\psi\|_{L^\infty} \le 1/2$.

It is sufficient to prove Theorem 1.1 for n even because applying it to φ and $\psi \circ f$ gives the case of odd n (we reduce the domain D if necessary). Using the invariance of μ, it is enough to show that

$$|\langle\mu, (\varphi \circ f^n)(\psi \circ f^{-n})\rangle - \langle\mu, \varphi\rangle\langle\mu, \psi\rangle| \le cd^{-n} \tag{3.1}$$

for some $c > 0$. It is equivalent to proving that

$$\langle\mu, (\varphi \circ f^n)(\psi \circ f^{-n})\rangle - \langle\mu, \varphi\rangle\langle\mu, \psi\rangle \le cd^{-n}$$

and

$$\langle\mu, (\varphi \circ f^n)(-\psi \circ f^{-n})\rangle - \langle\mu, \varphi\rangle\langle\mu, -\psi\rangle \le cd^{-n}.$$

For $j = 1, 2$, we define

$$\varphi^+_j := \varphi^2 + j\varphi + 6, \quad \varphi^-_j := \varphi^2 + j\varphi - 6, \quad \psi^+_j := \psi^2 + j\psi + 6, \quad \psi^-_j := -\psi^2 - j\psi + 6.$$

Consider the following eight functions on $\mathbb{C}^k \times \mathbb{C}^k$:

$$\Phi^+_{jl}(z, w) := \varphi^+_j(z)\psi^+_l(w), \quad \Phi^-_{jl}(z, w) := \varphi^-_j(z)\psi^-_l(w),$$

where $j, l = 1, 2$. We prove two lemmas first. □

LEMMA 3.1. *The functions* $\Phi^\pm_{jl}$ *are all p.s.h. on* $D \times D$.

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Proof. By a direct computation,

$$\begin{aligned}
i\partial\overline{\partial}\Phi_{jl}^{+} &= i\partial\overline{\partial}(\varphi^2 + j\varphi + 6)(\psi^2 + l\psi + 6)\\
&= (\psi^2 + l\psi + 6)i\partial\overline{\partial}(\varphi^2 + j\varphi + 6) + i\partial(\varphi^2 + j\varphi + 6) \wedge \overline{\partial}(\psi^2 + l\psi + 6)\\
&\quad + i\partial(\psi^2 + l\psi + 6) \wedge \overline{\partial}(\varphi^2 + j\varphi + 6) + (\varphi^2 + j\varphi + 6)i\partial\overline{\partial}(\psi^2 + l\psi + 6)\\
&= (\psi^2 + l\psi + 6)((2\varphi + j)i\partial\overline{\partial}\varphi + 2i\partial\varphi \wedge \overline{\partial}\varphi) + (2\varphi + j)(2\psi + l)i\partial\varphi \wedge \overline{\partial}\psi\\
&\quad + (2\varphi + j)(2\psi + l)i\partial\psi \wedge \overline{\partial}\varphi + (\varphi^2 + j\varphi + 6)((2\psi + l)i\partial\overline{\partial}\psi + 2i\partial\psi \wedge \overline{\partial}\psi).
\end{aligned}$$

Recalling our assumption $\|\varphi\|_{L^\infty} \leq 1/2$, $\|\psi\|_{L^\infty} \leq 1/2$, we have $2\varphi + j \geq 0$, $2\psi + l \geq 0$. Since $i\partial\overline{\partial}\varphi$, $i\partial\varphi \wedge \overline{\partial}\varphi$, $i\partial\overline{\partial}\psi$, $i\partial\psi \wedge \overline{\partial}\psi$ are all positive, we get

$$\begin{aligned}
i\partial\overline{\partial}\Phi_{jl}^{+} &\geq 10i\partial\varphi \wedge \overline{\partial}\varphi + 10i\partial\psi \wedge \overline{\partial}\psi + (2\varphi + j)(2\psi + l)(i\partial\varphi \wedge \overline{\partial}\psi + i\partial\psi \wedge \overline{\partial}\varphi)\\
&\geq 10i\partial\varphi \wedge \overline{\partial}\varphi + 10i\partial\psi \wedge \overline{\partial}\psi - 9(i\partial\varphi \wedge \overline{\partial}\varphi + i\partial\psi \wedge \overline{\partial}\psi) \geq 0.
\end{aligned}$$

The second inequality holds because

$$i\partial\varphi \wedge \overline{\partial}\varphi + i\partial\varphi \wedge \overline{\partial}\psi + i\partial\psi \wedge \overline{\partial}\varphi + i\partial\psi \wedge \overline{\partial}\psi = i\partial(\varphi + \psi) \wedge \overline{\partial}(\varphi + \psi) \geq 0.$$

