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II_1 FACTORS WITH EXOTIC CENTRAL SEQUENCE ALGEBRAS

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Abstract We provide a class of separable II_1 factors M whose central sequence algebra is not the 'tail' algebra associated with any decreasing sequence of von Neumann subalgebras of M. This settles a question of McDuff [On residual sequences in a II_1 factor, *J. Lond. Math. Soc. (2)* (1971), 273–280].

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1. Introduction and statement of main results

A uniformly bounded sequence (x_k) in a II_1 factor M is called *central* if $\lim_k \|x_k y - y x_k\|_2 = 0$, for every $y \in M$. Central sequences have played a fundamental role in the study of II_1 factors since the very beginning of the subject with Murray and von Neumann's property Gamma [29]. A separable II_1 factor M has *property Gamma* if it admits a central sequence (x_k) which is not trivial, in the sense that $\lim_k \|x_k - \tau(x_k)1\|_2 > 0$. Murray and von Neumann proved that the unique hyperfinite II_1 factor has property Gamma, while the free group factor $L(\mathbb{F}_2)$ does not, thus giving the first example of two non-isomorphic separable II_1 factors [29]. Over two decades later, in the late 1960s, the analysis of central sequences of [29] was refined to provide additional examples of non-isomorphic separable II_1 factors in [6, 10, 42, 49], culminating with McDuff's construction of a continuum of such factors [25, 26].

Shortly after, McDuff [27] defined the *central sequence algebra* of a II_1 factor M as the relative commutant, $M' \cap M^\omega$, of M into its ultrapower M^ω [41, 48], where ω is a free ultrafilter on $\mathbb{N}$. This has since allowed for a more structural approach to central sequences and led to significant progress in the study of II_1 factors. Indeed, the central sequence algebra was a crucial tool in Connes' famous classification of amenable II_1 factors [8]. Furthermore, the relative commutant $M' \cap \mathcal{M}^\omega$, for some von Neumann algebra $\mathcal{M} \supset M$, was used by Popa to formalize his influential spectral gap rigidity principle in [36, 37].

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Most recently, central sequence algebras and their subalgebras were used to provide a continuum of II_1 factors with non-isomorphic ultrapowers in [4] (adding to the four such factors noticed in [12–14]).

However, despite the progress the use of central sequence algebras has allowed, their structure remains fairly poorly understood. For instance, it is open whether any II_1 factor M whose central sequence algebra is abelian admits an abelian subalgebra A such that $M' \cap M^\omega \subset A^\omega$ (see [24]). In this article, we investigate the existence of a certain 'canonical form' for central sequence algebras. To make this precise, we recall the following notions introduced by McDuff in [28] in order to distill the key ideas of [26].

Definition 1.1 [28, Definition 2]. Let M be a separable II_1 factor. A von Neumann subalgebra A of M is called *residual* if $\lim_k \|x_k - E_A(x_k)\|_2 = 0$, for every central sequence (x_k) in M. A sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M is called a *residual sequence* if

(1) $A_{n+1} \subset A_n$, for every n;
(2) A_n is residual in M, for every n and
(3) if $x_k \in A_k$ and $\|x_k\| \leqslant 1$, for every k, then the sequence (x_k) is central in M.

Remark 1.2. A decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M is residual if and only if $M' \cap M^\omega = \bigcap_{n\in\mathbb{N}} A_n^\omega$. Thus, a separable II_1 factor M admits a residual sequence if and only if its central sequence algebra is equal to the 'tail' algebra, $\cap_{n\in\mathbb{N}} A_n^\omega$, associated with a decreasing sequence of von Neumann subalgebras $(A_n)_{n\in\mathbb{N}}$.

In [28], McDuff noted that it was unknown whether every II_1 factor admits a residual sequence. She gave examples of II_1 factors which do not admit any *strongly* residual sequence $(A_n)_{n\in\mathbb{N}}$ (i.e., ones satisfying, in addition to (1)–(3), the existence of a subalgebra $A^n \subset A_n$ such that $A_n = A_{n+1} \bar{\otimes} A^n$) but left open the case of residual sequences. The main goal of this article is to provide the first examples of II_1 factors with no residual sequence. Before stating our results in this direction, let us note that several large, well-studied classes of II_1 factors admit a residual sequence.

