ASYMPTOTIC BEHAVIOR AND QUASI-LIMITING DISTRIBUTIONS ON TIME-FRACTIONAL BIRTH AND DEATH PROCESSES

JORGE LITTIN CURINAO ⓘ,* *Universidad Católica del Norte*

Abstract

In this article we provide new results for the asymptotic behavior of a time-fractional birth and death process $N_\alpha(t)$, whose transition probabilities $\mathbb{P}[N_\alpha(t) = j \mid N_\alpha(0) = i]$ are governed by a time-fractional system of differential equations, under the condition that it is not killed. More specifically, we prove that the concepts of quasi-limiting distribution and quasi-stationary distribution do not coincide, which is a consequence of the long-memory nature of the process. In addition, exact formulas for the quasi-limiting distribution and its rate convergence are presented. In the first sections, we revisit the two equivalent characterizations for this process: the first one is a time-changed classic birth and death process, whereas the second one is a Markov renewal process. Finally, we apply our main theorems to the linear model originally introduced by Orsingher and Polito [23].

Keywords: Fractional processes; quasi-limiting distributions; inverse stable subordinators

2020 Mathematics Subject Classification: Primary 60G22
Secondary 60G18; 60K15

1. Introduction

The birth and death processes have been extensively studied in various areas of both probability theory and its applications in population models, epidemiology, queuing theory, and engineering, to name just a few. Two fundamental aspects related to its analysis are the representation of the transition probabilities that model the evolution of the system and the asymptotic behavior after a long time.

Since many processes exhibit the phenomenon of long memory, a Markov process does not seem at all appropriate, so that fractional models appear to be more precise. Time-fractional models in the context of anomalous diffusion were studied by Orsingher in [21] and [22], where he analyzed the time-fractional telegraph equation and a fractional diffusion equation, respectively. Previous results for fractional birth and death processes can be found in the articles of Orsingher [23] for the linear case, Meerschaert [20] for the fractional Poisson process, and Jumarie [13] for a pure birth and death process with multiple births. Surprisingly, none of them provide representations for an arbitrary time-fractional birth and death process, and consequently results concerning the asymptotic behavior for this class of process are not available in the literature.

For Markov processes, the study of the number of survivors after a long time started with the early work of Kolmogorov in 1938. Later in 1947, Yaglom [25] showed that the limit behavior of sub-critical branching processes conditioned to survival was given by a proper distribution. In 1965, Darroch and Seneta [7] began the study of quasi-stationary distributions (QSDs) on finite-state irreducible Markov chains, and Seneta and Vere-Jones [24] in 1966 did it for Markov chains with countable states. A very important publication is that of Van Doorn in 1991 [8], which states a criterion to determine the existence and uniqueness of QSDs for birth and death chains. More recent results about the existence and uniqueness of QSDs can be found in [9] and [10].

For diffusion processes on the half-line, the first work is due to Mandl [15], who studied the existence of a QSD on the half-line for $+\infty$ being a natural boundary according to Feller's classification. Subsequently many results on the existence of QSDs and limit laws for one-dimensional diffusions killed at 0 were provided by Ferrari [11], Collet, Martínez, and San Martín [5], and Martínez and San Martín [16, 17].

Most of these works are based on studying the spectral decomposition of the infinitesimal operator associated with the process. Applying similar ideas, we can study the asymptotic behavior of time-fractional models, which is precisely one of the main objectives of this article.

This article is organized as follows. In Section 2 we present the model description. More specifically, we introduce the system of time-fractional equations that governs the transition probabilities. In Section 3 two equivalent characterizations are shown: the first one is a time-changed birth and death process, whereas the second one is a Markov renewal process. In Section 4 we follow a different approach based on a spectral representation of the transition probabilities to study the quasi-limiting behavior of the process conditioned not to be killed. In Section 5.1 we study the concept of quasi-stationary distributions, proving that the quasi-limiting distribution and quasi-stationary distribution are not the same. Finally, in Section 6, we apply our main theorems to the linear model.

2. Model formulation

We let $N_\alpha(t)$, $t \geq 0$ denote the fractional birth and death process killed at zero. The transition probabilities denoted by

$$p_{i,j,\alpha}(t) = \mathbb{P}[N_\alpha(t) = j \mid N_\alpha(0) = i]$$

are governed by the time-fractional system of differential equations (commonly called the system of backward equations)

$$\mathbb{D}^\alpha p_{i,j,\alpha}(t) = \mu_i p_{i-1,j,\alpha}(t) - (\lambda_i + \mu_i) p_{i,j,\alpha}(t) + \lambda_i p_{i+1,j,\alpha}(t), \quad j \geq 0, \, i \geq 1,$$

$$p_{0,j,\alpha}(t) = 0, \quad j \geq 1. \tag{2.1}$$

As usual, the values $\lambda_i > 0$, $\mu_i > 0$ (with the convention $\mu_0 = \lambda_0 = 0$) are the birth rates and death rates respectively, whereas the parameter $\alpha \in (0, 1]$ determines the order of the Caputo fractional operator $\mathbb{D}^\alpha(\cdot)$, defined as

$$\mathbb{D}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(u)}{(t-u)^\alpha} \, \mathrm{d}u, \quad \alpha \in (0, 1),$$

$$\mathbb{D}^1 f(t) = f'(t), \quad \alpha = 1.$$

In particular, when $\alpha = 1$ the operator is just the derivative and $N_1(t)$ is the classical birth and death process. The matrix formulation for the system of equations (2.1) is

$$\mathbb{D}^\alpha P(t) = Q^{(a)} P(t),$$

where $P(t)$ is the matrix with coefficients $p_{i,j,\alpha}(t)$, $i \geq 1$, $j \geq 1$. The components $q_{i,j}$, $i \geq 1$, $j \geq 1$ of the matrix $Q^{(a)}$ are

$$q_{i,j} = \begin{cases} -(\lambda_i + \mu_i) & \text{if } i = j, \\ \mu_i & \text{if } i = j + 1, \\ \lambda_i & \text{if } i = j - 1. \end{cases} \tag{2.2}$$

We recall that $p_{0,j,\alpha}(t) = 0$ for all $j \geq 1$. This fact makes the matrix formulation consistent with the system of fractional differential equations (2.1). The initial condition is the Kronecker delta $p_{i,j,\alpha}(0) = \delta_{i,j}$.

3. Two equivalent characterizations

In this section we introduce the representation of the process $N_\alpha(t)$ as a standard birth and death chain changed in time, i.e. $N_\alpha(t) = N_1(L^{(\alpha)}(t))$, where $L^{(\alpha)}(t)$ is the inverse of a stable subordinator. Also, we get a representation as a Markov renewal process in the general case. To make this work self-contained, we first introduce a brief summary of the stable subordinators and its inverse.

3.1. The stable subordinator and its inverse

A subordinator $D^{(\alpha)}(t)$, $t \geq 0$ is a one-dimensional Levy process such that its trajectories are non-decreasing with probability 1. We say that a subordinator is stable when the Laplace transform of $D^{(\alpha)}(t)$ satisfies

$$\mathbb{E}\big[e^{-sD^{(\alpha)}(t)}\big] = e^{-ts^\alpha}. \tag{3.1}$$

Associated to a subordinator $D^{(\alpha)}(t)$, we define the inverse process $L^{(\alpha)}(t)$, $t \geq 0$ as follows:

$$L^{(\alpha)}(t) = \inf\{r \geq 0 \colon D^{(\alpha)}(r) > t\}. \tag{3.2}$$

The process $L^{(\alpha)}(t)$ denotes the first time that $D^{(\alpha)}(\cdot)$ exceeds a level $t > 0$. It is clear that the trajectories of the process $L^{(\alpha)}(t)$ are non-decreasing and moreover they are continuous (see Theorem 2 on page 642 of [2] for a technical proof concerning the modulus of continuity of $L^{(\alpha)}(\cdot)$). From (3.2) we can deduce that the finite-dimensional distributions of $D^{(\alpha)}(t)$ and $L^{(\alpha)}(t)$ satisfy the identity

$$\mathbb{P}[L^{(\alpha)}(t_i) > x_i, \, 1 \leq i \leq n] = \mathbb{P}[D^{(\alpha)}(x_i) < t_i, \, 1 \leq i \leq n]. \tag{3.3}$$

Equation (3.1) directly implies that the process $D^{(\alpha)}(t)$ is self-similar of index $1/\alpha$, that is,

$$\mathbb{P}[D^{(\alpha)}(cx_i) < t_i, \, 1 \leq i \leq n] = \mathbb{P}[c^{1/\alpha} D^{(\alpha)}(x_i) < t_i, \, 1 \leq i \leq n]. \tag{3.4}$$

Moreover, from (3.3) and (3.4) we have that the process $L^{(\alpha)}(t)$ is self-similar of index α:

$$\mathbb{P}[L^{(\alpha)}(ct_i) > x_i, \, 1 \leq i \leq n] = \mathbb{P}[c^\alpha L^{(\alpha)}(t_i) > x_i, \, 1 \leq i \leq n].$$

For all $t > 0$, the distribution of $D^{(\alpha)}(t)$ has only a density component, here denoted by $g_\alpha(\cdot, t)$. In the same way, the inverse $L^{(\alpha)}(t)$ has only a density component $h_\alpha(\cdot, t)$ satisfying

$$h_\alpha(u, t) = \frac{t}{\alpha} u^{-1-1/\alpha} g_\alpha(tu^{-1/\alpha}, t).$$

Concerning the Laplace transform of $h_\alpha(u, t)$ we have the identities

$$\int_0^\infty e^{-st} h_\alpha(u, t)\, dt = s^{\alpha-1}\, e^{-us^\alpha},$$

$$\int_0^\infty e^{-su} h_\alpha(u, t)\, du = \mathcal{E}_{\alpha,1}(-st^\alpha), \tag{3.5}$$

where $\mathcal{E}_{\alpha,1}(\cdot)$ is the Mittag–Leffler function formally defined as

$$\mathcal{E}_{\alpha,1}(z) = \sum_{k \geq 0} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}.$$

Finally, we emphasize that the increments of the process $L^{(\alpha)}(t)$ are dependent and non-stationary (see [18, Corollaries 3.3 and 3.4] for more details).

3.2. The time-changed process

The following theorem is a generalization from the previous ones given by Orsingher and Polito [23] for fractional linear birth and death process, and by Meerschaert [19, 20] in the context of the time-fractional Brownian motion and the fractional Poisson processes, respectively. The arguments given below do not differ too much in comparison with the aforementioned references. However, we consider it important to write a detailed formal proof for the general case in order to introduce some techniques that will be used in the forthcoming sections.

Theorem 3.1. *For all $\alpha \in (0, 1)$, the stochastic process $N_\alpha(t)$, $t \geq 0$ admits a representation (in the sense ot the finite-dimensional distributions) of the form*

$$N_\alpha(t) = N_1(L^{(\alpha)}(t)),$$

where $N_1(t)$ is a standard birth and death process and $L^{(\alpha)}(t)$ is the inverse of a stable subordinator independent of $N_1(t)$.

