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# A GENERALIZATION OF THE THEORY OF STANDARDLY STRATIFIED ALGEBRAS I: STANDARDLY STRATIFIED RINGOIDS

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**Abstract.** We extend the classical notion of standardly stratified $k$-algebra (stated for finite dimensional $k$-algebras) to the more general class of rings, possibly without 1, with enough idempotents. We show that many of the fundamental results, which are known for classical standardly stratified algebras, can be generalized to this context. Furthermore, new classes of rings appear as: ideally standardly stratified and ideally quasi-hereditary. In the classical theory, it is known that quasi-hereditary and ideally quasi-hereditary algebras are equivalent notions, but in our general setting, this is no longer true. To develop the theory, we use the well-known connection between rings with enough idempotents and skeletally small categories (ringoids or rings with several objects).

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**1. Introduction.** The notions of quasi-hereditary algebra and highest weight category were introduced and studied by Cline, Parshall and Scott [**9**, **11**, **47**]. Highest weight categories are a very special kind of abelian categories that arise in the representation theory of Lie algebras and algebraic groups. The highest weight categories with a finite number of simple objects are precisely the module categories of quasi-hereditary algebras. It is worth mentioning that quasi-hereditary algebras were originally defined through a special chain of ideals.

For the setting of finite dimensional algebras, quasi-hereditary algebras were amply studied, among others, by Dlab and Ringel in [**13**, **16**, **17**, **46**]. Dlab and Ringel introduced the set of standard modules ${}_{\Lambda}\Delta$ associated to a finite dimensional algebra $\Lambda$. Later on, M. Ringel established a relationship between quasi-hereditary algebras and tilting theory [**46**], which has been very fruitful for the study of quasi-hereditary algebras. For doing so, Ringel studied the homological properties of the category $\mathcal{F}({}_{\Lambda}\Delta)$ of ${}_{\Lambda}\Delta$-filtered $\Lambda$-modules and constructed the characteristic module ${}_{\Lambda}T$ (which turned out to be tilting) associated to

$\mathcal{F}({}_\Lambda\Delta)$. Moreover, Ringel proved that the endomorphism ring $\mathrm{End}({}_\Lambda T)$ is again a quasi-hereditary algebra. Since then, this tilting module is known as the Ringel's characteristic tilting module associated with a quasi-hereditary algebra.

Because of the success of the applications of the theory of quasi-hereditary algebras, it was natural to find useful generalizations of the notion of quasi-hereditary algebra. One step in this direction was given by Dlab, who introduced the concept of standardly stratified algebra [**14**]. These finite dimensional algebras have been amply studied [**1**, **4**, **20**, **21**, **22**, **23**, **35**, **36**, **38**, **39**, **40**, **41**, **48**, **50**] and have become an useful tool for different areas in mathematics.

Once we have the notion of standardly stratified algebra [**14**], in the context of finite dimensional algebras, a natural question is to find a more general class of algebras which has sense to define the notion of standardly stratified algebra. These are precisely the rings with enough idempotents. These kind of rings appear very naturally in different contexts, for example, as a generalization of Ringel's notion of species [**45**] or in connection with the Galois covering in the sense of Bongartz-Gabriel [**8**] or De la Peña-Martinez [**12**]. More generally, this type of rings appears as "Gabriel functor rings" (see discussion after proposition 1 on p. 346 in [**24**]) or "rings with several objects" in [**42**].

The context of rings with several objects (ringoids, in modern terminology) has become very fruitful as a tool that allows us to understand more deeply certain branches of mathematics. For example, motivated by the work on functor categories in [**6**, **7**], Martínez-Villa and Solberg studied the Auslander–Reiten components of finite dimensional algebras. They did so, in order to stablish when the category of graded functors is noetherian [**32**, **33**, **34**]. Recently, Martínez-Villa and Ortíz studied in [**31**, **30**] tilting theory in arbitrary functor categories. They proved that most of the properties that are satisfied by a tilting module over an Artin algebra also hold true for functor categories. To mention some, Brenner–Butler's Theorem and Happel's Theorem are valid on this more general context.

Inspired by the works mentioned above and the fact that in the theory of quasi-hereditary algebras the notion of tilting module is relevant, Ortíz introduced in [**43**] the concept of quasi-hereditary category. He did so, in order to study the Auslander–Reiten components of a finite dimensional algebra $\Lambda$. In a similar way, as the standard modules appear in the theory of quasi-hereditary algebras, Ortíz defined the concept of standard functors, which turned out to be a generalization of the notion of standard modules [**43**]. In particular, he established a connection between highest weight categories and quasi-hereditary categories. He did so by following the ideas introduced by Krause in [**27**], that is, Ortíz compared the notion of standard objects in an abelian length category and standard subcategories of the category of $\mathcal{C}$-modules over a quasi-hereditary category $\mathcal{C}$.

In this paper, we define the notions of standardly stratified ringoid and quasi-hereditary ringoid. These definitions generalize the notion of quasi-hereditary category given by Ortíz in [**43**]. To start with, we recall that for any class of objects $\mathcal{B}$ of a category $\mathcal{C}$, ind $\mathcal{B}$ denotes the class of iso-classes of local objects $B \in \mathcal{B}$, where $B$ local means that $\mathrm{End}_{\mathcal{C}}(B)$ is a local endomorphism ring.

Let $\mathbb{K}$ be a commutative ring and $\Lambda$ be a $\mathbb{K}$-algebra (possibly without 1) such that $\Lambda^2 = \Lambda$. For such algebra $\Lambda$, we denote by $\mathrm{Mod}(\Lambda)$ the category of all the unitary left $\Lambda$-modules $M$, where unitary means that $\Lambda M = M$. The finitely generated unitary left $\Lambda$-modules form a full subcategory of $\mathrm{Mod}(\Lambda)$ and it is usually denoted by $\mathrm{mod}(\Lambda)$. The class of finitely generated projective objects in $\mathrm{Mod}(\Lambda)$ is denoted by $\mathrm{proj}(\Lambda)$. We denote by $f.\ell(\mathbb{K})$ the class of all the $\mathbb{K}$-modules of finite length. In this context, in the category

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$\mathrm{Mod}(\Lambda)$ usually there exist infinitely many finitely generated indecomposable projective $\Lambda$-modules, in contrast to the case when $\Lambda$ is an Artin $R$-algebra. In order to define a right standardly stratified algebra, in the classical sense, we have to construct the family of standard modules $\Delta = \{\Delta(i)\}_{i=1}^n$, one standard module for each element in $\mathrm{ind}\,\mathrm{proj}(\Lambda^{op}) = \{P(i)\}_{i=1}^n$. In the general case of a $\mathbb{K}$-algebra without 1, it is not clear that $\mathrm{ind}\,\mathrm{proj}(\Lambda^{op})$ is even a set (at first glance) and we do not have a reasonable description of this class. In order to fix this problem, we consider a family $\{e_i\}_{i\in I}$ of orthogonal idempotents in $\Lambda$ satisfying mild conditions (sufficiency and Hom-finiteness). Using the family $\{e_i\}_{i\in I}$, we produce partitions $\tilde{\mathcal{A}}$ of the set $\mathrm{ind}\,\mathrm{proj}(\Lambda^{op})$ and each one of these partitions give us a sort of stratification of the class of finitely generated projective $\Lambda$-modules. Having a partition $\tilde{\mathcal{A}}$, as above, we can define the set of standard modules $\Delta := \{\Delta(i)\}_{i<\alpha}$, where $\alpha$ is an ordinal number giving the size of the partition $\tilde{\mathcal{A}}$.

A $\mathbb{K}$-algebra with enough idempotents (w.e.i $\mathbb{K}$-algebra, for short) is a pair $(\Lambda, \{e_i\}_{i\in I})$, where $\Lambda$ is a $\mathbb{K}$-algebra and $\{e_i\}_{i\in I}$ is a family of orthogonal idempotents of $\Lambda$ such that $\Lambda = \oplus_{i\in I}\, e_i\Lambda = \oplus_{i\in I}\Lambda e_i$. In this case, we have that $\Lambda^2 = \Lambda$. It is said that $(\Lambda, \{e_i\}_{i\in I})$ is Hom-finite if $\{e_j\Lambda e_i\}_{i,j\in I} \subseteq f.\ell(\mathbb{K})$.

Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i. $\mathbb{K}$-algebra. Then, by Corollary 6.5 (b), for each $i \in I$, there exists a unique (up to permutations) family $\overline{e}_i := \{e_{k,i}\}_{k=1}^{n_i}$ of primitive orthogonal idempotents in $\Lambda$ such that $e_i = \sum_{k=1}^{n_i} e_{k,i}$. Denote by $\mathrm{ind}\,\{e_i\}_{i\in I}$ the quotient of the set $\cup_{i\in I}\overline{e}_i$ by the equivalence relation $\sim$, where $f \sim g$ if, and only if, $f\Lambda \simeq g\Lambda$. Let $[e]$ be the equivalence class of $e \in \cup_{i\in I}\overline{e}_i$. Then, by Corollary 6.6 (b), we have

$$\mathrm{ind}\,\mathrm{proj}(\Lambda^{op}) = \{e\Lambda \,:\, [e] \in \mathrm{ind}\,\{e_i\}_{i\in I}\}.$$

The set of standard modules can be constructed by choosing a partition $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ of the set $\mathrm{ind}\,\{e_i\}_{i\in I}$, where $\alpha$ is an ordinal number (the size of the partition $\tilde{\mathcal{A}}$) and each ordinal $i < \alpha$ is the $i$th level of the given partition. Define ${}_{\Lambda^{op}}P_e(i) := e\Lambda$ for any $[e] \in \tilde{\mathcal{A}}_i$, and let ${}_{\Lambda^{op}}P := \{{}_{\Lambda^{op}}P(i)\}_{i<\alpha}$, where ${}_{\Lambda^{op}}P(i) := \{{}_{\Lambda^{op}}P_e(i)\}_{e\in\tilde{\mathcal{A}}_i}$. The family of $\tilde{\mathcal{A}}$-standard right $\Lambda$-modules ${}_{\Lambda^{op}}\Delta = \{\Delta(i)\}_{i<\alpha}$, where $\Delta(i) := \{\Delta_e(i)\}_{e\in\tilde{\mathcal{A}}_i}$, is defined as follows

$$\Delta_e(i) := \frac{{}_{\Lambda^{op}}P_e(i)}{\mathrm{Tr}_{\oplus_{j<i}\overline{P}(j)}({}_{\Lambda^{op}}P_e(i))},$$

where $\overline{P}(j) := \bigoplus_{r\in\tilde{\mathcal{A}}_j} {}_{\Lambda^{op}}P_r(j)$ and $\mathrm{Tr}_{\oplus_{j<i}\overline{P}(j)}({}_{\Lambda^{op}}P_e(i))$ is the trace of the $\Lambda$-module $\oplus_{j<i}\overline{P}(j)$ in ${}_{\Lambda^{op}}P_e(i)$. We say that the pair $(\Lambda, \tilde{\mathcal{A}})$ is a right standardly stratified algebra if $\mathrm{Tr}_{\oplus_{j<i}\overline{P}(j)}({}_{\Lambda^{op}}P_e(i)) \in \mathcal{F}_f(\bigcup_{j<i}\Delta(j))$, for any $i < \alpha$ and $e \in \tilde{\mathcal{A}}_i$. Here $\mathcal{F}_f(\bigcup_{j<i}\Delta(j))$ is the class of all the $\Lambda$-modules admitting a finite filtration in $\bigcup_{j<i}\Delta(j)$. Moreover, we say that $(\Lambda, \tilde{\mathcal{A}})$ is right quasi-hereditary if it is standardly stratified and $\mathrm{End}(\Delta_e(i))$ is a division ring for any $[e] \in \tilde{A}_i$ and $i < \alpha$.

Let $\Lambda$ be a basic Artin $\mathbb{K}$-algebra and let $\{e_i\}_{i=1}^n$ be a complete family of primitive orthogonal idempotents of $\Lambda$. Then, we have that $\mathrm{ind}\,\{e_i\}_{i=1}^n = \{e_i\}_{i=1}^n$. The classical notion of standardly stratified algebra for $\Lambda$ corresponds to the given one for the very particular pair $(\Lambda, \tilde{\mathbb{T}})$, where $\tilde{\mathbb{T}}$ is the one-point partition $\tilde{\mathbb{T}} = \{\tilde{\mathbb{T}}_i\}_{i<n}$, defined as $\tilde{\mathbb{T}}_i := \{e_{i+1}\}$ for $i \in [0, n)$. Note that we can choose different partitions $\tilde{\mathcal{A}}$ of the set $\{e_i\}_{i=1}^n$, not only the trivial one.

In this paper, we also define ideally standardly stratified and ideally quasi-hereditary $\mathbb{K}$-ringoids. We explain their meaning in terms of rings with enough idempotents. Let $(\Lambda, \{e_i\}_{i\in S})$ be a w.e.i $\mathbb{K}$-algebra. An ideal $I \trianglelefteq \Lambda$ is right stratifying if $I^2 = I$ and

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$eI \in \text{proj}(\Lambda^{op})$ for any $e \in \{e_i\}_{i\in S}$. We say that $I$ is right hereditary if it is right stratifying and $I\text{rad}(\Lambda)I = 0$. A right stratifying (respectively, hereditary) chain in $\Lambda$ is a chain $\{I_i\}_{i<\alpha}$ of ideals of $\Lambda$ such that $\sum_{i<\alpha} I_i = \Lambda$ and $I_i/I'_i$ is right stratifying (respectively, hereditary) in $\Lambda/I'_i$, where $I'_i := \sum_{j<i} I_j$.

Assume now that $(\Lambda, \{e_i\}_{i\in S})$ is Hom-finite. Let $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ be a partition of the set $\text{ind}\{e_i\}_{i\in S}$. The partition $\tilde{\mathcal{A}}$ induces a chain $\{I_{\mathcal{A}_i}\}_{i<\alpha}$ of ideals in $\Lambda$ satisfying that $\sum_{i<\alpha} I_{\mathcal{A}_i} = \Lambda$, where $I_{\mathcal{A}_i}$ is the ideal generated by the set of idempotents $\{e : [e] \in \mathcal{A}_i\}$ and $\mathcal{A}_i := \bigcup_{j\leq i} \tilde{\mathcal{A}}_j$ (see Lemma 5.2). We say that $(\Lambda, \tilde{\mathcal{A}})$ is right ideally standardly stratified (respectively, right ideally quasi-hereditary) if the associated chain $\{I_{\mathcal{A}_i}\}_{i<\alpha}$ of ideals in $\Lambda$ is right stratifying (respectively, hereditary).

The following question arises naturally: Are the definitions of ideally standardly stratified (respectively, ideally quasi-hereditary) and standardly stratified (respectively, quasi-hereditary) equivalent? In the case of an Artin algebra $\Lambda$ and the one-point partition $\tilde{\mathbb{T}}$, defined above, it is well known that both notions are equivalent. For the general case, we have the following results that are a consequence of Theorems 5.6 and 5.10. In order to state the following two theorems, we recall (see in Section 5) the notion of right noetherian partition. Let $(\Lambda, \{e_i\}_{i\in S})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra and $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ be a partition of the set $\text{ind}\{e_i\}_{i\in S}$. We say that $\tilde{\mathcal{A}}$ is right noetherian if for any $i < \alpha$ and $[e] \in \tilde{\mathcal{A}}_i$ the following statement holds true: the set $\{j < \alpha : eI_{\mathcal{A}_j}/eI'_{\mathcal{A}_j} \neq 0\}$ is finite and there is some $i_0 < \alpha$ such that $eI_{\mathcal{A}_j} = e\Lambda$ for any $j \geq i_0$.

Theorem A. *Let* $(\Lambda, \{e_i\}_{i\in S})$ *be a Hom-finite w.e.i* $\mathbb{K}$*-algebra and* $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ *be a partition of* $\text{ind}\{e_i\}_{i\in S}$. *Then, the following statements are equivalent.*

(a) $(\Lambda, \tilde{\mathcal{A}})$ *is right standardly stratified.*
(b) *The partition* $\tilde{\mathcal{A}}$ *is right noetherian and for any* $i < \alpha$, $[e] \in \tilde{\mathcal{A}}_i$ *and* $t < \alpha$, *we have that* $eI_{\mathcal{A}_t}/eI'_{\mathcal{A}_t}$ *is a finitely generated projective right* $\Lambda/I'_{\mathcal{A}_t}$*-module.*

As a consequence of the theorem above, it can be shown (see Corollary 5.9) the following result.

Corollary B. *Let* $(\Lambda, \{e_i\}_{i\in S})$ *be a Hom-finite w.e.i* $\mathbb{K}$*-algebra and* $\tilde{\mathcal{A}}$ *be a partition of* $\text{ind}\{e_i\}_{i\in S}$. *Then, the following statements are equivalent.*

(a) $(\Lambda, \tilde{\mathcal{A}})$ *is right standardly stratified.*
(b) *The partition* $\tilde{\mathcal{A}}$ *is right noetherian and* $(\Lambda, \tilde{\mathcal{A}})$ *is right ideally standardly stratified.*

Theorem C. *Let* $(\Lambda, \{e_i\}_{i\in S})$ *be a Hom-finite w.e.i* $\mathbb{K}$*-algebra and* $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ *be a partition of* $\text{ind}\{e_i\}_{i\in S}$. *Then, the following statements are equivalent.*

(a) $(\Lambda, \tilde{\mathcal{A}})$ *is right quasi-hereditary and* $\text{Hom}(\Delta_e(i), \Delta_{e'}(i)) = 0$ *for* $[e] \neq [e']$ *in* $\tilde{\mathcal{A}}_i$ *and* $i < \alpha$.
(b) *The partition* $\tilde{\mathcal{A}}$ *is right noetherian and* $(\Lambda, \tilde{\mathcal{A}})$ *is right ideally quasi-hereditary.*

Given a Hom-finite w.e.i $\mathbb{K}$-algebra $(\Lambda, \{e_i\}_{i\in S})$ and a partition $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ of $\text{ind}\{e_i\}_{i\in S}$, we have the family of standard modules $\Delta = \{\Delta(i)\}_{i<\alpha}$ and the category $\mathcal{F}_f(\Delta)$ of all the right $\Lambda$-modules which has a finite filtration through the objects of $\Delta$.

In this the paper, we also study some important properties of $\mathcal{F}_f(\Delta)$. As in the classic case, we prove (see Theorem 4.9) that if the standard modules $\Delta_e(i)$ are finitely presented, then $\mathcal{F}_f(\Delta)$ is a Krull–Schmidt skeletally small category and all the objects in this category are finitely presented. Furthermore, the multiplicity $[M : \Delta_e(i)]$ of each $\Delta_e(i)$, for a module $M \in \mathcal{F}_f(\Delta)$, does not depend on any $\Delta$-filtration of $M$. In order to prove that fact, we introduce an analogous of the trace filtration given by Dlab and Ringel [17, Lemma 1.4] and characterize the modules which belongs to $\mathcal{F}_f(\Delta)$ in terms of this trace filtration (see Theorem 4.7). It is worth mentioning that the proofs of these results use transfinite induction, in contrast with the classic case, where the usual induction is enough to handle the situation. As an application of the trace filtration, we show that $\mathcal{F}_f(\Delta)$ is closed under kernels of epimorphisms between its objects, a fact that is well known for the classical case.

**2. Preliminaries.** In this section, we introduce the notation and the basic results in functor categories which will be used in the development of the paper.

FUNCTOR CATEGORIES AND RINGOIDS. Let $\mathbb{K}$ be a commutative ring with 1. A category $\mathcal{C}$ is said to be a $\mathbb{K}$-category if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\mathbb{K}$-module for any $(X, Y) \in \mathcal{C}^2$, and the composition of morphisms in $\mathcal{C}$ is $\mathbb{K}$-bilinear. We denote by $[\mathcal{A}, \mathcal{B}]$ the category of additive (covariant) functors between two $\mathbb{K}$-categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ is skeletally small. For any $F, G \in [\mathcal{A}, \mathcal{B}]$, we have that $\mathrm{Hom}_{[\mathcal{A},\mathcal{B}]}(F, G)$ is the class $\mathrm{Nat}(F, G)$ of natural morphisms from $F$ to $G$. For the sake of simplicity, we write $(-, ?)$ instead of $\mathrm{Hom}(-, ?)$ wherever this $\mathrm{Hom}(-, ?)$ is defined. The term subcategory means full subcategory.

Let $\mathcal{C}$ be a $\mathbb{K}$-category. We say that an object $C \in \mathcal{C}$ is local if $\mathrm{End}_{\mathcal{C}}(C)$ is a local ring. For any subclass $\mathcal{B}$ of objects in $\mathcal{C}$, the class of iso-classes of local objects $B \in \mathcal{B}$ will be denoted by $\mathrm{ind}\,\mathcal{B}$. For any $B \in \mathcal{B}$, which is local, we write $[B]$ for the corresponding iso-class. That is, $\mathrm{ind}\,\mathcal{B} := \{[B] \text{ such that } B \in \mathcal{B} \text{ is local}\}$. For simplicity, sometimes we write $B$ instead of $[B]$. If $\mathcal{C}$ is an additive category, we denote by $\mathrm{add}\,(\mathcal{B})$ the class of all direct summands of finite coproducts of copies of objects in $\mathcal{B}$.

A very useful tool in the theory of categories is Yoneda's Lemma. We state this lemma for the case of $\mathbb{K}$-categories since this is precisely the context where we are working. Let $\mathcal{C}$ be a $\mathbb{K}$-category. Yoneda's Lemma states that Yoneda's functor

$$Y = Y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Ab}], \quad (a \xrightarrow{f} b) \mapsto (\mathrm{Hom}_{\mathcal{C}}(-, a) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,f)} \mathrm{Hom}_{\mathcal{C}}(-, b))$$

is full and faithful. Moreover, for any $c \in \mathcal{C}$, we have an isomorphism of abelian groups $\mathrm{Hom}(Y(c), F) \to F(c)$, $\alpha \mapsto \alpha_c(1_c)$.

Following B. Mitchell in [42], we recall that a $\mathbb{K}$-ringoid (or $\mathbb{K}$-algebra with several objects) is just a skeletally small $\mathbb{K}$-category. A ringoid is just a $\mathbb{Z}$-ringoid (or ring with several objects). Note that any $\mathbb{K}$-ringoid is in particular a ringoid.

