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Distributions of Order Patterns of Interval Maps

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A permutation σ describing the relative orders of the first n iterates of a point x under a self-map f of the interval $I = [0,1]$ is called an *order pattern*. For fixed f and n, measuring the points $x \in I$ (according to Lebesgue measure) that generate the order pattern σ gives a probability distribution $\boldsymbol{\mu}_n(f)$ on the set of length n permutations. We study the distributions that arise this way for various classes of functions f.

Our main results treat the class of measure-preserving functions. We obtain an exact description of the set of realizable distributions in this case: for each n this set is a union of open faces of the polytope of flows on a certain digraph, and a simple combinatorial criterion determines which faces are included. We also show that for general f, apart from an obvious compatibility condition, there is no restriction on the sequence $\{\boldsymbol{\mu}_n(f)\}_{n=1,2,\ldots}$.

In addition, we give a necessary condition for f to have *finite exclusion type*, that is, for there to be finitely many order patterns that generate all order patterns not realized by f. Using entropy we show that if f is piecewise continuous, piecewise monotone, and either ergodic or with points of arbitrarily high period, then f cannot have finite exclusion type. This generalizes results of S. Elizalde.

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1. Introduction

Given a function $f : [0,1] \to [0,1]$, it is natural to examine properties of the sequence of iterates of f beginning at some point $x \in [0,1]$:

$$x, f(x), f^2(x), \ldots .$$

The *order pattern* for a sequence of distinct reals $y_1, y_2, \ldots, y_n$ is the permutation $\sigma \in S_n$ that ranks the elements in increasing order; specifically, $y_i < y_j$ if and only if $\sigma(i) < \sigma(j)$. A number of authors have explored the relationship between functions f and the set of order patterns realized by the iterates of f. Work of C. Bandt, G. Keller, B. Pompe, J. M. Amigó, M. Kennel, and M. Misiurewicz [3, 4, 2, 8] relates the number of distinct order patterns arising from a function f to the entropy of f. S. Elizalde and others [1, 5, 6] have examined which and how many order patterns do not arise for particular functions and classes of functions.

Here we take a slightly broader view and investigate the collection of distributions of order patterns achieved by particular classes of functions. Specifically, if $I = [0, 1]$ is equipped with Lebesgue measure and f is almost aperiodic (meaning that the set of points with finite orbit has measure zero), then f induces a probability distribution $\boldsymbol{\mu}_n(f)$ on S_n in a natural way:

$$\boldsymbol{\mu}_n(f)(\sigma) = \mu_{\text{Leb}}\{x \mid \text{Order}(x, f(x), \ldots, f^{n-1}(x)) = \sigma\}.$$

We shall focus on the functions $\boldsymbol{\mu}_n$ as well as the function $\boldsymbol{\mu}$ which maps f to the sequence $(\boldsymbol{\mu}_1(f), \boldsymbol{\mu}_2(f), \ldots)$.

Throughout the paper we consider functions with the property that almost all orbits are infinite:

$$\mathcal{A} = \{f : I \to I \mid \mu_{\text{Leb}}(I_{ap}) = 1\},$$

where I_{ap} is the set of aperiodic points, *i.e.*, points with infinite orbit. If $\mathcal{C} \subset \mathcal{A}$ is a collection of functions, then we address the following natural questions.

Question 1.1. What is $\boldsymbol{\mu}_n(\mathcal{C})$?

Question 1.2. What is $\boldsymbol{\mu}(\mathcal{C})$?

We begin by answering both questions for the class $\mathcal{C} = \mathcal{A}$. For any f, the distributions $\boldsymbol{\mu}_n(f)$, $n = 2, 3, \ldots$ must satisfy a certain compatibility condition. In Theorem 2.1 we show that this is the only constraint on what is realizable for arbitrary $f \in \mathcal{A}$: that is, for any sequence $\{\mu_n\}_{n \geqslant 1}$ of compatible distributions on S_n, there is a function $f \in \mathcal{A}$ which simultaneously satisfies $\boldsymbol{\mu}_n(f) = \mu_n$.

We then turn our attention to the class of measure-preserving functions,

$$\mathcal{C} = \mathcal{A}^{\text{mp}} = \{f \in \mathcal{A} \mid f \text{ preserves } \mu_{\text{Leb}}\}.$$

Our main theorem (Theorem 6.8) provides a complete answer to Question 1.1 for $\mathcal{C} = \mathcal{A}^{\text{mp}}$. It is easy to see that the conclusion of Theorem 2.1 cannot hold for $\mathcal{C} = \mathcal{A}^{\text{mp}}$; in fact we observe that $\boldsymbol{\mu}_n(\mathcal{A}^{\text{mp}})$ is contained in a polytope P_n consisting of all (normalized) flows on a certain digraph, which we call a *permutation digraph*. We then show that $\boldsymbol{\mu}_n(\mathcal{A}^{\text{mp}})$ is a union of open faces of P_n including the top-dimensional face, and we give a combinatorial criterion for determining whether or not a given open face of P_n is contained in $\boldsymbol{\mu}_n(\mathcal{A}^{\text{mp}})$.

To prove the main theorem we introduce the fundamental notion of *drift*. Naively, if one wants to construct $f \in \mathcal{A}^{\mathrm{mp}}$ realizing a given distribution μ, one might chop the interval into several subintervals and define f to permute the intervals to produce the desired frequencies. Problems soon arise, however: for example, if we want half the mass of the interval to have iterates with order pattern (132) and the other half (213), then we quickly realize that this is impossible, because f^2 would move all the mass to the right, which is impossible for a measure-preserving function. This is the essence of drift, and the upshot of Theorem 6.8 is that this is the only obstruction: a face of P_n either has drift or not, and the faces contained in $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ are exactly those without drift.

Finally, we discuss the relationship between the entropy of f and a property we call *finite exclusion type*. The latter is equivalent to f having finitely many *basic forbidden patterns*, in the language introduced by Amigó, Elizalde, and Kennel [1]; these properties mean that there are finitely many fixed patterns such that every permutation either arises as an order pattern of iterates of f or contains one of the forbidden patterns. A function with finite entropy can realize at most exponentially many permutations of length N (see [3]), but using the notion of drift we show that quite often, a function with finite exclusion type must realize a super-exponential number of permutations. In particular, if either f is continuous and has points of arbitrarily large period or f is ergodic, then f cannot have both finite entropy and finite exclusion type: see Corollary 7.7. This generalizes results from [5].

The paper is organized as follows. We introduce some language and give our result for $\mathcal{C} = \mathcal{A}$ in Section 2, although we defer the proof to Section 8. Sections 3–5 develop the combinatorial ideas required for our main theorem, including several preliminary results about permutation digraphs and drift. The main theorem is stated and proved in Section 6. Our discussion of entropy and finite exclusion type makes up Section 7, and Section 8 contains the proof of Theorem 2.1. We close with some open questions in Section 9.

2. Generalities

In this section we introduce some language and notation which will be used throughout the paper, and we state our first result, Theorem 2.1, which says that if no restriction is placed on f, then one can always find f realizing a given compatible sequence of distributions of order patterns.

Order patterns

For a positive integer n we denote $\{1, \ldots, n\}$ by $[n]$ and the group of bijections of $[n]$ by S_n.

Let g be an injective map from a finite totally ordered set $X = \{x_1, \ldots, x_n\}$ (where $x_1 < x_2 < \cdots < x_n$) to a totally ordered set Y. Let $y_i = g(x_i)$. We define the *order pattern* Order(g) to be the unique permutation $\sigma \in S_n$ satisfying $y_i < y_j$ if and only if $\sigma(i) < \sigma(j)$. Equivalently $y_{\sigma^{-1}(1)} < \cdots < y_{\sigma^{-1}(n)}$. Note that if $\sigma \in S_n$ then Order$(\sigma) = \sigma$. The *order*

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pattern of an n-tuple of distinct real numbers $(x_1,\ldots,x_n)$ is $\mathrm{Order}(x_1,\ldots,x_n)=\mathrm{Order}(g)$, where $g:[n]\to\mathbb{R}$ takes i to x_i.

There is a restriction map $\rho: S_{n+1}\to S_n$ given by $\rho(\sigma)=\mathrm{Order}(\sigma|_{[n]})$. Using this we define S_∞ as $\{(\sigma_1,\sigma_2,\ldots)\mid \sigma_i\in S_i, \rho(\sigma_{i+1})=\sigma_i\ \forall i\geqslant 1\}$, which is equal to the inverse limit of the maps $\rho: S_{n+1}\to S_n$. Let $S=\cup_{n=1}^{\infty}S_n$. The set S is graded by n and we use notation like $S_{\geqslant n}$ to mean $\cup_{j=n}^{\infty}S_j$.

Distributions

Next, let Δ_n be the space of probability distributions on S_n. Note that Δ_n is the standard simplex in $\mathbb{R}^{S_n}\cong\mathbb{R}^{n!}$.

If $\mu\in\Delta_n$ and $\mu'\in\Delta_{n+1}$ we say μ and μ' are *compatible* if

$$\mu(\sigma)=\sum_{\rho(\sigma')=\sigma}\mu'(\sigma').$$

Then $\Delta_\infty=\{(\mu_1,\mu_2,\ldots)\mid \mu_i\in\Delta_i, \mu_{i+1}\text{ and }\mu_i\text{ are compatible }\forall i\geqslant 1\}$, and $\Delta=\cup_{n=1}^{\infty}\Delta_n$. As an example, the uniform distributions from each Δ_n form a compatible sequence, hence an element of Δ_∞.

Induced distributions

For $f\in\mathcal{A}$ and $x\in I_{ap}$, let $\boldsymbol{\sigma}^f(x)=(\boldsymbol{\sigma}_1^f(x),\boldsymbol{\sigma}_2^f(x),\ldots)\in S_\infty$, where

$$\boldsymbol{\sigma}_n^f(x)=\mathrm{Order}(x,f(x),\ldots,f^{n-1}(x)).$$

Let $\boldsymbol{\mu}_n:\mathcal{A}\to\Delta_n$ be the map taking a function $f\in\mathcal{A}$ to the distribution defined by

$$\boldsymbol{\mu}_n(f)(\sigma)=\mu_{\mathrm{Leb}}\{x\mid \boldsymbol{\sigma}_n^f(x)=\sigma\}.$$

Note that for any f and n, the distributions $\boldsymbol{\mu}_n(f)$ and $\boldsymbol{\mu}_{n+1}(f)$ are compatible; thus we may define $\boldsymbol{\mu}:\mathcal{A}\to\Delta_\infty$ by $\boldsymbol{\mu}(f)=(\boldsymbol{\mu}_1(f),\boldsymbol{\mu}_2(f),\ldots)\in\Delta_\infty$.