Proof. Given fixed $i > 1$ and $j \geq 0$, for all $t > 0$ we have

$$\mathbb{P}[N_1(L^{(\alpha)}(t)) = j \mid N_1(0) = i] = \int_0^\infty p_{i,j,1}(u) h_\alpha(u, t)\, du.$$

We will show that $\mathbb{P}[N_1(L^{(\alpha)}(t)) = j \mid N_1(0) = i]$ defined above is the solution to the system of fractional differential equations (2.1) through its Laplace transform. We notice first that

$$\Psi_{i,j}(s) = \int_0^\infty e^{-st} \mathbb{P}[N_1(L^{(\alpha)}(t)) = j \mid N_1(0) = i]\, dt$$

satisfies the identity

$$\Psi_{i,j}(s) = s^{\alpha-1} \Phi_{i,j}(s^\alpha),$$

where

$$\Phi_{i,j}(s) = \int_0^\infty e^{-st} p_{i,j,1}(t)\, dt$$

is the Laplace transform of $p_{i,j,1}(t)$. This follows from the simple computation

$$\begin{aligned}
\Psi_{i,j}(s) &= \int_0^\infty e^{-st}\left(\int_0^\infty p_{i,j,1}(u)h_\alpha(u,t)\, du\right) dt \\
&= \int_0^\infty p_{i,j,1}(u)\left(\int_0^\infty e^{-st}h_\alpha(u,t)\, dt\right) du \\
&= s^{\alpha-1}\int_0^\infty p_{i,j,1}(u)\, e^{-s^\alpha u}\, du \\
&= s^{\alpha-1}\Phi_{i,j}(s^\alpha).
\end{aligned} \tag{3.6}$$

By keeping $j \geq 0$ fixed, it is not difficult to check that the sequence $\Phi_{i,j}(s^\alpha)$ defined above satisfies the recurrence relation

$$s^\alpha \Phi_{i,j}(s^\alpha) - \delta_{i,j} = \mu_i \Phi_{i-1,j}(s^\alpha) - (\lambda_i + \mu_i)\Phi_{i,j}(s^\alpha) + \lambda_i \Phi_{i+1,j}(s^\alpha), \quad i \geq 1. \tag{3.7}$$

By combining (3.6) and (3.7) we get

$$\frac{s\Psi_{i,j}(s) - \delta_{i,j}}{s^{1-\alpha}} = \mu_i \Psi_{i-1,j}(s) - (\lambda_i + \mu_i)\Psi_{i,j}(s) + \lambda_i \Psi_{i+1,j}(s). \tag{3.8}$$

Since $\mathbb{P}[N_1(L^{(\alpha)}(0)) = j \mid N_1(0) = i] = \delta_{i,j}$, from (A.3) we have the identities

$$\begin{aligned}
\frac{s\Psi_{i,j}(s) - \delta_{i,j}}{s^{1-\alpha}} &= s^\alpha \Psi_{i,j}(s) - s^{\alpha-1}\mathbb{P}[N_1(L^{(\alpha)}(0)) = j \mid N_1(0) = i] \\
&= \int_0^\infty e^{-st}\mathbb{D}^\alpha(\mathbb{P}[N_1(L^{(\alpha)}(t)) = j \mid N_1(0) = i])\, dt.
\end{aligned}$$

By taking the inverse of the Laplace transform in (3.8), we conclude that $\mathbb{P}[N_1(L^{(\alpha)}(t)) = j \mid N_1(0) = i]$ is the solution to (2.1), finishing the proof. □

Remark 3.1. The trajectories of the process $L^{(\alpha)}(t)$, $t \geq 0$ are non-decreasing, so that Theorem 3.1 can be used to obtain the finite-dimensional distributions of $N_\alpha(t)$,

$$\begin{aligned}
&\mathbb{P}[N_\alpha(t_\ell) = j_\ell; 1 \leq \ell \leq n \mid N_\alpha(0) = i] \\
&= \mathbb{P}[N_1(L^{(\alpha)}(t_\ell)) = j_\ell; 1 \leq \ell \leq n \mid N_\alpha(0) = i] \\
&= \prod_{\ell=0}^{n-1}\mathbb{P}[N_1(L^{(\alpha)}(t_{\ell+1}) - L^{(\alpha)}(t_\ell)) = j_{\ell+1} \mid N_\alpha(t_\ell) = j_\ell],
\end{aligned}$$

with the convention $j_0 = i$, $t_0 = 0$.

3.3. Markov renewal process

Definition 3.1. Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with states in $\mathbb{N}_0$ and let $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ be a sequence of random times satisfying $\mathcal{S}_0 = 0$, $\mathcal{S}_n < \mathcal{S}_{n+1}$ for all $n \geq 0$. The stochastic process $(X_n, \mathcal{S}_n)_{n\in\mathbb{N}_0}$ is called a Markov renewal process with state space $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ if the identity

$$\mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1} \leq t \mid (X_\ell, \mathcal{S}_\ell)_{0\leq\ell\leq n}] = \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1} \leq t \mid X_n] \qquad (3.9)$$

holds for all $j \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, and $t \geq 0$. Here $\mathcal{T}_{n+1} = \mathcal{S}_{n+1} - \mathcal{S}_n$, $n \geq 0$ are called the inter-arrival times. The stochastic process $(Y(t))_{t\geq 0}$ defined as $Y(t) = \sum_{n\geq 0} X_n \mathbb{1}_{\mathcal{S}_n \leq t < \mathcal{S}_{n+1}}$ is called the minimal semi-Markov process associated with $(X_n, \mathcal{S}_n)_{n\in\mathbb{N}_0}$.

Connected to a Markov renewal process, we consider the transition probabilities

$$p_{i,j} = \mathbb{P}[X_{n+1} = j \mid X_n = i]$$

and the kernel

$$Q_{i,j}(t) = \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1} \leq t \mid X_n = i].$$

The transition probabilities are recovered in the limit $t \to \infty$,

$$\lim_{t\to\infty} Q_{i,j}(t) = \lim_{t\to\infty} \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1} \leq t \mid X_n = i] = p_{i,j}.$$

By introducing the notation

$$G_{i,j}(t) = \frac{Q_{i,j}(t)}{p_{i,j}},$$

from a direct computation we get

$$\mathbb{P}[\mathcal{T}_{n+1} \leq t \mid X_n = i, X_{n+1} = j] = G_{i,j}(t),$$

and more generally, for all finite collections of times $0 < t_1 < t_2 < \cdots < t_n$,

$$\mathbb{P}[\mathcal{T}_m \leq t_m; 1 \leq m \leq n \mid (X_\ell)_{0\leq\ell\leq n}] = \prod_{\ell=1}^{n} G_{X_{\ell-1}, X_\ell}(t_\ell). \qquad (3.10)$$

Equation (3.10) implies that the inter-arrival times $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ are conditionally independent given the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with a distribution $G_{X_\ell, X_{\ell+1}}(\cdot)$ depending only on X_ℓ and $X_{\ell+1}$. It is well known that a Markov renewal process is characterized by $G_{X_\ell, X_{\ell+1}}(t)$ and the transition probabilities $p_{i,j}$. In addition, this is a Markov process if and only if $G_{X_\ell, X_{\ell+1}}(t) = 1 - \mathrm{e}^{-r(X_\ell, X_{\ell+1})t}$, for some positive rate $r(X_\ell, X_{\ell+1})$. A more detailed review of Markov renewal processes can be found in [4, Chapter 8]. In the following theorem we provide a characterization for the sample paths of the process $(N_\alpha(t))_{t\geq 0}$.

Theorem 3.2. *For all $\alpha \in (0, 1)$ the process $(N_\alpha(t))_{t\geq 0}$ admits a representation of the form*

$$N_\alpha(t) = \sum_{n\geq 0} X_n \mathbb{1}_{\mathcal{S}_{n,\alpha} \leq t < \mathcal{S}_{n+1,\alpha}},$$

where $(X_n, \mathcal{S}_{n,\alpha})_{n\geq 0}$ is a Markov renewal process. The transition probabilities are

$$\mathbb{P}[X_{n+1} = i + 1 \mid X_n = i] = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad \mathbb{P}[X_{n+1} = i - 1 \mid X_n = i] = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i \geq 1, \quad (3.11)$$

and $\mathbb{P}[X_{n+1} = j \mid X_n = 0] = \delta_{0,j}, j \geq 0$. *Before $L_0 = \inf\{n > 0 \colon X_n = 0\}$, the distributions of the inter-arrival times*

$$\mathcal{T}_{m+1,\alpha} = \mathcal{S}_{m+1,\alpha} - \mathcal{S}_{m,\alpha}, \quad 0 \leq m \leq n,$$

conditioned on $(X_\ell)_{0 \leq \ell \leq n}$, $1 \leq n \leq L_0 - 1$, are independent and they follow a Mittag–Leffler distribution with parameter $\lambda_{X_n} + \mu_{X_n}$, that is,

$$G_{i,j}(t) = 1 - \mathcal{E}_{\alpha,1}(-(\lambda_i + \mu_i)t^\alpha), \quad i \geq 1, \ j \geq 1.$$

Proof. We first construct the process associated with $\alpha = 1$. Let $(X_n)_{n \in \mathbb{N}_0}$ be a Markov chain with transition probabilities given by (3.11) and the killing condition $\mathbb{P}[X_{n+1} = j \mid X_n = 0] = \delta_{0,j}, j \geq 1$. Let $(\mathcal{S}_{n,1})_{n \in \mathbb{N}_0}$ be an increasing sequence of random variables defined recursively as $\mathcal{S}_{0,1} = 0$, $\mathcal{S}_{n,1} = \sum_{\ell=1}^{n} \mathcal{T}_{\ell,1}$, $n \geq 1$. Here the sequence of random variables $\mathcal{T}_{n+1,1} = \mathcal{S}_{n+1,1} - \mathcal{S}_{n,1}$ satisfies

$$\mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,1} \leq t \mid (X_\ell, \mathcal{S}_{\ell,1})_{0 \leq \ell \leq n}] = \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,1} \leq t \mid X_n]$$

for all $t \geq 0$, $n \in \mathbb{N}_0$ and they are conditionally independent of $(X_n)_{n \in \mathbb{N}_0}$. Let us recall the definition $L_0 = \inf\{n > 0 \colon X_n = 0\}$. For $0 \leq n \leq L_0 - 1$, the distribution of $\mathcal{T}_{n+1,1}$ conditioned on (X_n, X_{n+1}) is

$$\mathbb{P}[\mathcal{T}_{n+1,1} \leq t \mid X_{n+1} = j, X_n = i] = 1 - e^{-(\lambda_i + \mu_i)t}, \quad t \geq 0, \ i \geq 1.$$

Note that the origin is an absorbing state, so we automatically have $X_\ell = 0$ for all $\ell \geq L_0$ and consequently $N_1(t) = 0$ for all $t \geq S_{L_0,1}$. This implies that the sequence of random variables $(\mathcal{T}_{\ell,1})_{\ell \geq L_0+1}$ has no influence on the evolution of $(N_1(t))_{t \geq S_{L_0,1}}$. The process $(N_1(t))_{t \geq 0}$ is defined by

$$N_1(t) := \sum_{n \geq 0} X_n \mathbb{1}_{\mathcal{S}_{n,1} \leq t < \mathcal{S}_{n+1,1}} = \sum_{n \geq 0} X_{n \wedge L_0} \mathbb{1}_{\mathcal{S}_{n,1}, \leq t < \mathcal{S}_{n+1,1}},$$

where $n \wedge L_0 = \min\{n, L_0\}$ is a Markov process (see [4, Chapter 8, Section 3] for more details). Now, from Theorem 3.1, we know that $(N_\alpha(t))_{t \geq 0} = (N_1(L^{(\alpha)}(t)))_{t \geq 0}$ is a time-changed version of $N_1(t)$, so

$$N_\alpha(t) := \sum_{n \geq 0} X_{n \wedge L_0} \mathbb{1}_{\mathcal{S}_{n,1} \leq L^{(\alpha)}(t) < \mathcal{S}_{n+1,1}},$$

where $L^{(\alpha)}(t)$ is the inverse of a stable subordinator $(D^{(\alpha)}(t))_{t \geq 0}$ independent of $(X_n, \mathcal{S}_{n,1})_{n \in \mathbb{N}_0}$. The sequence $(\mathcal{S}_{n,\alpha})_{n \in \mathbb{N}_0}$ is defined via the equivalence

$$\mathcal{S}_{n,\alpha} \leq t \iff \mathcal{S}_{n,1} \leq L^{(\alpha)}(t), \quad t \geq 0, \ n \geq 1.$$

Once the process $N_\alpha(t)$ is properly defined, the next step is to prove that the pairs $(X_n, \mathcal{S}_{n,\alpha})_{n \in \mathbb{N}_0}$ defined above form a Markov renewal process with the specified properties. To prove the condition (3.9), we first note that

$$\{\mathcal{S}_{\ell,\alpha} \leq t_\ell \colon 1 \leq \ell \leq n\} \iff \{\mathcal{S}_{\ell,1} \leq L^{(\alpha)}(t_\ell); 1 \leq \ell \leq n\}, \quad t \geq 0, \ n \geq 1.$$

for any collection $0 = t_0 < t_1 < \cdots t_n$. From the definition of an inverse subordinator we have $\mathcal{S}_{n,1} \leq L^{(\alpha)}(t) \iff D^{(\alpha)}(\mathcal{S}_{n,1}) \leq t$ for all $t \geq 0$, so that $\mathcal{S}_{n,\alpha} = D^{(\alpha)}(\mathcal{S}_{n,1})$ in the finite distributional sense. This implies that for all $n \geq 0$

$$\mathcal{S}_{n+1,\alpha} - \mathcal{S}_{n,\alpha} = D^{(\alpha)}(\mathcal{S}_{n+1,1}) - D^{(\alpha)}(\mathcal{S}_{n,1})$$

$$= D^{(\alpha)}(\mathcal{T}_{n+1,1} + \mathcal{S}_{n,1}) - D^{(\alpha)}(\mathcal{S}_{n,1}). \tag{3.12}$$