Let $\mathfrak{S}$ be a ringoid. A left $\mathfrak{S}$-module is an additive covariant functor $F : \mathfrak{S} \to \mathrm{Ab}$, where Ab is the category of abelian groups. The category of left $\mathfrak{S}$-modules is $\mathrm{Mod}\,(\mathfrak{S}) := [\mathfrak{S}, \mathrm{Ab}]$. Note that $\mathrm{Mod}\,(\mathfrak{S})$ is abelian and bicomplete, since Ab is so. We also consider the category of right $\mathfrak{S}$-modules $\mathrm{Mod}_\rho(\mathfrak{S}) := \mathrm{Mod}\,(\mathfrak{S}^{op})$, where $\mathfrak{S}^{op}$ is the opposite category of $\mathfrak{S}$.

We denote by $\mathrm{Proj}\,(\mathfrak{S})$ the class of projective left $\mathfrak{S}$-modules and $\mathrm{proj}\,(\mathfrak{S})$ denotes the class of finitely generated projective left $\mathfrak{S}$-modules. We also have the classes $\mathrm{Proj}_\rho(\mathfrak{S}) := \mathrm{Proj}\,(\mathfrak{S}^{op})$ and $\mathrm{proj}_\rho(\mathfrak{S}) := \mathrm{proj}\,(\mathfrak{S}^{op})$.

Let $\mathfrak{R}$ be a ringoid. Using Yoneda's functor $Y : \mathfrak{R} \to \mathrm{Mod}_\rho(\mathfrak{R})$, it can be proved that $M \in \mathrm{proj}_\rho(\mathfrak{R})$ iff $M$ is a direct summand of $\coprod_{i\in I} Y(a_i)$ for some finite family $\{a_i\}_{i\in I}$ of objects in $\mathfrak{R}$. Thus, the ringoid $\mathfrak{R}$ can be seen as a full subcategory of $\mathrm{proj}_\rho(\mathfrak{R})$.

Following Auslander [7], it is said that $M \in \mathrm{Mod}_\rho(\mathfrak{R})$ is finitely presented if there is an exact sequence $P_1 \to P_0 \to M \to 0$, where $P_1, P_0 \in \mathrm{proj}_\rho(\mathfrak{R})$. We denote by $\mathrm{fin.p}_\rho(\mathfrak{R})$ the full subcategory of $\mathrm{Mod}_\rho(\mathfrak{R})$ whose objects are all the finitely presented right $\mathfrak{R}$-modules. A projective cover of $M \in \mathrm{Mod}_\rho(\mathfrak{R})$ is an essential epimorphism $P \to M$ in

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$\mathrm{Mod}_\rho(\mathfrak{R})$ with $P \in \mathrm{Proj}_\rho(\mathfrak{R})$. A projective presentation $P_1 \xrightarrow{g} P_0 \xrightarrow{f} M \to 0$ is minimal if the epimorphisms $P_0 \xrightarrow{f} M$ and $P_1 \to \mathrm{Im}\,(g)$ are projective covers.

Let $\mathfrak{R}$ be an additive ringoid, that is, $\mathfrak{R}$ is a skeletally small additive category. It is well known (see [7], [19, Theorem 1.4] and [25]) that $\mathrm{fin.p}_\rho(\mathfrak{R})$ is a full abelian subcategory of $\mathrm{Mod}_\rho(\mathfrak{R})$ if and only if $\mathfrak{R}$ has pseudo kernels.

We say that a ringoid $\mathfrak{R}$ is thick if $\mathfrak{R}$ is an additive category whose idempotents split. In this case, Yoneda's functor $Y : \mathfrak{R} \to \mathrm{Mod}_\rho(\mathfrak{R})$ induces an equivalence of categories $\mathfrak{R} \simeq \mathrm{proj}_\rho(\mathfrak{R})$. A ringoid $\mathfrak{R}$ is Krull–Schmidt (KS ringoid, for short) if it is a Krull–Schmidt category (that is an additive category in which every nonzero object decomposes into a finite direct sum of objects having local endomorphism ring). It can be shown [26, Corollary 4.4] that any ringoid is Krull–Schmidt if it is thick and the endomorphism ring of every object is semiperfect.

LEMMA 2.1. *Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid. Then, any $M \in \mathrm{fin.p}_\rho(\mathfrak{R})$ has a minimal projective presentation in* $\mathrm{proj}_\rho(\mathfrak{R})$.

*Proof.* For any $C \in \mathfrak{R}$, we have that $R_C := \mathrm{End}_{\mathfrak{R}}(C)^{op}$ is a semi-perfect ring. Then, the result follows from [7, Corollary 4.13]. □

Let $\mathfrak{R}$ be a thick $\mathbb{K}$-ringoid. For any additive full subcategory $\mathcal{B}$ of $\mathfrak{R}$, we consider the class $I_{\mathcal{B}}$ of all the morphisms in $\mathfrak{R}$ which factor through objects of $\mathcal{B}$. Note that $I_{\mathcal{B}}$ is an ideal of $\mathfrak{R}$, and it is known as the ideal associated with $\mathcal{B}$. For $M, N \in \mathrm{Mod}_\rho(\mathfrak{R})$, we denote by $\mathrm{Tr}_M(N)$ the trace of $M$ in $N$.

We say that a $\mathbb{K}$-ringoid $\mathfrak{R}$ is Hom-finite if the $\mathbb{K}$-module $\mathrm{Hom}_{\mathfrak{R}}(a, b)$ is of finite length, for any $(a, b) \in \mathfrak{R}^2$. A locally finite $\mathbb{K}$-ringoid is a $\mathbb{K}$-ringoid which is Hom-finite and Krull–Schmidt. A locally finite $\mathbb{K}$-ringoid with pseudo kernels is called strong locally finite $\mathbb{K}$-ringoid.

LEMMA 2.2. *For a Krull–Schmidt $\mathbb{K}$-ringoid $\mathfrak{R}$, the following statements hold true.*

(a) $\mathrm{proj}_\rho(\mathfrak{R}) = \{P = \oplus_{i\in I} Y(a_i)$ *for a finite family* $\{a_i\}_{i\in I}$ *in* $\mathrm{ind}\,(\mathfrak{R})\}$.
(b) $\mathrm{ind}\,(\mathrm{proj}_\rho(\mathfrak{R})) = \{Y(a) \,:\, a \in \mathrm{ind}\,(\mathfrak{R})\}$.
(c) *For any $a, b \in \mathfrak{R}$, we have that $Y(a) \simeq Y(b)$ iff $a \simeq b$.*

*Proof.* We start by proving (c). Let $\eta : Y(a) \to Y(b)$ be an isomorphism of functors. Consider $f_a \in \mathrm{Hom}_{\mathfrak{R}}(a, b)$ and $g_b \in \mathrm{Hom}_{\mathfrak{R}}(b, a)$, where $f_a := \eta_a(1_a)$ and $\eta_b(g_b) = 1_b$. By using $\eta : Y(a) \to Y(b)$, it can be shown that $f_a \circ g_b = 1_b$ and $g_b \circ f_a = 1_a$.

The proof of (a) and (b) follows from (c), since $\mathfrak{R}$ is a Krull–Schmidt $\mathbb{K}$-ringoid and thus Yoneda's functor $Y : \mathfrak{R} \to \mathrm{Mod}_\rho(\mathfrak{R})$ gives an equivalence between $\mathfrak{R}$ and $\mathrm{proj}_\rho(\mathfrak{R})$. □

Let $\mathfrak{R}$ be a $\mathbb{K}$-ringoid and $M \in \mathrm{Mod}_\rho(\mathfrak{R})$. The support of $M$ is the set $\mathrm{Supp}\,(M) := \{e \in \mathrm{ind}(\mathfrak{R}) \,:\, M(e) \neq 0\}$. We say that $\mathfrak{R}$ is right support finite if $\mathrm{Supp}\,(Y(e))$ is finite for any $e \in \mathrm{ind}(\mathfrak{R})$, where $Y(e) := \mathrm{Hom}_{\mathfrak{R}}(-, e)$.

LEMMA 2.3. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid and $\mathcal{B}$ be an additive full subcategory of $\mathfrak{R}$. Then, the following statements hold true.*

(a) $\mathrm{Tr}_{\{Y(b)\}_{b\in\mathcal{B}}}(Y(e)) = I_{\mathcal{B}}(-, e)$, *for any* $e \in \mathfrak{R}$.
(b) *If $\mathfrak{R}$ is right support finite, then* $Y(e)/I_{\mathcal{B}}(-, e) \in \mathrm{fin.p}_\rho(\mathfrak{R})$, *for any* $e \in \mathrm{ind}(\mathfrak{R})$.

*Proof.* (a) The proof of [43, Lemma 3.1] can be adapted to get (a).
(b) Let $\mathfrak{R}$ be right support finite and $e \in \mathrm{ind}(\mathfrak{R})$. By (a), we get

$$I_{\mathcal{B}}(-, e) = \mathrm{Tr}_{\{Y(b)\}_{b\in\mathcal{B}}}(Y(e)) = \mathrm{Tr}_{\bigoplus_{b\in\mathrm{ind}(\mathcal{B})} Y(b)}(Y(e)).$$

Since $\mathrm{Hom}(\bigoplus_{b\in\mathrm{ind}(\mathcal{B})} Y(b), Y(e)) = \prod_{b\in\mathrm{ind}(\mathcal{B})} Y(e)(b)$ and $\mathfrak{R}$ is right support finite, there are some $b_1, b_2, \ldots, b_n$ in $\mathrm{ind}(\mathcal{B})$ and $Q := \oplus_{i=1}^{n} Y(b_i)$ such that

$$\mathrm{Hom}\left(\bigoplus_{b\in\mathrm{ind}(\mathcal{B})} Y(b), Y(e)\right) = \mathrm{Hom}(Q, Y(e)).$$

Note that $\mathrm{Hom}(Q, Y(e))$ is a $\mathbb{K}$-module of finite length, since $\mathfrak{R}$ is Hom-finite. Therefore, $I_{\mathcal{B}}(-, e) = \mathrm{Tr}_Q(Y(e))$ is a finitely generated right $\mathfrak{R}$-module. Finally, by [7, Proposition 4.2 (c)], we get (b). □

Proposition 2.4. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid. Then* $\mathrm{fin.p}_\rho(\mathfrak{R})$ *is a locally finite $\mathbb{K}$-ringoid.*

*Proof.* First, we prove that $\mathrm{fin.p}_\rho(\mathfrak{R})$ is Hom-finite. Indeed, let $F, G \in \mathrm{fin.p}_\rho(\mathfrak{R})$. Then, there are morphisms $a \xrightarrow{f} b$ and $a' \xrightarrow{f'} b'$ in $\mathfrak{R}$ and exact sequences in $\mathrm{Mod}_\rho(\mathfrak{R})$

$$Y(a) \xrightarrow{Y(f)} Y(b) \xrightarrow{\lambda} F \to 0, \tag{2.1}$$

$$Y(a') \xrightarrow{Y(f')} Y(b') \xrightarrow{\lambda'} G \to 0. \tag{2.2}$$

By (2.2) we get an epimorphism $\mathrm{Hom}_{\mathfrak{R}}(b, b') \xrightarrow{\lambda'_b} G(b)$ of $\mathbb{K}$-modules, and since $\mathfrak{R}$ is Hom-finite, we get that $G(b)$ is a $\mathbb{K}$-module of finite length. By applying the functor $(-, G)$ to the sequence (2.1), we obtain a monomorphism $(\lambda, G) : (F, G) \to (Y(b), G)$ of $\mathbb{K}$-modules. Therefore, $(F, G)$ is of finite length since $(Y(b), G) \simeq G(b)$. In particular, $\mathrm{End}(M)$ is a left Artin ring for any $M \in \mathrm{fin.p}_\rho(\mathfrak{R})$.

Now, we prove that $\mathrm{fin.p}_\rho(\mathfrak{R})$ is a Krull–Schmidt $\mathbb{K}$-category. By [7, Proposition 4.2 (d)], it follows that the idempotents in $\mathrm{fin.p}_\rho(\mathfrak{R})$ split, and, moreover, it is an additive category. Finally, from [26, Corollary 4.4], we get that $\mathrm{fin.p}_\rho(\mathfrak{R})$ is a Krull–Schmidt $\mathbb{K}$-category since $\mathrm{End}(M)$ is a semi-perfect ring, for any $M \in \mathrm{fin.p}_\rho(\mathfrak{R})$. □

Filtrations. Let $\mathcal{A}$ be an abelian category and $\mathcal{X} \subseteq \mathcal{A}$. We denote by $\mathcal{X}^{\oplus}$ the class of all the objects of $\mathcal{A}$ which are a finite direct sum of objects in $\mathcal{X}$.

We say that $M \in \mathcal{A}$ is $\mathcal{X}$-filtered if there exists a continuous chain $\{M_i\}_{i<\alpha}$ of subobjects of $M$, for some ordinal number $\alpha$, such that $M_{i+1}/M_i \in \mathcal{X}^{\oplus}$ for $i + 1 \leq \alpha$. In case $\alpha < \aleph_0$, we say that $M$ has a finite $\mathcal{X}$-filtration of length $\alpha$. We denote by $\mathcal{F}(\mathcal{X})$ the class of objects which are $\mathcal{X}$-filtered and by $\mathcal{F}_f(\mathcal{X})$ the class of objects having a finite filtration. Note that, for $M \in \mathcal{F}_f(\mathcal{X})$, the $\mathcal{X}$-length of $M$ can be defined as follows:

$$\ell_{\mathcal{X}}(M) := \min\{n \in \mathbb{N} \mid M \text{ has an } \mathcal{X}\text{-filtration of length } n\}.$$

By using the notion of $\mathcal{X}$-length and induction, it can be proven the following useful remark.

Remark 2.5. Let $\mathcal{X}$ be a class of objects in an abelian category $\mathcal{A}$. Then, the class $\mathcal{F}_f(\mathcal{X})$ is closed under extensions.

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**3. Standardly stratified ringoids.** In this section, we define the concept of standardly stratified algebra for the class of rings with several objects. We also prove some main properties which generalize several well-known facts from the classical theory of stardardly stratified algebras.

DEFINITION 3.1. Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be a class of objects of $\mathfrak{R}$ such that $\mathrm{add}\,(\mathcal{C}) = \mathcal{C}$. Let $\tilde{\mathcal{A}} := \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ be a partition of the set $\mathrm{ind}(\mathcal{C})$, where $\alpha$ is an ordinal number (the size of the partition $\tilde{\mathcal{A}}$). For each $i \in [0, \alpha)$, we set $\mathcal{A}_i := \bigcup_{j\leq i} \tilde{\mathcal{A}}_j$ and $\mathcal{B}_i(\mathcal{A}) := \mathrm{add}\,(\mathcal{A}_i)$. We say that $\mathcal{B}(\mathcal{A}) := \{\mathcal{B}_i(\mathcal{A})\}_{i<\alpha}$ is the family of subcategories of $\mathcal{C}$ related to the partition $\tilde{\mathcal{A}}$. We denote by $\wp(\mathcal{C})$ the class of all the partitions of the set $\mathrm{ind}\,(\mathcal{C})$.

DEFINITION 3.2. Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be such that $\mathrm{add}\,(\mathcal{C}) = \mathcal{C}$. Let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be a family of subcategories of $\mathcal{C}$, where $\alpha$ is an ordinal number (the size of the family $\mathcal{B}$). We say that $\mathcal{B}$ is *admissible* in $\mathcal{C}$, if the following conditions hold true:

(a) $\mathrm{add}\,(\mathcal{B}_i) = \mathcal{B}_i$ for any $i < \alpha$;
(b) $\mathcal{B}_i \subseteq \mathcal{B}_j$ if $i \leq j < \alpha$;
(c) $\mathcal{C} = \bigcup_{i<\alpha} \mathcal{B}_i$;
(d) $\sigma_i(\mathcal{B}) := \mathrm{ind}(\mathcal{B}_i) - \bigcup_{j<i} \mathrm{ind}(\mathcal{B}_i) \neq \emptyset$ for any $i < \alpha$.

We call $\sigma_i(\mathcal{B})$ the $i$th section of $\mathcal{B}$. An admissible family $\mathcal{B}$ in $\mathcal{C}$ is said to be *exhaustive* in $\mathfrak{R}$, if $\mathcal{C} = \mathfrak{R}$. We set $\sigma(\mathcal{B}) := \{\sigma_i(\mathcal{B})\}_{i<\alpha}$. The class of all the admissible families of subcategories of $\mathcal{C}$ will be denoted by $\mathtt{AF}(\mathcal{C})$.

PROPOSITION 3.3. *Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be a class of subobjects of $\mathfrak{R}$ such that* $\mathrm{add}\,(\mathcal{C}) = \mathcal{C}$. *Then, the correspondence $\sigma : \mathtt{AF}(\mathcal{C}) \to \wp(\mathcal{C})$, $\mathcal{B} \mapsto \sigma(\mathcal{B})$, is bijective with inverse $\tilde{\mathcal{A}} \mapsto \mathcal{B}(\mathcal{A})$.*

*Proof. From admissible families to partitions*: Let $\mathcal{B} = \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family in $\mathcal{C}$. We prove that $\sigma(\mathcal{B})$ is a partition of $\mathrm{ind}(\mathcal{C})$ and $\mathcal{B}(\sigma(\mathcal{B})) = \mathcal{B}$. By the definition of admissible families, we have that $\sigma_i(\mathcal{B})$ is not empty. Furthermore, by Definition 3.2 (b) and (d), we get that

$$\sigma_i(\mathcal{B}) = \bigcap_{j<i} (\mathrm{ind}(\mathcal{B}_i) - \mathrm{ind}(\mathcal{B}_j)), \text{ for any } i < \alpha.$$

Let us check that $\mathrm{ind}\,(\mathcal{C}) = \bigcup_{i<\alpha} \sigma_i(\mathcal{B})$. Consider $X \in \mathrm{ind}\,(\mathcal{C})$. Then, by Definition 3.2 (c), there is some $j < \alpha$ such that $X \in \mathrm{ind}\,(\mathcal{B}_j)$ and thus the set $S := \{j < \alpha \,:\, X \in \mathrm{ind}\,(\mathcal{B}_j)\}$ is not empty. Now, for $k := \min S$ it follows that $X \in \mathrm{ind}\,(\mathcal{B}_k)$ and $X \notin \mathrm{ind}\,(\mathcal{B}_j)$ for any $j < k$, which means that $X \in \sigma_k(\mathcal{B})$.

We show that $\sigma_k(\mathcal{B}) \cap \sigma_l(\mathcal{B}) = \emptyset$ for $k < l < \alpha$. Indeed, suppose that there is some $X \in \sigma_k(\mathcal{B}) \cap \sigma_l(\mathcal{B})$. In particular, $X \in \sigma_l(\mathcal{B})$ and thus for any $j < l$ $X \in \mathrm{ind}\,(\mathcal{B}_l)$ and $X \notin \mathrm{ind}\,(\mathcal{B}_j)$. But, for $j = k$, the former conditions say that $X \notin \mathrm{ind}\,(\mathcal{B}_k)$, contradicting that $X \in \sigma_k(\mathcal{B})$.

Let $D := \mathcal{B}(\sigma(\mathcal{B}))$. We assert that $D = \mathcal{B}$. Consider some $i < \alpha$. By definition, we have

$$D_i := \mathrm{add}\left(\bigcup_{j\leq i} \sigma_j(\mathcal{B})\right) = \mathrm{add}\left(\bigcup_{j\leq i} \left(\mathrm{ind}\,(\mathcal{B}_j) - \bigcup_{k<j} \mathrm{ind}\,(\mathcal{B}_k)\right)\right).$$

Therefore, in order to prove that $D_i = \mathcal{B}_i$, it is enough to show that

$$\mathrm{ind}\,(\mathcal{B}_i) = \bigcup_{j\leq i} \left(\mathrm{ind}\,(\mathcal{B}_j) - \bigcup_{k<j} \mathrm{ind}\,(\mathcal{B}_k)\right).$$

Let $X \in \operatorname{ind}(\mathcal{B}_i)$. Thus, the set $S_X := \{j \le i < \alpha \,:\, X \in \operatorname{ind}(\mathcal{B}_j)\}$ is not empty. Then for $k_0 := \min S_X$, we get that $X \in \operatorname{ind}(\mathcal{B}_{k_0})$ and $X \notin \operatorname{ind}(\mathcal{B}_l)$ for any $l < k_0$. Therefore, for $k_0 \le i$, we obtain that $X \in \operatorname{ind}(\mathcal{B}_{k_0})$ and $X \notin \bigcup_{l<k_0} \operatorname{ind}(\mathcal{B}_l)$. This says that $X \in \bigcup_{j \le i} \big(\operatorname{ind}(\mathcal{B}_j) - \bigcup_{k<j} \operatorname{ind}(\mathcal{B}_k)\big)$, proving that $D_i = \mathcal{B}_i$.

*From partitions to admissible families*: Let $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}$ be a partition of the set $\operatorname{ind}(\mathcal{C})$. Let $\mathcal{B}(\mathcal{A}) = \{\mathcal{B}_i(\mathcal{A})\}_{i<\alpha}$ be the family of subcategories of $\mathcal{C}$ related to the partition $\tilde{\mathcal{A}}$. Note that 3.2 (a), (b), and (c) hold true by construction. In order to prove that $\mathcal{B}(\mathcal{A}) \in \mathtt{AF}(\mathcal{C})$ and $\sigma(\mathcal{B}(\mathcal{A})) = \tilde{\mathcal{A}}$, it is enough to show that $\tilde{\mathcal{A}}_i = \sigma_i(\mathcal{B}(\mathcal{A}))$ for any $i < \alpha$.