Similarly, by using

$$i\partial\varphi \wedge \overline{\partial}\varphi - i\partial\varphi \wedge \overline{\partial}\psi - i\partial\psi \wedge \overline{\partial}\varphi + i\partial\psi \wedge \overline{\partial}\psi = i\partial(\varphi - \psi) \wedge \overline{\partial}(\varphi - \psi) \geq 0,$$

we obtain

$$i\partial\overline{\partial}\Phi_{jl}^{-} \geq 9i\partial\varphi \wedge \overline{\partial}\varphi + 9i\partial\psi \wedge \overline{\partial}\psi - 9(i\partial\varphi \wedge \overline{\partial}\varphi + i\partial\psi \wedge \overline{\partial}\psi) \geq 0.$$

The proof of this lemma is finished. □

LEMMA 3.2. *There exists a constant $c > 0$ such that*

$$\langle \mu, (\varphi_j^{+} \circ f^n)(\psi_l^{+} \circ f^{-n})\rangle - \langle \mu, \varphi_j^{+}\rangle\langle \mu, \psi_l^{+}\rangle \leq cd^{-n}$$

and

$$\langle \mu, (\varphi_j^{-} \circ f^n)(\psi_l^{-} \circ f^{-n})\rangle - \langle \mu, \varphi_j^{-}\rangle\langle \mu, \psi_l^{-}\rangle \leq cd^{-n}$$

for all j, l and n.

Proof. Without loss of generality, we only show the first inequality. Define $T_F := T_+^s \otimes T_-^s$. Using $F^*(T_F) = d^{2s}T_F$ and that $T_\pm$ have continuous potentials in $\mathbb{C}^k$, we get

$$\begin{aligned}
\langle \mu, (\varphi_j^{+} \circ f^n)(\psi_l^{+} \circ f^{-n})\rangle &= \langle T_+^s \wedge T_-^s, (\varphi_j^{+} \circ f^n)(\psi_l^{+} \circ f^{-n})\rangle\\
&= \langle T_F \wedge [\Delta], \Phi_{jl}^{+} \circ F^n\rangle\\
&= \langle d^{-4sn+2sm}(F^{2n-m})^* T_F \wedge [\Delta], \Phi_{jl}^{+} \circ F^n\rangle\\
&= \langle d^{-4sn+2sm}(F^{n-m})^* T_F \wedge (F^n)_*[\Delta], \Phi_{jl}^{+}\rangle\\
&= \langle d^{-4sn+4sm}(F^{n-m})^* T_F \wedge (F^{n-m})_* T_m, \Phi_{jl}^{+}\rangle,
\end{aligned}$$

where $T_m := d^{-2sm}(F^m)_*[\Delta]$ and m is a fixed and sufficiently large integer.

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Since $\mathbb{P}^k$ is homogeneous, using regularizations again, one can find two smooth currents T'_F and T'_m of mass 1 with support in small neighborhoods U of $K_+^F = K_+ \times K_-$ and V of $K_-^F = K_- \times K_+$, respectively, such that

$$\begin{aligned}\langle d^{-4sn+4sm}(F^{n-m})^* T_F \wedge (F^{n-m})_* T_m, \Phi_{jl}^+\rangle \\ - \langle d^{-4sn+4sm}(F^{n-m})^* T'_F \wedge (F^{n-m})_* T'_m, \Phi_{jl}^+\rangle \le d^{-n}. \end{aligned} \tag{3.2}$$

The sets U and V satisfy $U \cap V \Subset D \times D$ and they depend only on f. The choice of m depends on f as well. The currents T'_F and T'_m may depend on n.

Thus, we can apply Proposition 2.4 to F, $\mu \otimes \mu$ and Φ_{jl}^+ instead of f, μ and φ to get that for some constant $c > 0$,

$$\langle d^{-4sn+4sm}(F^{n-m})^* T'_F \wedge (F^{n-m})_* T'_m - \mu \otimes \mu, \Phi_{jl}^+\rangle \le cd^{-n} \tag{3.3}$$

for all n. We can choose c independent of φ and ψ because the $\|\Phi_{jl}^+\|_{L^\infty}$ are bounded by some constant independent of φ and ψ.

Since $\langle \mu \otimes \mu, \Phi_{jl}^+\rangle = \langle \mu, \varphi_j^+\rangle\langle \mu, \psi_l^+\rangle$, combining (3.2) and (3.3) gives

$$\langle \mu, (\varphi_j^+ \circ f^n)(\psi_l^+ \circ f^{-n})\rangle - \langle \mu, \varphi_j^+\rangle\langle \mu, \psi_l^+\rangle \le (c+1)d^{-n}$$

for all n. This finishes the proof of this lemma. □

Now we can finish the proof of Theorem 1.1.