Examples 1.3. The following II_1 factors admit a residual sequence:

(1) Any II_1 factor without property Gamma.
(2) The hyperfinite II_1 factor R. If we write $R = \bar{\bigotimes}_{k\in\mathbb{N}} \mathbb{M}_2(\mathbb{C})$ and let $R_n = \bar{\bigotimes}_{k\geqslant n} \mathbb{M}_2(\mathbb{C})$, then $(R_n)_{n\in\mathbb{N}}$ is a residual sequence in R.
(3) Any II_1 factor M which is strongly McDuff, i.e., can be written as $M = N \bar{\otimes} R$, where N is a II_1 factor without property Gamma. If $A_n = 1 \otimes R_n$, then Connes' characterization of property Gamma [8, Theorem 2.1] implies that $(A_n)_{n\in\mathbb{N}}$ is a residual sequence in M.
(4) Any infinite tensor product $M = \bar{\bigotimes}_{k\in\mathbb{N}} M_k$ of II_1 factors without property Gamma. If $A_n = \bar{\bigotimes}_{k\geqslant n} M_k$, then [8, Theorem 2.1] implies that $(A_n)_{n\in\mathbb{N}}$ is a residual sequence in M. Note that M is McDuff, i.e., $M \cong M \bar{\otimes} R$ but not strongly McDuff [39, Theorem 4.1].

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(5) The II_1 factors $L(T_0(\Gamma))$ and $L(T_1(\Gamma))$, where Γ is any countable group and the countable groups $T_0(\Gamma), T_1(\Gamma)$ are defined as in [10, 26] (see also [4, Section 1.1]). Then $T_0(\Gamma)$ and $T_1(\Gamma)$ both contain $\widetilde{\Gamma} := \bigoplus_{i\in\mathbb{N}} \Gamma_i$, where each Γ_i is a copy of Γ. If $A_n = L(\bigoplus_{i\geqslant n} \Gamma_i)$, then [4, Corollary 2.11] shows that $(A_n)_{n\in\mathbb{N}}$ is a residual sequence in both $L(T_0(\Gamma))$ and $L(T_1(\Gamma))$. In particular, the uncountably many II_1 factors which were shown to have non-isomorphic ultrapowers in [4] all admit residual sequences.

(6) Any tensor product $M = \bar{\bigotimes}_{k=1}^{N} M_k$, where $N \in \mathbb{N}\cup\{\infty\}$, and for every k, M_k is a II_1 factor admitting a residual sequence, $(A_{k,n})_{n\in\mathbb{N}}$. If $B_n = (\bar{\bigotimes}_{k=1}^{\min\{n,N\}} A_{k,n}) \bar{\otimes} (\bar{\bigotimes}_{k=\min\{n,N\}+1}^{N} M_k)$, then [24, Proposition 5.2] implies that $(B_n)_{n\in\mathbb{N}}$ is a residual sequence in M.

Remark 1.4. In [39, 40], Popa studied the class of II_1 factors M which arise as an inductive limit of sub-factors $(M_n)_{n\in\mathbb{N}}$ with spectral gap and noticed that $M' \cap M^\omega = \bigcap_n (M_n' \cap M)^\omega$ (see [39, Lemma 2.3]). Thus, every such II_1 factor M admits a residual sequence, $(M_n' \cap M)_{n\in\mathbb{N}}$. Conversely, although it is unclear whether any II_1 factor admitting a residual sequence must be an inductive limit of sub-factors with spectral gap, we note that this holds for the factors in Examples 1.3(1)–(5).

We are now ready to state our first main result which gives examples of II_1 factors with no residual sequences, and thereby settles McDuff's question [28].

