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Dependence of Friedrichs' constant on boundary integrals

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This note extends the results in 'Optimal coercivity inequalities in $W^{1,p}(\Omega)$' (G. Auchmuty, *Proc. R. Soc. Edinb.* A **135**, 915–933.) describing the dependence of the optimal constant in the p-version of Friedrichs' inequality on the boundary integral term. In particular, it is shown that this constant is continuous, increasing, concave and increases to the optimal constant for the Dirichlet problem as $s \to \infty$.

1. Introduction

Recently, in [2], the optimal constants in the inequality

$$\int_\Omega \sum_{j=1}^N |\mathrm{D}_j u|^p \, \mathrm{d}x + \int_{\partial\Omega} b|u|^p \, \mathrm{d}\sigma \geqslant C_\mathrm{F} \int_\Omega \rho |u|^p \, \mathrm{d}x. \tag{1.1}$$

for all $u \in W^{1,p}(\Omega)$ were studied. In particular C_F was characterized as the principal eigenvalue of an eigenvalue problem for the p-Laplacian with Robin boundary conditions (see [2, §§ 6 and 7]).

Here our interest is in the dependence of the constant C_F on the boundary integral term in (1.1). Specifically, we shall describe the behaviour of $C_\mathrm{F}(s)$ on $[0, \infty)$, where $C_\mathrm{F}(s)$ is the optimal constant in

$$\int_\Omega \sum_{j=1}^N |\mathrm{D}_j u|^p \, \mathrm{d}x + s \int_{\partial\Omega} b|u|^p \, \mathrm{d}\sigma \geqslant C_\mathrm{F}(s) \int_\Omega \rho |u|^p \, \mathrm{d}x. \tag{1.2}$$

Here we shall show that $C_\mathrm{F}(s)$ is increasing, locally Lipschitz continuous, and concave on $(0, \infty)$. Moreover,

$$\lim_{s\to\infty} C_\mathrm{F}(s) = C_\mathrm{D}, \tag{1.3}$$

where C_{D} is the least eigenvalue of the Dirichlet eigenproblem for the p-Laplacian on Ω. This p-Laplacian is slightly different from the usual one, as studied, for example, in [5], but it has many similar properties and defines an equivalent norm on $W^{1,p}(\Omega)$.

2. Definitions and notation

The definitions and notation of Auchmuty [2] will be used. Our essential assumptions include the following:

(i) Ω is a non-empty bounded connected open subset of $\mathbb{R}^N$;

(ii) $\partial\Omega$ is a finite union of disjoint Lipschitz surfaces with finite surface area;

(iii) σ represents Hausdorff $(N-1)$-dimensional surface measure on $\partial\Omega$.

We shall assume that the boundary is sufficiently regular that the Sobolev embedding theorem and the Rellich–Kondrachov theorem hold for $W^{1,p}(\Omega)$.

Assumption 2.1. The embedding $i : W^{1,p}(\Omega) \to C^0(\bar{\Omega})$ is compact when $p > N$ and $i : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact for $1 \leqslant q < q_c$ when $p \leqslant N$ and $q_c = Np/(N-p)$.

Criteria for this assumption are given in [1] and [3, ch. V].

Let Γ denote the boundary trace operator. We will require the following assumption to hold.

Assumption 2.2. The boundary trace operator $\Gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega, \mathrm{d}\sigma)$ is continuous.

See [4, ch. 4] for a discussion of this.

The standard norm on $W^{1,p}(\Omega)$ is denoted $\|u\|_{1,p}$ and is defined by

$$\|u\|_{1,p}^p := \int_\Omega \left[\sum_{j=1}^N |\mathrm{D}_j\, u|^p + |u|^p\right] \mathrm{d}x. \tag{2.1}$$

The following are our assumptions on the coefficient functions in (1.2).

Assumption 2.3. The function ρ is in $L^1(\Omega)$ when $p > N$ or else ρ is in $L^q(\Omega)$ for some $q > q_0$ with $q_0 := N/p$ when $1 < p \leqslant N$ and also $\rho(x) \geqslant \rho_0 > 0$ almost everywhere (a.e.) on Ω.

Assumption 2.4. $b : \partial\Omega \to [0, \infty)$ is in $L^\infty(\partial\Omega, \mathrm{d}\sigma)$ and $b(x) > 0$ σ-a.e. on $\partial\Omega$.