We can now state our first result.

Theorem 2.1. *For every $\mu=(\mu_1,\mu_2,\ldots)\in\Delta_\infty$ there exists a measurable function $f\in\mathcal{A}$ with $\boldsymbol{\mu}(f)=\mu$. That is, $\boldsymbol{\mu}(\mathcal{A})=\Delta_\infty$.*

The proof is constructive, a little involved, and unnecessary for the results that follow. Therefore we defer the proof to Section 8.

Convexity

Before we end this section we make an observation about convexity. Suppose $\mathcal{C}\subset\mathcal{A}$ is a collection of functions such that whenever $f,g\in\mathcal{C}$ and $t\in[0,1]$, the function

$$h(x)=\begin{cases} tf\left(\dfrac{x}{t}\right) & \text{if } x<t,\\ t+(1-t)g\left(\dfrac{x-t}{1-t}\right) & \text{if } t<x\leqslant 1\end{cases}$$

is also in $\mathcal{C}$. If in addition $h^{-1}(x)$ has measure zero for all $h\in\mathcal{C}$ and $x\in[0,1]$, then $\boldsymbol{\mu}_n(\mathcal{C})$ is a convex subset of Δ_n. This is because h is the 'block sum' of f scaled by t and g scaled by $1-t$, and so for all n, $\boldsymbol{\mu}_n(h)=t\boldsymbol{\mu}_n(f)+(1-t)\boldsymbol{\mu}_n(g)$.

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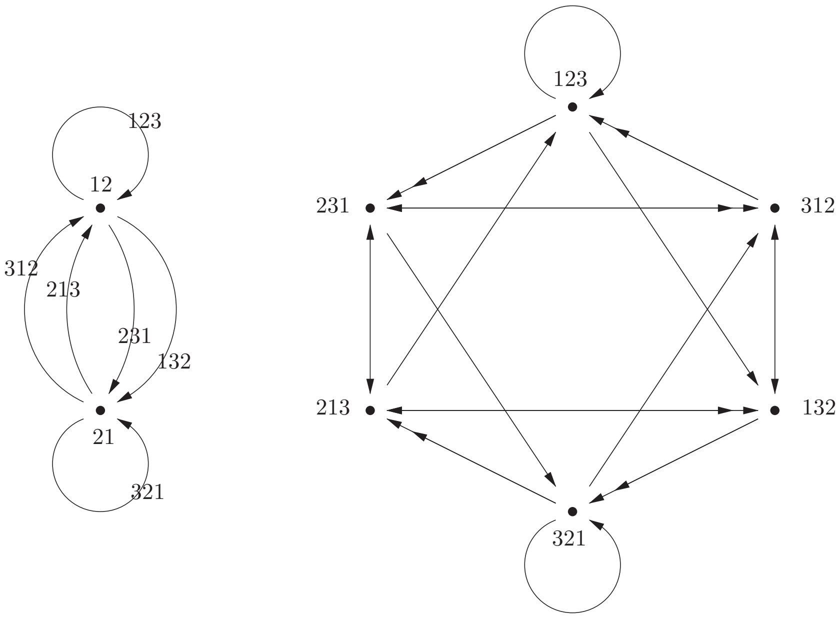

Figure 1. The digraphs G_2 and G_3. The edges of G_2 are shown with labels; the edges of G_3 are abbreviated and the labels omitted. For instance two directed edges go from 231 to 312, with labels 2413 and 3412. An edge labelled 4231 goes in the reverse direction.

This holds, for instance, if $\mathcal{C} = \mathcal{A}^{\mathrm{mp}}$, which we study in this paper. Adam Hesterberg has pointed out to us that the condition that preimages of points have measure zero is necessary; in particular he has studied the class $\mathcal{C} = \mathcal{A}^{\mathrm{pc}}$ of piecewise continuous functions, where he requires a slightly different argument to show that $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{pc}})$ is convex. See [7].

3. Digraphs

The next several sections develop the language used in the remainder of the paper. We begin with digraphs.

A *digraph* is a quadruple $G = (VG, EG, h, t)$ with VG the vertex set, EG the edge set, and h and t the head and tail maps from EG to VG.

Recall that $\rho : S_{n+1} \to S_n$ is defined by $\rho(\sigma) = \mathrm{Order}(\sigma|_{[n]})$. Similarly define $\rho' : S_{n+1} \to S_n$ by $\rho'(\sigma) = \mathrm{Order}(\sigma|_{[2,n+1]})$.

Definition 3.1. For $n \geqslant 1$ let G_n denote the *permutation digraph* $(S_n, S_{n+1}, \rho, \rho')$. The digraphs G_2 and G_3 are shown in Figure 1.

Paths

A *path* of length ℓ (where $0 \leqslant \ell < \infty$) in a digraph is an alternating sequence

$$p = (v_0, e_1, v_1, e_2, \ldots, v_\ell),$$

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with $v_i \in VG$ and $e_i \in EG$ such that $h(e_i) = v_{i-1}$ and $t(e_i) = v_i$. A path of length ∞ is $p = (v_0, e_1, v_1, \ldots)$ such that each finite initial segment ending with a vertex is a (finite) path. We write $\mathrm{Path}_\ell(G)$ for the set of all paths of length ℓ in G and $\mathrm{Path}(G)$ for the set of all paths in G. Note that $S_n = \mathrm{Path}_0(G_n)$. To define specific paths we sometimes abuse notation slightly by thinking of v_i and e_i as functions from $\mathrm{Path}_{\geqslant i}(G)$ to VG and EG.

For example, if p is a path of finite length ℓ and q is any path with $v_0(q) = v_\ell(p)$, then the *concatenation* pq of p and q has $v_i(pq) = v_i(p)$ and $e_i(pq) = e_i(p)$ for $i \leqslant \ell$ and $v_i(pq) = v_{i-\ell}(q)$ and $e_i(pq) = e_{i-\ell}(q)$ for $i > \ell$.

A digraph is *strongly connected* if there are paths connecting every ordered pair of vertices. A (finite) path is *embedded* if all $\ell + 1$ vertices are distinct, except possibly $v_0 = v_\ell$. A *loop* is a finite path with $v_0 = v_\ell$.

Projections

For each $n < \infty$ we define

$$\pi_n : S_\infty \cup \left(\bigcup_{m \geqslant n} \mathrm{Path}(G_m) \right) \to \mathrm{Path}(G_n)$$

as follows. First, π_n is the identity on $\mathrm{Path}(G_n)$. if $\sigma \in \mathrm{Path}_0(G_{n+1}) = S_{n+1}$, let $\pi_{n+1,n}(\sigma)$ be the path $(\rho(v), v, \rho'(v)) \in \mathrm{Path}_1(G_n)$. If $p = (v_0, e_0, \ldots, v_\ell) \in \mathrm{Path}_\ell(G_{n+1})$, then let $\pi_{n+1,n}(p)$ be the concatenation $\pi_{n+1,n}(v_0)\pi_{n+1,n}(v_1) \cdots \pi_{n+1,n}(v_\ell)$. (The result is an infinite path if $\ell = \infty$; otherwise the result is a path of length $\ell + 1$.) Thus $\pi_{n+1,n} : \mathrm{Path}(G_{n+1}) \to \mathrm{Path}(G_n)$. Let $\pi_{m,n} = \pi_{n+1,n} \circ \cdots \circ \pi_{m,m-1} : \mathrm{Path}(G_m) \to \mathrm{Path}(G_n)$ and let π_n be the union of the functions $\pi_{m,n}$ on $\bigcup_{m \geqslant n} \mathrm{Path}(G_m)$.

Finally, extend π_n further by defining $\pi_n(\sigma)$ for $\sigma = (\sigma_1, \sigma_2, \ldots) \in S_\infty$ to be the infinite path whose initial subpath of length ℓ is equal to $\pi_n(\sigma_{\ell+n})$.

Note that if $p \in \mathrm{Path}_\ell(G_m)$ then $\pi_n(p) \in \mathrm{Path}_{\ell+m-n}(G_n)$.

Lifts

The next lemma says that any path (of length > 0) on G_n can be lifted to G_{n+1} (where it becomes shorter if its length is finite). Note, however, that the edges of the lift are not determined; only the vertices are determined, because the edges of p do not appear in the definition of $\pi_{n+1,n}(p)$. The ambiguity in the lifting process will play an important role later.

Lemma 3.2 (path lifting). *The map $\pi_{m,n}$ is surjective. The image of $\pi_n|_{S_\infty}$ is* $\mathrm{Path}_\infty(G_n)$.

Proof. For the first part, it suffices to show that $\pi_{n+1,n}$ is surjective, as $\pi_{m,n}$ is a composition of maps of this form. If $p \in \mathrm{Path}(G_n)$ then each edge of p is a vertex of G_{n+1}. We only need to show that if $e, e' \in EG_n = VG_{n+1}$ with $t(e) = h(e')$, then there is an edge $f \in EG_{n+1} = S_{n+2}$ with $h(f) = e$ and $t(f) = e'$. Extend the function $e : [n+1] \to [n+1]$ to $\bar{e} : [n+2] \to \mathbb{R}$ by defining $e(n+2)$ such that $e' = \mathrm{Order}(e|_{[2,n+2]})$. Then $f = \mathrm{Order}(e) \in S_{n+2}$ is the desired edge.