Let $i < \alpha$. For any $X \in \operatorname{ind}(\mathcal{C})$, we assert that

$$(*) \quad X \in \sigma_i(\mathcal{B}(\mathcal{A})) \quad \Leftrightarrow \quad \forall\, j < i,\ \exists\, k \in (j, i] \text{ such that } X \in \tilde{\mathcal{A}}_k.$$

Indeed, the assertion above follows from the following sequel of equivalences

$$\begin{aligned} X \in \sigma_i(\mathcal{B}(\mathcal{A})) \quad &\Leftrightarrow \quad X \in \bigcap_{j<i} \big(\operatorname{ind} \mathcal{B}_i(\mathcal{A}) - \operatorname{ind} \mathcal{B}_j(\mathcal{A})\big) \\ &\Leftrightarrow \quad \forall\, j < i \quad X \in \bigcup_{k \le i} \tilde{\mathcal{A}}_k \quad \text{and} \quad X \notin \bigcup_{l \le j} \tilde{\mathcal{A}}_k \\ &\Leftrightarrow \quad \forall\, j < i,\ \exists\, k \in (j, i] \text{ such that } X \in \tilde{\mathcal{A}}_k. \end{aligned}$$

By $(*)$, it is clear that $\tilde{\mathcal{A}}_i \subseteq \sigma_i(\mathcal{B}(\mathcal{A}))$. Let $X \in \sigma_i(\mathcal{B}(\mathcal{A}))$. Then by $(*)$ there is some $k \in (j, i]$ such that $X \in \tilde{\mathcal{A}}_k$. Suppose that $k < i$. Then, again by $(*)$ there is some $k' \in (k, i]$ such that $X \in \tilde{\mathcal{A}}_{k'}$. Therefore, $X \in \tilde{\mathcal{A}}_k \cap \tilde{\mathcal{A}}_{k'}$, contradicting that $\tilde{\mathcal{A}}$ is a partition of $\operatorname{ind}(\mathcal{C})$. Then, $k = i$ and thus $X \in \tilde{\mathcal{A}}_i$. □

Associated to a partition $\tilde{\mathcal{A}}$ of $\operatorname{ind}(\mathcal{C})$, as above, we can compute the $(\tilde{\mathcal{A}}, \mathcal{C})$-standard right $\mathfrak{R}$-modules. These modules play an important role in the definition of a right standardly stratified ringoid. In order to define such modules, we consider the Yoneda's contravariant functor $Y : \mathfrak{R} \to \operatorname{Mod}_\rho(\mathfrak{R})$, where $Y(e) := \operatorname{Hom}_{\mathfrak{R}}(-, e)$.

Definition 3.4. Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be a class of objects of $\mathfrak{R}$ such that $\operatorname{add}(\mathcal{C}) = \mathcal{C}$, and let $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ be a partition of the set $\operatorname{ind}(\mathcal{C})$. Consider the projective right $\mathfrak{R}$-modules $P_e^{op}(i) := Y(e)$ for $e \in \tilde{\mathcal{A}}_i$ and $i < \alpha$. Let $P^{op} = P^{op}(\tilde{\mathcal{A}}) := \{P^{op}(i)\}_{i<\alpha}$ where $P^{op}(i) := \{P_e^{op}(i)\}_{e \in \tilde{\mathcal{A}}_i}$. We say that $P^{op}(\tilde{\mathcal{A}})$ is the family of projective modules associated with the partition $\tilde{\mathcal{A}}$. We define the family ${}_{(\tilde{\mathcal{A}},\mathcal{C})}\Delta = \{\Delta(i)\}_{i<\alpha}$ of $(\tilde{\mathcal{A}}, \mathcal{C})$-standard right $\mathfrak{R}$-modules, where $\Delta(i) := \{\Delta_e(i)\}_{e \in \tilde{\mathcal{A}}_i}$ is defined as follows:

$$\Delta_e(i) := \frac{P_e^{op}(i)}{\operatorname{Tr}_{\oplus_{j<i}\overline{P}(j)}(P_e^{op}(i))},$$

where $\overline{P}(j) := \bigoplus_{r \in \tilde{\mathcal{A}}_j} P_r^{op}(j)$ and $\operatorname{Tr}_{\oplus_{j<i}\overline{P}(j)}(P_e^{op}(i))$ is the trace of $\oplus_{j<i}\overline{P}(j)$ in $P_e^{op}(i)$. In case $\mathfrak{R} = \mathcal{C}$, we just write ${}_{\tilde{\mathcal{A}}}\Delta$ instead of ${}_{(\tilde{\mathcal{A}},\mathcal{C})}\Delta$, and we say that ${}_{(\tilde{\mathcal{A}},\mathcal{C})}\Delta$ is the family of $\tilde{\mathcal{A}}$-standard right $\mathfrak{R}$-modules.

Definition 3.5. Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be a class of objects of $\mathfrak{R}$ such that $\operatorname{add}(\mathcal{C}) = \mathcal{C}$. For any admissible family $\mathcal{B} = \{\mathcal{B}_i\}_{i<\alpha}$ of subcategories of $\mathcal{C}$, we know by Proposition 3.3 that $\sigma(\mathcal{B})$ is a partition of $\operatorname{ind}(\mathcal{C})$. Then, ${}_{(\mathcal{B},\mathcal{C})}\Delta := {}_{(\sigma(\mathcal{B}),\mathcal{C})}\Delta$ is called the family of $(\mathcal{B}, \mathcal{C})$-standard $\mathfrak{R}$-modules. In case $\mathfrak{R} = \mathcal{C}$, we just write ${}_{\mathcal{B}}\Delta$ instead of ${}_{(\mathcal{B},\mathcal{C})}\Delta$, and we say that ${}_{\mathcal{B}}\Delta$ is the family of $\mathcal{B}$-standard right $\mathfrak{R}$-modules.

Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be a class of objects of $\mathfrak{R}$ such that $\operatorname{add}(\mathcal{C}) = \mathcal{C}$. For any partition $\tilde{\mathcal{A}}$ of $\operatorname{ind}(\mathcal{C})$, we point out that by Proposition 3.3, it holds that ${}_{(\mathcal{B}(\tilde{\mathcal{A}}),\mathcal{C})}\Delta = {}_{(\tilde{\mathcal{A}},\mathcal{C})}\Delta$.

DEFINITION 3.6. A right standardly stratified $\mathbb{K}$-ringoid is a pair $(\mathfrak{R}, \tilde{\mathcal{A}})$, where $\mathfrak{R}$ is a Krull–Schmidt $\mathbb{K}$-ringoid and $\tilde{\mathcal{A}}$ is a partition of $\operatorname{ind}(\mathfrak{R})$ such that the $\tilde{\mathcal{A}}$-standard family $\Delta = {}_{\tilde{\mathcal{A}}}\Delta$ of right $\mathfrak{R}$-modules satisfies the following condition, for any $i < \alpha$ and $e \in \tilde{\mathcal{A}}_i$,

$$\operatorname{Tr}_{\oplus_{j<i}\overline{P}(j)}(P_e^{op}(i)) \in \mathcal{F}_f\left(\bigcup_{j<i} \Delta(j)\right).$$

DEFINITION 3.7. A right standardly stratified $\mathbb{K}$-ringoid $(\mathfrak{R}, \tilde{A})$ is quasi-hereditary if $\operatorname{End}(\Delta_e(i))$ is a division ring, for any $e \in \tilde{\mathcal{A}}_i$ and $i < \alpha$.

Let $\Lambda$ be a basic Artin $\mathbb{K}$-algebra and let $\{e_i\}_{i=1}^n$ be a complete family of primitive orthogonal idempotents of $\Lambda$. There is a $\mathbb{K}$-ringoid $\mathfrak{R}(\Lambda)$, associated to $\Lambda$, where the objects are $e_1, e_2, \ldots, e_n$ and the morphisms are $\operatorname{Hom}_{\mathfrak{R}(\Lambda)}(e_i, e_j) := e_j\Lambda e_i$ for any $1 \leq i, j \leq n$. The composition of morphism in $\mathfrak{R}(\Lambda)$ is just the multiplication in $\Lambda$. Note that $\operatorname{ind}\mathfrak{R}(\Lambda) = \{e_1, e_2, \ldots, e_n\}$. We have the canonical isomorphism of categories

$$\delta : \operatorname{Mod}_\rho(\mathfrak{R}(\Lambda)) \to \operatorname{Mod}(\Lambda^{op}), \quad M \mapsto \oplus_{i=1}^n M(e_i).$$

For the Yoneda's functor $Y : \mathfrak{R}(\Lambda) \to \operatorname{Mod}_\rho(\mathfrak{R}(\Lambda))$, we have

$$\delta(Y(e_i)) = \oplus_{j=1}^n \operatorname{Hom}_{\mathfrak{R}(\Lambda)}(e_j, e_i) = \oplus_{j=1}^n e_i\Lambda e_j = e_i\Lambda.$$

Let $\tilde{\mathcal{A}}_i := \{e_i\}$, $P(i) := Y(e_i)$, and $P := \{\{P(i)\}\}_{i=1}^n$. Consider the standard modules ${}_{\mathfrak{R}(\Lambda)}\Delta := {}_{\tilde{\mathcal{A}}}\Delta$. Note that $\delta({}_{\mathfrak{R}(\Lambda)}\Delta(i)) \simeq {}_\Lambda\Delta(i)$ for any $i \in [1, n]$. Therefore, $(\mathfrak{R}(\Lambda), \tilde{\mathcal{A}})$ is a right standardly stratified $\mathbb{K}$-ringoid if, and only if, $\Lambda$ is a right standardly stratified $\mathbb{K}$-algebra as in the classical sense.

We recall that for a given abelian category $\mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{A}$, $\mathcal{X}^\oplus$ denotes the class of all the objects of $\mathcal{A}$ which are a finite direct sum of objects in $\mathcal{X}$.

PROPOSITION 3.8. *Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid and $\mathcal{C} \subseteq \mathfrak{R}$ be a class of objects in $\mathfrak{R}$ such that* $\operatorname{add}(\mathcal{C}) = \mathcal{C}$. *For any admissible family $\mathcal{B} = \{\mathcal{B}_i\}_{i<\alpha}$ of subcategories of $\mathcal{C}$, the following statements hold true.*

(a) *For any $e \in \sigma_i(\mathcal{B})$, we have*

$${}_{(\mathcal{B},\mathcal{C})}\Delta_e(i) = \frac{Y(e)}{\operatorname{Tr}_{\{Y(t)\}_{t\in\bigcup_{j<i}\mathcal{B}_j}}(Y(e))}.$$

*Moreover, if $\mathfrak{R}$ is locally finite, then*

$$\operatorname{Tr}_{\{Y(t)\}_{t\in\bigcup_{j<i}\mathcal{B}_j}}(Y(e)) = I_{\bigcup_{j<i}\mathcal{B}_j}(-, e) \quad \textit{and} \quad {}_{(\mathcal{B},\mathcal{C})}\Delta_e(i) \neq 0.$$

(b) *If $\mathfrak{R}$ is locally finite and right support finite, then ${}_{(\mathcal{B},\mathcal{C})}\Delta^\oplus \subseteq \operatorname{fin.p}_\rho(\mathfrak{R})$.*

*Proof.* (a) We have ${}_{(\mathcal{B},\mathcal{C})}\Delta = {}_{(\sigma(\mathcal{B}),\mathcal{C})}\Delta$ and $\sigma_i(\mathcal{B}) = \operatorname{ind}(\mathcal{B}_i) - \bigcup_{j<i} \operatorname{ind}(\mathcal{B}_j)$. We assert that

$$(*) \quad \operatorname{add}\left(\bigcup_{j<i} \sigma_j(\mathcal{B})\right) = \bigcup_{j<i} \mathcal{B}_j.$$

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Indeed, $\mathrm{add}(\bigcup_{j<i}\mathcal{B}_j)=\bigcup_{j<i}\mathcal{B}_j$ since $\mathcal{B}_j\subseteq\mathcal{B}_{j'}$ if $j\leq j'$ and $\mathrm{add}(\mathcal{B}_j)=\mathcal{B}_j$ for every $j$. Now, using $\sigma_j(\mathcal{B})\subseteq\mathcal{B}_j$ for every $j$, it follows that $\bigcup_{j<i}\sigma_j(\mathcal{B})\subseteq\bigcup_{j<i}\mathcal{B}_j$. Then we have that $\mathrm{add}\big(\bigcup_{j<i}\ \sigma_j(\mathcal{B})\big)\subseteq\mathrm{add}(\bigcup_{j<i}\mathcal{B}_j)=\bigcup_{j<i}\mathcal{B}_j$. Now, let $X\in\bigcup_{j<i}\mathcal{B}_j$, then there exists $j'<i$ such that $X\in\mathcal{B}_{j'}$. Thus $X=\bigoplus_{k=1}^n X_k$ with $X_k\in B_{j'}$ and local. For each $X_k$ consider the set $S(X_k):=\{j<i\ :\ X_k\in\mathcal{B}_j\}$ which is not empty. For $j_k:=\min S(X_k)$, it follows that $X_k\in(\mathrm{ind}(\mathcal{B}_{j_k})-\bigcup_{j<j_k}\ \mathrm{ind}(\mathcal{B}_j))=\sigma_{j_k}(\mathcal{B})$ and therefore $X\in\mathrm{add}\big(\bigcup_{j<i}\ \sigma_j(\mathcal{B})\big)$; proving $(*)$.

Using $(*)$ and $\overline{P}(j)=\bigoplus_{r\in\sigma_j(\mathcal{B})}\ P_r^{op}(j)=\bigoplus_{r\in\sigma_j(\mathcal{B})}\ Y(r)$, we obtain the following sequence of equalities

$$\begin{aligned}\mathrm{Tr}_{\bigoplus_{j<i}\overline{P}(j)}(P_e^{op}(i))&=\mathrm{Tr}_{\left\{\bigoplus\limits_{j<i}\ \bigoplus\limits_{r\in\sigma_j(\mathcal{B})}Y(r)\right\}}(Y(e))\\&=\mathrm{Tr}_{\{Y(t)\}_{t\in\bigcup\limits_{j<i}\sigma_j(\mathcal{B})}}(Y(e))\\&=\mathrm{Tr}_{\{Y(t)\}_{t\in\mathrm{add}(\bigcup\limits_{j<i}\sigma_j(\mathcal{B}))}}(Y(e))\\&=\mathrm{Tr}_{\{Y(t)\}_{t\in\bigcup\limits_{j<i}\mathcal{B}_j}}(Y(e)).\end{aligned}$$

Let $\mathfrak{R}$ be locally finite. Then by Lemma 2.3 (a), $\mathrm{Tr}_{\{Y(t)\}_{t\in\bigcup_{j<i}\mathcal{B}_j}}Y(e)$ is equal to $I_{\bigcup_{j<i}\mathcal{B}_j}(-,e)$. We assert that $\Delta_e(i)\neq0$. In order to prove this, it is enough to see that $\Delta_e(i)(e)\neq0$.

Suppose that $\Delta_e(i)(e)=0$. Then, $I_{\bigcup_{j<i}\mathcal{B}_j}(e,e)=\mathfrak{R}(e,e)$ and thus $1_e\in I_{\bigcup_{j<i}\mathcal{B}_j}(e,e)$. Therefore, $1_e$ factorizes through some $X\in\mathcal{B}_j$, where $j<i$. Then $e$ is a direct summand of $X$ and so $e\in\mathcal{B}_j$, contradicting that $e\in\sigma_i(\mathcal{B})$.

(b) Let $\mathfrak{R}$ be locally finite and right support finite. By Lemma 2.3 (b), the item (a) and [7, Proposition 4.2 (d)], we get ${}_{\mathcal{B}}\Delta^{\oplus}\subseteq\mathrm{fin.p}_{\rho}(\mathfrak{R})$. □

DEFINITION 3.9. Let $\mathcal{B}=\{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathcal{C}\subseteq\mathfrak{R}$, for some Krull–Schmidt $\mathbb{K}$-ringoid $\mathfrak{R}$. Let $\Delta={}_{(\mathcal{B},\mathcal{C})}\Delta$ be the $(\mathcal{B},\mathcal{C})$-standard family of right $\mathfrak{R}$-modules. We say that $M\in\mathcal{F}'_f(\Delta)$ if there exists a filtration $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_{n-1}\subseteq M_n=M$ such that $M_i/M_{i-1}\in\Delta(s_i)^{\oplus}$, for some $s_i<\alpha$ and $i\in[1,n]$.

PROPOSITION 3.10. *Let $\mathcal{B}=\{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathcal{C}\subseteq\mathfrak{R}$, for some Krull–Schmidt $\mathbb{K}$-ringoid $\mathfrak{R}$, and let $\Delta={}_{(\mathcal{B},\mathcal{C})}\Delta$. Then, $\mathcal{F}_f(\Delta)=\mathcal{F}'_f(\Delta)$.*

*Proof.* It can be shown that $\mathcal{F}'_f(\Delta)$ is closed under extensions. Moreover, it is also clear that $\mathcal{F}'_f(\Delta)\subseteq\mathcal{F}_f(\Delta)$. Finally, the inclusion $\mathcal{F}_f(\Delta)\subseteq\mathcal{F}'_f(\Delta)$ can be obtained by induction on the $\Delta$-length of objects in $\mathcal{F}_f(\Delta)$. □

LEMMA 3.11. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid and $\mathcal{B}:=\{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$. Then, the family ${}_{\mathcal{B}}\Delta$, of $\mathcal{B}$-standard modules, satisfies the following conditions.*

(a) *If ${}_{\mathcal{B}}\Delta\subseteq\mathrm{fin.p}_{\rho}(\mathfrak{R})$ then $\Delta_e(i)$ is local, for any $i<\alpha$ and $e\in\sigma_i(\mathcal{B})$.*
(b) $\mathrm{Hom}(\Delta_e(i),\Delta_{e'}(i))\simeq\Delta_{e'}(i)(e)$, *for any $e,e'\in\sigma_i(\mathcal{B})$.*
(c) $\mathrm{Hom}(\Delta_e(i),\Delta_{e'}(i'))=0$ *if $i<i'$ and $e\in\sigma_i(\mathcal{B})$, $e'\in\sigma_{i'}(\mathcal{B})$.*
(d) $\mathrm{Ext}^1(\Delta_e(i),\Delta_{e'}(i'))=0$ *if $i\leq i'$ and $e\in\sigma_i(\mathcal{B})$, $e'\in\sigma_{i'}(\mathcal{B})$.*

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*Proof.* Let $e\in\sigma_i(\mathcal{B})$. By Proposition 3.8, $0\neq\Delta_e(i)=Y(e)/U_e(i)$, where $U_e(i):=I_{\cup_{j<i}\mathcal{B}_j}(-,e)=\mathrm{Tr}_{\bigoplus_{a\in\cup_{j<i}\mathcal{B}_j}Y(a)}(Y(e))$.

(a) Let ${}_{\mathcal{B}}\Delta\subseteq\mathrm{fin.p}_\rho(\mathfrak{R})$. Since $e$ is local in $\mathfrak{R}$ and $\mathrm{fin.p}_\rho(\mathfrak{R})$ is a Krull–Schmidt category (see Proposition 2.4), the epimorphism $Y(e)\to\Delta_e(i)$ is a projective cover. Let $\Delta_e(i)=\oplus_{k=1}^n M_k$ be a decomposition of $\Delta_e(i)$, where each $M_k$ is local. Consider the projective cover $P_k\to M_k$, for $k\in[1,n]$. Using the fact that a finite coproduct of projective covers is a projective cover, it follows that $Y(e)=\oplus_{k=1}^n P_k$. Therefore, $n=1$ since $Y(e)$ is local. Then we get that $\Delta_e(i)=M_1$, proving that $\Delta_e(i)$ is local.

(b), (c) and (d) Let $i,i'\in[0,\alpha)$ and $e\in\sigma_i(\mathcal{B})$, $e'\in\sigma_{i'}(\mathcal{B})$. Thus, we have the exact sequences of right $\mathfrak{R}$-modules

$$0\to I_{\cup_{t<i}\mathcal{B}_t}(-,e)\to Y(e)\to\Delta_e(i)\to 0,$$
$$\bigoplus_{a\in\cup_{t<i}\mathcal{B}_t}Y(a)\to I_{\cup_{t<i}\mathcal{B}_t}(-,e)\to 0.$$

Then, by applying $\mathrm{Hom}(-,\Delta_{e'}(i'))$ to the above exact sequences, we get the exact sequence of abelian groups

$$0\to(\Delta_e(i),\Delta_{e'}(i'))\to(Y(e),\Delta_{e'}(i'))\to(I_{\cup_{t<i}\mathcal{B}_t}(-,e),\Delta_{e'}(i')),$$

an epimorphism $\mathrm{Hom}(I_{\cup_{t<i}\mathcal{B}_t}(-,e),\Delta_{e'}(i'))\to\mathrm{Ext}^1(\Delta_e(i),\Delta_{e'}(i'))$ and a monomorphism $\mathrm{Hom}(I_{\cup_{t<i}\mathcal{B}_t}(-,e),\Delta_{e'}(i'))\to\mathrm{Hom}(\bigoplus_{a\in\cup_{t<i}\mathcal{B}_t}Y(a),\Delta_{e'}(i'))$.

By Yonedas's Lemma, we have that

$$\mathrm{Hom}\left(\bigoplus_{a\in\cup_{t<i}\mathcal{B}_t}Y(a),\Delta_{e'}(i')\right)\simeq\prod_{a\in\cup_{t<i}\mathcal{B}_t}\Delta_{e'}(i')(a).$$

On the other hand, we know that

$$\Delta_{e'}(i')(a)=\frac{\mathfrak{R}(a,e')}{I_{\cup_{t<i'}\mathcal{B}_t}(a,e')}.$$

Let $i\leq i'$. Then, we get $\cup_{t<i}\mathcal{B}_t\subseteq\cup_{t<i'}\mathcal{B}_t$ and hence $\Delta_{e'}(i')(a)=0$ for any $a\in\cup_{t<i}\mathcal{B}_t$. Therefore, we conclude that

$$\mathrm{Ext}^1(\Delta_e(i),\Delta_{e'}(i'))=0\text{ and }(\Delta_e(i),\Delta_{e'}(i'))\simeq(Y(e),\Delta_{e'}(i'))\simeq\Delta_{e'}(i')(e).$$

If $i<i'$, it follows that $e\in\sigma_i(\mathcal{B})\subseteq\mathcal{B}_i\subseteq\cup_{t<i'}\mathcal{B}_t$ and so $\Delta_{e'}(i')(e)=\frac{\mathfrak{R}(e,e')}{I_{\cup_{t<i'}\mathcal{B}_t}(e,e')}=0$; proving that $(\Delta_e(i),\Delta_{e'}(i'))=0$. □

LEMMA 3.12. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, and let $\mathcal{B}:=\{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$. Then, for the family ${}_{\mathcal{B}}\Delta$ of $\mathcal{B}$-standard right $\mathfrak{R}$-modules and any $i<\alpha$, the following statements are equivalent.*

(a) $\mathrm{End}(\Delta_e(i))$ *is a division ring, for any* $e\in\sigma_i(\mathcal{B})$.
(b) $I_{\cup_{t<i}\mathcal{B}_t}(e,e)=\mathrm{rad}_{\mathfrak{R}}(e,e)$, *for any* $e\in\sigma_i(\mathcal{B})$.
(c) $\mathrm{End}(\Delta_e(i))\simeq\mathrm{End}_{\mathfrak{R}}(e)/\mathrm{rad}\,\mathrm{End}_{\mathfrak{R}}(e)$, *for any* $e\in\sigma_i(\mathcal{B})$.