Theorem A. *Let Γ be a countable non-amenable group. For every $k \in \mathbb{N}$, let $\pi_k : \Gamma \to \mathcal{O}(\mathcal{H}_k)$ be an orthogonal representation such that*

(1) *$\pi_k^{\otimes l}$ is weakly contained in the left regular representation of Γ, for some $l = l(k) \in \mathbb{N}$; and*

(2) *there is an orthonormal sequence $(\xi_k^m)_{m\in\mathbb{N}} \subset \mathcal{H}_k$ such that $\sup_{m\in\mathbb{N}} \|\pi_k(g)(\xi_k^m) - \xi_k^m\| \to 0$, as $k \to \infty$, for every $g \in \Gamma$.*

Let $\Gamma \curvearrowright (B_k, \tau_k)$ be the Gaussian action associated with π_k and $\Gamma \curvearrowright (B, \tau) := \bar{\bigotimes}_{k\in\mathbb{N}} (B_k, \tau_k)$ be the diagonal product action. Define $M = B \rtimes \Gamma$.

Then the II_1 factor M does not admit a residual sequence of von Neumann subalgebras.

For the definition of Gaussian actions, we refer the reader to § 2.6. Next, we provide a class of examples to which Theorem A applies and discuss a connection with a problem posed in [23].

Example 1.5. Let $\Gamma = \mathbb{F}_n$ be the free group on $n \geqslant 2$ generators. Denote by $|g|$ the word length of an element $g \in \Gamma$ with respect to a fixed free set of generators. Let $t > 0$. By [15], the function $\varphi_t : \Gamma \to \mathbb{R}$ given by $\varphi_t(g) = e^{-t|g|}$ is positive definite. Let $\rho_t : \Gamma \to \mathcal{O}(\mathcal{H}_t)$ be the Gelfand Naimark Segal (GNS) orthogonal representation associated with φ_t and $\xi_t \in \mathcal{H}_t$ such that $\langle \rho_t(g)(\xi_t), \xi_t\rangle = \varphi_t(g)$, for all $g \in \Gamma$. Let $\tilde{\rho}_t = \rho_t \otimes \mathrm{Id}_{\ell^2(\mathbb{N})} : \Gamma \to \mathcal{O}(\mathcal{H}_t \otimes \ell^2(\mathbb{N}))$ be the direct sum of infinitely many copies of ρ_t.

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Let (t_k) be any sequence of positive numbers converging to 0 and put $\pi_k := \tilde{\rho}_{t_k} : \Gamma \to \mathcal{O}(\mathcal{H}_{t_k} \otimes \ell^2(\mathbb{N}))$. Then the representations $(\pi_k)_{k\in\mathbb{N}}$ satisfy the hypothesis of Theorem A. First, given $t > 0$, note that $\varphi_t^l \in \ell^2(\Gamma)$, and hence $\rho_t^{\otimes l}$ is contained in a multiple of the left regular representation of Γ, whenever $l > \log(2n-1)/(2t)$. This implies that $\pi_k^{\otimes l}$ is contained in a multiple of the left regular representation of Γ, for some integer $l = l(k) \geqslant 1$. Second, note that the vectors $\xi_k^m := \xi_{t_k} \otimes \delta_m \in \mathcal{H}_{t_k} \otimes \ell^2(\mathbb{N})$ satisfy $\sup_{m\in\mathbb{N}} \|\pi_k(g)(\xi_k^m) - \xi_k^m\| = \sqrt{2(1-\varphi_{t_k}(g))} \to 0$, as $k \to \infty$, for any $g \in \Gamma$.

Remark 1.6. Theorem A also sheds new light on a problem of Jones and Schmidt. In [23, Theorem 2.1], they proved that any ergodic but not strongly ergodic countable measure preserving equivalence relation $\mathcal{R}$ on a probability space (X, μ) admits a hyperfinite quotient. More specifically, there exists an ergodic hyperfinite measure preserving equivalence relation $\mathcal{R}_{\text{hyp}}$ on a probability space (Y, ν) together with a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that $(\pi \times \pi)(\mathcal{R}) = \mathcal{R}_{\text{hyp}}$, almost everywhere. In [23, Problem 4.3], Jones and Schmidt asked whether there is always such a quotient with the additional property that $\mathcal{R}_0 := \{(x_1, x_2) \in \mathcal{R} \mid \pi(x_1) = \pi(x_2)\}$ is strongly ergodic on almost all of its ergodic components. If such a quotient exists, then following [21, Definition 1.3], we say that $\mathcal{R}$ has the *Jones–Schmidt property*. If $\mathcal{R}$ has the Jones–Schmidt property and we let $M = L(\mathcal{R})$, $A = L^\infty(X)$, then there exists a decreasing sequence of von Neumann subalgebras $(B_n)_{n\in\mathbb{N}}$ of A such that $M' \cap A^\omega = \bigcap_n B_n^\omega$ and $B_{n+1} \subset B_n$ has finite index for every $n \in \mathbb{N}$ (see [21, Proposition 5.3 and the proof of Lemma 6.1]).