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For the second part, if $p \in \mathrm{Path}_\infty(G_n)$ then set $p = p_0$, and for each $i > 0$ let $p_i \in \mathrm{Path}_\infty(G_{n+i})$ be a lift of p_{i-1}. Then for $m \geqslant n$ let σ_m be the initial vertex of p_{m-n}, and note that $\sigma = (\sigma_1, \sigma_2, \ldots) \in S_\infty \cap (\pi_n)^{-1}(p)$. □

Example 3.3. Consider the infinite path $p \in \mathrm{Path}_\infty(G_2)$ that begins at the vertex (12) and traverses the edges (132) followed by (312) repeatedly. Then p projects to the path $q = \pi_1(p) \in \mathrm{Path}_\infty(G_1)$, which traverses the loop labelled (12) and then the loop labelled (21) and then repeats. There are infinitely many paths other than p in $\pi_2((\pi_1)^{-1}(q))$, since the vertices must alternate between (12) and (21) but there are two choices for each edge. By contrast, at the next step, $\pi_3((\pi_2)^{-1}(p))$ is the singleton consisting of the infinite path on G_3 that starts at the vertex (312) and traverses the edges (1423) and (4132) repeatedly. In fact $(\pi_2)^{-1}(p) \subseteq S_\infty$ is already a singleton, being the compatible sequence $(\sigma_1, \sigma_2, \ldots)$, where σ_n is the permutation $(1, n, 2, n-1, \ldots)$.

4. The poset of a path

To any path p on the digraph G_n, we will associate a partially ordered set Q_p that keeps track of the order relations imposed on any sequence $x_1, \ldots, x_m$ of real numbers whose order pattern projects to p.

Given a path $p = (v_0, e_1, v_1, \ldots, v_\ell)$ on G_n, consider the set

$$\overline{Q}_p = \big(\{v\} \times [0, \ell] \times [n]\big) \cup \big(\{e\} \times [1, \ell] \times [n+1]\big).$$

This set is (in one-to-one correspondence with) the disjoint union of the domains of all the permutations v_i and e_i. They are 'patched together' by the equivalence $\sim$ generated by

(i) $(v, a, c) \sim (e, a+1, c)$,
(ii) $(v, a, c) \sim (e, a, c+1)$.

The element (v, a, c) stands for the same thing as $v_a(c)$; likewise (e, a, c) really means $e_a(c)$. The equivalence class of (v, a, c) in $Q_p = \overline{Q}_p/\sim$ will be denoted by $x_{a+c}(p)$, or x_{a+c} if the path p is understood; note that (a) this is well-defined and (b) every element of Q_p is equal to x_i for some $1 \leqslant i \leqslant \ell + n := m$. By (a), if $x_i = x_j$ for $1 \leqslant i, j \leqslant m$ then $i = j$, and so $Q_p = \{x_1, \ldots, x_m\}$.

The set Q_p is easy to visualize; see Figure 2. Here is the partial ordering on it.

Consider the relation $\leqslant$ on Q_p generated by

(iii) $[(v, a, c)] \leqslant [(v, a, d)]$ if $v_a(c) \leqslant v_a(d)$,
(iv) $[(e, a, c)] \leqslant [(e, a, d)]$ if $e_a(c) \leqslant e_a(d)$,

and extended by transitivity.

We will show in a moment that $\leqslant$ is a partial ordering on Q_p. Again, the point of Q_p is to keep track of all order relationships which necessarily hold among $\sigma(1), \ldots, \sigma(m)$, if σ is a permutation in $\pi_n^{-1}(p)$.

Example 4.1. Consider the path p of length 5 in G_3 with edges

$$e_1 = (2134), \quad e_2 = (1342), \quad e_3 = (2314), \quad e_4 = (3241), \quad e_5 = (2314).$$

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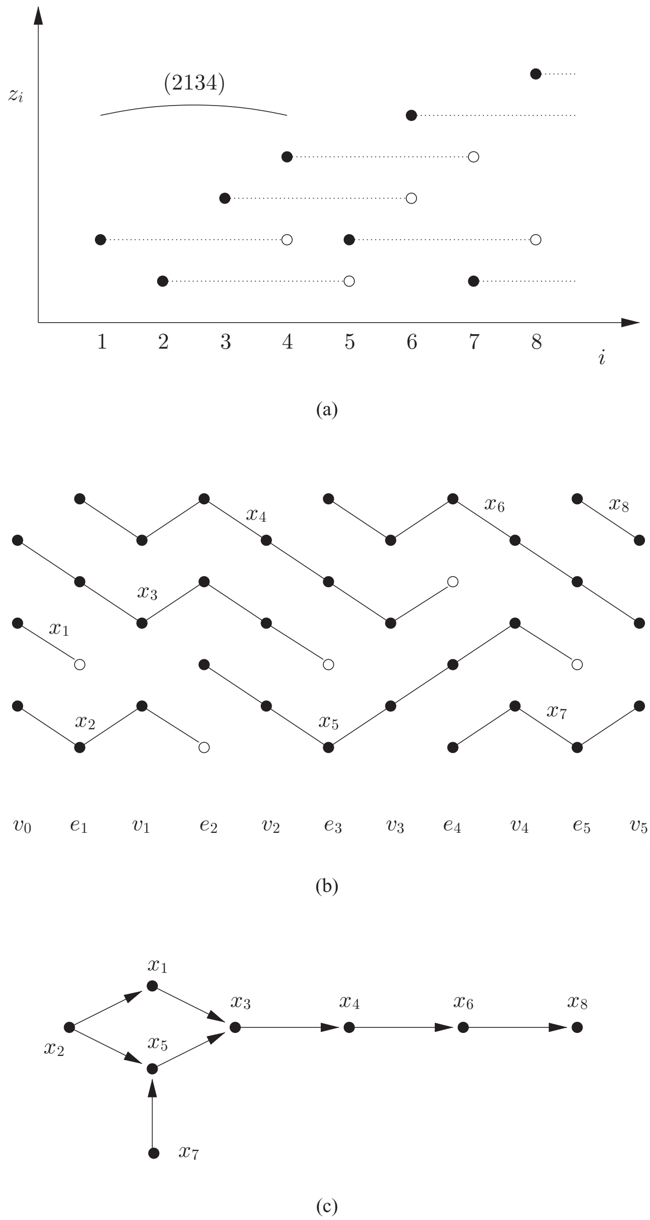

Figure 2. The path p is a loop of length 5 in G_3 traversing in this order the edges $2134, 1342, 2314, 3241, 2314$. Figure (a) is a plot of a sequence z_i (for $i = 1, \ldots, 8$) which maps to p under π_3. The elements of $\overline{Q}_p$, shown in (b) as dots, are the intersections of the above plot with vertical lines at $i = 3.5, 4, 4.5, \ldots, 8.5$. The elements $(v, 0, 3)$, $(v, 0, 1)$, $(v, 0, 2)$ of $\overline{Q}_p$ make up the leftmost column of dots (read from top to bottom). Equivalent elements are joined by an arc. Figure (c) depicts the poset Q_p whose elements are the arcs in either of the previous pictures.

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This is a loop based at (213). Attempts to construct real numbers $z_1, \ldots, z_8$ such that

$$\mathrm{Order}(z_i, z_{i+1}, z_{i+2}, z_{i+3}) = e_i$$

quickly lead one to draw pictures like Figure 2. Figure 2(a) is a plot of the desired z. Note that z_5 could be perturbed to be larger or smaller than z_1, and similarly for z_7 and z_2. The dotted lines indicate the duration of the influence of z_i on future z_j. This information is abstracted in Figure 2(b), in which the dots are the elements of $\overline{Q_p}$ and equivalent elements are joined by an arc. Each arc is an element of Q_p. Figure 2(c) shows the partial ordering: an edge pointing from x_i to x_j indicates that $x_i \leqslant x_j$.

Lemma 4.2. *Let p be a path on G_n, and let $\tilde{p} \in \pi_n^{-1}(p)$. If $x_i(p) \leqslant x_j(p)$ then $x_i(\tilde{p}) \leqslant x_j(\tilde{p})$.*

Proof. We may assume the length ℓ is not zero. It suffices to prove for $\tilde{p} \in \mathrm{Path}(G_{n+1})$, as we can lift multiple times.

The hypothesis $x_i(p) \leqslant x_j(p)$ implies there is a sequence

$$x_i \ni (y_0, a_0, c_0) \sim (y_1, a_1, c_1) \leqslant (y_2, a_2, c_2) \sim \cdots \leqslant (y_r, a_r, c_r) \in x_j \tag{4.1}$$

in $\overline{Q_p}$, where each step is one of the types (i)–(iv). In this sequence, if one of the inequalities has $y_i = y_{i+1} = v$ then $a_i = a_{i+1}$, and using (i) and (ii) we can replace y_i and y_{i+1} by e and either increase a_i and a_{i+1} by 1 (if $a_i \neq \ell$) or increase c_i and c_{i+1} by 1 (if $a_i \neq 0$). Since $\rho(e_i) = v_{i-1}$ and $\rho'(e_i) = v_i$, the inequality is preserved in either case. Thus we obtain a new sequence (4.1) with each $y_i = e$.

Now note that the elements (e, a, c) of $\overline{Q_p}$ are in one-to-one correspondence with the elements (v, a, c) of $\overline{Q_{\tilde{p}}}$. Thus, if we now switch every e to a v and subtract 1 from each a, we obtain a sequence in $\overline{Q_{\tilde{p}}}$ showing $x_i(\tilde{p}) \leqslant x_j(\tilde{p})$. □

Corollary 4.3. *The relation $\leqslant$ is a partial order on Q_p.*

Proof. The relation is reflexive and transitive by definition. We must show that if $x_i \leqslant x_j$ and $x_j \leqslant x_i$ then $i = j$. Lift p to a path $v_0 = \sigma \in \mathrm{Path}_0(G_m) = S_m$ (where $m = \ell + n$). There are no equivalences in $\overline{Q_\sigma}$, and $x_i \leqslant x_j$ in the poset Q_σ if and only if $\sigma(i) \leqslant \sigma(j)$. By Lemma 4.2, $x_i \leqslant x_j$ and $x_j \leqslant x_i$ in Q_σ, so $i = j$. □

Remark 4.4. Note that for any $1 \leqslant i \leqslant \ell$, the elements $x_i, x_{i+1}, \ldots, x_{i+n}$ are totally ordered in Q_p.

Let $\psi(v) = 1/2$ and $\psi(e) = 0$. If $x_i \leqslant x_j$ in Q_p then there is a sequence (4.1) with each $(y_k, a_k, c_k) \in \overline{Q_p}$. Call such a sequence *monotonic* if the function $\psi(y_k) + a_k$ is monotonic in k.