Since $D^{(\alpha)}(\cdot)$ is a stable subordinator, (3.12) implies that $\mathcal{T}_{n+1,\alpha} = \mathcal{S}_{n+1,\alpha} - \mathcal{S}_{n,\alpha}$ is independent of $\mathcal{S}_{\ell,\alpha} = D^{(\alpha)}(\mathcal{S}_{\ell,1})$, $1 \leq \ell \leq n$ and has the same distribution as $D^{(\alpha)}(\mathcal{T}_{n+1,1})$. But $D^{(\alpha)}(\mathcal{T}_{n+1,1}) \leq t$ is equivalent to $\mathcal{T}_{n+1,1} \leq L^{(\alpha)}(t)$. Consequently it follows that

$$\mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,\alpha} \leq t \mid (X_\ell = i_\ell, \mathcal{S}_{\ell,\alpha} \leq t_\ell)_{1 \leq \ell \leq n}]$$

$$= \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,1} \leq L^{(\alpha)}(t) \mid (X_\ell = i_\ell, \mathcal{S}_{\ell,1} \leq L^{(\alpha)}(t_\ell))_{0 \leq \ell \leq n}]$$

$$= \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,1} \leq L^{(\alpha)}(t) \mid X_n = i_n]$$

$$= \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,\alpha} \leq t \mid X_n = i_n].$$

We will now show that $N_\alpha(t)$ preserves the transition probabilities of the original process $N_1(t)$. Since $\lim_{t \to \infty} L^{(\alpha)}(t) = \infty$ with probability 1, we get for all $j \geq 0$, $i \geq 0$

$$p_{i,j}^\alpha = \lim_{t \to \infty} \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,\alpha} \leq t \mid X_n = i]$$

$$= \lim_{t \to \infty} \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,1} \leq L^{(\alpha)}(t) \mid X_n = i]$$

$$= \lim_{u \to \infty} \mathbb{P}[X_{n+1} = j, \mathcal{T}_{n+1,1} \leq u \mid X_n = i]$$

$$= p_{i,j}.$$

The next step is to prove that the random variables $\mathcal{T}_{n+1,\alpha}$, conditioned on (X_n, X_{n+1}), $1 \leq n \leq L_0 - 1$, follow a Mittag–Leffler distribution. For $i \geq 1$ we have

$$\mathbb{P}[\mathcal{T}_{n+1,\alpha} > t \mid X_{n+1} = j, X_n = i] = \mathbb{P}[\mathcal{T}_{n+1,1} > L^{(\alpha)}(t) \mid X_{n+1} = j, X_n = i]$$

$$= \int_0^\infty \mathbb{P}[\mathcal{T}_{n+1,1} > u \mid X_{n+1} = j, X_n = i] h_\alpha(u, t) \, du$$

$$= \int_0^\infty e^{-(\lambda_i + \mu_i)u} h_\alpha(u, t) \, du$$

$$= \mathbb{E}[e^{-(\lambda_i + \mu_i)L^{(\alpha)}(t)}]$$

$$= \mathcal{E}_{\alpha,1}(-(\lambda_i + \mu_i)t^\alpha).$$

The independence of $(\mathcal{T}_{\ell,\alpha})_{1 \leq \ell \leq n}$ conditioned on $(X_\ell)_{0 \leq \ell \leq n}$, is more delicate. We first notice that from (A.4), for all $s > 0$ and $i \geq 1$, we have

$$\int_0^\infty e^{-st} \mathbb{P}[\mathcal{T}_{n+1,\alpha} \leq t \mid X_{n+1} = j, X_n = i] \, dt = \int_0^\infty e^{-st}(1 - \mathcal{E}_{\alpha,1}(-(\lambda_i + \mu_i)t^\alpha)) \, dt$$

$$= \frac{1}{s} \frac{\lambda_i + \mu_i}{s^\alpha + \lambda_i + \mu_i}. \tag{3.13}$$

By integrating by parts, we can deduce that the identity (3.13) is the same as

$$\mathbb{E}[e^{-s\mathcal{T}_{n+1,\alpha}} \mid X_{n+1} = j, X_n = i] = \frac{\lambda_i + \mu_i}{s^\alpha + \lambda_i + \mu_i}. \tag{3.14}$$

Now we set

$$\varphi_{n,\alpha}(s) = \int_0^\infty e^{-st} \mathbb{P}[\mathcal{S}_{n,\alpha} \le t \mid (X_\ell)_{0 \le \ell \le n}] \, dt.$$

Similarly to (3.14), from the construction of $\varphi_{n,\alpha}(s)$ we can see that

$$\varphi_{n,\alpha}(s) = \frac{1}{s} \mathbb{E}\big[e^{-s\mathcal{S}_{n,\alpha}} \mid (X_\ell)_{0 \le \ell \le n} \big] = \frac{1}{s} \mathbb{E}\left[\prod_{m=1}^n e^{-s\mathcal{T}_{m,\alpha}} \mid (X_\ell)_{0 \le \ell \le n} \right]. \tag{3.15}$$

We need to compute $\varphi_{n,\alpha}(s)$ more explicitly. Since $N_\alpha(t)$ is a time-changed process, we have

$$\varphi_{n,\alpha}(s) = \int_0^\infty e^{-st} \mathbb{P}[\mathcal{S}_{n,1} \le L^{(\alpha)}(t) \mid (X_\ell)_{0 \le \ell \le n}] \, dt$$

$$= \int_0^\infty e^{-st} \left(\int_0^\infty \mathbb{P}[\mathcal{S}_{n,1} \le u \mid (X_\ell)_{0 \le \ell \le n}] h_\alpha(u, t) \, du \right) dt,$$

by using Fubini's theorem and recalling the identity

$$\int_0^\infty e^{-st} h_\alpha(u, t) \, dt = s^{\alpha-1} e^{-us^\alpha}.$$

Thus we obtain

$$\varphi_{n,\alpha}(s) = \int_0^\infty e^{-st} \left(\int_0^\infty \mathbb{P}[\mathcal{S}_{n,1} \le u \mid (X_\ell)_{0 \le \ell \le n}] h_\alpha(u, t) \, du \right) dt$$

$$= \int_0^\infty \mathbb{P}[\mathcal{S}_{n,1} \le u \mid (X_\ell)_{0 \le \ell \le n}] \left(\int_0^\infty e^{-st} h_\alpha(u, t) \, dt \right) du$$

$$= \int_0^\infty \mathbb{P}[\mathcal{S}_{n,1} \le u \mid (X_\ell)_{0 \le \ell \le n}] s^{\alpha-1} e^{-s^\alpha u} \, du$$

$$= s^{\alpha-1} \varphi_{n,1}(s^\alpha).$$

For $\alpha = 1$, conditioned on $(X_\ell)_{0 \le \ell \le n}$, the inter-arrival times are independent and exponentially distributed, so

$$\varphi_{n,1}(s^\alpha) = \frac{1}{s^\alpha} \prod_{\ell=0}^{n-1} \frac{\lambda_{X_\ell} + \mu_{X_\ell}}{s^\alpha + \lambda_{X_\ell} + \mu_{X_\ell}},$$

leading to the formula

$$\varphi_{n,\alpha}(s) = \frac{1}{s} \prod_{\ell=0}^{n-1} \frac{\lambda_{X_\ell} + \mu_{X_\ell}}{s^\alpha + \lambda_{X_\ell} + \mu_{X_\ell}}. \tag{3.16}$$

By combining (3.14), (3.15), and (3.16), we now get

$$s\varphi_{n,\alpha}(s) = \mathbb{E}\left[\prod_{m=1}^n e^{-s\mathcal{T}_{m,\alpha}} \mid (X_\ell)_{0 \le \ell \le n} \right] = \prod_{m=1}^n \mathbb{E}\big[e^{-s\mathcal{T}_{m,\alpha}} \mid (X_m, X_{m-1}) \big]. \tag{3.17}$$

Equation (3.17) directly implies that for all $1 \le n \le L_0 - 1$, conditioned on $(X_\ell)_{0 \le \ell \le n}$, the inter-arrival times $\mathcal{T}_{\ell+1,\alpha} = \mathcal{S}_{\ell+1,\alpha} - \mathcal{S}_{\ell,\alpha}$, $0 \le \ell \le n$ are independent, concluding the proof. $\qquad\square$

4. The spectral representation of the transition probabilities

4.1. Preliminaries

For all $i \geq 0$ fixed, we let

$$T_{i,\alpha} = \inf\{t > 0 \colon N_\alpha(t) = i\}$$

denote the first time the process $N_\alpha(t)$ attains the state i. In particular, for $i = 0$, we say that $T_{0,\alpha}$ is the absorption time or the extinction time of the process. For the sake of convenience, for $\alpha = 1$ we write T_0 instead of $T_{0,1}$. In the following we will use the notation $\mathbb{P}_i[\cdot]$ for $\mathbb{P}[\cdot \mid N_\alpha(0) = i]$ and $\mathbb{E}_i[\cdot]$ for $\mathbb{E}[\cdot \mid N_\alpha(0) = i]$ to denote the dependence on the initial condition $N_\alpha(0)$. For $n \geq 1$ we define the coefficients

$$\pi_1 = 1,$$

$$\pi_n = \prod_{i=1}^{n-1} \frac{\lambda_i}{\mu_{i+1}}, \quad n \geq 2.$$

Note that these coefficients satisfy the identity $\pi_{n+1}/\pi_n = \lambda_n/\mu_{n+1}$. This implies that $Q^{(a)}$ is reversible with respect to the measure π, that is,

$$\pi_i q_{i,j} = \pi_j q_{j,i} \quad \text{for all } i, j \geq 1.$$

The following series are essential to describe some properties of the process:

$$A = \sum_{i \geq 1} (\lambda_i \pi_i)^{-1}, \quad B = \sum_{i \geq 1} \pi_i, \quad D = \sum_{j \geq 1} \pi_j \sum_{k=1}^{j-1} \frac{1}{\lambda_k \pi_k}.$$

For $\alpha = 1$, some well-known results are as follows (see e.g. [6, Chapter 5]).

(1) The process is almost surely absorbed at zero, i.e. $\mathbb{P}_i[T_0 < \infty] = 1$, if and only if $A = \infty$.

(2) The absorption time has a finite mean, i.e. $\mathbb{E}_i[T_0] < \infty$, if and only if $B < \infty$.

(3) The process comes from infinity, i.e. $\sup_{i \geq 1} \mathbb{E}_i[T_0] < \infty$, if and only if $D < \infty$.

4.2. Main results

Here we present the approach of Karlin and McGregor's polynomials originally introduced in [14] ([6, Section 5.2] is also recommended for a detailed discussion). Given a fixed $\theta > 0$, we define recursively the sequence of polynomials

$$-\theta \psi_i(\theta) = \mu_i \psi_{i-1}(\theta) - (\lambda_i + \mu_i)\psi_i(\theta) + \lambda_i \psi_{i+1}(\theta), \quad i \geq 1,$$

$$\psi_0(\theta) = 0, \quad \psi_1(\theta) = 1. \tag{4.1}$$

These polynomials satisfy the orthogonality condition

$$\pi_j \int_{\theta^\star}^{\infty} \psi_j(\theta)\psi_k(\theta)\,\mathrm{d}\Gamma(\theta) = \delta_{j,k}, \tag{4.2}$$

where Γ is a probability measure supported in $[\theta^\star, \infty)$, for some $\theta^\star \geq 0$, and $\delta_{j,k}$ is the Kronecker delta.

Based on these polynomials, it can be shown that when $\alpha = 1$, the spectral representation of the transition probabilities is

$$p_{i,j,1}(t) = \pi_j \int_{\theta^\star}^\infty e^{-t\theta} \psi_i(\theta) \psi_j(\theta) \, d\Gamma(\theta), \quad j \geq 1, \ i \geq 1, \tag{4.3}$$

The following theorem generalizes the representation (4.3) to the complete case $\alpha \in (0, 1]$.