End of the proof of Theorem 1.1. Now consider $\alpha_{11}^+ = 2$, $\alpha_{22}^+ = \alpha_{11}^- = \alpha_{21}^- = \alpha_{12}^- = 1$ and $\alpha_{21}^+ = \alpha_{12}^+ = \alpha_{22}^- = 0$. A direct computation gives

$$\begin{aligned}\mathcal{A} := &\sum_{j,l=1,2} (\alpha_{jl}^+(\varphi_j^+ \circ f^n)(\psi_l^+ \circ f^{-n}) + \alpha_{jl}^-(\varphi_j^- \circ f^n)(\psi_l^- \circ f^{-n})) \\ &= (\varphi \circ f^n)(\psi \circ f^{-n}) + 36\,\varphi^2 \circ f^n + 36\,\psi^2 \circ f^{-n} + 48\,\varphi \circ f^n + 48\,\psi \circ f^{-n}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B} := &\sum_{j,l=1,2} (\alpha_{jl}^+\langle \mu, \varphi_j^+\rangle\langle \mu, \psi_l^+\rangle + \alpha_{jl}^-\langle \mu, \varphi_j^-\rangle\langle \mu, \psi_l^-\rangle) \\ &= \langle \mu, \varphi\rangle\langle \mu, \psi\rangle + 36\langle \mu, \varphi^2\rangle + 36\langle \mu, \psi^2\rangle + 48\langle \mu, \varphi\rangle + 48\langle \mu, \psi\rangle.\end{aligned}$$

The invariance of μ implies that

$$\langle \mu, \varphi^m \circ f^{\pm n}\rangle = \langle \mu, \varphi^m\rangle \quad \text{and} \quad \langle \mu, \psi^m \circ f^{\pm n}\rangle = \langle \mu, \psi^m\rangle.$$

Therefore,

$$\langle \mu, \mathcal{A}\rangle - \mathcal{B} = \langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n})\rangle - \langle \mu, \varphi\rangle\langle \mu, \psi\rangle.$$

Finally, by applying Lemma 3.2, since $\alpha_{jl}^\pm$ are all non-negative, we deduce that

$$\langle \mu, \mathcal{A}\rangle - \mathcal{B} \le \bigg(\sum_{j,l=1,2} (\alpha_{jl}^+ + \alpha_{jl}^-)\bigg) cd^{-n} = 6cd^{-n}$$

for the constant c in Lemma 3.2.

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Similarly, taking $\beta_{11}^- = 2$, $\beta_{11}^+ = \beta_{21}^+ = \beta_{12}^+ = \beta_{22}^- = 1$ and $\beta_{22}^+ = \beta_{21}^- = \beta_{12}^- = 0$, and repeating the above computation, we can obtain

$$\langle \mu, (\varphi \circ f^n)(-\psi \circ f^{-n})\rangle - \langle \mu, \varphi\rangle\langle \mu, -\psi\rangle \leq \bigg(\sum_{j,l=1,2} (\beta_{jl}^+ + \beta_{jl}^-)\bigg) cd^{-n} = 6cd^{-n}.$$

The above two inequalities prove inequality (3.1) and finish the proof of the main theorem. □

Remark 3.3. For the case $k \neq 2s$, one needs to work in the compactification $\mathbb{P}^k \times \mathbb{P}^k$ of $\mathbb{C}^{2k}$ (see also [15]). Similar estimates can be obtained. However, this does not improve the paper very much, so we choose not to present it here.

By combining Theorem 1.1 and the moderate property of μ, we can prove the second main theorem of this paper.

Proof of Theorem 1.3. We can assume that φ and ψ are p.s.h. and negative on D because constant functions obviously satisfy Theorem 1.3. After multiplying them by some constant, we can also assume that $\langle \mu, |\varphi|\rangle \leq 1$ and $\langle \mu, |\psi|\rangle \leq 1$. Let $M > 0$ be a constant whose value will be specified later. Define

$$\varphi_1 := \max\{\varphi, -M\}, \quad \psi_1 := \max\{\psi, -M\},$$

and

$$\varphi_2 := \varphi - \varphi_1, \quad \psi_2 := \psi - \psi_1.$$

Then φ_1 and ψ_1 are bounded and p.s.h. on D. Since μ is moderate and clearly $\{\varphi, \psi\}$ is a compact family of d.s.h. functions, by (1.1), there exist constants $c > 0$ and $\alpha > 0$ such that

$$\mu\{|\varphi| > M'\} \leq ce^{-\alpha M'} \quad \text{and} \quad \mu\{|\psi| > M'\} \leq ce^{-\alpha M'}.$$

For $t \in \mathbb{N}$, we compute the integral

$$\int_{|\varphi|>t} |\varphi|\, d\mu \leq \sum_{k=t}^{\infty} (k+1)\mu\{|\varphi| > k\} \leq \sum_{k=t}^{\infty} c(k+1)e^{-\alpha k}.$$