*Proof.* Let $e\in\sigma_i(\mathcal{B})$. Then, by Lemma 3.11 (b), it follows that

$$(*)\quad\mathrm{End}(\Delta_e(i))\simeq\mathrm{End}_{\mathfrak{R}}(e)/I_{\cup_{t<i}\mathcal{B}_t}(e,e).$$

(a) $\Rightarrow$ (b) Assume that $\mathrm{End}(\Delta_e(i))$ is a division ring. Let $f \in I_{\cup_{t<i}\mathcal{B}_t}(e, e)$. Then, there are morphisms $e \xrightarrow{v} b \xrightarrow{u} e$, with $b \in \cup_{t<i}\mathcal{B}_t$ and such that $f = uv$. Since $b \in \cup_{t<i}\mathcal{B}_t$, we get that $f$ is not an isomorphism and hence $f \in \mathrm{rad}_{\mathfrak{R}}(e, e)$.

Let $f \in \mathrm{rad}_{\mathfrak{R}}(e, e)$. Suppose that $f \notin I_{\cup_{t<i}\mathcal{B}_t}(e, e)$. Then, by $(*)$, the class $\overline{f} = f + I_{\cup_{t<i}\mathcal{B}_t}(e, e)$ is invertible in $\mathrm{End}(\Delta_e(i))$ and there is $g : e \to e$ such that $fg - 1_e \in I_{\cup_{t<i}\mathcal{B}_t}(e, e)$. Note that $fg \in \mathrm{rad}_{\mathfrak{R}}(e, e)$ and thus $fg - 1_e$ is invertible in $\mathrm{End}_{\mathfrak{R}}(e)$. As a consequence, $1_e \in I_{\cup_{t<i}\mathcal{B}_t}(e, e)$ and so $e \in \cup_{t<i}\mathcal{B}_t$, which is a contradiction. Therefore, $f \in I_{\cup_{t<i}\mathcal{B}_t}(e, e)$.

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) follow from $(*)$ and the fact that $e$ is local in $\mathfrak{R}$. □

LEMMA 3.13. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, and let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$. Then, for the family ${}_{\mathcal{B}}\Delta$ of $\mathcal{B}$-standard right $\mathfrak{R}$-modules and any $i < \alpha$, the following statements are equivalent.*

(a) $\mathrm{Hom}(\Delta_e(i), \Delta_{e'}(i)) = 0$, *for any $e \neq e'$ in $\sigma_i(\mathcal{B})$.*
(b) $I_{\cup_{t<i}\mathcal{B}_t}(e, e') = \mathrm{rad}_{\mathfrak{R}}(e, e')$, *for any $e \neq e'$ in $\sigma_i(\mathcal{B})$.*

*Proof.* It is straightforward from Lemma 3.11 (b) and Proposition 3.8. □

LEMMA 3.14. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, and let $\Delta = {}_{\mathcal{B}}\Delta$. Then, the following statements hold true.*

(a) *Let $L \subseteq M \subseteq N$ be a chain of right $\mathfrak{R}$-submodules, with $M/L \in \Delta(i')^{\oplus}$, $N/M \in \Delta(i)^{\oplus}$ and $i < i'$. Then, there exists a chain of right $\mathfrak{R}$-submodules $L \subseteq M' \subseteq N$ such that $M'/L \simeq N/M \in \Delta(i)^{\oplus}$ and $N/M' \simeq M/L \in \Delta(i')^{\oplus}$.*
(b) *Let $\{\eta_i : 0 \to M_{i-1} \to M_i \to X_i \to 0\}_{i=1}^{n}$ be a family of exact sequences in $\mathrm{Mod}_{\rho}(\mathfrak{R})$, where $X_i \in \Delta(j)^{\oplus}$, for every $i \in [1, n]$ and some $j < \alpha$. Then, for each $k \in [1, n]$, there exists an exact sequence of the form $\xi_k : 0 \to M_0 \to M_k \to Z_k \to 0$, where $Z_k = \oplus_{i=1}^{k} X_i \in \Delta(j)^{\oplus}$.*

*Proof.* (a) From the chain of submodules $L \subseteq M \subseteq N$, we construct the following exact and commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'i} & & & & \\
0 & \longrightarrow & M & \xrightarrow{i'} & N & \xrightarrow{d'} & \frac{N}{M} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & \frac{M}{L} & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & \frac{N}{M} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

By Lemma 3.11 (d), the bottom exact sequence, in the above diagram, splits. Thus, we have the exact sequence $\xi : 0 \longrightarrow \frac{N}{M} \xrightarrow{\beta_2} A \xrightarrow{\beta_1} \frac{M}{L} \longrightarrow 0$. Then, we get the exact and commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & \frac{N}{M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \beta_2} & & \\
0 & \longrightarrow & L & \xrightarrow{i'i} & N & \xrightarrow{\beta} & A & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \beta_1} & & \\
 & & & & \frac{M}{L} & = & \frac{M}{L} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Finally, we conclude that $L \subseteq M' \subseteq N$ and so $\frac{M'}{L} \simeq \frac{N}{M} \in \Delta(i)^{\oplus}$ and $\frac{N}{M'} \simeq \frac{M}{L} \in \Delta(i')^{\oplus}$.

(b) We proceed by induction on $k$. If $k = 1$, we set $\xi_1 := \eta_1$.
Let $k \geq 2$. Then, by induction, we have defined $\xi_{k-1}$ satisfying (b). We construct the following exact and commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_0 & = & M_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M_{k-1} & \longrightarrow & M_k & \longrightarrow & X_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \oplus_{s=1}^{k-1} X_s & \longrightarrow & L_k & \longrightarrow & X_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By Lemma 3.11 (d), we have that the bottom exact sequence splits. Then, we have that $L_k \simeq \oplus_{s=1}^{k} X_k$. Therefore, the second column in the above diagram is the required exact sequence. □

In the following definition, we use that $\mathcal{F}_f(\Delta) = \mathcal{F}'_f(\Delta)$, see Proposition 3.10.

Definition 3.15. Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, and let $\Delta = {}_{\mathcal{B}}\Delta$. For $M \in \mathcal{F}_f(\Delta)$, we consider a filtration

$$\xi : \quad 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M,$$

where $X_k := M_k/M_{k-1} \in \Delta(i_k)^{\oplus}$. In this case, we have the set

$$\Phi_\xi(i) := \{k \in [1, m] \mid 0 \neq X_k \in \Delta(i)^{\oplus}\}.$$

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(a) The $\xi$-ladder filtration multiplicity $[M:\Delta(i)]_\xi$, of $\Delta(i)$ in $M$, is the cardinality of $\Phi_\xi(i)$. In general, the ladder filtration multiplicity $[M:\Delta(i)]_\xi$ could be depending on $\xi$.

(b) We define, the $\xi$-ladder $\Delta$-length of $M$

$$\ell'_{\Delta,\xi}(M) = \sum_{\Delta(i)\in\Delta} [M:\Delta(i)]_\xi.$$

Observe that this sum is finite, since only a finite number of $\Delta(i)$ appears in $\xi$.

(c) For $i<\alpha$ and $k\in\Phi_\xi(i)$, we consider a decomposition

$$D_{k,i}(\xi): \qquad X_k = \bigoplus_{e\in J_k} \Delta_e(i)^{\mu_{e,k}}$$

of each $X_k$, where $J_k \subseteq \sigma_i(\mathcal{B})$ is finite. Let $D_i(\xi) := \{D_{k,i}(\xi)\}_{k\in\Phi_\xi(i)}$ be called the family of decompositions associated with the set $\Phi_\xi(i)$. We define the $\xi$-filtration multiplicity of $\Delta_e(i)$ in $M$ as follows:

$$[M:\Delta_e(i)]_{\xi,D_i(\xi)} := \begin{cases} 0 & \text{if } [M:\Delta(i)]_\xi = 0,\\ \sum_{k\in\Phi_\xi(i)} \mu_{e,k} & \text{if } [M:\Delta(i)]_\xi \neq 0.\end{cases}$$

REMARK 3.16. Note that $[M:\Delta_e(i)]_{\xi,D_i(\xi)}$ depends not only on $\xi$ but also on the chosen family $D_i(\xi)$ of decompositions associated with the set $\Phi_\xi(i)$. However, if $\Delta^\oplus \subseteq \text{fin.p}_\rho(\Delta)$, then $[M:\Delta_e(i)]_{\xi,D_i(\xi)}$ does not depend on $D_i(\xi)$, since by Lemma 3.11 (a) all the $\Delta_e(i)$ are local objects.

PROPOSITION 3.17. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, $\Delta = {}_{\mathcal{B}}\Delta$ and $M \in \mathcal{F}_f(\Delta)$. Consider a finite $\Delta$-filtration $\xi$ of $M$*

$$\xi: \quad 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M,$$

*such that $M_k/M_{k-1} \in \Delta(j_k)^\oplus$, and fix a family $D_i(\xi)$ of decompositions associated with the set $\Phi_\xi(i)$ for each $i$.*

*Then, there exist $\Delta$-filtrations $\eta$ and $\varepsilon$ of $M$, and decompositions $D_i(\eta)$, $D_i(\varepsilon)$, for each $i$, satisfying the following conditions:*

(a) $[M:\Delta_e(i)]_{\xi,D_i(\xi)} = [M:\Delta_e(i)]_{\eta,D_i(\eta)}$, *for any $e\in\sigma_i(\mathcal{B})$.*

(b) *The filtration $\eta$ is well ordered. That is, there is a family of exact sequences*

$$\eta = \{\eta_b:\ 0 \longrightarrow \overline{M}_{b-1} \longrightarrow \overline{M}_b \longrightarrow \overline{X}_b \longrightarrow 0\}_{b=1}^m$$

*with $\overline{M}_0 := 0$, $i_1 \le i_2 \le \cdots \le i_m$ and $\overline{X}_b \in \Delta(i_b)^\oplus$.*

(c) *If $M\neq 0$, the filtration $\varepsilon$ is strictly well ordered. That is, $\varepsilon$ has the form $\varepsilon: 0 = M'_0 \subsetneq M'_1 \subsetneq M'_2 \subsetneq \cdots \subsetneq M'_{a-1} \subsetneq M'_a = M$ where $M'_k/M'_{k-1} \in \Delta(i'_k)^\oplus$, for $k\in[1,a]$, $a\le m$ and $i'_1 < i'_2 < i'_3 < \cdots < i'_{a-1} < i'_a$. Moreover,*

$$[M:\Delta_e(i)]_{\varepsilon,D_i(\varepsilon)} = [M:\Delta_e(i)]_{\eta,D_i(\eta)}, \textit{ for any } e\in\sigma_i(\mathcal{B}).$$

*Proof.* If $M=0$, we have that (a) and (b) are trivial. Let $M\neq 0$.

Let $\xi$ be the given filtration of $M$. We may assume that

$$\xi: \quad 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{m-1} \subsetneq M_m = M,$$

where $X_k := M_k/M_{k-1} \in \Delta(i_k)^\oplus$.

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We prove (a) and (b), by induction on the $\xi$-ladder length $n := \ell'_{\Delta,\xi}(M)$. If $n = 1$, the filtration $\xi$ is already well ordered and hence $\eta := \xi$ and $\varepsilon := \xi$ satisfy the required properties.

Let $n \geq 2$. Consider the family of exact sequences induced by the filtration $\xi$ of $M$

$$\{\xi_b : \ 0 \longrightarrow M_{b-1} \longrightarrow M_b \longrightarrow X_b \longrightarrow 0\}_{b=1}^{m}.$$

Since $\xi' := \xi - \{\xi_m\}$ is a filtration of $M_{m-1}$ and $\ell'_{\Delta,\xi'}(M_{m-1}) = m-1$, by induction, there is a well-ordered filtration

$$\eta' = \{\eta'_b : \ 0 \longrightarrow M'_{b-1} \longrightarrow M'_b \longrightarrow Y_b \longrightarrow 0\}_{b=1}^{m-1}$$

of $M_{m-1}$ with $i'_1 \leq i'_2 \leq \cdots \leq i'_{m-1}$ and $[M_{m-1} : \Delta_e(i)]_{\xi',D_i(\xi')} = [M_{m-1} : \Delta_e(i)]_{\eta',D_i(\eta')}$, for any $e \in \sigma_i(\mathcal{B})$. If $i'_{m-1} \leq j_m$, then $\eta := \eta' \cup \{\xi_m\}$ satisfies the required conditions.

Suppose now that $j_m < i'_{m-1}$. Let $l := \max\{n \in [1, m-1] \mid j_m < i'_{m-n}\}$. Observe that the filtration $\eta' \cup \{\xi_m\}$ is almost the one we want, the only exact sequence that is not ordered is precisely the $\xi_m$. This can be rearranged by applying $l$-times Lemma 3.14 (a) to $\eta' \cup \{\xi_m\}$.

In order to construct $\varepsilon$, we use the well-ordered filtration $\eta$ from (b). We proceed as follows. For each $b$, we group the $i_b$ that are the same and rename them by $\lambda_a$. So we get $\lambda_1 < \lambda_2 < \cdots < \lambda_a$ and hence $\Delta(\lambda_1), \cdots, \Delta(\lambda_a)$ are the different $\Delta(j)$ appearing in the filtration $\eta$ of $M$. Define $s(i) := [M : \Delta(\lambda_i)]_\eta$, $\alpha(i) := \sum_{j=1}^{i} s(j)$ and $\alpha(0) := 0$.

We divide the filtration $\eta$ into the following pieces

$$\{\eta_b : \ 0 \longrightarrow M_{b-1} \longrightarrow M_b \longrightarrow Y_b \longrightarrow 0\}_{b=\alpha(l-1)+1}^{\alpha(l)},$$

with $l \in [1, a]$. For each $l \in [1, a]$, by Lemma 3.14 (b), we obtain the following exact sequence

$$\varepsilon_l : \quad 0 \longrightarrow M_{\alpha(l-1)} \longrightarrow M_{\alpha(l)} \longrightarrow Z_{\alpha(l)} \longrightarrow 0$$

Hence, by setting $M'_0 = 0$ and $M'_i := M_{\alpha(i)}$ for $i \in [1, a]$, we conclude that the filtration $\varepsilon = \{\varepsilon_i\}_{i=1}^{a}$ satisfies the required properties. Finally, we bring out that, in the construction of $\eta$ and $\varepsilon$, we have not added different factors as appearing in $\xi$. These factors have just been reordered and regrouped to obtain $\eta$ and $\varepsilon$. $\square$

**4. Filtration multiplicities in ringoids.** Let $\mathcal{A}$ be an abelian category. It is well known that a pre-radical $\mathfrak{r}$ of $\mathcal{A}$ is a subfunctor of the identity functor $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$. A pre-radical $\mathfrak{r}$ of $\mathcal{A}$ is additive if it is an additive functor.

Let $\mathcal{A}$ be an abelian category with arbitrary coproducts. Given a set $\mathcal{X}$ of objects in $\mathcal{A}$ and $M \in \mathcal{A}$, we recall that the trace of $M$, with respect to $\mathcal{X}$, is $\mathrm{Tr}_{\mathcal{X}}(M) := \sum_{\{f \in \mathrm{Hom}(X,M) \mid X \in \mathcal{X}\}} \mathrm{Im}(f)$. Note that, for any morphism $f : A \to B$ in $\mathcal{A}$, we have that $f(\mathrm{Tr}_{\mathcal{X}}(A)) \subseteq \mathrm{Tr}_{\mathcal{X}}(B)$. Thus, a pre-radical $\tau_{\mathcal{X}}$ of $\mathcal{A}$ can be defined as follows: $\tau_{\mathcal{X}}(Z) := \mathrm{Tr}_{\mathcal{X}}(Z)$ for any $Z \in \mathcal{A}$, and $\tau_{\mathcal{X}}(f) := f|_{\tau_{\mathcal{X}}(A)} : \tau_{\mathcal{X}}(A) \to \tau_{\mathcal{X}}(B)$, for any morphism $f : A \longrightarrow B$ in $\mathcal{A}$. Note that, the pre-radical $\tau_{\mathcal{X}}$ is additive. In case $\mathcal{X}$ has just one element, say $X$, we write $\tau_X$ instead of $\tau_{\mathcal{X}}$.

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LEMMA 4.1. *Let $\mathcal{A}$ be an abelian category with arbitrary coproducts, and let $M = N \oplus N'$ be a decomposition of $M \in \mathcal{A}$. Then, $\tau_N \circ \tau_M = \tau_N$ and thus $\tau_N$ is a subfunctor of $\tau_M$.*

*Proof.* Let $X \in \mathcal{A}$. Then $\mathrm{Tr}_M(X) \subseteq X$ and hence $\mathrm{Tr}_N(\mathrm{Tr}_M(X)) \subseteq \mathrm{Tr}_N(X)$.

Let $g \in \mathrm{Hom}_{\mathcal{A}}(N, X)$. Consider the factorization $N \xrightarrow{g'} \mathrm{Im}\,(g) \to X$ of $g$ through its image. Define the matrix morphism $f := (g\ \ 0) : M \to X$. Note that $\mathrm{Im}(f) = \mathrm{Im}(g) \subseteq \mathrm{Tr}_M(X)$. Let $\mathrm{Im}\,(f) \xrightarrow{j} \mathrm{Tr}_M(X)$ be the natural inclusion. Then, for the composition $N \xrightarrow{g'} \mathrm{Im}\,(f) \xrightarrow{j} \mathrm{Tr}_M(X)$, we have

$$\mathrm{Im}\,(g) = \mathrm{Im}\,(g') = \mathrm{Im}\,(j \circ g') \subseteq \mathrm{Tr}_N(\mathrm{Tr}_M(X)).$$

Therefore, $\mathrm{Tr}_N(\mathrm{Tr}_M(X)) = \mathrm{Tr}_N(X)$, proving the result. □

In what follows, we consider the abelian category $\mathcal{A} := \mathrm{Mod}_\rho(\mathfrak{R})$, where $\mathfrak{R}$ is a $\mathbb{K}$-ringoid. Note that $\mathcal{A}$ has arbitrary coproducts and then $\tau_{\mathcal{X}}$ is well defined, for any set $\mathcal{X}$ of objects in $\mathcal{A}$. We recall that $\mathrm{fin.p}_\rho(\mathfrak{R})$ denotes the category of finitely presented right $\mathfrak{R}$-modules.

DEFINITION 4.2. Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$. For each $i < \alpha$, we consider the additive pre-radicals

$$\tau_i(-) := \mathrm{Tr}_{\bigoplus_{j\le i} \overline{P}(j)}(-) \quad \text{and} \quad \overline{\tau_i}(-) := \mathrm{Tr}_{\bigoplus_{j<i} \overline{P}(j)}(-),$$

where $P^{op} = \{P^{op}(i)\}_{i<\alpha}$ is the family of projective right $\mathfrak{R}$-modules associated with the partition $\sigma(\mathcal{B})$, and $\overline{P}(j) := \bigoplus_{e\in\sigma_j(\mathcal{B})} P^{op}_e(j)$.

Let $M$ be a right $\mathfrak{R}$-module. The $i$th $\mathcal{B}$-trace of $M$ is $\tau_i(M)$ and $\tau_{\mathcal{B},M} := \{\tau_i(M)\}_{i<\alpha}$ is the $\mathcal{B}$-trace filtrationof $M$, which is a chain of submodules of $M$.

LEMMA 4.3. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, and let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$. Then, for any $i < \alpha$, the following statements hold true.*

(a) *$\overline{\tau_i}$ is a subfunctor of $\tau_i$ and $\overline{\tau_i} \circ \tau_i = \overline{\tau_i}$.*
(b) *$\overline{\tau_i} = \sum_{j<i} \tau_j$.*
(c) *$\tau_j \circ \tau_i = \tau_k$ for $k := \min\{i, j\}$.*

*Proof.* (a) follows from Lemma 4.1. To prove (b), let us consider $X \in \mathrm{Mod}_\rho(\mathfrak{R})$. Then, we have the following sequence of equalities

$$\begin{aligned}\sum_{j<i} \tau_j(X) &= \sum_{j<i} \mathrm{Tr}_{\bigoplus_{k\le j} \overline{P}(i)}(M) \\ &= \mathrm{Tr}_{\bigoplus_{j<i}\left(\bigoplus_{k\le j} \overline{P}(i)\right)}(X) \\ &= \overline{\tau_i}(X).\end{aligned}$$

Finally, for the proof of (c), let $M \in \mathrm{Mod}_\rho(\mathfrak{R})$. Note that $\tau_i(M) \subseteq M$ and thus $\tau_j(\tau_i(M)) \subseteq \tau_j(M)$. Let $j < i$. Then, $\tau_j(M) \subseteq \tau_i(M)$ and therefore $\tau_j^2(M) = \tau_j(M) \subseteq \tau_j(\tau_i(M))$. Hence, we conclude that $\tau_j(M) = \tau_j(\tau_i(M))$ for every $j < i$. Similarly for $j \ge i$, we can show that $\tau_j(\tau_i(M)) = \tau_i(M)$. □

LEMMA 4.4. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, $\Delta = {}_{\mathcal{B}}\Delta$ and let $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} E \to 0$ be an exact sequence with $E \in \Delta(i)^{\oplus}$ and $j < i$. Then, for every $f \in \mathrm{Hom}(\overline{P}(j), M)$, we have that $\mathrm{Im}\,(f) \subseteq N$.*

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*Proof.* We assert that $\mathrm{Hom}(\overline{P}(j), E) = 0$. Indeed, let $E = \bigoplus_{e\in J_i} \Delta_e(i)^{d_e}$, where $J_i \subseteq \sigma_i(\mathcal{B})$ is a finite subset. Since

$$\mathrm{Hom}(\overline{P}(j), E) = \prod_{e\in J_i} \mathrm{Hom}(\overline{P}(j), \Delta_e(i))^{d_e},$$

it is enough to see that $\mathrm{Hom}(\overline{P}(j), \Delta_e(i)) = 0$ for every $e \in J_i$.

Let $f : \overline{P}(j) \longrightarrow \Delta_e(i)$ be a morphism. Note that $\overline{P}(j)$ is projective, and thus, there exists a morphism $g : \overline{P}(j) \longrightarrow P_e^{op}(i)$ such that the following diagram commutes

$$\begin{array}{ccccccccc} & & & & & & \overline{P}(j) & & \\ & & & & & \swarrow^{g} & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & U_e(i) & \xrightarrow{\gamma_e(i)} & P_e^{op}(i) & \xrightarrow{\delta_e(i)} & \Delta_e(i) & \longrightarrow & 0, \end{array}$$

where $U_e(i) := \mathrm{Tr}_{\oplus_{j<i}\overline{P}(j)}(P_e^{op}(i))$. Then, there is $g' : \overline{P}(j) \longrightarrow U_e(i)$ such that $g = \gamma_e(i)g'$, and thus, we have that $f = \delta_e(i)g = \delta_e(i)\gamma_e(i)g' = 0$. Proving that $\mathrm{Hom}(\overline{P}(j), \Delta_e(i)) = 0$.