Theorem 4.1. *The solution to the system of equations* (2.1) *can be written as*

$$p_{i,j,\alpha}(t) = \pi_j \int_{\theta^\star}^\infty \mathcal{E}_{\alpha,1}(-\theta t^\alpha) \psi_i(\theta) \psi_j(\theta) \, d\Gamma(\theta), \quad i \geq 1, \ j \geq 1, \tag{4.4}$$

where $\mathcal{E}_{\alpha,1}(\cdot)$ is the Mittag–Leffler function with parameter α.

Proof. By combining Theorem 3.1 and equation (4.3), we have for $j \geq 1$, $i \geq 1$

$$\begin{aligned}
p_{i,j,\alpha}(t) &= \mathbb{P}_i[N_1(L^{(\alpha)}(t)) = j] \\
&= \int_0^\infty \mathbb{P}_i[N_1(u) = j] h_\alpha(u, t) \, du \\
&= \pi_j \int_0^\infty \left(\int_{\theta^\star}^\infty e^{-\theta u} \psi_i(\theta) \psi_j(\theta) \, d\Gamma(\theta) \right) h_\alpha(u, t) \, du. \tag{4.5}
\end{aligned}$$

We now define the integrals

$$I_{i,j}(t) := \int_{\theta^\star}^\infty e^{-\theta t} \psi_i(\theta) \psi_j(\theta) \, d\Gamma(\theta), \quad i \geq 1, \ j \geq 1.$$

Recalling that Γ is a probability measure, it is clear that $I_{j,j}(t) \leq I_{j,j}(0)$ for all $t \geq 0$, $j \geq 1$. In addition, from the Cauchy–Schwarz inequality and (4.2) we get

$$0 \leq \int_{\theta^\star}^\infty e^{-\theta t} |\psi_i(\theta) \psi_j(\theta)| \, d\Gamma(\theta) \leq \sqrt{I_{i,i}(t) I_{j,j}(t)} \leq \sqrt{I_{i,i}(0) I_{j,j}(0)} = \frac{1}{\sqrt{\pi_i \pi_j}} < \infty. \tag{4.6}$$

For each $t \geq 0$ fixed, $h_\alpha(u, t)$ is a probability density function supported on $\mathbb{R}_0^+$. By applying Fubini's theorem in (4.5) we get

$$p_{i,j,\alpha}(t) = \pi_j \int_{\theta^\star}^\infty \left(\int_0^\infty e^{-\theta u} h_\alpha(u, t) \, du \right) \psi_i(\theta) \psi_j(\theta) \, d\Gamma(\theta),$$

and recalling the identity

$$\mathcal{E}_{\alpha,1}(-\theta t^\alpha) = \int_0^\infty e^{-\theta u} h_\alpha(u, t) \, du,$$

we deduce (4.4), concluding the proof. $\qquad \square$

The following theorem provides an explicit form for the distribution of the absorption time.

Theorem 4.2. *Assume $A = \infty$. For all $i \geq 1$, the probability of non-extinction before the instant $t \geq 0$ is*

$$\mathbb{P}_i[T_{0,\alpha} > t] = \mu_1 \int_{\theta^\star}^\infty \mathcal{E}_{\alpha,1}(-\theta t^\alpha) \frac{\psi_i(\theta)}{\theta} \, d\Gamma(\theta).$$

Proof. If $A = \infty$, it is well known that the probability measure Γ satisfies $\Gamma(\{0\}) = 0$, and moreover $\mathrm{d}\Gamma^d(\theta) = \mu_1 \theta^{-1} \, \mathrm{d}\Gamma(\theta)$ is a probability measure with no atoms at zero (see formula (2.20) from [8]). In this case we have for $\alpha = 1$

$$\mathbb{P}_i[T_0 > t] = 1 - p_{i,0,1}(t) = 1 - \mu_1 \int_{\theta^\star}^{\infty} \mathrm{e}^{-t\theta} \frac{\psi_i(\theta)}{\theta} \, \mathrm{d}\Gamma(\theta), \quad i \geq 1.$$

We first check that

$$\int_{\theta^\star}^{\infty} \frac{1}{\theta} |\psi_i(\theta)| \, \mathrm{d}\Gamma(\theta) < \infty, \quad i \geq 1.$$

For any $\delta > 0$ we have

$$\int_{\theta^\star}^{\infty} \frac{1}{\theta} |\psi_i(\theta)| \, \mathrm{d}\Gamma(\theta) \leq \left(\max_{\theta \in [\theta^\star, \theta^\star + \delta]} |\psi_i(\theta)| \right) \int_{\theta^\star}^{\delta + \theta^\star} \frac{1}{\theta} \, \mathrm{d}\Gamma(\theta) + \frac{1}{\delta + \theta^\star} \int_{\delta + \theta^\star}^{\infty} |\psi_i(\theta)| \, \mathrm{d}\Gamma(\theta). \tag{4.7}$$

For the first term on the right-hand side of (4.7), we have that $\psi_i(\theta)$ is a polynomial of degree $i - 1$, so it is bounded on any closed interval. Since $\mathrm{d}\Gamma^d$ is a probability measure, we have

$$0 \leq \int_{\theta^\star}^{\delta + \theta^\star} \frac{1}{\theta} \, \mathrm{d}\Gamma(\theta) \leq \int_{\theta^\star}^{\infty} \frac{1}{\theta} \, \mathrm{d}\Gamma(\theta) = \frac{1}{\mu_1}.$$

By choosing $j = 1$ in (4.6) we get

$$0 \leq \int_{\delta + \theta^\star}^{\infty} |\psi_i(\theta)| \, \mathrm{d}\Gamma(\theta) \leq \int_{\theta^\star}^{\infty} |\psi_i(\theta)| \, \mathrm{d}\Gamma(\theta) \leq \frac{1}{\sqrt{\pi_i}} < \infty.$$

The proof finishes by following the same approach as Theorem 4.1. $\qquad \square$

Remark 4.1. The transition probabilities $p_{i,j,\alpha}(t)$ are also the solution to the forward system of equations

$$\mathbb{D}^\alpha p_{i,j,\alpha}(t) = \lambda_{j-1} p_{i,j-1,\alpha}(t) - (\lambda_j + \mu_j) p_{i,j,\alpha}(t) + \mu_{j+1} p_{i,j+1,\alpha}(t), \quad j \geq 1, \ i \geq 1, \tag{4.8}$$

$$\mathbb{D}^\alpha p_{i,0,\alpha}(t) = \mu_1 p_{i,1,\alpha}(t), \tag{4.9}$$

with the convention $\lambda_0 = \mu_0 = 0$. In fact, by combining (2.1), (4.1), and (4.4), for $i \geq 1, j \geq 1$ we have

$$\mathbb{D}^\alpha p_{i,j,\alpha}(t) = \mu_i p_{i-1,j,\alpha}(t) - (\lambda_i + \mu_i) p_{i,j,\alpha}(t) + \lambda_i p_{i+1,j,\alpha}(t)$$

$$= \pi_j \int_{\theta^\star}^{\infty} \mathcal{E}_{\alpha,1}(-\theta t^\alpha)(\mu_i \psi_{i-1}(\theta) - (\lambda_i + \mu_i)\psi_i(\theta) + \lambda_i \psi_{i+1}(\theta))\psi_j(\theta) \, \mathrm{d}\Gamma(\theta)$$

$$= \pi_j \int_{\theta^\star}^{\infty} \mathcal{E}_{\alpha,1}(-\theta t^\alpha)(-\theta \psi_i(\theta))\psi_j(\theta) \, \mathrm{d}\Gamma(\theta).$$

We now apply the recursive formula (4.1) to $-\theta \psi_j(\theta)$,

$$\mathbb{D}^\alpha p_{i,j,\alpha}(t) = \pi_j \int_{\theta^\star}^{\infty} \mathcal{E}_{\alpha,1}(-\theta t^\alpha)\psi_i(\theta)(\mu_j \psi_{j-1}(\theta) - (\lambda_j + \mu_j)\psi_j(\theta) + \lambda_j \psi_{j+1}(\theta)) \, \mathrm{d}\Gamma(\theta),$$

and we use (4.4) to obtain

$$\mathbb{D}^{\alpha} p_{i,j,\alpha}(t) = \frac{\pi_j \mu_j}{\pi_{j-1}} p_{i,j-1,\alpha}(t) - (\mu_j + \lambda_j) p_{i,j,\alpha}(t) + \frac{\lambda_j \pi_j}{\pi_{j+1}} p_{i,j+1,\alpha}(t).$$

This is equivalent to (4.8), since $\pi_j \mu_j = \pi_{j-1} \lambda_{j-1}$ and $\pi_j \lambda_j = \pi_{j+1} \mu_{j+1}$. For $j = 0$, (4.9) can be deduced in a similar way. From Theorem 4.2 we get for all $i \geq 1$

$$p_{i,0,\alpha}(t) = 1 - \mathbb{P}_i[T_{0,\alpha} > t] = 1 - \mu_1 \int_{\theta^\star}^{\infty} \mathcal{E}_{\alpha,1}(-\theta t^\alpha) \frac{\psi_i(\theta)}{\theta} \, \mathrm{d}\Gamma(\theta). \qquad (4.10)$$

By replacing (4.10) in the backward equation (2.1) for $j = 0$, $i \geq 1$,

$$\mathbb{D}^{\alpha} p_{i,0,\alpha}(t) = \mu_i p_{i-1,0,\alpha}(t) - (\lambda_i + \mu_i) p_{i,0,\alpha}(t) + \lambda_i p_{i+1,0,\alpha}(t)$$

$$= -\mu_1 \int_{\theta^\star}^{\infty} \mathcal{E}_{\alpha,1}(-\theta t^\alpha) \frac{\mu_i \psi_{i-1}(\theta) - (\lambda_i + \mu_i) \psi_i(\theta) + \lambda_i \psi_{i+1}(\theta)}{\theta} \, \mathrm{d}\Gamma(\theta),$$

and from the recurrence relation (4.1),

$$\mathbb{D}^{\alpha} p_{i,j,\alpha}(t) = \mu_1 \int_{\theta^\star}^{\infty} \mathcal{E}_{\alpha,1}(-\theta t^\alpha) \psi_i(\theta) \, \mathrm{d}\Gamma(\theta) = \mu_1 p_{i,1,\alpha}(t).$$

5. Asymptotic behavior and quasi-limiting distributions

Theorem 4.2 allows us to deduce some new results related to the quasi-limiting behavior. We start by defining the integrals

$$C_{i,j,k} = \int_{\theta^\star}^{\infty} \frac{\psi_i(\theta) \psi_j(\theta)}{\theta^k} \, \mathrm{d}\Gamma(\theta), \quad i \geq 1, \ j \geq 1, \ k \geq 0.$$

When $\theta^\star > 0$, the coefficients $C_{i,j,k}$ are finite. In fact

$$C_{i,j,k} \leq \frac{1}{(\theta^\star)^k} \left(\int_{\theta^\star}^{\infty} \psi_i^2(\theta) \, \mathrm{d}\Gamma(\theta) \right)^{1/2} \left(\int_{\theta^\star}^{\infty} \psi_j^2(\theta) \, \mathrm{d}\Gamma(\theta) \right)^{1/2} < \infty.$$

From a probabilistic point of view, the parameter $\theta^\star$ can be identified as

$$\theta^* := \sup\{\theta : \mathbb{E}_i(\mathrm{e}^{\theta T_0}) < \infty\} = -\liminf_{t \to \infty} \frac{1}{t} \ln \mathbb{P}_i(T_0 > t), \quad i \geq 1.$$

We remark that the definition of $\theta^\star$ does not depend on i. In particular, the condition $\theta^\star > 0$ implies that the absorption time T_0 has finite exponential moments for all $\theta \in (0, \theta^\star)$ (see [6, Lemma 4.1]). In the following proposition, we establish the asymptotic behavior of both the transition probabilities and the distribution of the absorption time $T_{0,\alpha}$.