In [21, Theorems E and F], the authors settled in the negative [23, Problem 4.3] by providing examples of equivalence relations $\mathcal{R}$ without the Jones–Schmidt property. This was achieved by showing that for certain $\mathcal{R}$, in the above notation, $M' \cap A^\omega$ is not equal to $\cap_n B_n^\omega$, for any decreasing sequence of von Neumann subalgebras $(B_n)_{n\in\mathbb{N}}$ of A with $B_{n+1} \subset B_n$ of finite index for every $n \in \mathbb{N}$.

Theorem A allows us to strengthen the negative solution to [23, Problem 4.3] given in [21]. More precisely, in the context of Theorem A, assume that Γ is not inner amenable and let $\mathcal{R}$ be the equivalence relation associated with the action $\Gamma \curvearrowright B$. Since $M = L(\mathcal{R}) = B \rtimes \Gamma$ has no residual sequence by Theorem A, while $M' \cap A^\omega = M' \cap M^\omega$ by [7], we deduce that $M' \cap A^\omega$ cannot be written as $\bigcap_n B_n^\omega$, for *any* decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of A.

Our second main result shows that the conclusion of Theorem A also holds if we replace Gaussian by free Bogoljubov actions (see § 2.6). Moreover, we establish the following stronger statement.

Theorem B. *Let Γ be a countable non-inner amenable group. For every $k \in \mathbb{N}$, let $\pi_k : \Gamma \to \mathcal{O}(\mathcal{H}_k)$ be an orthogonal representation such that*

(1) *$\pi_k^{\otimes l}$ is weakly contained in the left regular representation of Γ, for some $l = l(k) \in \mathbb{N}$, and*

(2) *there are orthogonal unit vectors $\xi_k^1, \xi_k^2 \in \mathcal{H}_k$ such that $\max_{m\in\{1,2\}} \|\pi_k(g)(\xi_k^m) - \xi_k^m\| \to 0$, as $k \to \infty$, for every $g \in \Gamma$.*

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Let $\Gamma \curvearrowright (B_k, \tau_k)$ be the free Bogoljubov action associated with π_k and $\Gamma \curvearrowright (B, \tau) := \bar{\bigotimes}_{k\in\mathbb{N}}(B_k, \tau_k)$ be the diagonal product action. Define $M = B \rtimes \Gamma$.

Then the II_1 factor M does not admit a residual sequence of von Neumann subalgebras.

Moreover, there exists a separable von Neumann subalgebra $P \subset M' \cap M^\omega$ such that there is no sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M satisfying $P \subset \prod_\omega A_n \subset M' \cap M^\omega$.

Since $\Gamma = \mathbb{F}_n$ is not inner amenable for any $n \geqslant 2$ and the representations $(\pi_k)_{k\in\mathbb{N}}$ from Example 1.5 satisfy the hypothesis of Theorem B, its conclusion holds for those examples. Moreover, in the notation from Example 1.5, $\pi_k = \rho_{t_k} \oplus \rho_{t_k}$ also satisfy the hypothesis of Theorem B.

In order to put Theorem B into a better perspective and to contrast it with Theorem A, we note the following result.

Proposition C. *Let (M_n, τ_n), $n \in \mathbb{N}$, be a sequence of tracial von Neumann algebras. Let P, Q be commuting separable von Neumann subalgebras of $\prod_\omega M_n$. Assume that P is amenable.*

Then there exist commuting von Neumann subalgebras P_n, Q_n of M_n, for every $n \in \mathbb{N}$, such that $P \subset \prod_\omega P_n$ and $Q \subset \prod_\omega Q_n$.