Let $f : \overline{P}(j) \longrightarrow M$ be a morphism. Hence $\beta f \in \mathrm{Hom}(\overline{P}(j), E) = 0$ and therefore $\mathrm{Im}\,(f) \subseteq N$. □

PROPOSITION 4.5. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, $\Delta = {}_{\mathcal{B}}\Delta$, and $0 \neq M \in \mathcal{F}_f(\Delta)$. Consider a strictly well-ordered filtration*

$$\xi : \quad 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{a-1} \subsetneq M_a = M,$$

*where $M_k/M_{k-1} \in \Delta(i_k)^{\oplus}$ and $i_1 < i_2 < \cdots < i_{a-1} < i_a$. Then, for any morphism $f : \overline{P}(j) \longrightarrow M$, with $j \leq i_k$ and $k \in [1, a]$, we have that* $\mathrm{Im}\,(f) \subseteq N$ *where*

$$N = \begin{cases} M_k & \text{if } j = i_k, \\ M_{k-1} & \text{if } j < i_k. \end{cases}$$

*Proof.* Let $f : \overline{P}(j) \longrightarrow M$ with $j \leq i_k$ and $k \in [1, a]$. We consider the following diagram

$$\begin{array}{ccccccccc} & & & & \overline{P}(j) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & M_{a-1} & \xrightarrow{u_a} & M_a & \xrightarrow{\pi_a} & X_a & \longrightarrow & 0, \end{array}$$

where $X_a \in \Delta(i_a)^{\oplus}$. Since $j \leq i_k < i_a$, by Lemma 4.4, there exists a morphism $v_a : \overline{P}(j) \longrightarrow M_{a-1}$ such that $f = u_a v_a$.

Now, consider the diagram

$$\begin{array}{ccccccccc} & & & & \overline{P}(j) & & & & \\ & & & & \downarrow{\scriptstyle v_a} & & & & \\ 0 & \longrightarrow & M_{a-2} & \xrightarrow{u_{a-1}} & M_{a-1} & \xrightarrow{\pi_{a-1}} & X_{a-1} & \longrightarrow & 0, \end{array}$$

where $X_{a-1} \in \Delta(i_{a-1})^{\oplus}$. Since $j \le i_k < i_{a-1} < i_a$, by Lemma 4.4 there exists a morphism $v_{a-1} : \overline{P}(j) \longrightarrow M_{a-2}$ such that $v_a = u_{a-1}v_{a-1}$. By iterating the same argument, we get the following diagram

$$\begin{array}{ccccccccc} & & & & \overline{P}(j) & & & & \\ & & & & \downarrow{\scriptstyle v_{k+2}} & & & & \\ 0 & \longrightarrow & M_k & \xrightarrow{u_{k+1}} & M_{k+1} & \xrightarrow{\pi_{k+1}} & X_{k+1} & \longrightarrow & 0, \end{array}$$

where $X_{k+1} \in \Delta(i_{k+1})^{\oplus}$. Since $j \le i_k < i_{k+1}$, by Lemma 4.4, there exists $v_{k+1} : \overline{P}(j) \longrightarrow M_k$ such that $v_{k+2} = u_{k+1}v_{k+1}$. Then, by taking $\overline{f} := v_{k+1}$, we have that $f = u_a u_{a-1} \ldots u_{k+1}\overline{f}$. Therefore, $\mathrm{Im}(f) \subseteq M_k$.

Now if $j < i_k$, we consider the diagram

$$\begin{array}{ccccccccc} & & & & \overline{P}(j) & & & & \\ & & & & \downarrow{\scriptstyle v_{k+1}} & & & & \\ 0 & \longrightarrow & M_{k-1} & \xrightarrow{u_k} & M_k & \xrightarrow{\pi_k} & X_k & \longrightarrow & 0, \end{array}$$

where $X_k \in \Delta(i_k)^{\oplus}$. Since $j < i_k$, there exists $v_k : \overline{P}(j) \longrightarrow M_{k-1}$ such that $v_{k+1} = u_k v_k$. Then, by taking $\overline{f} := v_k$ we have that $f = u_a u_{a-1} \ldots u_k \overline{f}$. Therefore $\mathrm{Im}(f) \subseteq M_{k-1}$. □

Definition 4.6. Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, and let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$. For any $M \in \mathrm{Mod}_{\rho}(\mathfrak{R})$, the $i$th $\tau$-section of $M$ is the quotient $\tau_i/\overline{\tau}_i(M)$. The support of the $\mathcal{B}$-trace filtration of $M$ is the set

$$\mathrm{Supp}(\tau_{\mathcal{B},M}) := \{i < \alpha \ : \ \tau_i/\overline{\tau}_i(M) \neq 0\}.$$

Theorem 4.7. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, $\Delta = {}_{\mathcal{B}}\Delta$, and $M \in \mathrm{Mod}_{\rho}(\mathfrak{R})$. Then, the following statements are equivalent.*

(a) *$M$ has a finite $\Delta$-filtration.*
(b) *There exist some $i_0 < \alpha$ such that $\tau_j(M) = M$ for any $j \ge i_0$, $\mathrm{Supp}(\tau_{\mathcal{B},M})$ is finite and $\tau_i/\overline{\tau}_i(M) \in \Delta(i)^{\oplus}$, for any $i < \alpha$.*

*Proof.* (a) ⇒ (b) Let $0 \neq M \in \mathcal{F}_f(\Delta)$. Consider a strictly well-ordered filtration

$$\xi : \quad 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{a-1} \subsetneq M_a = M,$$

where $M_k/M_{k-1} \in \Delta(i_k)^{\oplus}$ and $i_1 < i_2 < \cdots < i_{a-1} < i_a$. We have the following filtration $\Omega$, which is composed of the following pieces

$$\begin{array}{rll} \Omega_0 : & 0 = N_0 = N_1 = \cdots = N_{i'_1} = M_0 & \forall\, i'_1 \in [0, i_1), \\ \Omega_1 : & \subsetneq N_{i_1} = N_{i_1+1} = \cdots = N_{i'_2} = M_1 & \forall\, i'_2 \in [i_1, i_2), \\ \Omega_2 : & \subsetneq N_{i_2} = N_{i_2+1} = \cdots = N_{i'_3} = M_2 & \forall\, i'_3 \in [i_2, i_3), \\ & \cdots\cdots\cdots\cdots & \\ \Omega_{a-2} : & \subsetneq N_{i_{a-2}} = N_{i_{a-2}+1} = \cdots = N_{i'_{a-1}} = M_{a-2} & \forall\, i'_{a-1} \in [i_{a-2}, i_{a-1}), \\ \Omega_{a-1} : & \subsetneq N_{i_{a-1}} = N_{i_{a-1}+1} = \cdots = N_{i'_a} = M_{a-1} & \forall\, i'_a \in [i_{a-1}, i_a), \\ \Omega_a : & \subsetneq N_{i_a} := M_a. & \end{array}$$

In order to prove the result, it is enough to see that $\Omega = \tau_{\mathcal{B},M}$ and $\tau_j(M) = M$ for any $j \geq i_a$.

Note that, for $j > i_a$, we have $\tau_j(M) = \tau_{i_a}(M) + \mathrm{Tr}_{\bigoplus_{i_a<k\leq j}\overline{P}(k)}(M)$. Thus, we only need to check that $\tau_{i_a}(M) = M$ and $N_i = \tau_i(M)$, for all $0 \leq i \leq i_a$. To perform that, we will follow a series of steps as follows.

(i) $\tau_{i'_1}(M) = M_0 = 0 \quad \forall\, i'_1 \in [0, i_1)$.
Indeed, since $\tau_0(M) \subseteq \tau_{i'_1}(M)$, it is enough to see that $\tau_{i'_1}(M) = 0$. Let $f : \overline{P}(j) \longrightarrow M$ with $j \leq i'_1 < i_1$. By Proposition 4.5, it follows that $\mathrm{Im}\,(f) \subseteq M_0 = 0$, proving that $\tau_{i'_1}(M) = \mathrm{Tr}_{\bigoplus_{j\leq i'_1}\overline{P}(j)}(M) = 0$.

(ii) $\tau_{i'_2}(M) = M_1 \in \Delta(i_1)^{\oplus} \quad \forall\, i'_2 \in [i_1, i_2)$.
First, we assert that $\tau_{i_1}(M) = M_1$. Indeed, note that $M_1 \in \Delta(i_1)^{\oplus}$, since $M_1/M_0 \in \Delta(i_1)^{\oplus}$. Thus, $M_1 = \bigoplus_{e\in J_{i_1}} \Delta_e(i_1)^{\mu_{e,1}}$ for some finite subset $J_{i_1} \subseteq \sigma_{i_1}(\mathcal{B})$. Observe now, that there exists an epimorphism

$$\bigoplus_{e\in J_{i_1}} P_e^{op}(i_1)^{\mu_{e,1}} \longrightarrow \bigoplus_{e\in J_{i_1}} \Delta_e(i_1)^{\mu_{e,1}} = M_1 \subseteq M,$$

and therefore $M_1 \subseteq \mathrm{Tr}_{\bigoplus_{j\leq i_1}\overline{P}(j)}(M) = \tau_{i_1}(M)$. On the other hand,

$$\tau_{i_1}(M) = \mathrm{Tr}_{\overline{P}(i_1)}(M) + \mathrm{Tr}_{\bigoplus_{j<i_1}\overline{P}(j)}(M) = \mathrm{Tr}_{\overline{P}(i_1)}(M)$$

since, by Proposition 4.5, we know that $\mathrm{Tr}_{\{\overline{P}(j)|j<i_1\}}(M) = 0$. Let $f \in \mathrm{Hom}(\overline{P}(i_1), M)$. Then, by Proposition 4.5, we have that $\mathrm{Im}\,(f) \subseteq M_1$ and so $\tau_{i_1}(M) \subseteq M_1$; proving that $\tau_{i_1}(M) = M_1$.

At this point, we have $M_1 = \tau_{i_1}(M) \subseteq \tau_{i'_2}(M)$. To finish the proof of (ii), we only have to see that $\tau_{i'_2}(M) \subseteq \tau_{i_1}(M)$. Let $j \leq i'_2 < i_2$ and $f \in \mathrm{Hom}(\overline{P}(j), M)$. Then, by Proposition 4.5, it follows that $\mathrm{Im}\,(f) \subseteq M_1 = \tau_{i_1}(M)$ and thus $\tau_{i'_2}(M) \subseteq \tau_{i_1}(M)$.

(iii) $\tau_{i'_3}(M) = M_2 \quad \forall\, i'_3 \in [i_2, i_3)$.
First, we assert that $\tau_{i_2}(M) = M_2$. Indeed, consider the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow X_2 \longrightarrow 0,$$

where $X_2 \in \Delta(i_2)^{\oplus}$. By (ii), we know that $M_1 = \mathrm{Tr}_{Q_1}(M)$, where $Q_1 := \bigoplus_{j\leq i_1}\overline{P}(j)$. There exists an epimorphism $f : Q_1^{(I_1)} \to M_1$, for the set $I_1 := \mathrm{Hom}(Q_1, M_1)$. On the other hand, since $X_2 \in \Delta(i_2)$, there exists an epimorphism $h : \overline{P}(i_2)^{m_2} \longrightarrow X_2$. Then, we have the following exact and commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_1^{(I_1)} & \longrightarrow & Q_1^{(I_1)} \bigoplus \overline{P}(i_2)^{m_2} & \longrightarrow & \overline{P}(i_2)^{m_2} & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & X_2 & \longrightarrow & 0, \end{array}$$

where $g$ is an epimorphism. Therefore,

$$M_2 \subseteq \mathrm{Tr}_{\bigoplus_{j\leq i_2}\overline{P}(j)}(M) = \tau_{i_2}(M).$$

Now, let $f : \overline{P}(j) \longrightarrow M$ with $j \leq i_2$. Then, by Proposition 4.5, we get that $\mathrm{Im}\,(f) \subseteq N$, where $N = M_1$ or $N = M_2$. In any case, we conclude that $\mathrm{Im}\,(f) \subseteq M_2$, since $M_1 \subseteq M_2$. Hence, $\tau_{i_2}(M) \subseteq M_2$ and so $\tau_{i_2}(M) = M_2$. Now, by following the same arguments as we did in (ii), we can show that $\tau_{i'_3}(M) = \tau_{i_2}(M)$; proving (iii).

Note that the above procedure can be repeated in order to get that $N_i = \tau_i(M)$, for all $0 \le i \le i_a$. Finally, by following the process we did in (iii), we obtain an epimorphism $Q_{a-1}^{(I_{a-1})} \bigoplus \overline{P}(i_a)^{m_a} \to M$ and thus $\tau_{i_a}(M) = M$.

(b) $\Rightarrow$ (a) Assume the hypothesis of (b). If $\mathrm{Supp}(\tau_{\mathcal{B},M}) = \emptyset$, by using transfinite induction, it can be shown that $M = 0$ and thus $M \in \mathcal{F}_f(\Delta)$.

Let $\mathrm{Supp}(\tau_{\mathcal{B},M}) = \{i_1 < i_2 < \cdots < i_a\}$. Consider $M_0 := \tau_0(M)$ and $M_k := \tau_{i_k}(M)$ for $k \in [1, a]$. Note that, for any $i \notin \mathrm{Supp}(\tau_{\mathcal{B},M})$, Lemma 4.3 implies that $\tau_i(M) = \sum_{j<i} \tau_j(M)$.

In the proofs of the following assertions, we use transfinite induction.

(0) $\tau_{i'_1}(M) = M_0 = 0$ for any $i'_1 \in [0, i_1)$.

Indeed, let $S_{i_1} = \{i'_1 \in [0, i_1) : \tau_{i'_1}(M) = 0\}$. Note that $0 \in S_{i_1}$ since $\tau_0(M) = \sum_{j<0} \tau_j(M) = 0$. Let $\beta + 1 \in [0, i_1)$ and $\beta \in S_{i_1}$. Since $j < \beta + 1$ implies that $j \le \beta$, it follows that $\tau_{\beta+1}(M) = \sum_{j<\beta+1} \tau_j(M) \subseteq \tau_\beta(M) = 0$, and thus $\beta + 1 \in S_{i_1}$.

Let $\gamma \in [0, i_1)$ be a limit ordinal and let $\delta \in S_{i_1}$ for any $\delta < \gamma$. Then, $\tau_\gamma(M) = \sum_{\delta<\gamma} \tau_j(M) = 0$. Thus, by transfinite induction, we get that (0) holds.

(1) $\tau_{i'_2}(M) = M_1$ for any $i'_2 \in [i_1, i_2)$.

Indeed, let $S_{i_2} = \{i'_2 \in [i_1, i_2) : \tau_{i'_2}(M) = M_1\}$. It is clear that $i_1 \in S_{i_2}$. Let $i_1 < \beta + 1 < i_2$ and $\tau_\beta(M) = M_1$. Then, $M_1 \subseteq \tau_{\beta+1}(M) = \sum_{j<\beta+1} \tau_j(M) \subseteq \tau_\beta(M) = M_1$ and hence $\beta + 1 \in S_{i_2}$.

Let $\gamma \in [i_1, i_2)$ be a limit ordinal and let $\delta \in S_{i_2}$ for any $\delta \in [i_1, \gamma)$. Then, by using (0), we can get the following equalities

$$\begin{aligned} \tau_\gamma(M) &= \sum_{j<\gamma} \tau_j(M) \\ &= \sum_{j<i_1} \tau_j(M) + \sum_{i_1 \le \delta < \gamma} \tau_\delta(M) \\ &= M_1. \end{aligned}$$

Thus, by transfinite induction, we get that (1) holds.

Note that the above procedure in (0) and (1) can be repeated to obtain that $\tau_{i'_k}(M) = M_k$ for any $i'_k \in [i_{k-1}, i_k)$, and $\tau_j(M) = M$ for $j \ge i_a$. Thus, we have a finite chain of submodules $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_a = M$ such that $M_t/M_{t-1} = \tau_{i_t}/\overline{\tau}_{i_t}(M) \in \Delta(i_t)$. Therefore, $M \in \mathcal{F}_f(\Delta)$. □

REMARK 4.8. Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, $\Delta = {}_{\mathcal{B}}\Delta$, and $M \in \mathrm{Mod}_\rho(\mathfrak{R})$ be such that $\mathrm{Supp}(\tau_{\mathcal{B},M}) = \{i_1 < i_2 < \cdots < i_a\}$, for some finite ordinal $a$. In the proof of Theorem 4.7, we have shown the following:

(a) $\tau_j(M) = 0$ for all $j \in [0, i_1)$;
(b) $\tau_j(M) = M_k := \tau_{i_k}(M)$ for all $j \in [i_k, i_{k+1})$ and $k \in [1, a)$;
(c) the finite chain of submodules $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_a = M$ satisfies that $M_t/M_{t-1} = \tau_{i_t}/\overline{\tau}_{i_t}(M)$.

THEOREM 4.9. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$ and $\Delta = {}_{\mathcal{B}}\Delta$. If $\Delta \subseteq \mathrm{fin.p}_\rho(\mathfrak{R})$, then all the objects $\Delta_e(i)$ are local and the following statements hold true.*

(a) *For any $M \in \mathcal{F}_f(\Delta)$, the filtration multiplicity $[M : \Delta_e(i)]$ does not depend on a given $\Delta$-filtration of $M$.*
(b) *$\mathcal{F}_f(\Delta) \subseteq \mathrm{fin.p}_\rho(\mathfrak{R})$ and it is a locally finite $\mathbb{K}$-ringoid.*

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*Proof.* Let $\Delta \subseteq \text{fin.p}_\rho(\mathfrak{R})$. By Proposition 2.4, we have that $\text{fin.p}_\rho(\mathfrak{R})$ is a locally finite $\mathbb{K}$-ringoid. Moreover, by [7, Proposition 4.2 (d)], we have that $\Delta^\oplus \subseteq \text{fin.p}_\rho(\mathfrak{R})$ and thus all $\Delta_e(i)$ are local objects (see Lemma 3.11 (a)).

(a) Let $0 \neq M \in \mathcal{F}_f(\Delta)$. Since $\Delta^\oplus \subseteq \text{fin.p}_\rho(\mathfrak{R})$ and all the objects $\Delta_e(i)$ are local, the proof of Theorem 4.7 implies that

$$[M:\Delta_e(i)]_\xi = [M:\Delta_e(i)]_{\xi'} = [M:\Delta_e(i)]_{\tau_{\mathcal{B},M}},$$

where $\xi'$ is the strictly well-ordered filtration of $M$, constructed by the proof of Propositions 3.10 and 3.17. Note that $[M:\Delta_e(i)]_{\tau_{\mathcal{B},M}}$ does not depend on any given filtration. Therefore, the proof of (a) is complete.

(b) Let $M \in \mathcal{F}_f(\Delta)$. Since $\mathcal{F}_f(\Delta)$ is closed under extensions and $\Delta^\oplus \subseteq \text{fin.p}_\rho(\mathfrak{R})$, by induction on the $\xi$-ladder length $\ell'_{\Delta,\xi}(M)$, we can show that $M \in \text{fin.p}_\rho(\mathfrak{R})$. Therefore, $\mathcal{F}_f(\Delta) \subseteq \text{fin.p}_\rho(\mathfrak{R})$.

Assume now that $M = L \bigoplus N$ in $\text{Mod}_\rho(\mathfrak{R})$. Since $M \in \text{fin.p}_\rho(\mathfrak{R})$, it follows that $M = L \bigoplus N$ in $\text{fin.p}_\rho(\mathfrak{R})$. Consider the split exact sequence

$$\xi: \quad 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

given by the decomposition $M = L \bigoplus N$. Thus, we have the following split exact sequence

$$\varepsilon_i(\xi): \quad 0 \longrightarrow \varepsilon_i(L) \xrightarrow{\varepsilon_i(f)} \varepsilon_i(M) \xrightarrow{\varepsilon_i(g)} \varepsilon_i(N) \longrightarrow 0,$$

for $\varepsilon_i = \tau_i$ or $\varepsilon_i = \overline{\tau_i}$. Note that

$$(*) \quad \tau_i/\overline{\tau_i}(M) = \frac{\tau_i(L) \oplus \tau_i(N)}{\overline{\tau}_i(L) \oplus \overline{\tau_i}(N)} = \frac{\tau_i(L)}{\overline{\tau}_i(L)} \bigoplus \frac{\tau_i(N)}{\overline{\tau}_i(N)}.$$

Since $M \in \mathcal{F}_f(\Delta)$, by Theorem 4.7, there exists $i_0 \in \mathbb{N}$ such that $\tau_j(M) = M$ for any $j \geq i_0$. Moreover, by $(*)$ we have $\tau_i/\overline{\tau_i}(M) \in \Delta(i)^\oplus$ for any $i < \alpha$.

From the split-exact sequences of the form $\varepsilon_i(\xi)$, for $\varepsilon_i = \tau_i$ or $\varepsilon_i = \overline{\tau_i}$, we get the following commutative and exact diagram, where all the rows are split exact sequences

$$(**) \quad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \tau_i/\overline{\tau}_i(L) & \longrightarrow & \tau_i/\overline{\tau}_i(M) & \longrightarrow & \tau_i/\overline{\tau}_i(N) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \tau_i(L) & \xrightarrow{\tau_i(f)} & \tau_i(M) & \xrightarrow{\tau_i(g)} & \tau_i(N) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \overline{\tau}_i(L) & \xrightarrow{\overline{\tau}_i(f)} & \overline{\tau}_i(M) & \xrightarrow{\overline{\tau}_i(g)} & \overline{\tau}_i(N) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

If $\tau_i/\overline{\tau}_i(M) = 0$, we conclude that $\tau_i/\overline{\tau}_i(L) = 0 = \tau_i/\overline{\tau}_i(N)$. In particular, $\text{Supp}(\tau_{\mathcal{B},L}) \cup \text{Supp}(\tau_{\mathcal{B},N}) \subseteq \text{Supp}(\tau_{\mathcal{B},M})$.