Proposition 5.1. *Assume $\theta^\star > 0$. For all $0 < \alpha < 1$ the following limits are fulfilled:*

$$\lim_{t \to \infty} t^\alpha \mathbb{P}_i[T_{0,\alpha} > t] = \frac{\mu_1}{\Gamma(1-\alpha)} \int_{\theta^\star}^{\infty} \frac{\psi_i(\theta)}{\theta^2} \, \mathrm{d}\Gamma(\theta), \qquad (5.1)$$

$$\lim_{t \to \infty} t^\alpha p_{i,j,\alpha}(t) = \frac{\pi_j}{\Gamma(1-\alpha)} \int_{\theta^\star}^{\infty} \frac{\psi_i(\theta) \psi_j(\theta)}{\theta} \, \mathrm{d}\Gamma(\theta). \qquad (5.2)$$

Proof. For all $\theta > 0$ the following limit applies (see (A.8) from the Appendix):

$$\lim_{t \to \infty} t^{\alpha} \mathcal{E}_{\alpha,1}(-\theta t^{\alpha}) = \frac{1}{\theta} \frac{1}{\Gamma(1-\alpha)}. \tag{5.3}$$

In addition, we know that

$$\int_{\theta^{\star}}^{\infty} \frac{\psi_j(\theta)}{\theta^k} \, d\Gamma(\theta) < \infty \quad \text{for all } k \geq 1.$$

Given $\epsilon > 0$, the limit (5.3) in particular implies that

$$\mathcal{E}_{\alpha,1}(-\theta t^{\alpha}) \leq \frac{1+\epsilon}{\theta \Gamma(1-\alpha)} t^{-\alpha}$$

for $t > t_{\epsilon}$ large enough. This means that the limit (5.1) follows from the dominated convergence theorem

$$\lim_{t \to \infty} t^{\alpha} \mathbb{P}_i[T_{0,\alpha} > t] = \mu_1 \int_{\theta^{\star}}^{\infty} \left(\lim_{t \to \infty} t^{\alpha} \mathcal{E}_{\alpha,1}(-\theta t^{\alpha}) \right) \frac{\psi_i(\theta)}{\theta} \, d\Gamma(\theta)$$

$$= \frac{\mu_1}{\Gamma(1-\alpha)} \int_{\theta^{\star}}^{\infty} \frac{\psi_i(\theta)}{\theta^2} \, d\Gamma(\theta).$$

The limit (5.2) is proved by using the same argument. □

Proposition 5.1 leads us to the following theorem, interpreted as a Yaglom limit for the fractional case.

Theorem 5.1. *Assume $\theta^{\star} > 0$. For all $0 < \alpha < 1$ we have*

$$\lim_{t \to \infty} \mathbb{P}_i[N_{\alpha}(t) = j \mid T_{0,\alpha} > t] = \frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}}, \tag{5.4}$$

where

$$P_{i,j} = \pi_j \left(P_{i,1} + \sum_{k=1}^{\min\{i-1,j-1\}} \frac{1}{\lambda_k \pi_k} \right), \quad j \geq 2, \tag{5.5}$$

$$P_{i,1} = \frac{1}{\mu_1}.$$

Remark 5.1. Before proving Theorem 5.1, let us make some comments.

1. The limit (5.4) is the same for the complete interval $\alpha \in (0, 1)$ and strongly depends on the initial condition $i \geq 1$. This behavior is completely different to the non-fractional case $\alpha = 1$,

$$\lim_{t \to \infty} \mathbb{P}_i[N_1(t) = j \mid T_0 > t] = \nu_j(\theta^{\star}),$$

because the limit $\nu_j(\theta^{\star})$ is independent of the initial condition $i \geq 1$. Here $\nu_j(\theta^{\star})$ is a probability measure, also called the quasi-stationary distribution or quasi-limiting distribution.

2. We analyze in more detail the two extreme cases for the limit (5.4).

- For $i = 1$, the second term in (5.5) vanishes, so

$$\lim_{t \to \infty} \mathbb{P}_i[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{\pi_j}{\sum_{j \geq 1} \pi_j}.$$

- For $i \to \infty$ (assuming that the limit exists) we get

$$\lim_{i \to \infty} \lim_{t \to \infty} \mathbb{P}_i[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{\pi_j\left(\frac{1}{\mu_1} + \sum_{k=1}^{j-1} \frac{1}{\lambda_k \pi_k}\right)}{\sum_{j \geq 1} \pi_j\left(\frac{1}{\mu_1} + \sum_{k=1}^{j-1} \frac{1}{\lambda_k \pi_k}\right)}.$$

The condition $\sum_{j \geq 1} \pi_j < \infty$ is equivalent to the positive recurrence of the process, i.e. $\mathbb{E}_i[T_0] < \infty$ for all $i \geq 1$. On the other hand, the series

$$\sum_{j \geq 1} \pi_j \sum_{k=1}^{j-1} \frac{1}{\lambda_k \pi_k} = \sum_{k \geq 1} \frac{1}{\lambda_k \pi_k} \sum_{j \geq k+1} \pi_j$$

converges if and only if the process comes down from infinity, i.e. $\sup_{j \geq 1} \mathbb{E}_j[\mathrm{e}^{\theta T_{0,1}}] < \infty$ for all $\theta \in (0, \theta^\star)$. According to Theorem 3.2 of [8], this is equivalent to the existence of a unique quasi-stationary distribution, which is associated with $\theta^\star$. As we mention below, in the fractional model (i.e. $\alpha \in (0, 1)$) the quasi-limiting behavior strongly depends on the initial condition, so its behavior changes drastically compared to the Markovian case. This fact is not at all intuitive, and constitutes an interesting result on asymptotic behavior associated with a long memory process.

Proof of Theorem 5.1. As a direct consequence of Proposition 5.1, we have

$$\lim_{t \to \infty} \mathbb{P}_i[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}}. \tag{5.6}$$

Since $p_{i,j,1}(t) = \mathbb{P}_i[N_1(t) = j, T_0 > t]$, we get from (4.3)

$$\mathbb{P}_i[N_1(t) = j, T_0 > t] = \pi_j \int_{\theta^\star}^{\infty} \mathrm{e}^{-t\theta} \psi_i(\theta) \psi_j(\theta) \, \mathrm{d}\Gamma(\theta).$$

By taking the integral over $t \geq 0$, Fubini's theorem yields

$$\int_0^{\infty} \mathbb{P}_i[N_1(t) = j, T_0 > t] \, \mathrm{d}t = \pi_j \int_0^{\infty} \int_{\theta^\star}^{\infty} \mathrm{e}^{-t\theta} \psi_i(\theta) \psi_j(\theta) \, \mathrm{d}\Gamma(\theta) \, \mathrm{d}t$$

$$= \pi_j \int_{\theta^\star}^{\infty} \frac{\psi_i(\theta) \psi_j(\theta)}{\theta} \, \mathrm{d}\Gamma(\theta), \tag{5.7}$$

and analogously

$$\int_0^{\infty} \mathbb{P}_i[T_0 > t] \, \mathrm{d}t = \mu_1 \int_{\theta^\star}^{\infty} \frac{\psi_i(\theta)}{\theta^2} \, \mathrm{d}\Gamma(\theta), \tag{5.8}$$

so the limit (5.6) is equivalent to

$$\lim_{t \to \infty} \mathbb{P}_i[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{\int_0^{\infty} \mathbb{P}_i[N_1(t) = j, T_0 > t] \, \mathrm{d}t}{\int_0^{\infty} \mathbb{P}_i[T_0 > t] \, \mathrm{d}t}.$$

We recall the system of equations (4.8) for $\alpha = 1$

$$p'_{i,j,1}(t) = \lambda_{j-1}p_{i,j-1,1}(t) - (\lambda_j + \mu_j)p_{i,j,1}(t) + \mu_{j+1}p_{i,j+1,1}(t), \quad j \geq 1. \tag{5.9}$$

By taking the integral over $t \geq 0$, from the right-hand side of (5.9), we obtain for $j \geq 1$

$$\int_0^\infty p'_{i,j,1}(t)\,\mathrm{d}t = \lim_{t\to\infty} p_{i,j,1}(t) - p_{i,j,1}(0) = -\delta_{i,j}.$$

When $j = 0$, from (4.9) we get for $\alpha = 1$ that $p'_{i,0,1}(t) = \mu_1 p_{i,1,\alpha}(t)$, and consequently

$$\int_0^\infty \mathbb{P}_i[N_1(t) = 1, T_0 > t] = \frac{1}{\mu_1}.$$

By introducing the notation

$$P_{i,j} = \int_0^\infty \mathbb{P}_i[N_1(t) = j, T_0 > t]\,\mathrm{d}t,$$

with the convention $P_{i,0} = 0$ we get the recurrence formula

$$-\delta_{i,j} = \lambda_{j-1}P_{i,j-1} - (\lambda_j + \mu_j)P_{i,j} + \mu_{j+1}P_{i,j+1}, \quad j \geq 1,$$

whose solution can be computed explicitly. Let us rearrange some terms,

$$-\delta_{i,j} = (\lambda_{j-1}P_{i,j-1} - \lambda_j P_{i,j}) + (\mu_{j+1}P_{i,j+1} - \mu_j P_{i,j}),$$

by taking the sum over $1 \leq k \leq j$:

$$-\sum_{k=1}^j \delta_{i,k} = \lambda_0 P_{i,0} - \lambda_j P_{i,j} + \mu_{j+1}P_{i,j+1} - \mu_1 P_{i,1}, \quad j \geq 1.$$

Since $P_{i,0} = 0$ and $P_{i,1} = 1/\mu_1$, we get

$$1 - \sum_{k=1}^j \delta_{i,k} = -\lambda_j P_{i,j} + \mu_{j+1}P_{i,j+1}. \tag{5.10}$$

We notice that

$$1 - \sum_{k=1}^j \delta_{i,k} = \begin{cases} 0 & \text{if } j \geq i, \\ 1 & \text{if } j < i, \end{cases}$$

equation (5.10) becomes

$$\mathbb{1}_{j<i} = \mu_{j+1}P_{i,j+1} - \lambda_j P_{i,j}.$$

Recalling the identity

$$\frac{\mu_{j+1}}{\lambda_j \pi_j} = \frac{1}{\pi_{j+1}},$$

we now get

$$\frac{\mathbb{1}_{j<i}}{\lambda_j \pi_j} = \frac{P_{i,j+1}}{\pi_{j+1}} - \frac{P_{i,j}}{\pi_j},$$

whose solution is

$$P_{i,j} = \pi_j \left(P_{i,1} + \sum_{k=1}^{j-1} \frac{\mathbb{1}_{k<i}}{\lambda_k \pi_k} \right), \quad j \geq 2,$$

$$P_{i,1} = \frac{1}{\mu_1}.$$

For $j \geq 2$ this is the same as

$$P_{i,j} = \pi_j \left(\frac{1}{\mu_1} + \sum_{k=1}^{\min\{i-1,j-1\}} \frac{1}{\lambda_k \pi_k} \right), \tag{5.11}$$

so the limit is

$$\lim_{t \to \infty} \mathbb{P}_i[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}},$$

concluding the proof. □

The following theorem provides the convergence rate of the limit obtained in (5.6).