Lemma 4.5. *If $x_i \leqslant x_j$ in Q_p, then there is a monotonic sequence of the form* (4.1).

Proof. Choose a sequence of the form (4.1); one exists by definition. Note that only rules (1) and (2) change ψ, and that ψ increases (by 1/2) if either of these rules is applied

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by replacing the left side with the right. Suppose the given sequence is not monotonic. Specifically suppose that ψ increases and later decreases; the other case is virtually identical. Choosing an innermost such backtrack, we find a subsequence of one of the following two forms:

(i) $(v,a,c) \sim (e,a+1,c) \leqslant (e,a+1,d) \sim (v,a,d)$,
(ii) $(e,a,c+1) \sim (v,a,c) \leqslant (v,a,d) \sim (e,a,d+1)$.

In case (i), we have $e_{a+1}(c) \leqslant e_{a+1}(d)$. But $v_a = \rho(e_{a+1})$ so we also have $v_a(c) \leqslant v_a(d)$. Thus we can delete the middle two terms of (i) and eliminate the backtracking.

Case (ii) is similar: $v_a(c) \leqslant v_a(d)$ but this time $v_a = \rho'(e_a)$. Now it follows that $e_a(c+1) \leqslant e_a(d+1)$, so again we can eliminate the backtracking. □

Referring again to Figure 2, Lemma 4.5 says that it is very easy to determine whether $x_i \leqslant x_j$. If $i < j$, one just sees whether it is possible to connect the right endpoint of x_i to any point above x_j with a path that passes the vertical line test. If not, then $x_i \leqslant x_j$.

Lemma 4.6. *Let p, q be paths on G_n of lengths ℓ, ℓ' such that the concatenation pq is defined. For $1 \leqslant i, j \leqslant \ell' + n$, if $x_i(q) \leqslant x_j(q)$, then $x_{\ell+i}(pq) \leqslant x_{\ell+j}(pq)$.*

Proof. Choose a monotone sequence of the form (4.1), and add ℓ to the second coordinate of each term. The new sequence proves the result. □

Remark 4.7. Let p be a path of length ℓ on G_n, and let $m = \ell + n$. A choice of lift $\sigma \in \pi_{m,n}^{-1}(p)$ amounts to a choice of extension of $\leqslant$ to a total order on $\{x_1, \dots, x_m\}$. That this can be done is well known; the process is sometimes called a 'topological sort'. In particular, for a subset $\{i_1, \dots, i_k\} \subset [m]$ of indices, if the x_{i_j} are pairwise incomparable in Q_p then for any permutation $v \in S_k$ there is an extension of $\leqslant$ to a total order on $\{x_1, \dots, x_m\}$ satisfying $x_{i_{v(1)}} < \cdots < x_{i_{v(k)}}$. In the terminology of lifts this becomes the following statement, which bears on the discussion of entropy in a later section.

Corollary 4.8. *Let p be a path of length ℓ on G_n, and let $m = \ell + n$. If the elements $\{x_{i_1}, \dots, x_{i_k}\}$ of Q_p are pairwise incomparable, then for any permutation $v \in S_k$ there is a lift $\sigma \in S_m \subset \pi_n^{-1}(p)$ such that $\sigma(i_{v(1)}) < \cdots < \sigma(i_{v(k)})$. In particular $|\pi_n^{-1}(p) \cap S_m| \geqslant k!$.*

As a special case of this we also note the following.

Corollary 4.9. *Let p be a path of length ℓ on G_n. Then $x_i \leqslant x_j$ if and only if $\sigma(i) \leqslant \sigma(j)$ for every $\sigma \in S_m \cap \pi_n^{-1}(p)$ (where necessarily $m = \ell + n$).*

5. Drift

If γ is a loop of length $\ell < \infty$ on G_n, then the elements $x_1, \dots, x_n$ of Q_p are totally ordered, as are the elements $x_{\ell+1}, \dots, x_{\ell+n}$, and if we set $y_i = x_{\ell+i}$, then we have $x_i \leqslant x_j$ if and only if $y_i \leqslant y_j$, for all $i, j \in [n]$. The notion of drift is based on how the x_i compare to the y_j, as measured by the following two functions. Let $\langle n\rangle$ be the totally ordered set $[n]\cup\{-\infty,\infty\}$ (with $-\infty<1$ and $n<\infty$), and for $i\in\langle n\rangle$ define

$$\mathrm{Max}_\gamma(i)=\begin{cases} j & \text{if } x_i\leqslant y_j,\ \text{and for } k\in[n], x_i\leqslant y_k \text{ implies } y_j\leqslant y_k,\\ \infty & \text{if } i=\infty \text{ or } i\in[n] \text{ and there is no } j \text{ such that } x_i\leqslant y_j,\\ -\infty & \text{if } i=-\infty,\end{cases}$$

$$\mathrm{Min}_\gamma(i)=\begin{cases} j & \text{if } x_i\geqslant y_j,\ \text{and for } k\in[n], x_i\geqslant y_k \text{ implies } y_j\geqslant y_k,\\ -\infty & \text{if } i=-\infty \text{ or } i\in[n] \text{ and there is no } j \text{ such that } x_i\geqslant y_j,\\ \infty & \text{if } i=\infty.\end{cases}$$

Lemma 5.1. *If $x_i\leqslant x_j$ then*

$$\mathrm{Max}_\gamma(i)\leqslant\mathrm{Max}_\gamma(j)\quad\textit{and}\quad\mathrm{Min}_\gamma(i)\leqslant\mathrm{Min}_\gamma(j).$$

Proof. This is immediate from the definitions, and from the fact that $x_i\leqslant x_j$ if and only if $y_i\leqslant y_j$. □

Lemma 5.2. *Suppose p and q are finite paths such that pq is a path. Then*

$$\mathrm{Max}_{pq}=\mathrm{Max}_q\,\mathrm{Max}_p\quad\textit{and}\quad\mathrm{Min}_{pq}=\mathrm{Min}_q\,\mathrm{Min}_p\,.$$

Proof. We give the verification for Max. For Min, flip the argument upside down. Let ℓ be the length of p and ℓ' the length of q.

It is clear that the two functions are equal on $\pm\infty$. Let $i\in[n]$, and let $j=\mathrm{Max}_p(i)$. If $j=\infty$ then by Lemma 4.5 it is impossible to have $x_i\leqslant x_{\ell+\ell'+k}$ for any $k\in[n]$, so $\mathrm{Max}_{pq}(i)=\infty$. We may therefore assume $j\neq\infty$, and let $k=\mathrm{Max}_q(j)$. If $k=\infty$, then again by Lemma 4.5 it is impossible to have $x_i\leqslant x_{\ell+\ell'+k'}$ for any $k'\in[n]$, so $\mathrm{Max}_{pq}(i)=\infty$. Thus we may assume $k\neq\infty$. We want to show that $\mathrm{Max}_{pq}(i)=k$.

In the poset Q_{pq}, we have $x_i\leqslant x_{\ell+j}\leqslant x_{\ell+\ell'+k}$. Also, if $x_i\leqslant x_{\ell+\ell'+k'}$ then there is a monotonic sequence in $\overline{Q}_{pq}$ showing $x_i\leqslant x_{\ell+\ell'+k'}$. This sequence must contain a point of the form (v,ℓ,j'), so $x_i\leqslant x_{\ell+j'}$. By definition of j, we have $x_{\ell+j}\leqslant x_{\ell+j'}$, hence $x_{\ell_j}\leqslant x_{\ell+\ell'+k'}$. By definition of k we now have $x_{\ell+\ell'+k}\leqslant x_{\ell+\ell'+k'}$. Thus $k=\mathrm{Max}_{pq}(i)$, as desired. □

Let γ be a loop of length ℓ on G_n. For $i,j\in[n]$ let

$$\mathrm{Drift}_\gamma(i,j)=\begin{cases} + & \text{if } x_i\leqslant y_j \text{ in } Q_\gamma,\\ - & \text{if } x_i\geqslant y_j \text{ in } Q_\gamma,\\ 0 & \text{otherwise.}\end{cases}$$

We will write $\mathrm{Drift}_\gamma(i)$ for $\mathrm{Drift}_\gamma(i,i)$.

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Definition 5.3. A loop γ is *partially driftless* if $\text{Drift}_\gamma(i) = 0$ for some $i \in [n]$. A loop γ is *driftless* if $\text{Drift}_\gamma(i) = 0$ for all $i \in [n]$. A loop γ is *totally driftless* if $\text{Drift}_\gamma(i, j) = 0$ for all $i, j \in [n]$.

Thus γ is totally driftless if and only if $\text{Max}_\gamma(i) = \infty$ and $\text{Min}_\gamma(i) = -\infty$ for all $i \in [n]$, and there are similar descriptions of driftless and partially driftless loops.

Example 5.4. The loop p in Figure 2 is partially driftless. In Q_p, we have $x_1 \leqslant x_6 = y_1$ and $x_3 \leqslant x_8 = y_3$, so $\text{Drift}_p(1) = \text{Drift}_p(3) = +$. However, x_2 and $x_7 = y_2$ are incomparable, so $\text{Drift}_p(2) = 0$. Note that the number z_6 is necessarily greater than z_1, but z_7 can be chosen to be greater than or less than z_2.

Lemma 5.5. *Let β and γ be (partially driftless) loops based at v, with*

$$\text{Drift}_\beta(j) = \text{Drift}_\gamma(j) = 0.$$

Then $\text{Drift}_{\beta\gamma}(j) = 0$.