Proposition C implies that for any tracial von Neumann algebra (M, τ) and any separable amenable von Neumann subalgebra $P \subset M' \cap M^\omega$, there is a sequence $(P_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M such that $P \subset \prod_\omega P_n$ and $M \subset \prod_\omega (P_n' \cap M)$, and, therefore, $P \subset \prod_n P_n \subset M' \cap M^\omega$. Consequently, the moreover part of Theorem B cannot hold if P is amenable. In particular, if $M = B \rtimes \Gamma$ is as in Theorem A and Γ is not inner amenable, then M will not satisfy the moreover assertion of Theorem B. Indeed, in this case, $M' \cap M^\omega$ is abelian, being a subalgebra of B^ω by [7].

In recent years, there has been growing interest in the study of the notion of stability for groups (see the survey [44]). As a byproduct of the methods developed in this article, we obtain two applications to the notion of tracial stability for countable groups, formalized recently in [17] (see also [16]).

Definition 1.7 [17, Definition 3]. A countable group Γ is *W^*-tracially stable* if for any sequence (M_n, τ_n), $n \in \mathbb{N}$, of tracial von Neumann algebras and any homomorphism $\varphi : \Gamma \to \mathcal{U}(\prod_\omega M_n)$, there exist homomorphisms $\varphi_n : \Gamma \to \mathcal{U}(M_n)$, for every $n \in \mathbb{N}$, such that $\varphi = (\varphi_n)_n$.

The class of W*-tracially stable groups contains all abelian and free groups as well as other classes of both amenable and non-amenable groups; see [17]. As an immediate consequence of Proposition C, we deduce that the class of W*-tracially stable groups is closed under taking the direct product with a W^*-tracially stable amenable group. For the case of the direct product with an abelian group, this result is part of [17, Theorem 1].

Corollary D. *Let Γ and Σ be W^*-tracially stable groups. Assume that Σ is amenable. Then $\Gamma \times \Sigma$ is W^*-tracially stable.*

In contrast to Corollary D, we show that the class of W*-tracially stable groups is not closed under taking the direct product. More precisely, we prove that any direct product of non-abelian free groups is not W*-tracially stable, thereby answering a question of Atkinson in the negative (see [1, Question 4.16]).

Theorem E. *$\mathbb{F}_l \times \mathbb{F}_m$ is not W*-tracially stable, for any $2 \leqslant l, m \leqslant +\infty$.*

Moreover, there exist a II_1 factor M and a trace preserving $*$-homomorphism $\varphi : L(\mathbb{F}_2 \times \mathbb{F}_2) \to M^\omega$ such that there is no sequence of homomorphisms $\varphi_n : \mathbb{F}_2 \times \mathbb{F}_2 \to \mathcal{U}(M)$ satisfying $\varphi_{|\mathbb{F}_2\times\mathbb{F}_2} = (\varphi_n)_n$.

Structure of the paper

Besides the introduction, there are four other sections in this paper. In § 2, we recall some preliminaries and prove a few useful lemmas needed in the remainder of the paper. In § 3, inspired by Boutonnet's work [2, 3], we prove a structural result concerning II_1 factors associated with Gaussian and free Bogoljubov actions. In § 4, this is used to prove Theorems A and B. Finally in § 5, we prove Proposition C and use the established machinery from the previous sections to deduce Theorem E.

2. Preliminaries

2.1. Tracial von Neumann algebras

We begin this section by recalling several notions and constructions involving tracial von Neumann algebras.

A *tracial von Neumann algebra* (M, τ) is a von Neumann algebra M equipped with a faithful normal tracial state $\tau : M \to \mathbb{C}$. We denote by $L^2(M)$ the completion of M with respect to the 2-norm $\|x\|_2 = \sqrt{\tau(x^*x)}$ and consider the standard representation $M \subset \mathbb{B}(L^2(M))$. We also denote by $\mathcal{U}(M)$ the group of unitary elements of M, by $(M)_1 = \{x \in M \mid \|x\| \leqslant 1\}$ the unit ball of M and by $\mathcal{Z}(M) = M \cap M'$ the center of M. It follows from von Neumann's bicommutant theorem that a self-adjoint set $S \subset M$ generates M as a von Neumann algebra if and only if $S'' = M$.