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Let $\tau_i/\overline{\tau}_i(M) \neq 0$. Since $\tau_i/\overline{\tau}_i(M) \in \Delta(i)^{\oplus}$, it follows that

$$\tau_i/\overline{\tau}_i(L) \bigoplus \tau_i/\overline{\tau}_i(N) \in \Delta(i)^{\oplus}.$$

Since $\text{fin.p}_{\rho}(\mathfrak{R})$ is a Krull–Schmidt category, we get that $\tau_i/\overline{\tau}_i(L) \in \Delta(i)^{\oplus}$ and $\tau_i/\overline{\tau}_i(N) \in \Delta(i)^{\oplus}$. Furthermore, from $(**)$, Theorem 4.7 and the fact that $\tau_j(M) = M$ for every $j \geq i_0$, we get that $M = \tau_{i_0}(L) \oplus \tau_{i_0}(N)$ and $\tau_{i_0}(L), \tau_{i_0}(N) \in \mathcal{F}_f(\Delta)$. But $M = L \oplus N$ and thus $0 = (L/\tau_{i_0}(L)) \oplus (N/\tau_{i_0}(N))$. Therefore, $L = \tau_{i_0}(L)$ and $N = \tau_{i_0}(N)$, proving that $\mathcal{F}_f(\Delta)$ is closed under direct summands. □

COROLLARY 4.10. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, which is right support finite, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$ and $\Delta = {}_{\mathcal{B}}\Delta$. Then all the objects $\Delta_e(i)$ are local and the following statements hold true.*

(a) *For any $M \in \mathcal{F}_f(\Delta)$, the filtration multiplicity $[M : \Delta_e(i)]$ does not depend on a given $\Delta$-filtration of $M$.*
(b) *$\mathcal{F}_f(\Delta) \subseteq \text{fin.p}_{\rho}(\mathfrak{R})$ and it is a locally finite $\mathbb{K}$-ringoid.*

*Proof.* It follows from Proposition 3.8 and Theorem 4.9. □

We recall that a class $\mathcal{X}$ of objects, in an abelian category $\mathcal{A}$, is pre-resolving if it is closed under extensions and for any exact sequence $0 \to A \to B \to C \to 0$, with $B, C \in \mathcal{X}$, it follows that $A \in \mathcal{X}$. We prove that $\mathcal{F}_f(\Delta)$ is a pre-resolving class, and in order to do that, we start with the following lemma.

LEMMA 4.11. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, and $\Delta = {}_{\mathcal{B}}\Delta \subseteq \text{fin.p}_{\rho}(\mathfrak{R})$. Then, the following statements hold true.*

(a) *Let $u : L \longrightarrow M$ be a monomorphism with $M \in \Delta(i)^{\oplus}$. For $e \in \sigma_i(\mathcal{B})$, we have that $\text{Hom}(U_e(i), L) = 0$, where $U_e(i) := \text{Tr}_{\oplus_{j<i}\overline{P}(j)}(P_e^{op}(i))$.*

(b) *Let $\xi : \ 0 \longrightarrow L \xrightarrow{u} M \xrightarrow{\pi} N \longrightarrow 0$ be an exact sequence with $M, N \in \Delta(i)^{\oplus}$. Then, $L \in \Delta(i)^{\oplus}$.*

*Proof.* (a) First, we show that $\text{Hom}(P_{e'}^{op}(j), L) = 0$ for $j < i$ and $e' \in \sigma_j(\mathcal{B})$. Indeed, we have that $M = \bigoplus_{e \in J_i} \Delta_e(i)^{\mu_e}$ with $J_i \subseteq \sigma_i(\mathcal{B})$ a finite subset. Then,

$$\text{Hom}(P_{e'}^{op}(j), M) \simeq \bigoplus_{e \in J_i} \text{Hom}(P_{e'}^{op}(j), \Delta_e(i))^{\mu_e} = 0$$

since $\text{Hom}(P_{e'}^{op}(j), \Delta_e(i)) = 0$ for $j < i$ and for every $e' \in \sigma_j(\mathcal{B})$. Now, let $\alpha : P_{e'}^{op}(j) \longrightarrow L$ be a morphism. Then, $u\alpha \in \text{Hom}(P_{e'}^{op}(j), M) = 0$. Since $u$ is a monomorphism, we have that $\alpha = 0$. This proves that $\text{Hom}(P_{e'}^{op}(j), L) = 0$ for $j < i$ and $e' \in \sigma_j(\mathcal{B})$.

Therefore, for $j < i$, it follows that

$$\text{Hom}(\overline{P}(j), L) \simeq \prod_{e' \in \sigma_j(\mathcal{B})} \text{Hom}(P_{e'}^{op}(j), L) = 0$$

since $\overline{P}(j) := \bigoplus_{e' \in \sigma_j(\mathcal{B})} P_{e'}^{op}(j)$. Consider $X := \text{Hom}\Big(\bigoplus_{j<i} \overline{P}(j), U_e(i)\Big)$. From the equality $U_e(i) = \text{Tr}_{\oplus_{j<i}\overline{P}(j)}(P_e(i))$, there exists an epimorphism

$$\lambda : \left(\bigoplus_{j<i} \overline{P}(j)\right)^{(X)} \longrightarrow U_e(i).$$

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Let $\gamma \in \operatorname{Hom}(U_e(i), L)$. Then, $\gamma\lambda \in \operatorname{Hom}\left(\left(\bigoplus_{j<i}\overline{P}(j)\right)^{(X)}, L\right)$. But

$$\operatorname{Hom}\left(\left(\bigoplus_{j<i}\overline{P}(j)\right)^{(X)}, L\right) \simeq \prod_{j<i}\prod_{x\in X}\operatorname{Hom}(\overline{P}(j), L) = 0.$$

Hence, $\gamma\lambda = 0$ and thus $\gamma = 0$, since $\lambda$ is an epimorphism, proving that $\operatorname{Hom}(U_e(i), L) = 0$.

(b) Since $N \in \Delta(i)^{\oplus}$, we have that $N = \bigoplus_{e\in K_i}\Delta_e(i)^{\nu_e}$ with $K_i \subseteq \sigma_i(\mathcal{B})$ a finite subset. For each $e \in K_i$, there is an exact sequence

$$0 \longrightarrow U_e(i) \longrightarrow P_e(i) \longrightarrow \Delta_e(i) \longrightarrow 0.$$

By applying $\operatorname{Hom}(-, L)$ to the above sequence, we obtain the exact sequence

$$\operatorname{Hom}(U_e(i), L) \longrightarrow \operatorname{Ext}^1(\Delta_e(i), L) \longrightarrow \operatorname{Ext}^1(P_e(i), L).$$

Since $\operatorname{Hom}(U_e(i), L) = 0$ by (a), and $\operatorname{Ext}^1(P_e(i), L) = 0$, it follows that $\operatorname{Ext}^1(\Delta_e(i), L) = 0$ for each $e \in K_i$. Then,

$$\operatorname{Ext}^1(N, L) = \prod_{e\in K_i}\operatorname{Ext}^1(\Delta_e(i), L)^{\nu_e} = 0.$$

We conclude that $\xi$ splits and thus $L \oplus N = M \in \Delta(i)^{\oplus}$. Finally, from the fact that $\operatorname{fin.p}_{\rho}(\mathfrak{R})$ is a Krull–Schmidt category, we get that $L \in \Delta(i)^{\oplus}$.

□

PROPOSITION 4.12. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an admissible family of subcategories of $\mathfrak{R}$, and let $\Delta = {}_{\mathcal{B}}\Delta \subseteq \operatorname{fin.p}_{\rho}(\mathfrak{R})$. Then, $\mathcal{F}_f(\Delta)$ is a pre-resolving class.*

*Proof.* By Remark 2.5, we know that $\mathcal{F}_f(\Delta)$ is closed under extensions. It remains to show that $\mathcal{F}_f(\Delta)$ is closed under kernels of epimorphisms between its objects.

Let $\xi: \ 0 \longrightarrow L \xrightarrow{u} M \xrightarrow{\pi} N \longrightarrow 0$ be an exact sequence with $M, N \in \mathcal{F}_f(\Delta)$. Let $\{\tau_i(M)\}_{i<\alpha}$ and $\{\tau_i(N)\}_{i<\alpha}$ be the $\mathcal{B}$-trace filtrations of $M$ and $N$, respectively. By Theorem 4.7, we have that $\operatorname{Supp}(\tau_{\mathcal{B},M})$ is finite, that is, $\operatorname{Supp}(\tau_{\mathcal{B},M}) = \{i_1 < i_2 < \cdots < i_a\}$. Then, $\tau_j(M) = M$ for every $j \geq i_a$. Since $\pi$ is an epimorphism, we have that $\tau_j(N) = N$ for every $j \geq i_a$. Moreover, by using that $\pi$ is an epimorphism and the fact that $\bigoplus_{j\leq i}\overline{P}(j)$ and $\bigoplus_{j<i}\overline{P}(j)$ are projectives, we conclude that $\pi(\epsilon_i(M)) = \epsilon_i(N)$ for every $i < \alpha$, where $\epsilon_i = \tau_i$ or $\epsilon_i = \overline{\tau}_i$ (that is, $\epsilon_i(\pi) := \pi|_{\epsilon_i(M)} : \epsilon_i(M) \longrightarrow \epsilon_i(N)$ is an epimorphism). Let $\overline{L}_i := \operatorname{Ker}(\overline{\tau}_i(\pi))$ and $L_i := \operatorname{Ker}(\tau_i(\pi))$. Then, for each $i < \alpha$, we obtain the following commutative and exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{L}_i & \longrightarrow & \overline{\tau}_i(M) & \xrightarrow{\overline{\tau}_i(\pi)} & \overline{\tau}_i(N) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle w_i} & & \\
0 & \longrightarrow & L_i & \longrightarrow & \tau_i(M) & \xrightarrow{\tau_i(\pi)} & \tau_i(N) & \longrightarrow & 0,
\end{array}$$

where $u_i$, $v_i$, and $w_i$ are monomorphisms. By the Snake's Lemma, there exists the following exact sequence

$$0 \longrightarrow \frac{L_i}{\overline{L}_i} \longrightarrow \frac{\tau_i(M)}{\overline{\tau}_i(M)} \longrightarrow \frac{\tau_i(N)}{\overline{\tau}_i(N)} \longrightarrow 0.$$

Since $M, N \in \mathcal{F}_f(\Delta)$, by Theorem 4.7, we obtain that $\frac{\tau_i(M)}{\overline{\tau}_i(M)}, \frac{\tau_i(N)}{\overline{\tau}_i(N)} \in \Delta(i)^{\oplus}$. By Lemma 4.11, it follows that $\frac{L_i}{\overline{L}_i} \in \Delta(i)^{\oplus}$.

Recall that $\mathrm{Supp}(\tau_{\mathcal{B},M}) = \{i_1 < i_2 < \cdots < i_a\}$. Hence, by Remark 4.8, the following statements hold true

(a) $\tau_j(M) = 0$ for every $j \in [0, i_1)$,
(b) $\tau_j(M) = M_k := \tau_{i_k}(M) = \overline{\tau}_{i_{k+1}}(M)$ for every $j \in [i_k, i_{k+1})$ and $k \in [1, a-1)$,
(c) the finite chain of submodules $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_a = M$ satisfies that $M_t/M_{t-1} = \tau_{i_t}/\tau_{i_{t-1}}(M) = \tau_{i_t}/\overline{\tau_{i_t}}(M)$.

For $i = i_a$, we have that $\tau_i(M) = M$ and hence $\tau_i(\pi) = \pi$. Therefore, $L_{i_a} = L$. We set $L_k := L_{i_k}$ for $k \in [1, a]$. Hence, we have the following filtration

$$0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_{a-1} \subseteq L_a = L.$$

By the item (b) $\tau_{i_k}(M) = \overline{\tau}_{i_{k+1}}(M)$ for $k \in [1, a-1)$, and so $\tau_{i_k}(\pi) = \overline{\tau}_{i_{k+1}}(\pi)$. Therefore, we conclude that $L_{i_k} = \overline{L}_{i_{k+1}}$ for every $k \in [1, a-1)$. Then, $\frac{L_k}{L_{k-1}} = \frac{L_{i_k}}{L_{i_{k-1}}} = \frac{L_{i_k}}{\overline{L}_{i_k}} \in \Delta(i_k)^{\oplus}$ for $k \in [1, a]$. This give us a finite filtration of $L$ proving that $L \in \mathcal{F}_f(\Delta)$. □

**5. Stratifying ideals in ringoids.** In this section, we introduce and study the notion of ideally standardly stratified ringoid. We prove that standardly stratified ringoids and ideally standardly stratified ringoids are only equivalent notions under a specific condition. It is also shown that certain equivalent characterizations of standardly stratified algebras and quasi-hereditary algebras are not necessarily equivalent any more in the realm of ringoids.

DEFINITION 5.1. Let $\mathfrak{R}$ be a ringoid. An ideal $I \trianglelefteq \mathfrak{R}$ is right stratifying if $I^2 = I$ and $I(-, a) \in \mathrm{proj}_\rho(\mathfrak{R})$ for any $a \in \mathfrak{R}$. We say that $I$ is right hereditary if it is right stratifying and $I\mathrm{rad}_{\mathfrak{R}}(-, ?)I = 0$. A right stratifying (respectively, hereditary) chain in $\mathfrak{R}$ is a chain $\{I_i\}_{i<\alpha}$ of ideals of $\mathfrak{R}$ such that $\sum_{i<\alpha} I_i = \mathfrak{R}$ and $I_i/I_i'$ is right stratifying (respectively, hereditary) in $\mathfrak{R}/I_i'$, where $I_i' := \sum_{j<i} I_j$.

LEMMA 5.2. *Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid, and let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an exhaustive family of subcategories of $\mathfrak{R}$. Then, the following statements hold true.*

(a) $\sum_{j<i} I_{\mathcal{B}_j} = I_{\bigcup_{j<i} \mathcal{B}_j}$, *where* $I_\emptyset(a, b) := \{0\}$ *for* $a, b \in \mathfrak{R}$.
(b) $\sum_{j<\alpha} I_{\mathcal{B}_j} = \mathfrak{R}$.

*Proof.* (a) Let $f \in I_{\bigcup_{j<i} \mathcal{B}_j}(x, y)$. Then, $f$ factorizes through some $b \in \bigcup_{j<i} \mathcal{B}_j$. Therefore, $f \in I_{\mathcal{B}_j}(x, y)$ for some $j < i$, and thus $f \in \sum_{j<i} I_{\mathcal{B}_j}(x, y)$, proving that $I_{\bigcup_{j<i} \mathcal{B}_j} \subseteq \sum_{j<i} I_{\mathcal{B}_j}$.

Let $f \in \sum_{j<i} I_{\mathcal{B}_j}(x, y)$. Then, $f = \sum_{k=1}^n f_k$ for some $f_k \in I_{\mathcal{B}_{j_k}}(x, y)$ with $j_k < i$. In particular, each $f_k$ is the composition of morphisms $x \xrightarrow{t_k} b_{j_k} \xrightarrow{h_k} y$, where $b_{j_k} \in \mathcal{B}_{j_k}$. Let $b := \bigoplus_{k=1}^n b_{j_k}$. Then, we have the matrix morphisms $x \xrightarrow{t} b \xrightarrow{h} x$ such that

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$f = ht$. Since $b \in \mathcal{B}_j$ for $j := \max\{j_1, j_2, \ldots, j_n\} < i$, it follows that $f \in I_{\bigcup_{j<i} \mathcal{B}_j}(x, y)$, proving that $\sum_{j<i} I_{\mathcal{B}_j} \subseteq I_{\bigcup_{j<i} \mathcal{B}_j}$.

(b) It follows from (a), since $\bigcup_{j<\alpha} \mathcal{B}_j = \mathfrak{R}$. □

DEFINITION 5.3. Let $\mathfrak{R}$ be a Krull–Schmidt $\mathbb{K}$-ringoid. We say that $\mathfrak{R}$ is a right ideally standardly stratified (respectively, quasi-hereditary) $\mathbb{K}$-ringoid, with respect to an exhaustive family $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ of subcategories of $\mathfrak{R}$, if the associated chain $\{I_{\mathcal{B}_i}\}_{i<\alpha}$ of ideals of $\mathfrak{R}$ is right stratifying (respectively, hereditary).

REMARK 5.4. A right ideally quasi-hereditary $\mathbb{K}$-ringoid $\mathfrak{R}$, with respect to an exhaustive family of subcategories $\mathcal{B} = \{B_i\}_{i<\alpha}$ of $\mathfrak{R}$ such that $\alpha \leq \aleph_0$, is called quasi-hereditary category in [43].

LEMMA 5.5. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid, and let $\mathcal{B} = \{\mathcal{B}_i\}_{i<\alpha}$ be an exhaustive family of subcategories of $\mathfrak{R}$ such that*

$$(I_{\mathcal{B}_j}/I'_{\mathcal{B}_j})\ \mathrm{rad}_{\mathfrak{R}/I'_{\mathcal{B}_j}}(-, ?)\ (I_{\mathcal{B}_j}/I'_{\mathcal{B}_j}) = 0,$$

*for any $j < \alpha$. Then, the following statements hold true.*

(a) $\mathrm{rad}_{\mathfrak{R}}(e, e') = I'_{\mathcal{B}_i}(e, e')$ *for any $e, e' \in \sigma_i(\mathcal{B})$ and $i < \alpha$.*
(b) $\mathrm{Hom}(\Delta_e(i), \Delta_{e'}(i)) = 0$ *for any $e \neq e'$ in $\sigma_i(\mathcal{B})$ and $i < \alpha$.*
(c) $\mathrm{End}\,(\Delta_e(i)) \simeq \frac{\mathrm{End}_{\mathfrak{R}}(e)}{\mathrm{rad}(\mathrm{End}_{\mathfrak{R}}(e))}$ *for any $e \in \sigma_i(\mathcal{B})$ and $i < \alpha$.*

*Proof.* Let $e, e'$ in $\sigma_i(\mathcal{B})$. Since $I'_{\mathcal{B}_i} := \sum_{j<i} I_{\mathcal{B}_j} = I_{\bigcup_{j<i} \mathcal{B}_j}$, we can adapt some part of the proof given in [43, Theorem 3.6 (i)] to get (a). Finally, (b) and (c) follow from (a), Lemmas 3.12 and 3.13. □

THEOREM 5.6. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid and $\mathcal{B} = \{\mathcal{B}_i\}_{i<\alpha}$ be an exhaustive family of subcategories of $\mathfrak{R}$. Then, the following statements are equivalent, for $i < \alpha$ and $e \in \sigma_i(\mathcal{B})$.*

(a) $\mathrm{Tr}_{\bigoplus_{j<i} \overline{P}(j)}\,(P_e(i)) \in \mathcal{F}_f(\bigcup_{j<i}\ \Delta(j))$.
(b) *The set $\{j < \alpha\ :\ I_{\mathcal{B}_j}(-, e)/I'_{\mathcal{B}_j}(-, e) \neq 0\}$ is finite, there is some $i_0 < \alpha$ such that $I_{\mathcal{B}_j}(-, e) = \mathfrak{R}(-, e)$ for $j \geq i_0$, and*

$$I_{\mathcal{B}_t}(-, e)/I'_{\mathcal{B}_t}(-, e) \in \mathrm{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_t})$$

*for any $t < \alpha$.*

*Proof.* Let $e \in \sigma_i(\mathcal{B})$ and $t < \alpha$. By Lemma 5.2 and Proposition 3.8, we have $I_{\mathcal{B}_t}(-, e) = \tau_t(P_e^{op}(i))$ and $I'_{\mathcal{B}_t}(-, e) = \overline{\tau}_t(P_e^{op}(i))$. In particular, $\overline{\tau}_i(P_e^{op}(i)) = \mathrm{Tr}_{\bigoplus_{j<i} \overline{P}(j)}\,(P_e^{op}(i))$ and $\mathrm{Supp}(\tau_{\mathcal{B}, P_e^{op}(i)}) = \{j < \alpha\ :\ I_{\mathcal{B}_j}(-, e)/I'_{\mathcal{B}_j}(-, e) \neq 0\}$.

(a) $\Rightarrow$ (b) By (a) and the following exact sequence

$$0 \longrightarrow \overline{\tau}_i(P_e^{op}(i)) \longrightarrow P_e^{op}(i) \longrightarrow \Delta_e(i) \longrightarrow 0,$$

it follows that $P_e^{op}(i) \in \mathcal{F}_f\Big(\bigcup_{j\leq i} \Delta(j)\Big)$. Then, by Theorem 4.7, we get that $\mathrm{Supp}(\tau_{\mathcal{B}, P_e^{op}(i)})$ is finite, there is some $i_0 < \alpha$ such that $\tau_j(P_e^{op}(i)) = P_e^{op}(i)$ for $j \geq i_0$, and $\tau_k/\overline{\tau}_k(P_e^{op}(i)) \in \Delta(k)^{\oplus}$ for any $k < \alpha$.

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Let $t < \alpha$. For each $h \in \sigma_t(\mathcal{B})$, we have $\Delta_h(t) = \mathfrak{R}(-, h)/I'_{\mathcal{B}_t}(-, h)$ and thus $\Delta_h(t) \in \text{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_t})$. Then, $\tau_t/\overline{\tau}_t(P_e^{op}(i)) \in \Delta(t)^{\oplus}$ implies that $\tau_t/\overline{\tau}_t(P_e^{op}(i)) \in \text{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_t})$.

(b) $\Rightarrow$ (a) Let (b) holds true. We need to show that $\overline{\tau}_i(P_e^{op}(i)) \in \mathcal{F}_f(\bigcup_{j<i} \Delta)$. We may assume that $\overline{\tau}_i(P_e^{op}(i)) \neq 0$.

By hypothesis, there is some $k_0 < \alpha$ such that $I_{\mathcal{B}_k}(-, e) = I_{\mathcal{B}_{k_0}}(-, e) = \mathfrak{R}(-, e)$, for any $k \geq k_0$. Consider the set $S := \{k \leq k_0 \,:\, I_{\mathcal{B}_k}(-, e) = I_{\mathcal{B}_{k_0}}(-, e)\}$. Since $S \neq \emptyset$ there exists $k_1 := \min S$. Therefore, $I_{\mathcal{B}_k}(-, e) = I_{\mathcal{B}_{k_1}}(-, e) = \mathfrak{R}(-, e)$ for any $k \geq k_1$, and $I_{\mathcal{B}_j}(-, e) \subsetneq I_{\mathcal{B}_{k_1}}(-, e)$ for $j < k_1$.