Proof. Let ℓ be the length of β. Suppose $x_j(\beta\gamma) \leqslant y_j(\beta\gamma)$. Then there is a monotonic sequence (4.1) proving this. In the sequence there must be a representative of x_i for some $\ell + 1 \leqslant i \leqslant \ell + n$. Now $x_i = x_{\ell+j}$ or $x_i \leqslant x_{\ell+j}$ would contradict $\text{Drift}_\beta(j) = 0$, and $x_i \geqslant x_{\ell+j}$ would contradict $\text{Drift}_\gamma(j) = 0$. By Remark 4.4 these are the only possibilities. Thus $\text{Drift}_{\beta\gamma}(j) \neq +$. Similarly $\text{Drift}_{\beta\gamma}(j) \neq -$. $\square$

Lemma 5.6. *Let β and γ be loops on G_n based at the vertex v. If γ is totally driftless then $\beta\gamma$ is totally driftless.*

Proof. Suppose not; then without loss of generality there exists $i \in [n]$ with $\text{Max}_{\beta\gamma}(i) = j < \infty$. Thus $x_i(\beta\gamma) \leqslant y_j(\beta\gamma)$ and by Lemma 4.5 there is a monotonic sequence proving this inequality. This sequence must contain (v, ℓ, k) for some k, where ℓ is the length of β. Starting there, the remainder of the sequence (in combination with Lemma 4.6) shows that $\text{Max}_\gamma(k) \leqslant j < \infty$, a contradiction. $\square$

Lemma 5.7. *Cyclic permutations of driftless loops are driftless.*

Proof. Let $\gamma = (v_0, e_1, \ldots, v_\ell)$ be a driftless loop and let γ_k be a cyclic permutation of γ starting at v_k. Suppose $x_i(\gamma_k) \leqslant x_{\ell+i}(\gamma_k)$. Fix a monotonic sequence showing this inequality, and add ℓ to the second coordinate of each element to obtain a new sequence, and concatenate the original sequence with the new one. This longer sequence shows $x_i(\gamma_k^2) \leqslant x_{2\ell+i}(\gamma_k^2)$, but it contains a subsequence showing for some j that $\text{Drift}_\gamma(j) \neq 0$. $\square$

Definition 5.8. A *face subgraph* of G_n is a subgraph H such that every edge of H is contained in a loop in H. Equivalently H is a face subgraph if each connected component of H is strongly connected.

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Definition 5.9. A strongly connected subgraph $H \subseteq G_n$ *drifts* if there exist $v \in VH$, $j \in [n]$ and $\epsilon \in \{+,-\}$ such that for every loop γ in H based at v, $\mathrm{Drift}_\gamma(j) = \epsilon$. Otherwise H is *driftless.*

A face subgraph $H \subseteq G_n$ *drifts* if any of its connected components drifts; otherwise H is driftless.

Proposition 5.10. *Let H be a strongly connected subgraph of G_n. The following are equivalent:*

(1) *H is driftless,*
(2) *there exists a totally driftless loop γ with support contained in H,*
(3) *there exists a totally driftless loop γ with support equal to H.*

□

Proof. The last two statements are equivalent by Lemma 5.6: if γ is a totally driftless loop with support contained in H, and β is any loop with support equal to H, then $\beta\gamma$ is a totally driftless loop with support equal to H.

Statement (3) easily implies statement (1): for fixed v, j, ϵ let γ_v be a cyclic permutation of γ which starts at v. By Lemma 5.7, $\mathrm{Drift}_{\gamma_v}(j) = 0 \neq \epsilon$.

Last, we show that (1) implies (2). Let γ_0 be a loop based at σ and supported in H. Let $i = \sigma^{-1}(1)$ and $j = \sigma^{-1}(n)$, so that $x_i \leqslant x_k \leqslant x_j$ for all $k \in [n]$.

Suppose $\mathrm{Max}_{\gamma_0}(i) = k \neq \infty$. As H is driftless, we may pick a loop γ_1 based at v and supported in H such that $\mathrm{Drift}_{\gamma_1}(k) \neq +$. Thus $\mathrm{Max}_{\gamma_0\gamma_1}(i) > k$, and we can continue this process until we have a loop β_0 with $\mathrm{Max}_{\beta_0}(i) = \infty$.

Then, similarly, we concatenate loops on to the end of β_0 to create a loop β with $\mathrm{Min}_\beta(j) = -\infty$. Note that $\mathrm{Max}_\beta(i) = \infty$ (by Lemma 5.2). Now, by Lemma 5.1, β is totally driftless, with support contained in H. □

Corollary 5.11. *If K and H are strongly connected, $K \subseteq H$, and K is driftless, then H is driftless.*

6. Measure-preserving functions

In this section we analyse the distributions of order patterns arising from (almost aperiodic) measure-preserving functions

$$\mathcal{A}^{\mathrm{mp}} = \{f \in \mathcal{A} \mid \mu_{\mathrm{Leb}}(f^{-1}(S)) = \mu_{\mathrm{Leb}}(S) \text{ for all measurable sets } S\}.$$

Our main theorem is that the image $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ is a union of open faces of a polytope $P_n \subset \Delta_n$ of dimension $n! - (n-1)!$, and that there is an easily checkable combinatorial criterion for determining whether a particular face of P_n is in the image.

Remark 6.1. For most of these results it is not essential that μ_{Leb} be the measure preserved by f. That is, given a function $f \in \mathcal{A}$ one could choose an invariant measure λ and proceed with this section, everywhere replacing μ_{Leb} with λ. For some steps it may be necessary to assume λ has no atoms.

We start by observing that Theorem 2.1 would not hold if $\mathcal{A}$ were replaced by $\mathcal{A}^{\mathrm{mp}}$. For $\sigma \in S_n$, let χ_σ denote the distribution that has mass 1 at σ and 0 elsewhere. Thus the vertex set of Δ_n is $\{\chi_\sigma \| \sigma \in S_n\}$.

Lemma 6.2. *If $J \subseteq I$ has positive measure and $f : J \to J$ is aperiodic and measure-preserving, then both $J_+ = \{x \in J \mid f(x) > x\}$ and $J_- = \{x \in J \mid f(x) < x\}$ have positive measure.*

In particular, there is no $f \in \mathcal{A}^{\mathrm{mp}}$ such that $\boldsymbol{\mu}_2(f) = \chi_{(12)}$ or $\chi_{(21)}$.

Proof. Suppose $\mu_{\mathrm{Leb}}(J_-) = 0$, *i.e.*, $f(x) > x$ for almost all $x \in J$. Then there is some $\epsilon > 0$ such that $\mu_{\mathrm{Leb}}\{x \mid f(x) - x > \epsilon\} > 0$, hence $\int_J f(x) - x > 0$. But f measure-preserving implies $\int_J f(x) - x = 0$, and similarly for $\mu_{\mathrm{Leb}}(J_+)$. □

Note that $\chi_{(12)}$ and $\chi_{(21)}$ are in the closure of $\boldsymbol{\mu}_2(\mathcal{A}^{\mathrm{mp}})$ since $\boldsymbol{\mu}_2(f_\epsilon)$ can be made arbitrarily close to these distributions by choosing $f_\epsilon(x) = x + \epsilon \mod 1$.

The flow polytope P_n

Lemma 6.2 notwithstanding, there is a much more serious reason for the failure of Theorem 2.1 in the measure-preserving category. For $f \in \mathcal{A}^{\mathrm{mp}}$, there is an additional set of constraints on $\boldsymbol{\mu}(f)$ beyond compatibility of the measures $\boldsymbol{\mu}_n(f)$. Namely, the order pattern of $(fx, f^2x, \ldots)$ must be distributed in the same way as the order pattern of $(x, fx, f^2x, \ldots)$. More precisely, if $I_\sigma = \{x \in I \mid \boldsymbol{\sigma}_n^f(x) = \sigma\}$ then

$$\boldsymbol{\mu}_n(f)(\sigma) = \mu_{\mathrm{Leb}}(I_\sigma) = \mu_{\mathrm{Leb}}(f^{-1}(I_\sigma)) = \mu_{\mathrm{Leb}}(\{x \in I \mid \boldsymbol{\sigma}_n^f(f(x)) = \sigma\}).$$

Thus, if $f \in \mathcal{A}^{\mathrm{mp}}$ we necessarily have

$$\mu_n(f)(\sigma) = \sum_{\rho'(\sigma')=\sigma} \mu_{n+1}(f)(\sigma'). \tag{6.2}$$

(Recall that $\rho'(\sigma) = \mathrm{Order}(\sigma|_{[2,n+1]})$.)

The functions ρ and ρ', now thought of as maps $S_n \to S_{n-1}$, induce maps $\rho_*, \rho'_* : \Delta_n \to \Delta_{n-1}$. Explicitly, for $\mu \in \Delta_n$,

$$\rho_*(\mu)(\sigma) = \sum_{\sigma' \in \rho^{-1}(\sigma)} \mu(\sigma'),$$
$$\rho'_*(\mu)(\sigma) = \sum_{\sigma' \in \rho'^{-1}(\sigma)} \mu(\sigma').$$

Thus by (6.2) and compatibility, $\rho_*(\boldsymbol{\mu}(f)) = \rho'_*(\boldsymbol{\mu}(f))$ for $f \in \mathcal{A}^{\mathrm{mp}}$.

Definition 6.3. Set $P_n = \{\mu \in \Delta_n \mid \rho_*(\mu) = \rho'_*(\mu)\}$.

As each condition (6.2) is linear, P_n is a polytope contained in the simplex Δ_n, and $P_n \cap \partial\Delta_n = \partial P_n$. We have already proved the following lemma.

Lemma 6.4. *If $f \in \mathcal{A}^{\mathrm{mp}}$ then $\boldsymbol{\mu}_n(f) \in P_n$.*

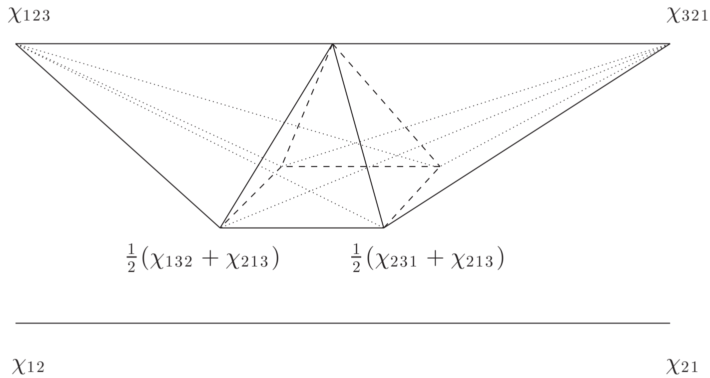

Figure 3. The polytopes P_2 (below) and P_3 (above). Fibres of the (vertical) projection are three-dimensional square pyramids; the preimage of the midpoint of P_2 is shown. The whole polytope P_3 is the join of an interval and a square.