To investigate the inequality (1.2), variational methods will be used. Define $\mathcal{F} : W^{1,p}(\Omega) \times [0, \infty) \to [0, \infty)$ by

$$\mathcal{F}(u, s) := \int_\Omega \sum_{j=1}^N |\mathrm{D}_j u|^p \, \mathrm{d}x + s \int_{\partial\Omega} b|u|^p \, \mathrm{d}\sigma. \tag{2.2}$$

Let $\mathcal{B} : W^{1,p}(\Omega) \to [0,\infty)$ and $\mathcal{P} : W^{1,p}(\Omega) \to [0,\infty)$ be defined by

$$\mathcal{B}(u) := \int_{\partial\Omega} b|\Gamma u|^p \,\mathrm{d}\sigma \tag{2.3}$$

and

$$\mathcal{P}(u) := \int_{\Omega} \rho(x)|u(x)|^p \,\mathrm{d}x. \tag{2.4}$$

3. Description of Friedrichs' constants

The constant $C_{\mathrm{F}}(s)$ in (1.2) is said to be *optimal* if it is the largest number such that (1.2) holds. A non-zero function $\hat{u}$ in $W^{1,p}(\Omega)$ *optimizes* (1.2), provided equality holds in (1.2).

When $s = 0$, constant functions optimize this inequality and $C_{\mathrm{F}}(0) = 0$. Henceforth we will consider $s \in (0,\infty)$.

The optimal constant in (1.2) can be characterized by a variational principle. Let $S_1 := \{u \in W^{1,p}(\Omega) : \mathcal{P}(u) = 1\}$. When assumption 2.3 holds then S_1 is a weakly closed subset of $W^{1,p}(\Omega)$, from [2, proposition 3.2].

Consider the family of variational principles of minimizing $\mathcal{F}(\cdot\,,s)$ on S_1. Then

$$C_{\mathrm{F}}(s) := \inf_{u \in S_1} \mathcal{F}(u,s). \tag{3.1}$$

Some properties of this value function of these principles may be summarized as follows. In the following a function g is said to be increasing on an interval I provided $g(t_1) \leqslant g(t_2)$ whenever $t_1 \leqslant t_2$ in I.

THEOREM 3.1. *Assume that assumptions 2.1–2.4 hold,* $1 < p < \infty$ *and* $s \in (0,\infty)$. *Then there are optimal functions* $\pm u_1(s)$ *for this variational principle. Moreover,* $C_{\mathrm{F}}(s)$ *is strictly positive, increasing, locally Lipschitz and concave on* $(0,\infty)$.

Proof. The existence of solutions is proved in [2, theorem 6.2]. In the proof of that theorem it is shown that $C_{\mathrm{F}}(s) \in (0,\infty)$ when $s > 0$. For each $u \in S_1$, $\mathcal{F}(u,s_1) \leqslant \mathcal{F}(u,s_2)$ whenever $s_1 < s_2$. Hence, $C_{\mathrm{F}}(s_1) \leqslant C_{\mathrm{F}}(s_2)$.

The functionals $\mathcal{F}(u,\cdot)$ are affine functions of s on $(0,\infty)$, so their infimum on S_1 will be a concave function of s, as the infimum of any family of concave functions is concave. Since $C_{\mathrm{F}}(s)$ is concave and finite on $(0,\infty)$ it is locally Lipschitz there. □

4. Optimal functions as $s \to \infty$

We now wish to prove (1.3). The optimal functions in (1.2) were characterized in [2, § 7]. They are the non-zero functions in $W^{1,p}(\Omega)$ that satisfy

$$\int_{\Omega} \left[\sum_{j=1}^{N} |\mathrm{D}_j u|^{p-2}\, \mathrm{D}_j u\, \mathrm{D}_j h - \mu_1 \rho |u|^{p-2} u h \right] \mathrm{d}x + \int_{\partial\Omega} sb|u|^{p-2} uh \,\mathrm{d}\sigma = 0 \tag{4.1}$$

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for all $h \in W^{1,p}(\Omega)$. Here μ_1 is the least eigenvalue of this problem. This is the weak form of the p-Laplacian eigenproblem

$$-\Delta_p u = -\sum_{j=1}^{N} D_j(|\mathrm{D}_j u|^{p-2}\,\mathrm{D}_j u) = \mu_1 \rho |u|^{p-2} u \quad \text{in } \Omega, \tag{4.2}$$

$$\sum_{j=1}^{N}(|\mathrm{D}_j u|^{p-2}\,\mathrm{D}_j u)\nu_j + sb|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

To treat the limiting case as s increases, let $t := s/(1+s)$, so that this boundary condition becomes

$$(1-t)\sum_{j=1}^{N}(|\mathrm{D}_j u|^{p-2}\,\mathrm{D}_j u)\nu_j + tb|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

Let $\mu_1(t)$ be the least eigenvalue of (4.1) with s replaced by $t/(1-t)$ and $0 \leqslant t < 1$, and let $u_1(t)$ be a corresponding minimizer which exists from theorem 3.1. Then [2, theorem 7.1] says that $\mu_1(t) = C_{\mathrm{F}}(t/(1-t))$.