We assert that $i < k_1$. Indeed, suppose that $k_1 \leq i$. Then,

$$\begin{aligned}
\overline{\tau}_i(P_e^{op}(i)) &= \sum_{j<i} \tau_j(P_e^{op}(i)) \\
&= \sum_{j<k_1} \tau_j(P_e^{op}(i)) + \sum_{k_1 \leq j < i} \tau_j(P_e^{op}(i)) \\
&= \sum_{j<k_1} \tau_j(P_e^{op}(i)) + P_e^{op}(i) \\
&= P_e^{op}(i),
\end{aligned}$$

and thus $\Delta_e(i) = P_e^{op}(i)/\overline{\tau}_i(P_e^{op}(i)) = 0$, contradicting Proposition 3.8 (a); proving that $i < k_1$. Let $\text{Supp}(\tau_{\mathcal{B}, P_e^{op}(i)}) = \{i_1 < i_2 < \cdots < i_a\}$. Note that $i_a < k_1$.

We assert that $\overline{\tau}_i(P_e^{op}(i)) = I_{\mathcal{B}_{i_k}}(-, e)$ for some $k \in [1, a]$ with $i_k < i$.

Indeed, we have two cases to consider: (1) Let $i = i_k$ for some $k \in [1, a]$. Since $\overline{\tau}_i(P_e^{op}(i)) \neq 0$, we have that $k \geq 2$. Then, by Remark 4.8, we obtain

$$\begin{aligned}
\overline{\tau}_i(P_e^{op}(i)) &= \sum_{j<i_k} I_{\mathcal{B}_j}(-, e) \\
&= \sum_{j<i_{k-1}} I_{\mathcal{B}_j}(-, e) + \sum_{i_{k-1} \leq j < i_k} I_{\mathcal{B}_j}(-, e) \\
&= \sum_{j<i_{k-1}} I_{\mathcal{B}_j}(-, e) + I_{\mathcal{B}_{i_{k-1}}}(-, e) \\
&= I_{\mathcal{B}_{i_{k-1}}}(-, e).
\end{aligned}$$

(2) Let $i \neq i_k$ for any $k \in [1, a]$. In particular, $\overline{\tau}_i(P_e^{op}(i)) = \tau_i(P_e^{op}(i)) = I_{\mathcal{B}_i}(-.e)$. Moreover, there is some $k \in [1, a)$ such that $i \in [i_k, i_{k+1})$. Then, by Remark 4.8, we have that $I_{\mathcal{B}_i}(-.e) = I_{\mathcal{B}_{i_k}}(-, e)$, proving our assertion in both cases.

Once we have that $\overline{\tau}_i(P_e^{op}(i)) = I_{\mathcal{B}_{i_k}}(-, e)$ for some $k \in [1, a]$. In order to see that $\overline{\tau}_i(P_e^{op}(i)) \in \mathcal{F}_f(\Delta)$, by Remark 4.8, it is enough to prove that $\frac{I_{\mathcal{B}_k}(-,e)}{I'_{\mathcal{B}_k}(-,e)} \in \Delta(k)^{\oplus}$ for any $k < \alpha$.

Let $k < \alpha$. By hypothesis we have that

$$\frac{I_{\mathcal{B}_k}(-, e)}{I'_{\mathcal{B}_k(-,e)}} \in \text{proj}_\rho\Big(\frac{\mathfrak{R}}{I'_{\mathcal{B}_k}}\Big).$$

Then by [43, Lemma 3.5], there is some $e' \in \mathcal{B}_k$ such that $\frac{I_{\mathcal{B}_k}(-,e)}{I'_{\mathcal{B}_k}(-,e)} \simeq \frac{\mathfrak{R}(-,e')}{I'_{\mathcal{B}_k}(-,e')}$. Moreover, since $e' \in \mathcal{B}_k$ and $\mathfrak{R}$ is locally finite, it follows that $e' = \oplus_{i=1}^{n_e} t_j^{m_j}$, where $t_1, \ldots, t_{n_e}$ are locally and pairwise non-isomorphic objects in $\mathcal{B}_k$. In case, some $t_j \in \mathcal{B}_l$ and $l < k$, we have that $\mathfrak{R}(-, t_j) = \mathcal{B}_l(-, t_j)$. Thus, we may assume that $t_j \in \sigma_k(\mathcal{B})$, for any $j \in [1, n_e]$. Then,

$$\frac{I_{\mathcal{B}_k}(-, e)}{I'_{\mathcal{B}_k}(-, e)} \simeq \bigoplus_{i=1}^{n_e} \left(\frac{\mathfrak{R}(-, t_j)}{I'_{\mathcal{B}_k}(-, t_j)}\right)^{m_j} = \bigoplus_{i=1}^{n_e} \Delta_{t_j}(k)^{m_j};$$

proving that $\overline{\tau}_i(P_e^{op}(i)) \in \mathcal{F}_f(\bigcup_{j<i} \Delta(j))$. □

Corollary 5.7. *Let $(\mathfrak{R}, \tilde{\mathcal{A}})$ be a right standardly stratified $\mathbb{K}$-ringoid, with $\mathfrak{R}$ locally finite, and let $\Delta = {}_{\tilde{\mathcal{A}}}\Delta$ be the $\tilde{\mathcal{A}}$-standard family of right $\mathfrak{R}$-modules. Then, all the standard modules $\Delta_e(i)$ are local and the following statements hold true.*

(a) *For any $M \in \mathcal{F}_f(\Delta)$, the filtration multiplicity $[M : \Delta_e(i)]$ does not depend on a given $\Delta$-filtration of $M$.*
(b) *$\mathcal{F}_f(\Delta) \subseteq \mathrm{fin.p}_\rho(\mathfrak{R})$ and it is a locally finite $\mathbb{K}$-ringoid.*

*Proof.* Let $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ be the given partition of ind $(\mathfrak{R})$. By Proposition 3.3, we have the exhaustive family $\mathcal{B}(\mathcal{A}) := \{\mathcal{B}_i(\mathcal{A})\}_{i<\alpha}$ of $\mathfrak{R}$. Then, ${}_{\tilde{\mathcal{A}}}\Delta = {}_{\mathcal{B}(\mathcal{A})}\Delta$ since $\sigma(\mathcal{B}(\mathcal{A})) = \tilde{\mathcal{A}}$. For simplicity, we write $\mathcal{B} = \mathcal{B}(\mathcal{A})$ and $\mathcal{B}_i = \mathcal{B}_i(\mathcal{A})$ for any $i < \alpha$. Since $(\mathfrak{R}, \tilde{\mathcal{A}})$ is a standardly stratified $\mathbb{K}$-ringoid, the conditions in Theorem 5.6 (b) hold.

We start by proving that $\Delta \subseteq \mathrm{fin.p}_\rho(\mathfrak{R})$. Let $i < \alpha$ and $e \in \sigma_i(\mathcal{B})$. If $\overline{\tau}_i(P_e^{op}(i)) = 0$, then $\Delta_e(i)$ is equal to $P_e^{op}(i)$, which is finitely presented. Assume that $\overline{\tau}_i(P_e^{op}(i)) \neq 0$ and let $\mathrm{Supp}(\tau_{\mathcal{B},P_e^{op}(i)}) = \{i_1 < i_2 < \cdots < i_a\}$.

We assert that $I_{\mathcal{B}_{i_k}}(-, e)$ is finitely generated for any $k \in [1, a]$.

Indeed, by Remark 4.8, we have $I'_{\mathcal{B}_{i_1}}(-, e) = \sum_{j<i_1} I_{\mathcal{B}_j}(-, e) = 0$, and thus, by hypothesis, $I_{\mathcal{B}_{i_1}}(-, e) \in \mathrm{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_{i_1}})$. Then, there is some $e' \in \mathfrak{R}$ such that $I_{\mathcal{B}_{i_1}}(-, e) = \mathfrak{R}(-, e')/I'_{\mathcal{B}_{i_1}}(-, e')$, proving that $I_{\mathcal{B}_{i_1}}(-, e)$ is a finitely generated right $\mathfrak{R}$-module. As before, we have that $I'_{\mathcal{B}_{i_2}}(-, e) = \sum_{j<i_2} I_{\mathcal{B}_j}(-, e) = I_{\mathcal{B}_{i_1}}(-, e)$ and $I_{\mathcal{B}_{i_2}}(-, e)/I'_{\mathcal{B}_{i_2}}(-, e) \in \mathrm{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_{i_2}})$. Therefore, we get that the quotient $I_{\mathcal{B}_{i_2}}(-, e)/I_{\mathcal{B}_{i_1}}(-, e)$ is a finitely generated right $\mathfrak{R}$-module. Then, the exact sequence $0 \to I_{\mathcal{B}_{i_1}}(-, e) \to I_{\mathcal{B}_{i_2}}(-, e) \to I_{\mathcal{B}_{i_2}}(-, e)/I_{\mathcal{B}_{i_1}}(-, e) \to 0$ implies that $I_{\mathcal{B}_{i_2}}(-, e)$ is finitely generated. It is clear, by induction, that the assertion above holds.

In the proof of Theorem 5.6, we proved that $\overline{\tau}_i(P_e^{op}(i)) = I_{\mathcal{B}_{i_k}}(-, e)$ for some $k \in [1, a]$. Thus, $\overline{\tau}_i(P_e^{op}(i))$ is finitely generated. Therefore from the exact sequence $0 \to \overline{\tau}_i(P_e^{op}(i)) \to P_e(i) \to \Delta_e(i) \to 0$ and [7, Proposition 4.2 (c) i)], we conclude that $\Delta_e(i)$ is finitely presented, and thus $\Delta \subseteq \mathrm{fin.p}_\rho(\mathfrak{R})$. Hence, the result follows from Theorem 4.9. □

Definition 5.8. Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid and let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an exhaustive family of subcategories of $\mathfrak{R}$. We say that $\mathcal{B}$ is right noetherian if for any $i < \alpha$ and $e \in \sigma_i(\mathcal{B})$ the following statement holds true: $\mathrm{Supp}(\tau_{\mathcal{B},P_e^{op}(i)})$ is finite and there is some $i_0 < \alpha$ such that $I_{\mathcal{B}_j}(-, e) = P_e^{op}(i)$ for any $j \geq i_0$.

Corollary 5.9. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid and let $\mathcal{B} := \{\mathcal{B}_i\}_{i<\alpha}$ be an exhaustive family of subcategories of $\mathfrak{R}$. Then, the following statements are equivalent.*

(a) *$\mathcal{B}$ is right noetherian and $\mathfrak{R}$ is right ideally standardly stratified with respect to $\mathcal{B}$.*
(b) *For the partition $\sigma(\mathcal{B})$ of* ind $(\mathfrak{R})$, *related with the family $\mathcal{B}$, we have that $(\mathfrak{R}, \sigma(\mathcal{B}))$ is a right standardly stratified $\mathbb{K}$-ringoid.*

*Proof.* (a) $\Rightarrow$ (b) It follows directly from Theorem 5.6.

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(b) $\Rightarrow$ (a) By hypothesis, we have that Theorem 5.6 (b) holds for any $i<\alpha$ and $e\in\sigma_i(\mathcal{B})$. We need to show that

$$\forall\, t<\alpha\ \forall\, a\in\mathfrak{R}\quad I_{\mathcal{B}_t}(-,a)/I'_{\mathcal{B}_t}(-,a)\in\operatorname{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_t}).$$

Let $t<\alpha$ and $a\in\mathfrak{R}$. We may assume that $a\in\operatorname{ind}(\mathfrak{R})$. Since $\sigma(\mathcal{B})$ is a partition of $\operatorname{ind}(\mathfrak{R})$, by Proposition 3.3, there is some $i<\alpha$ such that $a\in\sigma_i(\mathcal{B})$. Then, by Theorem 5.6 (b), we get that $I_{\mathcal{B}_t}(-,a)/I'_{\mathcal{B}_t}(-,a)\in\operatorname{proj}_\rho(\mathfrak{R}/I'_{\mathcal{B}_t})$.

□

THEOREM 5.10. *Let $\mathfrak{R}$ be a locally finite $\mathbb{K}$-ringoid and let $\mathcal{B}:=\{\mathcal{B}_i\}_{i<\alpha}$ be an exhaustive family of subcategories of $\mathfrak{R}$. Then, the following statements are equivalent.*

(a) *$\mathcal{B}$ is right noetherian and $(\mathfrak{R},\mathcal{B})$ is a right ideally quasi-hereditary $\mathbb{K}$-ringoid.*
(b) *For the partition $\sigma(\mathcal{B})$ of $\operatorname{ind}(\mathfrak{R})$, we have that $(\mathfrak{R},\sigma(\mathcal{B}))$ is a right quasi-hereditary $\mathbb{K}$-ringoid and $\operatorname{Hom}(\Delta_e(i),\Delta_{e'}(i))=0$ for $e\neq e'$ in $\sigma_i(\mathcal{B})$.*

*Proof.* (a) $\Rightarrow$ (b) Since $\mathfrak{R}$ is right ideally quasi-hereditary, it follows from Lemma 5.5 that $I'_{\mathcal{B}_i}(e,e')=\operatorname{rad}_{\mathfrak{R}}(e,e')$ for any $e,e'\in\sigma_i(\mathcal{B})$ and $i<\alpha$. Then, by Corollary 5.9, Lemmas 3.12 and 3.13, we get (b).

(b) $\Rightarrow$ (a) Since $(\mathfrak{R},\sigma(\mathcal{B}))$ is a right quasi-hereditary $\mathbb{K}$-ringoid and

$$\operatorname{Hom}(\Delta_e(i),\Delta_{e'}(i))=0\quad\text{for}\quad e,e'\in\sigma_i(\mathcal{B})$$

for any $i<\alpha$, it follows from Lemmas 3.12 and 3.13 that $I'_{\mathcal{B}_i}(e,e')=\operatorname{rad}_{\mathfrak{R}}(e,e')$ for any $e,e'\in\sigma_i(\mathcal{B})$ and $i<\alpha$. We assert that

$$(*)\quad \operatorname{rad}_{\mathfrak{R}}(e,e')\subseteq I'_{\mathcal{B}_i}(e,e')\quad \forall\, e,e'\in\operatorname{ind}(\mathcal{B}_i),\ \forall\, i<\alpha.$$

Indeed, let $i<\alpha$ and $e,e'\in\operatorname{ind}(\mathcal{B}_i)$. If $e,e'\in\sigma_i(\mathcal{B})$, then $\operatorname{rad}_{\mathfrak{R}}(e,e')=I'_{\mathcal{B}_i}(e,e')$. Assume that one of them, say $e$, belongs to $\mathcal{B}_j$ for some $j<i$. Thus, $I_{\mathcal{B}_j}(e,e')=\mathfrak{R}(e,e')$ and therefore $I'_{\mathcal{B}_i}(e,e')=\sum_{k<i}I_{\mathcal{B}_k}(e,e')=\mathfrak{R}(e,e')$, proving that $\operatorname{rad}_{\mathfrak{R}}(e,e')\subseteq I'_{\mathcal{B}_i}(e,e')$.

Let $e,e'\in\operatorname{ind}(\mathcal{B}_i)$ and $x,y\in\mathfrak{R}$. Then, by $(*)$ we get

$$I_{\mathcal{B}_i}(e',x)\operatorname{rad}_{\mathfrak{R}}(e,e')\,I_{\mathcal{B}_i}(y,e)\subseteq I_{\mathcal{B}_i}(e',x)I'_{\mathcal{B}_i}(e,e')I_{\mathcal{B}_i}(y,e)\subseteq I'_{\mathcal{B}_i}(y,x).$$

Therefore, we conclude that $I_{\mathcal{B}_i}\operatorname{rad}_{\mathfrak{R}} I_{\mathcal{B}_i}\subseteq I'_{\mathcal{B}_i}$ for any $i<\alpha$. Then, as in the proof of [43, Theorem 3.6 (i)]) and using that $I'_{\mathcal{B}_i}=I_{\bigcup_{j<i}\mathcal{B}_j}$, we obtain that $I_{\mathcal{B}_i}/I'_{\mathcal{B}_i}\operatorname{rad}_{\mathfrak{R}} I_{\mathcal{B}_i}/I'_{\mathcal{B}_i}=0$ for any $i<\alpha$. Then, by Corollary 5.9 we get (a). □

**6. Rings with enough idempotents.** In this section, we define and study the terms "standardly stratified" and "quasi-hereditary" for a $\mathbb{K}$-algebra with enough idempotents (w.e.i $\mathbb{K}$-algebra, for short) which is a pair $(\Lambda,\{e_i\}_{i\in I})$, where $\Lambda$ is a $\mathbb{K}$-algebra and $\{e_i\}_{i\in I}$ is a family of orthogonal idempotents of $\Lambda$ such that $\Lambda=\oplus_{i\in I}e_i\Lambda=\oplus_{i\in I}\Lambda e_i$. Note that $\Lambda^2=\Lambda$ and $\Lambda=\oplus_{(i,j)\in I^2}e_i\Lambda e_j$. Moreover, it is said that $(\Lambda,\{e_i\}_{i\in I})$ is Hom-finite if $\{e_j\Lambda e_i\}_{i,j\in I}\subseteq f.\ell(\mathbb{K})$.

Let $(\Lambda,\{e_i\}_{i\in I})$ be an w.e.i $\mathbb{K}$-algebra. The $\mathbb{K}$-ringoid $\mathfrak{R}(\Lambda)$ associated with $(\Lambda,\{e_i\}_{i\in I})$ is defined as follows: the objects of $\mathfrak{R}(\Lambda)$ is the set $\{e_i\}_{i\in I}$, and the set of morphisms from $e_i$ to $e_j$ is $\operatorname{Hom}_{\mathfrak{R}(\Lambda)}(e_i,e_j):=e_j\Lambda e_i$. The composition of morphism in $\mathfrak{R}(\Lambda)$ is given by the multiplication of $\Lambda$. We recall that $Y:\mathfrak{R}(\Lambda)\to\operatorname{Mod}_\rho(\mathfrak{R}(\Lambda))$ is the Yoneda's contravariant functor, where $Y(e):=\operatorname{Hom}_{\mathfrak{R}(\Lambda)}(-,e)$.

The following result is more or less known in the mathematical folklore, but for completeness and the benefit or the reader, we state it and give a proof.

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PROPOSITION 6.1. *Let* $(\Lambda, \{e_i\}_{i\in I})$ *be a w.e.i* $\mathbb{K}$*-algebra. Then, the functor*

$$\delta : \mathrm{Mod}_\rho(\mathfrak{R}(\Lambda)) \to \mathrm{Mod}(\Lambda^{op}), \quad M \mapsto \oplus_{i\in I} M(e_i)$$

*is an isomorphism of categories, and* $\delta(Y(e_i)) = e_i\Lambda$ *for any* $i \in I$.

*Proof.* Let $f : M \to N$ in $\mathrm{Mod}_\rho(\mathfrak{R}(\Lambda))$. For each $i \in I$, we have $f_{e_i} : M(e_i) \to N(e_i)$ and thus $\delta(f) := \oplus_{i\in I} f_{e_i}$.

The structure of $\Lambda$-module on $\delta(M)$ :

Let $\lambda \in \Lambda = \oplus_{i,j} e_j\Lambda e_i$ and $m \in \delta(M) = \oplus_{i\in I} M(e_i)$. Then, we have that $\lambda = \sum_{i,j} \lambda_{i,j}$ and $m = \sum_i m_i$, where $\lambda_{i,j} \in e_j\Lambda e_i$ and $m_i \in M(e_i)$. Since $\lambda_{i,j} : e_i \to e_j$ is a morphism in $\mathfrak{R}(\Lambda)$, we obtain $M(\lambda_{i,j}) : M(e_j) \to M(e_i)$. We set $(m \cdot \lambda)_t := \sum_i M(\lambda_{t,i})(m_i)$. It is a routine calculation to show that $\delta(M)$ is a right $\Lambda$-module. Observe that $m \cdot e_j = m_j$ and thus $\delta(M) \cdot e_j = M(e_j)$, for any $j \in I$. Let us consider $e := \sum_{i\in \mathrm{Supp}(m)} e_i$, where $\mathrm{Supp}(m) := \{i \in I\ :\ m_i \neq 0\}$. Since $m \cdot e_j = m_j$, it follows that $m \cdot e = m$ and thus $\delta(M) \cdot \Lambda = \delta(M)$.

Consider the correspondence

$$\varepsilon : \mathrm{Mod}(\Lambda^{op}) \to \mathrm{Mod}_\rho(\mathfrak{R}(\Lambda)), \quad X \mapsto (e_i \mapsto Xe_i).$$

Let $g : X \to Y$ in $\mathrm{Mod}(\Lambda^{op})$ and $\lambda_{i,j} : e_i \to e_j$ in $\mathfrak{R}(\Lambda)$. Le $X(\lambda_{i,j}) : Xe_j \to Xe_i$ and $\varepsilon_{e_j}(g) : Xe_j \to Ye_j$ be defined as $X(\lambda_{i,j})(xe_j) := xe_j\lambda_{i,j}$ and $\varepsilon_{e_j}(g)(xe_j) := g(xe_j)$. It can be seen that $\varepsilon(g) : \varepsilon(X) \to \varepsilon(X)$ is a morphism in $\mathrm{Mod}_\rho(\mathfrak{R}(\Lambda))$, and moreover, it is a functor.

Let $M \in \mathrm{Mod}_\rho(\mathfrak{R}(\Lambda)$. We know that $\delta(M) \cdot e_j = M(e_j)$. Therefore,

$$(\varepsilon\delta(M))(e_j) = \delta(M) \cdot e_j = M(e_j).$$

Let $X \in \mathrm{Mod}(\Lambda^{op})$. Since $X\Lambda = X$ and $\Lambda = \oplus_{i\in I}\Lambda e_i$, we get

$$\varepsilon\delta(X) = \bigoplus_{i\in I} \varepsilon\delta(X)e_i = \bigoplus_{i\in I} \delta(X) \cdot e_i = \oplus_{i\in I}Xe_i = X.$$

Thus, $\delta$ is an isomorphism of categories with inverse $\varepsilon$. Finally, we have

$$\delta(Y(e_i)) = \bigoplus_{j\in I} Y(e_i)(e_j) = \bigoplus_{j\in I} e_i\Lambda e_j = e_i\Lambda. \qquad \square$$

REMARK 6.2. Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra. Let $\overline{\mathfrak{R}}(\Lambda) := \mathrm{proj}_\rho(\mathfrak{R}(\Lambda))$. Then, $\overline{\mathfrak{R}}(\Lambda)$ is a locally finite $\mathbb{K}$-ringoid. Moreover, it is well known [42] that the restriction functor

$$\Psi : \mathrm{Mod}_\rho(\overline{\mathfrak{R}}(\Lambda)) \to \mathrm{Mod}_\rho(\mathfrak{R}(\Lambda)), \quad F \mapsto F|_{\mathfrak{R}(\Lambda)}$$

is an equivalence of categories and $\Psi((-, Y(e_i))) = Y(e_i)$ for any $i \in I$. Therefore $\Psi(\mathrm{proj}_\rho(\overline{\mathfrak{R}}(\Lambda))) = \mathrm{proj}_\rho(\mathfrak{R}(\Lambda))$. Thus, by using that $\overline{\mathfrak{R}}(\Lambda)$ is a locally finite $\mathbb{K}$-ringoid, we can translate in terms of $\mathfrak{R}(\Lambda)$ (and also in terms of $\Lambda$) all the results that we have proven for locally finite $\mathbb{K}$-ringoids.