Example 6.5. The polytope P_2 is all of Δ_2; this is a line segment connecting $\chi_{(12)}$ to $\chi_{(21)}$. The preimage of a point $a\chi_{(12)} + (1-a)\chi_{(21)} \in \mathrm{Int}(P_2)$ under the map ρ_* is a three-dimensional square pyramid with apex $a\chi_{(123)} + (1-a)\chi_{(321)}$. If $0 < a < 1/2$ the vertices of the square base are $a(\chi_\sigma + \chi_\tau) + (1-2a)\chi_{(321)}$, where $\sigma \in \{(132),(231)\}$ and $\tau \in \{(213),(312)\}$, whereas if $1/2 < a < 1$ then the vertices are $(1-a)(\chi_\sigma + \chi_\tau) + 2a\chi_{(321)}$, with the same choices for σ and τ. If $a = 1/2$ then the square base is a (two-dimensional) face of P_3; it corresponds to the face subgraph $H \subset G_2$ consisting of all the edges except the loops (123) and (321).

The entire polytope P_3 is four-dimensional; it resembles a suspension of the (middle) square pyramid, except that the apex of the pyramid lies on the segment connecting the suspension points $\chi_{(123)}$ and $\chi_{(321)}$, so that P_3 has six vertices rather than seven. See Figure 3 (in which ρ_* projects vertically).

Dictionary between P_n and G_{n-1}

Before we get to the main theorem we establish several connections between P_n and G_{n-1}.

An *edge weighting* on a digraph G is a map $\phi : EG \to [0,1]$ such that $\sum \phi(e) = 1$. A *flow* on G is an edge weighting ϕ such that, for every $v \in VG$,

$$\sum_{\{e|h(e)=v\}} \phi(e) = \sum_{\{e|t(e)=v\}} \phi(e).$$

Note that the set of all edge weightings on G_{n-1} is exactly Δ_n, and the set of all flows on G_{n-1} is exactly P_n.

A flow supported on an embedded loop in G_{n-1} is a vertex of P_n. The set of all flows supported on a face subgraph $H \subset G_{n-1}$ is a face F_H of P_n. The assignment $H \mapsto F_H$ is an inclusion-preserving bijection between the set of face subgraphs of G_{n-1} and the set of faces of P_n. The dimension of F_H is one less than the rank of the first homology of H. In particular, if $H = G_{n-1}$ then $F_H = P_n$ has dimension $n! - (n-1)!$.

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If two face subgraphs $H, K \subset G_{n-1}$ are disjoint, then $F_{H\cup K} = F_H * F_K$, where $*$ denotes the join.

Example 6.6. By counting the face subgraphs of various ranks in, say, G_2, one determines the number and structure of faces of P_3 of each dimension. It is instructive to compare this with the earlier description of P_3 given in Example 6.5.

Remark 6.7. The dimension of Δ_n is $n! - 1$, and the conditions (6.2) impose $(n-1)!$ additional linear constraints. These constraints are obviously not independent, since their sum is zero; the fact that P_n has dimension $n! - (n-1)!$ shows that the constraints are otherwise linearly independent.

Realizable faces

Here is our main theorem, which we prove after a sequence of lemmas.

Theorem 6.8.

(1) *The set* $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ *is a union of open faces of* P_n.
(2) *Let* F *be a face of* P_n *and let* H *be the corresponding face subgraph of* G_{n-1}, *so that* $F = F_H$. *Then* $\mathrm{Int}(F) \subset \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ *if and only if* H *is driftless.*
(3) *The closure of* $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ *is* P_n.

Example 6.9. The set $\boldsymbol{\mu}_2(\mathcal{A}^{\mathrm{mp}})$ is equal to the interior of P_2. The set $\boldsymbol{\mu}_3(\mathcal{A}^{\mathrm{mp}})$ consists of $\mathrm{Int}(P_3)$ (which is four-dimensional) together with all six of its open three-dimensional facets, nine of its thirteen open two-dimensional faces (including the square face), and two of its thirteen open edges. None of the six vertices of P_3 is in $\boldsymbol{\mu}_3(\mathcal{A}^{\mathrm{mp}})$.

There are sometimes vertices of P_n in $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$. For example, the embedded loop in G_4 with edges (23451), (34512), (45132), (41325), (13254), (31542), (15423), (54123), (51234) is driftless, as is easily seen by computing its poset Q. Hence by Theorem 6.8 the corresponding vertex of P_5 is realizable.

Lemma 6.10. *Let* γ *be a driftless loop in* G_{n-1}. *Then there is an* $f \in \mathcal{A}^{\mathrm{mp}}$ *such that* $\mu_n(f)$ *equals the counting measure induced on* EG_{n-1} *by* γ. *In particular* $\mu_n(f)$ *is in the interior of the face* F_H, *where* H *is the (edge) support of* γ.

Proof. For any permutation σ we build the *permutation function* corresponding to σ: if $\sigma \in S_\ell$, we take the argument of the function σ mod ℓ, and set

$$f_\sigma(x) = x + \frac{\sigma(i+1) - \sigma(i)}{\ell}, \quad \text{where} \quad i = \sigma^{-1}(1 + \lfloor \ell x \rfloor).$$

This is an interval exchange map in the usual sense, and it is periodic. The graph of f_σ looks like an $\ell \times \ell$ permutation matrix with empty space in place of zeros and diagonal line segments in place of ones.

Now if ℓ is the length of γ, then lift γ to a path in G_ℓ beginning at a vertex σ. The function f_σ has the distribution μ_n that we want, but because f_σ is periodic, we let f equal

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f_σ composed with a measure-preserving ergodic function on the sub-interval $[0, 1/\ell]$. Now $f \in \mathcal{A}^{\mathrm{mp}}$ and $\mu_n(f) = \mu_n(f_\sigma)$ so f has the desired properties. □

Lemma 6.11 (Balayage). *Let H be a connected face subgraph of G_n. Then H is driftless if and only if* $\mathrm{Int}(F_H) \cap \mu_n(\mathcal{A}^{\mathrm{mp}}) \neq \emptyset$.

Proof. Assume H driftless. By Lemma 5.10 there is a totally driftless loop γ with support H. By Lemma 6.10 there is an $f \in \mathcal{A}^{\mathrm{mp}}$ with $\boldsymbol{\mu}_n(f) \in \mathrm{Int}(F_H)$.

Conversely, assume H drifts. Let $f \in \mathcal{A}^{\mathrm{mp}}$, and suppose that $\boldsymbol{\mu}_n(f) \in \mathrm{Int}(F_H)$. Using the drift, we will construct from f a positive measure subset of I and a measure-preserving function g on that subset such that either $g(x) > x$ for all x or $g(x) < x$ for all x. This will contradict Lemma 6.2.

Let $J = \{x \in I_{ap} \mid \forall u \in VG_n, |\{i \mid \boldsymbol{\sigma}_n^f(f^i(x)) = u\}| \in \{0, \infty\}\}$. Note that $\mu_{\mathrm{Leb}}(J) = 1$, since $J \subseteq f^{-1}(J)$ and $I_{ap} = \cup_i f^{-i}(J)$. Let $v \in VH, j \in [n], \epsilon \in \{+, -\}$ be as asserted in the definition of drift. Set $J_v = \{x \in J \mid \boldsymbol{\sigma}_n^f(x) = v\}$ and $J_{v,j} = J \cap f^{(j-1)}(J_v)$. Note that $\mu_{\mathrm{Leb}}(J_{v,j}) \geqslant \mu_{\mathrm{Leb}}(J_v)$ are positive by hypothesis. For $x \in J_v$, let $i(x)$ be the smallest $j > 0$ such that $f^j(x) \in J_v$. For $y \in J_{v,j}$ we write $y = f^{j-1}(x)$ with $x \in J_v$, and now define $g : J_{v,j} \to J_{v,j}$ by $g(y) = f^{i(x)}(y)$.

Note that g is measure-preserving. To see this, consider $A \subseteq J_{v,j}$ measurable and write $B_r = f^{-r}(A) - \cup_{i \in [0, r-1]} f^{-i}(J_{v,j}) \subseteq \cup_i f^{-i}(J_{v,j}) - \cup_{i<n} f^{-i}(J_{v,j})$, a sequence with measure decreasing to 0. Write $A_r = B_r \cap J_{v,j}$. Note that $g^{-1}(A) = \cup_r A_r$ is a disjoint decomposition and for every n there is a $\mu_{\mathrm{Leb}}(A) = \mu_{\mathrm{Leb}}(\cup_{r<n} A_r) + \mu_{\mathrm{Leb}}(B_n)$ so that $\mu_{\mathrm{Leb}}(A) = \mu_{\mathrm{Leb}}(g^{-1}(A))$.

Now if $\epsilon = +$, then $g(y) > y$ for all $y \in J_{v,j}$, and if $\epsilon = -$, then $g(y) < y$ for all $y \in J_{v,j}$. Either case contradicts Lemma 6.2. □

Lemma 6.12. *For any face subgraph H of G_n,* $\mathrm{Int}(F_H) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}}) \neq \emptyset$ *if and only if* $\mathrm{Int}(F_K) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}}) \neq \emptyset$ *for every connected component K of H.*

Proof. Suppose $f \in \mathcal{A}^{\mathrm{mp}}$ and $\boldsymbol{\mu}_n(f) = \mu \in \mathrm{Int}(F_H)$. Let K be a connected component of H, and let $I_K = \{x \in I \mid \boldsymbol{\sigma}_n^f(x) \in VK\}$. Note that $\mu_{\mathrm{Leb}}(I_K) \neq 0$; defining g to be a scaled-up version of $f|_{I_K}$ so that $g : I \to I$, we have $\boldsymbol{\mu}_n(g) \in \mathrm{Int}(F_K)$.

The converse implication follows from the convexity of $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$. □

We now prove Theorem 6.8.

Proof of main theorem. To prove (1), let $\mathrm{Int}(F_H)$ denote the open face F_H. We will show that if $\mathrm{Int}(F_H) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ is non-empty, then for each vertex of F_H there are points of $\mathrm{Int}(F_H) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ arbitrarily close to v. By convexity of $\mu_n(\mathcal{A}^{\mathrm{mp}})$ it follows that $\mathrm{Int}(F_H) \subset \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$, thus proving (1).