There is a similar variational principle for the first eigenvalue of the Dirichlet eigenproblem. Let $\mathcal{F}_0(u) := \mathcal{F}(u, 0)$ be defined by (2.2) and $S_0 := \{u \in W_0^{1,p}(\Omega) : \mathcal{P}(u) = 1\}$. Consider the variational problem of minimizing $\mathcal{F}_0$ on S_0 and define

$$C_{\mathrm{D}} := \inf_{u \in S_0} \mathcal{F}_0(u). \tag{4.5}$$

Just as for the previous problems, C_{D} is the least eigenvalue $\hat{\mu}_1$ of the problem of finding non-zero functions in $W_0^{1,p}(\Omega)$ and eigenvalues μ satisfying

$$\int_\Omega \sum_{j=1}^{N} |\mathrm{D}_j u|^{p-2}\,\mathrm{D}_j u\,\mathrm{D}_j h\,\mathrm{d}x = \mu \int_\Omega \rho |u|^{p-2} u h\,\mathrm{d}x \tag{4.6}$$

for all $h \in W_0^{1,p}(\Omega)$.

Theorem 4.1. *Assume that $1 < p < \infty$ and that assumptions 2.1–2.4 hold. Then $\lim_{t\to 1^-} \mu_1(t) = \hat{\mu}_1$, and (1.3) holds.*

Proof. For $0 \leqslant t < 1$, since $\hat{u}_1 \in W_0^{1,p}(\Omega)$, we see that

$$\mu_1(t) \leqslant \mathcal{F}\left(\hat{u}_1, \frac{t}{1-t}\right) = \hat{\mu}_1. \tag{4.7}$$

From theorem 3.1, we see that $\mu_1(t)$ is increasing on $(0, 1)$, so there exists a $\mu^* := \lim_{t\to 1^-} \mu_1(t)$. The preceding inequality shows that $\mu^* \leqslant \hat{\mu}_1$.

Let $\{t_k : k \geqslant 1\}$ be a sequence which increases to 1 and $\{u_k : k \geqslant 1\}$ be a corresponding sequence of eigenfunctions in S_1. From (4.7), we see that

$$0 \leqslant \int_{\partial\Omega} b|\Gamma u_k|^p\,\mathrm{d}\sigma \leqslant \hat{\mu}_1 \frac{1-t_k}{t_k} \tag{4.8}$$

for all $k \geqslant 1$. Thus, $\mathcal{B}(u_k) \to 0$ as $t_k \to 1^-$.

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From (2.1), (4.7), assumption 2.3 and the definition of S_1, we obtain

$$\|u_k\|_{1,p}^p \leqslant \hat{\mu}_1 + \rho_0^{-1} \quad \text{for } k \geqslant 1.$$

Thus, this sequence has a weakly convergent subsequence, which will again be denoted u_k. Let u^* be the weak limit of this sequence. From assumption 2.2, Γu_k converges weakly to Γu^* in $L^p(\partial\Omega, \mathrm{d}\sigma)$. Thus, $\mathcal{B}(\Gamma u^*) = 0$ from (4.8) and [2, proposition 3.2], as $\mathcal{B}$ will be weakly lower semi-continuous on $L^p(\partial\Omega, \mathrm{d}\sigma)$. This and assumption 2.4 imply that $u^* = 0$ σ-a.e. on $\partial\Omega$ or $u^* \in W_0^{1,p}(\Omega)$.

Assumption 2.1 implies that u_k converges strongly to u^* in $L^p(\Omega)$ so $\mathcal{P}(u^*) = 1$ and thus $u^* \in S_0$. Finally, $\mathcal{F}_0$ is weakly lower semi-continuous on $W^{1,p}(\Omega)$, so

$$\mathcal{F}_0(u^*) \leqslant \liminf_{k\to\infty} \mathcal{F}_0(u_k) \leqslant \mu^*. \tag{4.9}$$

Thus, $\hat{\mu}_1 \leqslant \mu^*$ as $u^* \in S_0$, so $\hat{\mu}_1 = \mu^*$ and the theorem is proved. □

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