LEMMA 6.3. *Let* $(\Lambda, \{e_i\}_{i\in I})$ *be a Hom-finite w.e.i.* $\mathbb{K}$*-algebra and let* $f$ *and* $g$ *be idempotents in* $\Lambda$*. Then, the following statements hold true:*

(a) $g\Lambda f \subseteq f.\ell(\mathbb{K})$;
(b) $f\Lambda \simeq g\Lambda \Leftrightarrow \Lambda f \simeq \Lambda g$.

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*Proof.* (a) We have the finite sums $f=\sum_{k,l} e_{k,l}$ and $g=\sum_{i,j} e_{i,j}$, where $e_{k,l}\in e_k\Lambda e_l$ and $e_{i,j}\in e_i\Lambda e_j$, and thus $g\Lambda f=\sum_{i,j,k,l} e_{i,j}\Lambda e_{k,l}$. Moreover, each $e_{i,j}\Lambda e_{k,l}\subseteq e_i\Lambda e_l$ and so it has finite length as $\mathbb{K}$-module. Therefore, $g\Lambda f$ has finite length as $\mathbb{K}$-module.

(b) It follows by applying the functor $\mathrm{Hom}(-\Lambda)$ to the given isomorphism and by using that $\mathrm{Hom}(\Lambda e,\,\Lambda)\simeq e\Lambda$ and $\mathrm{Hom}(e\Lambda,\,\Lambda)\simeq\Lambda e$, for any $e^2=e\in\Lambda$. □

PROPOSITION 6.4. *For a w.e.i $\mathbb{K}$-algebra $(\Lambda,\{e_i\}_{i\in I})$, the following statements are equivalent.*

(a) $(\Lambda,\{e_i\}_{i\in I})$ *is Hom-finite.*
(b) $\mathrm{proj}(\Lambda^{op})$ *is a locally finite $\mathbb{K}$-ringoid.*
(c) $\mathrm{proj}_\rho(\mathfrak{R}(\Lambda))$ *is a locally finite $\mathbb{K}$-ringoid.*
(d) $\mathrm{proj}(\Lambda)$ *is a locally finite $\mathbb{K}$-ringoid.*
(e) $\mathrm{proj}(\mathfrak{R}(\Lambda))$ *is a locally finite $\mathbb{K}$-ringoid.*

*Proof.* Let $i,j\in I$. Then, we have the isomorphisms of $\mathbb{K}$-modules

$$e_j\Lambda e_i=\mathrm{Hom}_{\mathfrak{R}(\Lambda)}(e_i,\,e_j)\simeq\mathrm{Hom}_\Lambda(e_i\Lambda,\,e_j\Lambda).$$

Therefore, the fact that (b) (respectively, (d)) implies (a) follows easily, and the equivalence between (b) (respectively, (d)) and (c) (respectively, (e)) can be obtained from Proposition 6.1. Let us prove that (a) implies (b).

Assume that $(\Lambda,\{e_i\}_{i\in I})$ is Hom-finite. Then, it is clear that $\mathrm{proj}(\Lambda^{op})$ is a Hom-finite $\mathbb{K}$-ringoid. In order to prove that $\mathrm{proj}(\Lambda^{op})$ is a Krull–Schmidt category, it is enough by [**49**, 49.10] to see that $e\Lambda e$ is a semiperfect ring for any $e^2=e\in\Lambda$. Indeed, let $e^2=e\in\Lambda$. Then, by Lemma 6.3 (a), we get that $e\Lambda e$ has finite length as $\mathbb{K}$-module, and thus it is an Artin ring. In particular, $e\Lambda e$ is semiperfect. The fact that (a) implies (d) can be shown in a similar way. □

COROLLARY 6.5. *For a Hom-finite w.e.i $\mathbb{K}$-algebra $(\Lambda,\{e_i\}_{i\in I})$, the following statements hold true.*

(a) $\mathrm{End}(e\Lambda)$ *and* $\mathrm{End}(\Lambda e)$ *are Artin rings, for any $e^2=e\in\Lambda$.*
(b) *For each $i\in I$, there exists a unique (up to permutations) family $\overline{e}_i:=\{e_{k,i}\}_{k=1}^{n_i}$ of primitive orthogonal idempotents in $\Lambda$ such that $e_i=\sum_{k=1}^{n_i}e_{k,i}$.*

*Proof.* (a) Let $e^2=e\in\Lambda$. Then, by Lemma 6.3 $e\Lambda e\in f.\ell(\mathbb{K})$. Finally, since $\mathrm{End}(e\Lambda)\simeq e\Lambda e\simeq\mathrm{End}(\Lambda e)$ as $\mathbb{K}$-modules, we get (a).

(b) Let $i\in I$. By Proposition 6.4 (b), there is a decomposition

$$(*)\quad e_i\Lambda=\oplus_{k=1}^{n_i}P_{k,i}\text{ with }P_{k,i}\text{ local, for all }k,\,i.$$

Since $e_i=e_i^2\in e_i\Lambda$, we get from $(*)$ the unique decomposition $e_i=\sum_{k=1}^{n_i}e_{k,i}$ of $e_i$. Therefore, the family $\{e_{k,i}\}_{k=1}^{n_i}$ consists of orthogonal idempotents in $\Lambda$. Hence, $P_{k,i}=e_{k,i}\Lambda$ for each $k,\,i$. But now, since each $P_{k,i}$ is local, we get that $e_{k,i}\Lambda e_{k,i}\simeq\mathrm{End}(e_{k,i}\Lambda)$ has only trivial idempotents. But the latest condition is equivalent that $e_{k,i}$ be primitive.

□

COROLLARY 6.6. *Let $(\Lambda,\{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra and* $\mathrm{ind}\,\{e_i\}_{i\in I}$ *be the quotient of the set $\cup_{i\in I}\overline{e}_i$ (see Corollary 6.5) by the equivalence relation $\sim$, where $f\sim g$ if, and only if, $f\Lambda\simeq g\Lambda$. Denote by $[e]$ the equivalence class of $e\in\cup_{i\in I}\overline{e}_i$. Then, the following statements hold true*

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(a) $\operatorname{ind}\overline{\mathfrak{R}}(\Lambda) = \{\delta^{-1}(e\Lambda)\ :\ [e] \in \operatorname{ind}\{e_i\}_{i\in I}\}$;
(b) $\operatorname{ind}\operatorname{proj}(\Lambda^{op}) = \{e\Lambda\ :\ [e] \in \operatorname{ind}\{e_i\}_{i\in I}\}$;
(c) $\operatorname{ind}\operatorname{proj}(\Lambda) = \{\Lambda e\ :\ [e] \in \operatorname{ind}\{e_i\}_{i\in I}\}$.

*Proof.* By Proposition 6.1 $\operatorname{proj}(\Lambda^{op}) = \delta(\overline{\mathfrak{R}}(\Lambda))$. Then, by Remark 6.2, Proposition 6.4, Lemma 2.2 and Corollary 6.5, we get (a) and (b). In order to show (c), by Corollary 6.5, we have that $\Lambda e_i = \oplus_{k=1}^{n_i} \Lambda e_{k,i}$ and $\Lambda e_{k,i}$ is local, for all $i, k$. Consider the relation on $\cup_{i\in I}\overline{e}_i$ given by: $f \approx g$ if and only if $\Lambda f \simeq \Lambda g$. By Lemma 6.3 (b), we have that $\approx$ coincide with $\sim$. Thus, we obtain (c) in a similar way as we did for (b). □

Definition 6.7. Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra. For $M \in \operatorname{Mod}(\Lambda^{op})$, the support of $M$ is

$$\operatorname{Supp}(M) := \{e \in \operatorname{ind}\{e_i\}_{i\in I}\ :\ Me \neq 0\}.$$

We say that $\Lambda$ is right support finite if $\operatorname{Supp}(e\Lambda)$ is finite for any $e \in \operatorname{ind}\{e_i\}_{i\in I}$. Dually, $\Lambda$ is left support finite if $\operatorname{Supp}(\Lambda e)$ is finite for any $e \in \operatorname{ind}\{e_i\}_{i\in I}$. Finally, $\Lambda$ is support finite if it is right and left support finite.

Remark 6.8. Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra.

(1) We say that $(\Lambda, \{e_i\}_{i\in I})$ is basic if $e_i$ is primitive for each $i$ and $e_i\Lambda \not\simeq e_j\Lambda$ for $e_i \neq e_j$. Note that $(\Lambda, \{e_i\}_{i\in I})$ is basic if, and only if, $\operatorname{ind}\{e_i\}_{i\in I} = \{e_i\}_{i\in I}$.
(2) By Proposition 6.1, Remark 6.2 and Corollary 6.6, we can see that $\Lambda$ is right (resp. left) support finite if, and only if, the ringoid $\overline{\mathfrak{R}}(\Lambda)$ is right (resp. left) support finite.

In what follows, we show a natural way to construct basic Hom-finite w.e.i. $\mathbb{K}$-algebras, which are also support finite. By following Bongartz and Gabriel [8], let $\mathbb{K}$ be a field and $Q$ be a quiver (which may be infinite), $Q_0$ is the set of vertices, and $Q_1$ is the set of arrows. A path $\gamma$ in $Q$, of length $n \geq 1$, is of the form $\gamma = \alpha_n\alpha_{n-1}\cdots\alpha_1$ for arrows $\alpha_i \in Q_1$ and can be visualised as $a_0 \xrightarrow{\alpha_1} a_1 \to \cdots \to a_{n-1} \xrightarrow{\alpha_n} a_n$. We say that $\gamma$ starts at the vertex $a_0$ and ends at the vertex $a_n$. The vertices in $Q$ can be seen as paths of length $0$, and for each $a \in Q_0$, its corresponding path of length zero will be denoted by $\varepsilon_a$. For each nonnegative integer $n$, we denote by $Q_n$ the set of all paths of length $n$. Let $\mathbb{K}Q_n$ be the $\mathbb{K}$-vector space whose base is the set $Q_n$.

The path $\mathbb{K}$-algebra is the $\mathbb{K}$-vector space $\mathbb{K}Q := \bigoplus_{n\geq 0} \mathbb{K}Q_n$ whose product of two basis vectors is given by the concatenation of paths. Note that $\varepsilon_Q := \{\varepsilon_a\}_{a\in Q_0}$ is a family of orthogonal idempotents in $\mathbb{K}Q$, and $\varepsilon_b Q_n \varepsilon_a$ is the set of all paths of length $n$, which start at $a$ and end at $b$. Moreover, the pair $(\mathbb{K}Q, \varepsilon_Q)$ is a $\mathbb{K}$-algebra with enough idempotents. We denote by $J_Q$ the ideal in $\mathbb{K}Q$ generated by the set $Q_1$. An ideal $I$ of $\mathbb{K}Q$ is admissible if $I \subseteq J_Q^2$ and for each $x \in Q_0$ there is a natural number $n_x$ such that $I$ contains each path of length $\geq n_x$ which starts or ends at $x$. For any admissible ideal $I$ of $Q$, we consider the quotient path $\mathbb{K}$-algebra $\mathbb{K}(Q, I) := \mathbb{K}Q/I$ and the set of orthogonal idempotents $e_{Q,I} := \{e_a\}_{a\in Q_0}$, where $e_a := \varepsilon_a + I$. We recall that a quiver $Q$ is locally finite if for each vertex $x \in Q_0$ there is a finite number of arrows in $Q_1$, which start or end at $x$. The main properties, from our point of view, of quotient path $\mathbb{K}$-algebras can be summarized in the following proposition.

Proposition 6.9. *Let $Q$ be a locally finite quiver (which may be infinite), $\mathbb{K}$ be a field, and $I$ be an admissible ideal of $\mathbb{K}Q$. Then, the following statements hold true.*

(a) *The pair $(\mathbb{K}(Q, I), e_{Q,I})$ is a basic Hom-finite w.e.i $\mathbb{K}$-algebra.*
(b) *$\mathbb{K}(Q, I)$ is support finite.*
(c) $\operatorname{proj}(\mathbb{K}(Q, I))$ *and* $\operatorname{proj}(\mathbb{K}(Q, I)^{op})$ *are locally finite $\mathbb{K}$-ringoids.*

(d) $\operatorname{ind}\operatorname{proj}(\mathbb{K}(Q,I)) = \{\mathbb{K}(Q,I)\, e_a\}_{a\in Q_0}$.
(e) $\operatorname{ind}\operatorname{proj}(\mathbb{K}(Q,I)^{op}) = \{e_a\, \mathbb{K}(Q,I)\}_{a\in Q_0}$.

*Proof.* For a proof of (a) and (b), see [8, 2.1]. The items (c), (d), and (e) can be obtained from Corollary 6.6. □

By Corollary 6.6, we know that the rings with a nice setting, where we can define the standard modules, are precisely the Hom-finite $\mathbb{K}$-algebras with enough idempotents.

Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra. Then, $\operatorname{ind}\operatorname{proj}(\Lambda^{op}) = \{e\Lambda \,:\, [e] \in \operatorname{ind}\{e_i\}_{i\in I}\}$. Choose a partition $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_i\}_{i<\alpha}$ of the set $\operatorname{ind}\{e_i\}_{i\in I}$. Define ${}_{\Lambda^{op}}P_e(i) := e\Lambda$, for any $[e] \in \tilde{\mathcal{A}}_i$. Let ${}_{\Lambda^{op}}P := \{{}_{\Lambda^{op}}P(i)\}_{i\leq\alpha}$, where ${}_{\Lambda^{op}}P(i) := \{{}_{\Lambda^{op}}P_e(i)\}_{e\in\tilde{\mathcal{A}}_i}$. The family of $\tilde{\mathcal{A}}$-standard right $\Lambda$-modules ${}_{\Lambda^{op}}\Delta = \{\Delta(i)\}_{i<\alpha}$, where $\Delta(i) := \{\Delta_e(i)\}_{e\in\tilde{\mathcal{A}}_i}$ is defined as follows:

$$\Delta_e(i) := \frac{{}_{\Lambda^{op}}P_e(i)}{\operatorname{Tr}_{\oplus_{j<i}\overline{P}(j)}({}_{\Lambda^{op}}P_e(i))},$$

where $\overline{P}(j) := \bigoplus_{r\in\tilde{\mathcal{A}}_j} {}_{\Lambda^{op}}P_r(j)$. Let $P := \delta^{-1}({}_{\Lambda^{op}}P)$, where $\delta : \operatorname{Mod}_\rho(\mathfrak{R}(\Lambda)) \to \operatorname{Mod}(\Lambda^{op})$ is the isomorphism of Proposition 6.1. Then, by Corollary 6.6 (a), it can be shown that $\delta({}_{\tilde{\mathcal{A}}}\Delta_e(i)) = {}_{\Lambda^{op}}\Delta_e(i)$.

Definition 6.10. Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra. We say that the pair $(\Lambda, \tilde{\mathcal{A}})$ is a right standardly stratified $\mathbb{K}$-algebra if $\tilde{\mathcal{A}}$ is a partition of $\operatorname{ind}\{e_i\}_{i\in I}$ such that $\operatorname{Tr}_{\oplus_{j<i}\overline{P}(j)}({}_{\Lambda^{op}}P_e(i)) \in \mathcal{F}_f(\bigcup_{j<i}\Delta(j))$, for any $i<\alpha$ and $e\in\tilde{\mathcal{A}}_i$.

Remark 6.11. Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra. Consider $\overline{\mathfrak{R}}(\Lambda) := \operatorname{proj}_\rho(\mathfrak{R}(\Lambda))$ as we did in Remark 6.2. Then, $\overline{\mathfrak{R}}(\Lambda)$ is a locally finite $\mathbb{K}$-ringoid such that the restriction functor

$$\Psi : \operatorname{Mod}_\rho(\overline{\mathfrak{R}}(\Lambda)) \to \operatorname{Mod}_\rho(\mathfrak{R}(\Lambda)), \qquad F \mapsto F|_{\mathfrak{R}(\Lambda)}$$

is an equivalence of categories and $\Psi((-, Y(e_i))) = Y(e_i)$, for any $i\in I$.

Let $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_j\}_{j<\alpha}$ be a partition of the set

$$\operatorname{ind}(\overline{\mathfrak{R}}(\Lambda)) = \{E_e := \delta^{-1}(e\Lambda) \,:\, [e]\in\operatorname{ind}\{e_i\}_{i\in I}\} \text{ (see Corollary 6.6 (a)).}$$

Then, $\Psi(\tilde{\mathcal{A}})$ is a partition of $\operatorname{ind}\{e_i\}_{i\in I}$. Moreover, for $E = \delta^{-1}(e\Lambda)\in\tilde{\mathcal{A}}_j$, we have $\Psi({}_{\tilde{\mathcal{A}}}\Delta_E(i)) = {}_{\Psi(\tilde{\mathcal{A}})}\Delta_e(i)$. Therefore, by using that $\overline{\mathfrak{R}}(\Lambda)$ is a locally finite $\mathbb{K}$-ringoid, we can translate in terms of $\Lambda$ all the results that we have proven for locally finite $\mathbb{K}$-ringoids.

Corollary 6.12. *Let $(\Lambda, \{e_i\}_{i\in I})$ be a Hom-finite w.e.i $\mathbb{K}$-algebra, and let $\tilde{\mathcal{A}}$ be a partition of* $\operatorname{ind}\{e_i\}_{i\in I}$ *such that $(\Lambda, \tilde{\mathcal{A}})$ is a right standardly stratified $\mathbb{K}$-algebra. Then, all the standard modules $\Delta_e(i)$ are local and the following statements hold true.*

(a) *For any $M\in\mathcal{F}_f(\Delta)$, the filtration multiplicity $[M : \Delta_e(i)]$ does not depend on a given $\Delta$-filtration of $M$.*
(b) *$\mathcal{F}_f(\Delta) \subseteq \operatorname{fin.p}(\Lambda^{op})$ and it is a locally finite $\mathbb{K}$-ringoid.*

*Proof.* It follows from Remark 6.11 and Corollary 5.7. □

Corollary 6.13. *Let $Q$ be a locally finite quiver (which may be infinite), $\mathbb{K}$ be a field, and $I$ be an admissible ideal of $\mathbb{K}Q$. Then, for any partition $\tilde{\mathcal{A}}$ of $e_{Q,I}$, each of the standard module $\Delta_e(i)$ is local and the following statements hold true.*

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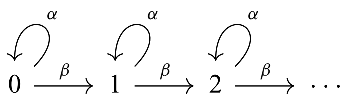
(a) *For any* $M \in \mathcal{F}_f(\Delta)$, *the filtration multiplicity* $[M : \Delta_e(i)]$ *does not depend on a given* $\Delta$*-filtration of* $M$.

(b) $\mathcal{F}_f(\Delta) \subseteq \mathrm{fin.p}(\mathbb{K}(Q, I)^{op})$ *and it is a locally finite* $\mathbb{K}$*-ringoid.*

*Proof.* By Proposition 6.9, we have that $(\mathbb{K}(Q, I), e_{Q,I})$ is a basic Hom-finite w.e.i $\mathbb{K}$-algebra, which is also support finite. Then, the result follows from Remark 6.11 and Corollary 4.10. □

EXAMPLE 6.14. *Let* $Q$ *be the following locally finite quiver*




*Consider the quotient path* $\mathbb{K}$*-algebra* $\Lambda := \mathbb{K}(Q, I)$, *where* $\mathbb{K}$ *is a field and* $I$ *is the admissible ideal* $< \alpha^2, \beta^2, \alpha\beta, \beta\alpha >$. *For each* $i \in Q_0$, *we have the idempotent* $e_i := \varepsilon_i + I$ *of* $\Lambda$. *In what follows, we choose different partitions of* $\{e_i\}_{i\in Q_0}$, *and we will see if* $\Lambda$ *is standardly stratified (or not) with respect to these partitions.*

(1) *Consider* $\tilde{\mathcal{A}} = \{\tilde{A}_i\}_{i\in<\aleph_0}$, *where* $\tilde{A}_i := \{e_i\}$. *In this case, we have that* $\Delta(i) = \{\Delta_{e_i}(i) = \Lambda e_i\}$ *and thus* $(\Lambda, \tilde{\mathcal{A}})$ *is standardly stratified. However, it is not quasi-hereditary since* $\mathrm{End}(\Delta_{e_0}(0)) \simeq e_0\Lambda e_0$ *is not a division ring.*

(2) *Consider* $\tilde{\mathcal{B}} = \{\tilde{B}_0, \tilde{B}_1, \tilde{B}_2\}$, *where* $\tilde{B}_0 := \{e_1\}$, $\tilde{B}_1 := \{e_0\}$ *and* $\tilde{B}_2 := \{e_i\}_{i\geq 2}$. *In this case, we get* $\Delta_{e_1}(0) = \Lambda e_1$, $\Delta_{e_0}(1) = \Lambda e_0/S(1)$ *and* $\Delta_{e_i}(2) = \Lambda e_i$, *for any* $i \geq 2$, *where* $S(1) = \Lambda e_1/\mathrm{rad}\,(\Lambda e_1)$. *Note that* $\tilde{\mathcal{B}}$ *is finite; however,* $(\Lambda, \tilde{\mathcal{B}})$ *is not standardly stratified, since* $\Lambda e_0 \notin \mathcal{F}_f(\Delta)$.

(3) *Consider* $\tilde{\mathcal{C}} = \{\tilde{C}_i\}_{i\in<\aleph_0}$, *where* $\tilde{C}_0 := \{e_1\}$, $\tilde{C}_1 := \{e_0\}$ *and* $\tilde{C}_i := \{e_i\}$, *for* $i \geq 2$. *In this case, we get* $\Delta_{e_1}(0) = \Lambda e_1$, $\Delta_{e_0}(1) = \Lambda e_0/S(1)$ *and* $\Delta_{e_i}(i) = \Lambda e_i$, *for any* $i \geq 2$. *Note that* $\tilde{\mathcal{C}}$ *is infinite; however* $(\Lambda, \tilde{\mathcal{C}})$ *is not standardly stratified, since* $\Lambda e_0 \notin \mathcal{F}_f(\Delta)$.

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