Suppose $\mathrm{Int}(F_H) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ is non-empty. If H is connected, then by Lemma 6.11 H is driftless, and by Lemma 5.10 there is a totally driftless loop γ with support H. Let v be a vertex of F_H and let β be an embedded loop in H such that $v = F_\beta$. By Lemma 4.6 the loop $\beta^N \gamma$ is driftless, so by Lemma 6.10 there is an $f \in \mathcal{A}^{\mathrm{mp}}$ with $\boldsymbol{\mu}_n(f)$ equal to the counting measure on the loop $\beta^N \gamma$. As N grows this sequence of measures approaches v.

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If H is not connected, then by Lemma 6.12, for each connected component K of H there is an $f_K \in \mathcal{A}^{\mathrm{mp}}$ with $\boldsymbol{\mu}(f_K) \in \mathrm{Int}(F_K)$. We apply the argument from the previous paragraph to each face F_K, obtaining points of $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ close to the vertices of F_K. As each vertex of F_H is a vertex of one of the F_K, we are done.

As for (2), by (1) we know that $\mathrm{Int}(H) \subset \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ if and only if $\mathrm{Int}(H) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}}) \neq \emptyset$. If H is connected, Lemma 6.11 finishes it. If H is not connected, then for any connected component K of H we have $\mathrm{Int}(F_K) \cap \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}}) \neq \emptyset$ if and only if K is driftless. So by Lemma 6.12, $\mathrm{Int}(H) \subset \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ if and only if each K is driftless, *i.e.*, if and only if H is driftless.

To prove (3), it suffices to show that $\mathrm{Int}(P_n) \subset \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$. This is easy: as $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$ is not empty, there must exist a (connected) driftless face subgraph. By Corollary 5.11, the whole graph G_{n-1} is driftless. Since $\mathrm{Int}(P_n) = F_{G_{n-1}}$, the result is implied by (2). □

Corollary 6.13. *For each n, there exists $f \in \mathcal{A}^{\mathrm{mp}}$ such that $\boldsymbol{\mu}_n(f)$ is uniform on S_n.*

Remark 6.14. We have answered Question 1.1 for $\mathcal{C} = \mathcal{A}^{\mathrm{mp}}$. However, Question 1.2 remains open. In particular, we do not know if there is an $f \in \mathcal{A}^{\mathrm{mp}}$ such that $\boldsymbol{\mu}_n(f)$ is uniform for all n. See Section 9.

7. Entropy and finite exclusion type

In this section we change our focus from the distribution $\boldsymbol{\mu}_n(f)$ to a coarser statistic, namely the number of permutations of length n realized by f. We relate two notions about a continuous piecewise monotone function f: finite entropy and finite exclusion type. The basic idea is that these two concepts imply opposite things for the number of length N permutations realized by iterates of f as N gets large. Roughly speaking, finite entropy implies that the number of permutations realized by f grows (at most) exponentially in the length. On the other hand, finite exclusion type means that the only restrictions on the permutations realized by f are given by looking at permutations of a fixed finite length. Often, this will imply that the number of realizable permutations in S_N grows super-exponentially in N.

Define $\boldsymbol{\sigma}_n(f)$ to be the image of $\boldsymbol{\sigma}_n^f$ in S_n.

Continuous functions and entropy

For (piecewise) continuous functions, several classical definitions of the topological entropy $h(f)$ are possible. The reader is referred to [10] for details. A new notion of entropy called *topological permutation entropy* has been studied recently by several people; the following combines Theorem 1 of [3] with Theorem 2.1 of [8].

Theorem 7.1. *If $f : I \to I$ is piecewise continuous and piecewise monotone, then $h(f) = \lim_{n\to\infty} \frac{1}{n-1} \log(|\boldsymbol{\sigma}_n(f)|)$ and $h(f)$ is finite.*

Finite exclusion type

Definition 7.2. A function $f \in \mathcal{A}$ has *exclusion type* n if there exists $H \subseteq G_n$ such that $\boldsymbol{\sigma}_m(f) = \pi_{m,n}^{-1}(\mathrm{Path}_{m-n} H)$ for all $m \geqslant n$ and *finite exclusion type* if it has exclusion type n for some n.

Note that this says not only that every path in G_n realized by f is supported on H, but also that every lift of every path supported on H is realized by f. A condition equivalent to finite exclusion type is that there are finitely many *basic forbidden patterns* for f, in the language of [5]. This means that there are finitely many permutations $\sigma_1, \ldots, \sigma_k$ such that any permutation σ (of any length m) either occurs as $\boldsymbol{\sigma}_m^f(x)$ for some x or else satisfies $\mathrm{Order}(\sigma|_J) = \sigma_i$ for some interval $J \subset [m]$ and some i. Elizalde has proposed the problem of characterizing those functions which have finite exclusion type. We will give some necessary conditions.

Theorem 7.3. *Suppose f has finite exclusion type n, and let $H \subset G_n$ be the associated subgraph. If H contains a partially driftless loop then $|\boldsymbol{\sigma}_N(f)|$ grows super-exponentially; i.e., for any $c \in \mathbb{R}$, we have*

$$|\boldsymbol{\sigma}_N(f)| > c^N, \quad \textit{for sufficiently large } N.$$

Proof. Let γ be a loop on H with $\mathrm{Drift}_\gamma(j) = 0$ for some particular $j \in [n]$. Let ℓ be the length of γ, and set $m_k = k\ell + n$. By the hypothesis of finite exclusion type we have

$$\sigma_{m_k}(f) = \pi_{m_k,n}^{-1}(\mathrm{Path}_{k\ell}(H)) \supset \pi_{m_k,n}^{-1}(\gamma^k).$$

Now since $\mathrm{Drift}_\gamma(j) = 0$, the i elements $x_j, x_{\ell+j}, \ldots, x_{(k-1)\ell+j}$ of the poset Q_{γ^k} are pairwise incomparable, by Lemma 5.5. Thus the number of lifts of γ^k to S_{m_k} is at least $k!$, by Corollary 4.8. Therefore $|\boldsymbol{\sigma}_{k\ell+n}(f)| \geqslant k!$ for all k and the result follows. □

Remark 7.4. Elizalde and Liu [6] have shown that there is no piecewise monotonic function $f : I \to I$ of finite exclusion type with associated graph $H \subset G_2$ where

$$EH = \{(123), (321), (213), (312)\}.$$

This does not follow from the preceding theorem, as this H contains no partially driftless loop.

For a given function f, denote by $H_n(f)$ the subgraph of G_n with edge set $\boldsymbol{\sigma}_{n+1}(f)$.

Theorem 7.5. *If $f : I \to I$ is ergodic, then for every n, $H_n(f)$ contains a partially driftless loop.*

Proof. Consider the graph H with vertex set $VH = VG_n$, edge set $EH = \{(x, N) \in I \times \mathbb{N} \mid N > n, \forall 1 \leqslant m \leqslant N + n, d(f^m(x), x) \geqslant d(f^N(x), x)\}$, and head and tail maps the restrictions to the initial and final segments of $\boldsymbol{\sigma}_N^f(x)$. Note that any directed cycle in H yields a

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partially driftless loop in $H_n(f)$ and that H has finitely many vertices so it suffices to construct an infinite path in H.

Consider

$$J = \left\{ x \in I \mid \forall y \in I, \epsilon > 0, \lim_{n\to\infty} \frac{|\{m < n | d(f^m(x), y) < \epsilon\}|}{n} \in \left(\frac{\epsilon}{2}, 4\epsilon\right) \right\}.$$

By the compactness of I and ergodicity of f, $\mu_{\mathrm{Leb}}(J) = \mu_{\mathrm{Leb}}(I) = 1$. Since $f(J) \subseteq J$ there will be an infinite path in H if $J \subseteq \pi_1(EH)$; this is shown next. For any $x \in J$ choose $\epsilon < \frac{1}{4n}$ so that if $1 \leqslant m \leqslant n$ then $d(f^m(x), x) > \epsilon$. Choose $r > n$ with $d(f^r(x), x) < \epsilon$. Choose $N > r > n$ with $d(f^m(x), x) \geqslant d(f^N(x), x)$ for every $1 \leqslant m \leqslant N + n$ (so that $(x, N) \in EH$). Such an N exists since there is always eventually another sequence of length n avoiding the ϵ ball around x. □

Theorem 7.6. *If $f : I \to I$ is piecewise continuous and if x_0 is a periodic point of period $p > n$ such that f is continuous at every iterate of x_0, then $H_n(f)$ contains a partially driftless loop.*

Proof. Using continuity, choose $\epsilon > 0$ so that for any $x \in I_{ap}$ within ϵ of x_0, the balls $B_\epsilon(f^i(x))$ are pairwise disjoint for $i = 0, \ldots, p-1$ and the iterates satisfy $|f^i(x) - f^{p+i}(x)| < \epsilon$ for $0 \leqslant i \leqslant n-1$. Then the image in G_n of $\boldsymbol{\sigma}^f_{p+n}(x)$ is a partially driftless loop. □

Corollary 7.7. *If $f : I \to I$ is piecewise continuous and piecewise monotonic, and either*

- *there is an f-invariant subinterval of I on which f is measure-preserving and ergodic, or*
- *f has arbitrarily large finite orbits on which it is continuous,*

then f does not have finite exclusion type.

Recall that by Sarkovskii's Theorem [9], a continuous function has points of arbitrarily large period as long as there is a periodic point whose period is not a power of 2.

8. Proof of Theorem 2.1

We now give the promised proof of Theorem 2.1. Given $\mu = (\mu_1, \mu_2, \ldots)$ we will construct f with $\boldsymbol{\mu}(f) = \mu$. Our construction will involve several layers of Cantor sets, and the resulting functions will be nowhere near continuous or measure-preserving.

Recall that $\rho(\sigma) = \mathrm{Order}(\sigma|_{[n-1]})$ if $\sigma \in S_n$.