Theorem 5.2. *For all $0 < \alpha < 1$, $\alpha \neq 1/2$, we have*

$$\lim_{t \to \infty} t^\alpha \left(\frac{p_{i,j,\alpha}(t)}{\mathbb{P}_i[T_{0,\alpha} > t]} - \frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}} \right) = \frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \left(\frac{C_{i,1,3}}{C_{i,1,2}} - \frac{C_{i,j,2}}{C_{i,j,1}} \right),$$

and similarly for $\alpha = 1/2$,

$$\lim_{t \to \infty} t \left(\frac{p_{i,j,\alpha}(t)}{\mathbb{P}_i[T_{0,\alpha} > t]} - \frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}} \right) = \frac{1}{2} \frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}} \left(\frac{C_{i,1,4}}{C_{i,1,2}} - \frac{C_{i,j,3}}{C_{i,j,1}} \right).$$

Proof. For $\alpha \neq 1/2$ we recall the asymptotic expansion of the Mittag–Leffler function for t large enough (see (A.5) from the Appendix):

$$\mathcal{E}_{\alpha,1}(-\theta t^\alpha) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{\theta t^\alpha} - \frac{1}{\Gamma(1-2\alpha)} \frac{1}{\theta^2 t^{2\alpha}} + \mathrm{o}(t^{-2\alpha}).$$

The constants $C_{i,j,k}$ are finite, so the following asymptotic expansions are valid,

$$p_{i,j,\alpha}(t) = \pi_j \left(\frac{C_{i,j,1}}{\Gamma(1-\alpha)} \frac{1}{t^\alpha} - \frac{C_{i,j,2}}{\Gamma(1-2\alpha)} \frac{1}{t^{2\alpha}} + \mathrm{o}(t^{-2\alpha}) \right),$$

$$\mathbb{P}_i[T_{0,\alpha} > t] = \mu_1 \left(\frac{C_{i,1,2}}{\Gamma(1-\alpha)} \frac{1}{t^\alpha} - \frac{C_{i,1,3}}{\Gamma(1-2\alpha)} \frac{1}{t^{2\alpha}} + \mathrm{o}(t^{-2\alpha}) \right),$$

and after some algebraic manipulations,

$$\frac{p_{i,j,\alpha}(t)}{\mathbb{P}_i[T_{0,\alpha} > t]} = \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}} \left(\frac{1 - \frac{C_{i,j,2}}{C_{i,j,1}} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \frac{1}{t^\alpha} + \mathrm{o}(t^{-\alpha})}{1 - \frac{C_{i,1,3}}{C_{i,1,2}} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \frac{1}{t^\alpha} + \mathrm{o}(t^{-\alpha})} \right).$$

For $|z|$ small enough, we know that $(1-z)^{-1} = 1 + z + \mathrm{o}(z)$, so

$$\frac{p_{i,j,\alpha}(t)}{\mathbb{P}_i[T_{0,\alpha} > t]} = \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}} \left(1 - \frac{1}{t^\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \left(\frac{C_{i,j,2}}{C_{i,j,1}} - \frac{C_{i,1,3}}{C_{i,1,2}} \right) \right) + \mathrm{o}(t^{-\alpha})$$

and consequently

$$t^\alpha \left(\frac{p_{i,j,\alpha}(t)}{\mathbb{P}_i[T_{0,\alpha} > t]} - \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}} \right) = \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}} \left(\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \left(\frac{C_{i,1,3}}{C_{i,1,2}} - \frac{C_{i,j,2}}{C_{i,j,1}} \right) \right) + o(1).$$

The limit is obtained by letting $t \to \infty$ and recalling the identity

$$\frac{P_{i,j}}{\sum_{j \geq 1} P_{i,j}} = \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}}.$$

For $\alpha = 1/2$ we now have to consider the asymptotic expansion (A.7),

$$\mathcal{E}_{1/2,1}(-\theta t^{1/2}) = \frac{1}{\theta t^{1/2}} \frac{1}{\Gamma(1/2)} + \frac{1}{\theta^3 t^{3/2}} \frac{1}{\Gamma(-1/2)} + o(t^{-3/2}).$$

Following the same approach we now obtain

$$\frac{p_{i,j,\alpha}(t)}{\mathbb{P}_i[T_{0,\alpha} > t]} = \frac{\pi_j}{\mu_1} \frac{C_{i,j,1}}{C_{i,1,2}} \left(1 + \frac{1}{t} \frac{\Gamma(1/2)}{\Gamma(-1/2)} \left(\frac{C_{i,j,3}}{C_{i,j,1}} - \frac{C_{i,1,4}}{C_{i,1,2}} \right) \right) + o(t^{-1}).$$

Since

$$\frac{\Gamma(1/2)}{\Gamma(-1/2)} = -\frac{1}{2},$$

the proof concludes by using the same argument as the case $\alpha \neq 1/2$. $\square$

5.1. Quasi-stationary distributions

In this section we suppose that the initial state $N_\alpha(0)$ is random, with a distribution ν supported on $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$. In this case we set

$$\mathbb{P}_\nu[N_\alpha(t) = j] := \sum_{i \geq 1} \mathbb{P}[N_\alpha(0) = i]\mathbb{P}[N_\alpha(t) = j \mid N_\alpha(0) = i]$$

$$= \sum_{i \geq 1} \nu_i p_{i,j,\alpha}(t),$$

where $p_{i,j,\alpha}(t)$ are the transition probabilities defined in the previous sections. More generally, given $A \subseteq \mathbb{N}^+$ we write

$$\mathbb{P}_\nu[N_\alpha(t) \in A] = \sum_{j \in A} \mathbb{P}_\nu[N_\alpha(t) = j].$$

It can be shown that

$$p_{j,\alpha}(t) := \mathbb{P}_\nu[N_\alpha(t) = j]$$

is the unique 'honest' solution, that is, it is a non-negative solution to the system of equations

$$\mathbb{D}^\alpha p_{j,\alpha}(t) = \lambda_{j-1} p_{j-1,\alpha}(t) - (\lambda_j + \mu_j)p_{j,\alpha}(t) + \mu_{j+1}p_{j+1,\alpha}(t), \quad j \geq 2, \tag{5.12}$$

$$\mathbb{D}^\alpha p_{1,\alpha}(t) = -(\lambda_1 + \mu_1)p_{1,\alpha}(t) + \mu_2 p_{2,\alpha}(t), \tag{5.13}$$

$$\mathbb{D}^\alpha p_{0,\alpha}(t) = \mu_1 p_{1,\alpha}(t), \tag{5.14}$$

satisfying $\sum_{j\geq 0} p_{j,\alpha}(t) = 1$ and $p_{j,\alpha}(0) = \nu_j$, for all $t \geq 0$, $j \geq 1$. In fact, for $\alpha \in (0, 1)$ we know from Theorem 3.1 that

$$p_{i,j,\alpha}(t) = \int_0^\infty p_{i,j,1}(u)h_\alpha(u, t)\, \mathrm{d}u, \quad i \geq 1, \; j \geq 0.$$

This implies

$$p_{j,\alpha}(t) = \sum_{i\geq 0} \nu_i \int_0^\infty p_{i,j,1}(u)h_\alpha(u, t)\, \mathrm{d}u = \int_0^\infty p_{j,1}(u)h_\alpha(u, t)\, \mathrm{d}u.$$

Since $p_{j,1}(t) = \sum_{i\geq 1} \nu_i p_{i,j,1}(t)$ is the unique honest solution for $\alpha = 1$ (see [8, pages 685–686] for more details on the non-fractional case), necessarily $p_{j,\alpha}(t)$ defined above is the unique solution for $\alpha \in (0, 1)$.

Definition 5.1. We say that a probability measure ν is a quasi-stationary distribution (QSD) if, for all $A \subseteq \mathbb{N}^+$ and $t \geq 0$, we have the identity

$$\mathbb{P}_\nu[N_\alpha(t) \in A, T_{0,\alpha} > t] = \nu(A)\mathbb{P}_\nu[T_{0,\alpha} > t]. \tag{5.15}$$

When $\alpha = 1$, a probability measure ν is a QSD if and only if it is a solution to the system $\nu^t Q^{(a)} = -\theta\nu$ for some $\theta \in (0, \theta^*]$, where $Q^{(a)}$ is the matrix defined in (2.2). Moreover,

$$\mathbb{P}_\nu[T_0 > t] = \mathrm{e}^{-\theta t}, \tag{5.16}$$

$$\mathbb{P}_\nu[N_1(t) = j, T_0 > t] = \nu_j\, \mathrm{e}^{-\theta t}. \tag{5.17}$$

Similar properties appear in the fractional case. They are enunciated in the following results.

Proposition 5.2. *Let ν be a QSD for $(N_1(t))_{t\geq 0}$. Then, for all $\alpha \in (0, 1)$, the following identities are satisfied:*

$$\mathbb{P}_\nu[T_{0,\alpha} > t] = \mathcal{E}_{\alpha,1}(-\theta t^\alpha), \tag{5.18}$$

$$\mathbb{P}_\nu[N_\alpha(t) = j, T_{0,\alpha} > t] = \nu_j \mathcal{E}_{\alpha,1}(-\theta t^\alpha) \tag{5.19}$$

for some $\theta \in (0, \theta^\star]$. Consequently, ν is a QSD for $(N_\alpha(t))_{t\geq 0}$, $\alpha \in (0, 1)$.

Proof. If ν is a QSD for $(N_1(t))_{t\geq 0}$, then (5.16) and (5.17) hold for some $\theta \in (0, \theta^\star]$. To prove (5.18), from Theorem 3.1 we have

$$\mathbb{P}_\nu[T_{0,\alpha} > t] = \int_0^\infty \mathbb{P}_\nu[T_0 > u]h_\alpha(u, t)\, \mathrm{d}u$$

$$= \int_0^\infty \mathrm{e}^{-\theta u}h_\alpha(u, t)\, \mathrm{d}u$$

$$= \mathcal{E}_{\alpha,1}(-\theta t^\alpha)$$

for all $t \geq 0$ and $\alpha \in (0, 1)$ fixed. Equation (5.19) is obtained from (5.17) by using the same argument. The proof concludes by noticing that (5.15) can be directly deduced from (5.18) and (5.19). □

Proposition 5.2 asserts that if the initial distribution coincides with some QSD of $(N_1(t))_{t \geq 0}$, then the distribution of the absorption time follows a Mittag–Leffler distribution. The following theorem establishes that the family of quasi-stationary distributions for $(N_\alpha(t))_{t \geq 0}$ coincides for all $\alpha \in (0, 1]$.

Theorem 5.3. *A probability measure ν is a QSD for $(N_\alpha(t))_{t \geq 0}$ if and only if it solves the system of equations*

$$-\theta \nu_j = \lambda_{j-1} \nu_{j-1} - (\lambda_j + \mu_j) \nu_j + \mu_{j+1} \nu_{j+1}, \quad j \geq 2, \tag{5.20}$$

$$-\theta \nu_1 = -(\lambda_1 + \mu_1) \nu_1 + \mu_2 \nu_2, \tag{5.21}$$

where $\theta = \mu_1 \nu_1$. Consequently ν is a QSD for $(N_\alpha(t))_{t \geq 0}$ if and only if ν is a QSD for $(N_1(t))_{t \geq 0}$.

Proof. From Proposition 5.2 we know that any QSD associated with $N_1(t)$ is necessarily a QSD for $N_\alpha(t)$, $\alpha \in (0, 1]$, so the system of equations (5.20), (5.21) is a sufficient condition to be a QSD for $(N_\alpha(t))_{t \geq 0}$.

To prove the converse, we first notice that if ν is a QSD, then for all $j \geq 1$

$$p_{j,\alpha}(t) = \mathbb{P}_\nu[N_\alpha(t) = j, T_{0,\alpha} > t] = \nu_j \mathbb{P}_\nu[T_{0,\alpha} > t].$$

By taking the operator $\mathbb{D}^\alpha$ on both sides we now get

$$\mathbb{D}^\alpha p_{j,\alpha}(t) = \nu_j \mathbb{D}^\alpha(\mathbb{P}_\nu[T_{0,\alpha} > t]).$$

Since $p_{j,\alpha}(t)$ is a solution to (5.12) satisfying $p_{j,\alpha}(0) = \nu_j$, $j \geq 1$, we get for all $t > 0$

$$\nu_j \mathbb{D}^\alpha(\mathbb{P}_\nu[T_{0,\alpha} > t]) = \lambda_{j-1} p_{j-1,\alpha}(t) - (\lambda_j + \mu_j) p_{j,\alpha}(t) + \mu_{j+1} p_{j+1,\alpha}(t), \quad j \geq 2, \tag{5.22}$$

$$\nu_1 \mathbb{D}^\alpha(\mathbb{P}_\nu[T_{0,\alpha} > t]) = -(\lambda_1 + \mu_1) p_{1,\alpha}(t) + \mu_2 p_{2,\alpha}(t). \tag{5.23}$$

The existence of the limit $\nu_1 \mu_1 = \lim_{t \to 0^+} \mathbb{D}^\alpha(\mathbb{P}_\nu[T_{0,\alpha} > t])$ can be deduced by noticing first that

$$\mathbb{D}^\alpha p_{0,\alpha}(t) = \mathbb{D}^\alpha(1 - \mathbb{P}_\nu[T_{0,\alpha} > t]) = -\mathbb{D}^\alpha(\mathbb{P}_\nu[T_{0,\alpha} > t]).$$

By letting $t \to 0^+$ in the equation $\mathbb{D}^\alpha p_{0,\alpha}(t) = \mu_1 p_{1,\alpha}(t)$, we get

$$\theta = \lim_{t \to 0^+} \mathbb{D}^\alpha p_{0,\alpha}(t) = \mu_1 \lim_{t \to 0^+} p_{1,\alpha}(t) = \mu_1 \nu_1.$$

Finally, we take the limit $t \to 0^+$ in (5.22), (5.23) to conclude that any QSD ν is a solution to (5.20), (5.21) for $\theta = \nu_1 \mu_1$. $\qquad \square$

Since the family of quasi-stationary distributions is the same for the complete interval $\alpha \in (0, 1]$, its characterization coincides with the one originally presented by Van Doorn, enunciated below.