Note that for $M' \geq 1$, $([M'] + 1)e^{-\alpha[M']} \lesssim e^{-\alpha M'/2}$, where the symbol $\lesssim$ stands for an inequality up to a multiplicative constant. Thus, we have

$$\int_{|\varphi|>M} |\varphi|\, d\mu \lesssim \sum_{k=[M]}^{\infty} e^{-\alpha k/2} \lesssim e^{-\alpha M/2}.$$

The same estimate holds for ψ. By the definitions of φ_2 and ψ_2, we obtain

$$\|\varphi_2\|_{L^1(\mu)} \lesssim e^{-\alpha M/2} \quad \text{and} \quad \|\psi_2\|_{L^1(\mu)} \lesssim e^{-\alpha M/2}.$$

Repeating the preceding arguments for φ^2 and ψ^2 gives

$$\|\varphi_2\|_{L^2(\mu)} \lesssim e^{-\alpha M/2} \quad \text{and} \quad \|\psi_2\|_{L^2(\mu)} \lesssim e^{-\alpha M/2}.$$

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On the other hand, applying Theorem 1.1 to φ_1 and ψ_1, we get

$$\bigg| \int (\varphi_1 \circ f^n)\psi_1 \, d\mu - \bigg(\int \varphi_1 \, d\mu \bigg)\bigg(\int \psi_1 \, d\mu \bigg) \bigg| \lesssim d^{-n/2} M^2.$$

From the invariance of μ, we have that

$$\|\varphi_2 \circ f^n\|_{L^p(\mu)} = \|\varphi_2\|_{L^p(\mu)} \quad \text{and} \quad \|\psi_2 \circ f^n\|_{L^p(\mu)} = \|\psi_2\|_{L^p(\mu)}$$

for $1 \leq p \leq \infty$. We proceed as follows:

$$\begin{aligned}
&|\langle \mu, (\varphi \circ f^n)\psi\rangle - \langle \mu, \varphi\rangle\langle \mu, \psi\rangle| \\
&\quad = |\langle \mu, (\varphi_1 \circ f^n + \varphi_2 \circ f^n)(\psi_1 + \psi_2)\rangle - \langle \mu, \varphi_1 + \varphi_2\rangle\langle \mu, \psi_1 + \psi_2\rangle| \\
&\quad \leq |\langle \mu, (\varphi_1 \circ f^n)\psi_1\rangle - \langle \mu, \varphi_1\rangle\langle \mu, \psi_1\rangle| + |\langle \mu, (\varphi_1 \circ f^n)\psi_2\rangle| + |\langle \mu, (\varphi_2 \circ f^n)\psi_1\rangle| \\
&\qquad + |\langle \mu, (\varphi_2 \circ f^n)\psi_2\rangle| + |\langle \mu, \varphi_2\rangle\langle \mu, \psi_1\rangle| + |\langle \mu, \varphi_1\rangle\langle \mu, \psi_2\rangle| + |\langle \mu, \varphi_2\rangle\langle \mu, \psi_2\rangle| \\
&\quad \leq |\langle \mu, (\varphi_1 \circ f^n)\psi_1\rangle - \langle \mu, \varphi_1\rangle\langle \mu, \psi_1\rangle| + M\|\varphi_2\|_{L^1(\mu)} + M\|\psi_2\|_{L^1(\mu)} \\
&\qquad + \|\varphi_2\|_{L^2(\mu)}\|\psi_2\|_{L^2(\mu)} + \|\varphi_2\|_{L^1(\mu)} + \|\psi_2\|_{L^1(\mu)} + \|\varphi_2\|_{L^1(\mu)}\|\psi_2\|_{L^1(\mu)} \\
&\quad \lesssim d^{-n/2}M^2 + (2M+2)e^{-\alpha M/2} + 2e^{-\alpha M}.
\end{aligned}$$

Taking $M := (n \log d)/\alpha$, we obtain the estimate

$$d^{-n/2}M^2 + (2M+2)e^{-\alpha M/2} + 2e^{-\alpha M} \lesssim n^2 d^{-n/2}.$$

Therefore,

$$\bigg| \int (\varphi \circ f^n)\psi \, d\mu - \bigg(\int \varphi \, d\mu \bigg)\bigg(\int \psi \, d\mu \bigg) \bigg| \lesssim n^2 d^{-n/2}.$$

The proof is finished. □

Remark 3.4. The constant c in Theorem 1.3 can be made more explicit, but this requires a long and complicated calculation as it corresponds to the two constants c, α in (1.1) and also the constant c in Theorem 1.1. For example, the last one depends on the geometry of the open sets U_1 and V_1. Therefore, we choose not to work in this direction in the present paper.

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