Lemma 8.1. *Given $\mu \in \Delta_\infty$, there exist intervals $\{I_\sigma \subset (\frac{1}{4}, \frac{3}{4}]\}_{\sigma \in \cup_n S_n}$, open at the left endpoint and closed at the right endpoint, with the properties that:*

(i) $I_\sigma \cap I_\tau = \emptyset$ *for all* $\sigma \neq \tau \in S_n$,
(ii) $I_\sigma \subset I_{\rho(\sigma)}$,
(iii) $\mu_{Leb}(I_\sigma) = \frac{1}{2}\mu_n(\sigma)$ *for all* $\sigma \in S_n$,
(iv) *for each* n, $\cup_{\sigma \in S_n} I_\sigma = (\frac{1}{4}, \frac{3}{4}]$.

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Proof. We define the I_σ inductively as follows. First set $I_{(1)} = (\frac{1}{4}, \frac{3}{4}]$. Now let $n > 1$ and assume that intervals I_τ have been constructed for all $\tau \in S_{n-1}$. Since $\mu_{n-1}(\tau) = \sum_\sigma \mu_n(\sigma)$, summed over all $\sigma \in S_n$ such that $\rho(\sigma) = \tau$, we may subdivide each I_τ into half-open intervals I_σ of length $\frac{1}{2}\mu_n(\sigma)$. □

Lemma 8.2. *There exist disjoint intervals* $\{J_\sigma\}_{\sigma \in \cup_n S_n}$ *such that for all compatible sequences* $(\sigma_1, \dots \sigma_n)$, *and any* $(x_1, \dots, x_n)$ *with* $x_i \in J_{\sigma_i}$, $\text{Order}(x_1, \dots, x_n) = \sigma_n$.

Proof. Again the construction is inductive. Suppose that the J_σ have been constructed for $\sigma \in \cup_{n=1}^k S_n$, and assume further that gaps of positive lengths exist between these intervals and at both endpoints. Order the permutations in $\sigma \in S_{k+1}$ arbitrarily, and for each such σ, let J_σ be an arbitrary open interval disjoint from the previously chosen intervals and with positive length gaps away from them, subject to the further condition that J_σ should lie in the correct gap as determined by the value of $\sigma(k+1)$. □

Proof of Theorem 2.1. Let $\mu = (\mu_1, \mu_2, \dots) \in \Delta_\infty$ be given; we will construct a function $f \in \mathcal{A}$ with $\boldsymbol{\mu}(f) = \mu$. The construction proceeds in a sequence of steps.

Step 1. Let C denote the (usual) Cantor set in $[0, 1]$. By applying an order-preserving transformation we can assume that the $\{J_\sigma\}$ given by Lemma 8.2 have the additional properties that $J_1 = [\frac{1}{4}, \frac{3}{4}]$ and $J_\sigma \subset [\frac{1}{8}, \frac{7}{8}]$ for all permutations σ. For each permutation σ choose an order-preserving injection $\phi_\sigma : C \to J_\sigma$ with $\mu_{\text{Leb}}(\phi_\sigma(C)) = 0$. Let I_σ be as in Lemma 8.1. Finally let β be an order-preserving bijection from the complement of a countable set in $[\frac{1}{4}, \frac{3}{4}]$ to C; we have in mind letting β be a scaled and shifted version of the usual order-preserving map $[0, 1] \to C$, the latter being defined and bijective on the complement of a countable subset of the interval.

We define the function f_1 on a subset of $[0, 1]$ recursively, as follows.

- First, on $(\frac{1}{4}, \frac{3}{4}] = I_{(1)} = I_{(12)} \cup I_{(21)}$: for each $\sigma \in S_2$, if $x \in I_\sigma$ then set $f_1(x) = \phi_\sigma \beta(x)$. Thus for $\sigma \in S_2$, we have $f_1(I_\sigma) \subset J_\sigma$.
- Next, assuming f_1 is defined on $f_1^{i-1}(I_{(1)})$, we define f_1 on $f_1^i(I_{(1)})$ as follows. Notice that $f_1^i(I_{(1)}) = \sqcup_{\sigma \in S_{i+2}} f_1^i(I_\sigma)$. For all $\sigma \in S_{i+2}$ and for all $x \in f_1^i(I_\sigma)$, define $f_1(x) = \phi_\sigma(\phi_{\rho(\sigma)}^{-1}(x))$. Thus we have $f_1^i(I_\sigma) \in \phi_\sigma(C) \subset J_\sigma$.

We have now defined every power of f_1 on a domain D which is a measure zero set union $(\frac{1}{4}, \frac{3}{4}]$ minus a countable set. The purpose of this construction is that for any $\sigma \in S_n$ and $x \in I_\sigma$, we now have $\boldsymbol{\sigma}_n^{f_1}(x) = \sigma$. We set $f = f_1$ on D.

Step m, m $\geqslant 2$. Denote by K_m the measure zero set in $(0, 2^{-m}] \cup (1 - 2^{-m}, 1]$ for which $f(x)$ has already been defined. Define $g_m : (0, 2^{-m}] \cup (1 - 2^{-m}, 1] \to (0, 1]$ to be the map

$$g_m(x) = \begin{cases} 2^{m-1}x & \text{for } x \in (0, 2^{-m}], \\ 1 - 2^{m-1}(1-x) & \text{for } x \in (1 - 2^{-m}, 1]. \end{cases}$$

For all $x \in g_m^{-1}(D) \setminus K_m$, define $f(x) = g_m^{-1}(f_1(g_m(x)))$. Note that if $\sigma \in S_n$ and $x \in g_m^{-1}(I_\sigma) \setminus K_m$, we now have

$$\boldsymbol{\sigma}_n^f(x) = \sigma. \tag{8.1}$$

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After step m, the domain of f includes the interval $(2^{-m}, 1-2^{-m}]$, so the iterative process defines f on $(0,1)$. It remains to show that $\boldsymbol{\mu}_n(f) = \mu_n$ for all n.

For $\sigma \in S_n$, define $\overline{I}_\sigma = I_\sigma \cup (\cup_{i\geqslant 2}\ g_i^{-1}(I_\sigma))$. By (8.1) we have $\boldsymbol{\sigma}_n^f(x) = \sigma$ for all $x \in \overline{I}_\sigma \setminus (\cup_m K_m)$. Thus $\boldsymbol{\mu}_n(f)(\sigma) = |\overline{I}_\sigma \setminus (\cup_m K_m)|$. Since $(\cup_m K_m)$ is of measure zero,

$$|\overline{I}_\sigma \setminus (\cup_m K_m)| = |\overline{I_\sigma}| = \frac{1}{2}\mu_n(\sigma) + \sum_{i\geqslant 2} 2^{-(i-1)}\left(\frac{1}{2}\mu_n(\sigma)\right) = \mu_n(\sigma).$$

We conclude that $\boldsymbol{\mu}_n(f) = \mu_n$ for each n, and $\boldsymbol{\mu}(f) = \mu$. This completes the proof. □

9. Questions

Many interesting open questions remain about the relationship between functions and their distributions of order patterns.

Measure-preserving functions

The bulk of the work presented here has focused on the class of measure-preserving functions; however, to date we have been unable to answer Question 1.2 for this class.

Question 9.1. What is $\boldsymbol{\mu}(\mathcal{A}^{\mathrm{mp}})$?

There is an infinite version of the polytope, P_∞, consisting of compatible sequences $(\mu_1, \mu_2, \ldots)$ with $\mu_i \in P_i$. We do not know if the 'interior' of P_∞ is realizable by some $f \in \mathcal{A}^{\mathrm{mp}}$ (where the meaning of 'interior' depends on the topology on P_∞), or if there is a drift condition for faces. One concrete question is as follows.

Question 9.2. Is there an $f \in \mathcal{A}^{\mathrm{mp}}$ with $\boldsymbol{\mu}_n(f)$ uniform for all n?

Corollary 6.13 asserts that such an f exists for any particular n, and of course by Theorem 2.1 there is an $f \in \mathcal{A}$ that works for all n. Yet there is no piecewise monotonic $f \in \mathcal{A}$ that works for all n, because such an f would have finite entropy (by [3], or Theorem 7.1), and hence $|\boldsymbol{\sigma}_N(f)|$ would grow at most exponentially in N. Note that such a function might be desirable as a random number generator, since from the point of view of order patterns, its iterates would look perfectly random.

In a somewhat different direction, if λ is a reasonably nice measure on I, then the results of Section 6 hold with $\mathcal{C} = \mathcal{A}^{\mathrm{mp}}$ replaced by the collection $\mathcal{C} = \mathcal{A}^{\lambda}$ of functions which preserve λ. (See Remark 6.1.)

Question 9.3. Are there measures λ for which $\boldsymbol{\mu}_n(\mathcal{A}^{\lambda}) \neq \boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{mp}})$?

Other functions

Returning to the broader Questions 1.1 and 1.2, there are several interesting classes of functions $\mathcal{C}$ to study, such as (piecewise) continuous functions, polynomials, interval exchange maps (a subclass of $\mathcal{A}^{\mathrm{mp}}$), *etc.* For example, if $\mathcal{C} = \mathcal{A}^{\mathrm{pc}}$ is the collection of

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piecewise continuous functions, then it is easy to see that the only vertices of Δ_n contained in $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{pc}})$ are $\chi_{(12\cdots n)}$ and $\chi_{(n\cdots 21)}$.

Question 9.4. Is the closure of $\boldsymbol{\mu}_n(\mathcal{A}^{\mathrm{pc}})$ equal to Δ_n?

Question 9.5. Is there a drift criterion which applies to $\mathcal{A}^{\mathrm{pc}}$?

We have recently learned that, after reading a preliminary version of this paper, Adam Hesterberg [7] has answered Questions 9.4 and 9.5.

Finally, it would be natural to study the extent to which the distributions $\boldsymbol{\mu}_n(f)$ determine f, for f in a given class $\mathcal{C}$.

Question 9.6. For $\mu \in \boldsymbol{\mu}_n(\mathcal{C})$, what is $\boldsymbol{\mu}_n^{-1}(\mu) \cap \mathcal{C}$?

Question 9.7. For $\mu = (\mu_1, \mu_2, \dots) \in \boldsymbol{\mu}(\mathcal{C})$, what is $\boldsymbol{\mu}^{-1}(\mu) \cap \mathcal{C}$?

These questions are in a sense converse to Questions 1.1 and 1.2.

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