Theorem 5.4. (Van Doorn [8].) *If the series*

$$\sum_{j \geq 1} \pi_j \sum_{k=1}^{j-1} \frac{1}{\lambda_k \pi_k} = \sum_{k \geq 1} \frac{1}{\lambda_k \pi_k} \sum_{j \geq k+1} \pi_j$$

diverges, then either $\theta^\star = 0$ and there is no QSD distribution, or $\theta^\star > 0$, in which case there is a family of QSD distributions indexed by ν_θ, $\theta \in (0, \theta^\star]$. If the D series converges, then there is a unique distribution QSD indexed by $\theta^\star$.

6. The linear process

We revisit the linear model, previously studied by Orsingher and Polito [23]. The birth rates and death rates are $\lambda_i = i\lambda$ and $\mu_i = i\mu$ respectively. To make our analysis simpler, we assume first that the initial condition is $N_\alpha(0) = 1$. It is well known that in the case $\alpha = 1$, the transition probabilities are

$$
p_{1,j,1}(t) = \begin{cases}
\dfrac{(\lambda t)^{j-1}}{(1 + \lambda t)^{j+1}}, & \lambda = \mu, \\[3mm]
\dfrac{(\lambda(1 - e^{-(\lambda-\mu)t}))^{j-1}}{(\lambda - \mu e^{-(\lambda-\mu)t})^{j+1}}(\lambda - \mu)^2\, e^{-(\lambda-\mu)t}, & \lambda \neq \mu.
\end{cases}
$$

Similarly, the probabilities of non-extinction are

$$
\mathbb{P}_1[T_0 > t] = \begin{cases}
\dfrac{\lambda - \mu}{\lambda}\left(1 - \displaystyle\sum_{m \geq 1}\left(\dfrac{\mu}{\lambda}\right)^m e^{-(\lambda-\mu)mt}\right), & \lambda > \mu, \\[5mm]
\dfrac{\mu - \lambda}{\lambda}\displaystyle\sum_{m \geq 1}\left(\dfrac{\lambda}{\mu}\right)^m e^{-(\mu-\lambda)mt}, & \lambda < \mu, \\[5mm]
\dfrac{1}{1 + \lambda t}, & \lambda = \mu
\end{cases}
$$

(see Section 8.6 of [1]). A widely known fact is that the asymptotic behavior depends on the ratio λ/μ. In the fractional case the same occurs, so we study the three cases separately.

The case $\lambda < \mu$. As mentioned above, we first study the asymptotic behavior of the process with initial state $N_\alpha(0) = 1$. When $\alpha \in (0, 1)$, the probability of non-extinction is

$$
\mathbb{P}_1[T_{0,\alpha} > t] = \left(\frac{\mu - \lambda}{\lambda}\right)\sum_{m \geq 1}(\lambda/\mu)^m \mathcal{E}_{\alpha,1}(-(\mu - \lambda)mt^\alpha)
$$

(see formula (2.20) from [23]). We recall that from (A.8) we have for all $m \geq 1$

$$
\lim_{t \to \infty} t^\alpha \mathcal{E}_{\alpha,1}(-(\mu - \lambda)mt^\alpha) = \frac{1}{\Gamma(1 - \alpha)}\frac{1}{(\mu - \lambda)m}. \tag{6.1}
$$

From (6.1) and the dominated convergence theorem, we obtain

$$
\lim_{t \to \infty} t^\alpha \mathbb{P}_1[T_{0,\alpha} > t] = \left(\frac{\mu - \lambda}{\lambda}\right)\lim_{t \to \infty} t^\alpha \sum_{m \geq 1}(\lambda/\mu)^m \mathcal{E}_{\alpha,1}(-(\mu - \lambda)mt^\alpha)
$$

$$
= \left(\frac{\mu - \lambda}{\lambda}\right)\sum_{m \geq 1}(\lambda/\mu)^m \lim_{t \to \infty} t^\alpha \mathcal{E}_{\alpha,1}(-(\mu - \lambda)mt^\alpha)
$$

$$
= \frac{1}{\Gamma(1 - \alpha)}\frac{1}{\lambda}\sum_{m \geq 1}\frac{(\lambda/\mu)^m}{m}
$$

$$
= -\frac{1}{\Gamma(1 - \alpha)}\frac{1}{\lambda}\ln\left(1 - \frac{\lambda}{\mu}\right).
$$

The limit can be deduced in an alternative way. We now consider the representation

$$\mathbb{P}_1[T_0 > t] = \frac{\mu - \lambda}{\lambda} \frac{(\lambda/\mu)\, e^{-(\mu-\lambda)t}}{1 - (\lambda/\mu)\, e^{-(\mu-\lambda)t}}.$$

Equations (5.1) and (5.8) jointly imply

$$\lim_{t\to\infty} t^{\alpha}\mathbb{P}_i[T_{0,\alpha} > t] = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \mathbb{P}_i[T_0 > u]\, du.$$

By using this equivalence we get for $i = 1$

$$\lim_{t\to\infty} t^{\alpha}\mathbb{P}_1[T_{0,\alpha} > t] = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\mu - \lambda}{\lambda} \frac{(\lambda/\mu)\, e^{-(\mu-\lambda)u}}{1 - \dfrac{\lambda}{\mu}\, e^{-(\mu-\lambda)u}}\, du$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda} \ln\left(1 - \frac{\lambda}{\mu}\, e^{-(\mu-\lambda)u}\right)\Big|_0^{\infty}$$

$$= -\frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda} \ln\left(1 - \frac{\lambda}{\mu}\right). \tag{6.2}$$

Similarly, from (5.2) and (5.7) we have for all $j \geq 1$

$$\lim_{t\to\infty} t^{\alpha}\mathbb{P}_1[N_{\alpha}(t) = j, T_{0,\alpha} > t] = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \mathbb{P}_1[N_1(u) = j, T_0 > u]\, du.$$

We obtain in the linear case

$$\int_0^{\infty} \mathbb{P}_1[N_1(u) = j, T_0 > u]\, du = (\lambda - \mu)^2 \int_0^{\infty} \frac{(\lambda(1 - e^{-(\lambda-\mu)u}))^{j-1}}{(\lambda - \mu e^{-(\lambda-\mu)u})^{j+1}}\, e^{-(\lambda-\mu)u}\, du$$

$$= (\lambda - \mu)^2 \int_0^{\infty} \frac{(\lambda(e^{(\lambda-\mu)u} - 1))^{j-1}}{(\lambda\, e^{(\lambda-\mu)u} - \mu)^{j+1}}\, e^{(\lambda-\mu)u}\, du.$$

By taking the change of variable $z = e^{(\lambda-\mu)u}$, after some manipulation we find

$$(\lambda - \mu)^2 \int_0^{\infty} \frac{(\lambda(e^{(\lambda-\mu)u} - 1))^{j-1}}{(\lambda\, e^{(\lambda-\mu)u} - \mu)^{j+1}}\, e^{(\lambda-\mu)u}\, du = (\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \int_0^1 \frac{(1-z)^{j-1}}{(1 - \lambda z/\mu)^{j+1}}\, dz.$$

For $z \in [0, 1]$, it is certain that

$$0 \leq \frac{\lambda z}{\mu} \leq \frac{\lambda}{\mu} < 1,$$

so the expansion series

$$\frac{1}{(1 - \lambda z/\mu)^{j+1}} = \sum_{\ell \geq 0} \binom{\ell+j}{\ell} (\lambda z/\mu)^{\ell}$$

holds, and by applying Tonelli's theorem we get

$$(\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \int_0^1 \frac{(1-z)^{j-1}}{(1-\lambda z/\mu)^{j+1}} \, \mathrm{d}z = (\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \int_0^1 (1-z)^{j-1} \sum_{\ell \geq 0} \binom{\ell+j}{\ell} (\lambda z/\mu)^\ell \, \mathrm{d}z$$

$$= (\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \sum_{\ell \geq 0} (\lambda/\mu)^\ell \binom{\ell+j}{\ell} \int_0^1 (1-z)^{j-1} z^\ell \, \mathrm{d}z$$

$$= (\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \sum_{\ell \geq 0} (\lambda/\mu)^\ell \binom{\ell+j}{\ell} \frac{\Gamma(j)\Gamma(\ell+1)}{\Gamma(\ell+j+1)}.$$

Since

$$\binom{\ell+j}{\ell} = \frac{(\ell+j)!}{\ell!j!} = \frac{\Gamma(\ell+j+1)}{\Gamma(\ell+1)\Gamma(j+1)},$$

we can simplify some terms in the series, obtaining

$$(\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \sum_{\ell \geq 0} (\lambda/\mu)^\ell \binom{\ell+j}{\ell} \frac{\Gamma(j)\Gamma(\ell+1)}{\Gamma(\ell+j+1)} = \frac{1}{j}(\mu - \lambda)\frac{\lambda^{j-1}}{\mu^{j+1}} \sum_{\ell \geq 0} (\lambda/\mu)^\ell = \frac{1}{j\lambda}(\lambda/\mu)^j.$$

For $j \geq 1$, from a direct computation we have shown the limit

$$\int_0^\infty \mathbb{P}_1[N_1(u) = j, T_0 > u] \, \mathrm{d}u = \frac{1}{\lambda}\frac{(\lambda/\mu)^j}{j}. \tag{6.3}$$

For $N_\alpha(0) = 1$, the quasi-limiting distribution can be deduced from (6.2) and (6.3), by noticing that

$$-\ln\left(1 - \frac{\lambda}{\mu}\right) = \sum_{j \geq 1} \frac{(\lambda/\mu)^j}{j},$$

so that

$$\lim_{t \to \infty} \mathbb{P}_1[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{(\lambda/\mu)^j/j}{\sum_{j \geq 1} (\lambda/\mu)^j/j}. \tag{6.4}$$

Since $P_{1,j} = (\lambda/\mu)^{j-1}/j$, the limit (6.4) can also be written in the form

$$\lim_{t \to \infty} \mathbb{P}_1[N_\alpha(t) = j \mid T_{0,\alpha} > t] = \frac{P_{1,j}}{\sum_{j \geq 1} P_{1,j}},$$

which is consistent with our theorems. Now, the quasi-limiting behavior when the initial condition is $N_\alpha(0) > 1$ can be obtained directly from (5.4). We recall the expressions $\pi_j = (\lambda/\mu)^{j-1}/j$ and $\lambda_j = j\lambda$, so $\lambda_j\pi_j = \mu(\lambda/\mu)^j$ and (5.11) becomes

$$P_{i,j} = \frac{(\lambda/\mu)^{j-1}}{j}\left[\frac{1}{\mu} + \sum_{k=1}^{\min\{i-1,j-1\}} \frac{1}{\mu(\lambda/\mu)^k}\right]$$

$$= \frac{(\lambda/\mu)^j}{\lambda j} \sum_{k=0}^{\min\{i-1,j-1\}} \frac{1}{(\lambda/\mu)^k}$$

$$= \frac{(\lambda/\mu)^j}{\lambda j} \frac{(\lambda/\mu)^{-\min\{i,j\}} - 1}{(\lambda/\mu)^{-1} - 1}. \tag{6.5}$$

Equation (6.5) is the same as

$$P_{i,j} = \begin{cases} \dfrac{1}{\lambda j} \dfrac{1 - (\lambda/\mu)^j}{(\lambda/\mu)^{-1} - 1}, & j \le i, \\[3mm] \dfrac{1}{\lambda j} \dfrac{(\lambda/\mu)^{j-i} - (\lambda/\mu)^j}{(\lambda/\mu)^{-1} - 1}, & j \ge i+1. \end{cases}$$

The quasi-limiting distribution is

$$\frac{P_{i,j}}{\sum_{j \ge 1} P_{i,j}} = \frac{j^{-1}((\lambda/\mu)^{\max\{0, j-i\}} - (\lambda/\mu)^j)}{\sum_{j \ge 1} j^{-1}((\lambda/\mu)^{\max\{0, j-i\}} - (\lambda/\mu)^j)}.$$

We notice that in the limit case $i \to \infty$ we have

$$P_{\infty, j} = \frac{1}{\lambda j} \frac{1 - (\lambda/\mu)^j}{(\lambda/\mu)^{-1} - 1},$$

which is not a finite measure since $\sum_{j \ge 1} j^{-1} = \infty$.

The case $\lambda \ge \mu$. We first study the case $\lambda = \mu$. The probability of non-extinction is

$$\mathbb{P}_1[T_{0,\alpha} > t] = \int_0^\infty e^{-u} \mathcal{E}_{\alpha,1}(-\lambda t^\alpha u)\, du$$

(see [23, formula (2.26)]). Alternatively, for $\alpha = 1$ we have the identity $\mathbb{P}_1[T_0 > t] = 1/(1 + \lambda t)$. From Theorem 3.1 we get the formula

$$\mathbb{P}_1[T_{0,\alpha} > t] = \mathbb{E}\left[\frac{1}{1 + \lambda L^{(\alpha)}(t)} \right].$$

In order to study the asymptotic behavior, we introduce the function

$$f(t) = t^\alpha (\log \log t^\alpha)^{1-\alpha}.$$

It is well known that (see [3, equation 4.2, page 31])

$$\limsup_{t \to \infty} \frac{L^{(\alpha)}(t)}{f(t)} = C_\alpha \tag{6.6}$$

with probability 1, for some positive constant C_α depending only on α. By taking the limit $t \to \infty$, we now get

$$\liminf_{t \to \infty} f(t) \mathbb{P}_1[T_{0,\alpha} > t] = \liminf_{t \to \infty} \mathbb{E}\left[\frac{f(t)}{1 + \lambda L^{(\alpha)}(t)} \right]$$

$$\ge \mathbb{E}\left[\liminf_{t \to \infty} \frac{f(t)}{1 + \lambda L^{(\alpha)}(t)} \right]$$

$$\ge \mathbb{E}\left[\frac{1}{\limsup_{t \to \infty} \frac{1}{f(t)} + \lambda \frac{L^{(\alpha)}(t)}{f(t)}} \right].$$

From (6.6) and noticing that $\lim_{t \to \infty} f(t) = \infty$, we conclude

$$\liminf_{t \to \infty} f(t) \mathbb{P}_1[T_{0,\alpha} > t] \geq \frac{1}{\lambda C_\alpha}.$$

Similarly, we compute

$$\limsup_{t \to \infty} \frac{p_{1,j,\alpha}(t)}{\mathbb{P}_1[T_{0,\alpha} > t]} = \limsup_{t \to \infty} \frac{f(t) p_{1,j,\alpha}(t)}{f(t) \mathbb{P}_1[T_{0,\alpha} > t]}$$

$$\leq \frac{\limsup_{t \to \infty} f(t) p_{1,j,\alpha}(t)}{\liminf_{t \to \infty} f(t) \mathbb{P}_1[T_{0,\alpha} > t]}$$

$$\leq \lambda C_\alpha \limsup_{t \to \infty} f(t) p_{1,j,\alpha}(t).$$

The identity

$$p_{1,j,\alpha}(t) = \mathbb{E}\left[\frac{(\lambda L^{(\alpha)}(t))^{j-1}}{(1 + \lambda L^{(\alpha)}(t))^{j+1}} \right]$$

allows us to compute the limit

$$\limsup_{t \to \infty} f^2(t) p_{1,j,\alpha}(t) = \limsup_{t \to \infty} f^2(t) \mathbb{E}\left[\frac{(\lambda L^{(\alpha)}(t))^{j-1}}{(1 + \lambda L^{(\alpha)}(t))^{j+1}} \right]$$

$$\leq \mathbb{E}\left[\limsup_{t \to \infty} f^2(t) \frac{(\lambda L^{(\alpha)}(t))^{j-1}}{(1 + \lambda L^{(\alpha)}(t))^{j+1}} \right]$$

$$\leq \frac{1}{\lambda^2} \mathbb{E}\left[\limsup_{t \to \infty} \frac{1}{(L^{(\alpha)}(t)/f(t))^2} \right]$$

$$= \frac{1}{(\lambda C_\alpha)^2}.$$

In conclusion,

$$\limsup_{t \to \infty} f(t) p_{1,j,\alpha}(t) \leq \lim_{t \to \infty} \frac{1}{f(t)} \limsup_{t \to \infty} f^2(t) p_{1,j,\alpha}(t) = 0.$$

Consequently there is no quasi-limiting distribution as expected. Finally, when $\lambda > \mu$ the probability of non-extinction is

$$\mathbb{P}_1[T_{0,\alpha} > t] = \frac{\mu}{\lambda} - \frac{\lambda - \mu}{\lambda} \sum_{m \geq 1} \left(\frac{\mu}{\lambda} \right)^m \mathcal{E}_{\alpha,1}(-(\lambda - \mu) m t^\alpha),$$

which is a strictly positive value (see [23, formula (2.13)]), so there is no quasi-limiting distribution. By following an argument similar to the previous case, we get the limit

$$\lim_{t \to \infty} t^\alpha \left(\mathbb{P}_1[T_{0,\alpha} > t] - \frac{\lambda - \mu}{\lambda} \right) = \frac{1}{\Gamma(1 - \alpha)} \frac{\lambda - \mu}{\lambda} \sum_{m \geq 1} \frac{(\lambda/\mu)^m}{m}.$$

Appendix A. The Mittag–Leffler function

We present a summary concerning the basic properties of the Mittag–Leffler function.

Definition A.1. The complex-valued Mittag–Leffler function with parameter $\alpha \in (0, 1]$ is defined as

$$\mathcal{E}_{\alpha,1}(z) = \sum_{k \geq 0} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}.$$

In particular, when $\alpha = 1$ we have that $\mathcal{E}_{1,1}(z) = e^z$ is the exponential function.

In the real-valued case it is well known (see [12, Proposition 3.23]) that the Mittag–Leffler function with negative argument $\mathcal{E}_{\alpha,1}(-x)$, $x > 0$ is completely monotonic for all $0 \leq \alpha \leq 1$, that is,

$$(-1)^n \frac{d^n}{dx^n}(\mathcal{E}_{\alpha,1}(-x)) \geq 0.$$

This is equivalent to the existence of a representation of $\mathcal{E}_{\alpha,1}(-x)$ in the form of a Laplace–Stieltjes integral with non-decreasing density and non-negative measure $d\mu$:

$$\mathcal{E}_{\alpha,1}(-x) = \int_0^\infty e^{-xu} \, d\mu(u).$$

Since $\mathcal{E}_{\alpha,1}(0) = 1$ we have from the dominated convergence theorem that μ is a probability measure. In fact we know from (3.5), for $s = x$ and $t = 1$, that

$$\mathcal{E}_{\alpha,1}(-x) = \mathbb{E}\big[e^{-xL^{(\alpha)}(1)}\big] = \int_0^\infty e^{-xu} h_\alpha(u, 1) \, du, \tag{A.1}$$

and consequently the measure is $d\mu(u) = h_\alpha(u, 1) \, du$. The density $h_\alpha(u, 1)$ is known as the Lamperti distribution obtained as the law of the ratio of two independent subordinators of order α. In particular, the integral representation (A.1) implies that the function $\mathcal{E}_{\alpha,1}(x)$ is strictly positive for $x \geq 0$.

Proposition A.1. *For all $\lambda > 0$, $f(x) = \mathcal{E}_{\alpha,1}(-\lambda x^\alpha)$ is the unique solution to the equation*

$$\mathbb{D}^\alpha f(x) = -\lambda f(x), \quad f(0) = 1. \tag{A.2}$$

Proof. It proceeds by direct computation by taking the Laplace transform

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x) \, dx$$

on both sides of (A.2). On the right-hand side we just have $\mathcal{L}[-\lambda f](s) = -\lambda \mathcal{L}[f](s)$, whereas on the left-hand side we have

$$\mathcal{L}[\mathbb{D}^\alpha f](s) = \frac{1}{\Gamma(1-\alpha)} \mathcal{L}\left[\int_0^x \frac{f'(u)}{(x-u)^\alpha} \, du\right](s)$$

$$= \mathcal{L}[f'](s)\mathcal{L}\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right](s)$$

$$= (s\mathcal{L}[f](s) - f(0))s^{\alpha-1}. \tag{A.3}$$

This implies

$$\mathcal{L}[f](s) = \frac{s^{\alpha-1}}{\lambda + s^{\alpha}}. \tag{A.4}$$

By taking the inverse

$$f(x) = \sum_{k \geq 0} \frac{(-\lambda x^{\alpha})^k}{\Gamma(\alpha k + 1)},$$

we conclude $f(x) = \mathcal{E}_{\alpha,1}(-\lambda x^{\alpha})$. $\qquad \square$

Concerning the asymptotic behavior of the complex-valued Mittag–Leffler function, we take formulas 4.7.4 and 4.7.5 from page 75 of [12]. Let $\alpha \in (0, 2)$ and let $\pi\alpha/2 < \theta < \min\{\pi, \alpha\pi\}$, where $\theta = \arg(z)$. For all $|z|$ large enough we have

$$\mathcal{E}_{\alpha,1}(z) = \frac{1}{\alpha} e^{z^{1/\alpha}} - \sum_{m=1}^{N-1} \frac{z^{-m}}{\Gamma(1-m\alpha)} + \mathrm{O}\left(|z|^{-N}\right), \quad |\arg(z)| \leq \theta,$$

$$\mathcal{E}_{\alpha,1}(z) = -\sum_{m=1}^{N-1} \frac{z^{-m}}{\Gamma(1-m\alpha)} + \mathrm{O}\left(|z|^{-N}\right), \quad \theta \leq |\arg(z)| \leq \pi.$$

In particular, when $\arg(z) = 0$ and $\arg(z) = \pi$, we recover the asymptotic expansion for the real-valued case. In fact when $|x|$ is large enough we get the following approximations:

$$\mathcal{E}_{\alpha,1}(x) = \frac{1}{\alpha} e^{x^{1/\alpha}} - \frac{1}{x} \frac{1}{\Gamma(1-\alpha)} - \frac{1}{x^2} \frac{1}{\Gamma(1-2\alpha)} + \mathrm{O}(x^{-3}),$$
$$\mathcal{E}_{\alpha,1}(-x) = \frac{1}{x} \frac{1}{\Gamma(1-\alpha)} - \frac{1}{x^2} \frac{1}{\Gamma(1-2\alpha)} + \mathrm{O}(x^{-3}). \tag{A.5}$$

The variable x can be replaced by $\pm \lambda t^{\alpha}$, so that for $\alpha \neq 1/2$, $\lambda > 0$ fixed and $t > 0$ large enough, we obtain the following asymptotic formulas:

$$\mathcal{E}_{\alpha,1}(\lambda t^{\alpha}) = \frac{1}{\alpha} e^{\lambda^{1/\alpha} t} - \frac{1}{\lambda t^{\alpha}} \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\lambda^2 t^{2\alpha}} \frac{1}{\Gamma(1-2\alpha)} + \mathrm{O}(t^{-3\alpha}),$$
$$\mathcal{E}_{\alpha,1}(-\lambda t^{\alpha}) = \frac{1}{\lambda t^{\alpha}} \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\lambda^2 t^{2\alpha}} \frac{1}{\Gamma(1-2\alpha)} + \mathrm{O}(t^{-3\alpha}). \tag{A.6}$$

For $\alpha = 1/2$, the second term in (A.6) vanishes, yielding

$$\mathcal{E}_{\alpha,1}(-\lambda t^{1/2}) = \frac{1}{\lambda t^{1/2}} \frac{1}{\Gamma(1/2)} + \frac{1}{\lambda^3 t^{3/2}} \frac{1}{\Gamma(-1/2)} + \mathrm{O}(t^{-2}). \tag{A.7}$$

Moreover, the following limits are valid for all $\lambda > 0$:

$$\lim_{t \to \infty} t^{\alpha} \mathcal{E}_{\alpha,1}(-\lambda t^{\alpha}) = \frac{1}{\lambda} \frac{1}{\Gamma(1-\alpha)},$$
$$\lim_{t \to \infty} e^{-\lambda^{1/\alpha} t} \mathcal{E}_{\alpha,1}(\lambda t^{\alpha}) = \frac{1}{\alpha}. \tag{A.8}$$

Acknowledgements

Funding information

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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