Let $P \subset M$ be a unital von Neumann subalgebra. *Jones' basic construction* of the inclusion $P \subset M$ is defined as the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and the orthogonal projection $e_P : L^2(M) \to L^2(P)$ and is denoted by $\langle M, e_P\rangle$. The basic construction $\langle M, e_P\rangle$ carries a canonical semi-finite trace $\hat{\tau}$ defined by $\hat{\tau}(xe_Py) = \tau(xy)$, for all $x, y \in M$. We further denote by $E_P : M \to P$ the conditional expectation onto P, by $P' \cap M = \{x \in M \mid xy = yx$, for all $y \in P\}$ the *relative commutant of P in M* and by $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) \mid uPu^* = P\}$ the *normalizer of P in M*. We say that P is *regular* in M if $\mathcal{N}_M(P)$ generates M as a von Neumann algebra.

Any trace preserving action $\Gamma \curvearrowright^\sigma (M, \tau)$ extends to a unitary representation $\sigma : \Gamma \to \mathcal{U}(L^2(M))$ called the *Koopman representation of σ*.

Let ω be a free ultrafilter on $\mathbb{N}$. Consider the C*-algebra $\ell^\infty(\mathbb{N}, M) = \{(x_n) \in M^{\mathbb{N}} \mid \sup \|x_n\| < \infty\}$ together with its closed ideal $\mathcal{I} = \{(x_n) \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\omega} \|x_n\|_2 = 0\}$. Then $M^\omega := \ell^\infty(\mathbb{N}, M)/\mathcal{I}$ is a tracial von Neumann algebra, called the *ultrapower* of

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M, whose canonical trace is given by $\tau_\omega(x) = \lim_{n\to\omega} \tau(x_n)$, for all $x = (x_n) \in M^\omega$. If $(M_n)_n$ is a sequence of von Neumann subalgebras of M, then their *ultraproduct*, denoted by $\prod_\omega M_n$, can be realized as the von Neumann subalgebra of M^ω consisting of $x = (x_n)$ such that $\lim_{n\to\omega} \|x_n - E_{M_n}(x_n)\|_2 = 0$.

Lemma 2.1. *Let (M, τ) be a tracial von Neumann algebra and $(A_n)_n$ be a sequence of von Neumann subalgebras of M such that $\prod_\omega A_n \subset M' \cap M^\omega$. Then $\lim_{n\to\omega} \|x - E_{A_n'\cap M}(x)\|_2 = 0$, for every $x \in M$.*

Proof. Let $x \in M$. If $n \in \mathbb{N}$, we can find $u_n \in \mathcal{U}(A_n)$ such that $\|x - u_n x u_n^*\|_2 \geqslant \|x - E_{A_n'\cap M}(x)\|_2$ (see, e.g., the proof of [21, Theorem 2.5]). Since $(u_n) \in \prod_\omega A_n$ and $\prod_\omega A_n \subset M' \cap M^\omega$, we get that $\lim_{n\to\omega} \|x - u_n x u_n^*\|_2 = 0$ and hence $\lim_{n\to\omega} \|x - E_{A_n'\cap M}(x)\|_2 = 0$. □

2.2. Hilbert bimodules

Let (M_1, τ_1) and (M_2, τ_2) be two tracial von Neumann algebras. An M_1-M_2-*bimodule* is a Hilbert space $\mathcal{H}$ endowed with two normal, commuting $*$-homomorphisms $\pi_1 : M_1 \to \mathbb{B}(\mathcal{H})$ and $\pi_2 : M_2^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$. We define a $*$-homomorphism $\pi_{\mathcal{H}} : M_1 \otimes M_2^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$ by $\pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}}) = \pi_1(x)\pi_2(y^{\mathrm{op}})$ and write $x\xi y = \pi_1(x)\pi_2(y^{\mathrm{op}})\xi$, for all $x \in M_1$, $y \in M_2$ and $\xi \in \mathcal{H}$. We also write ${}_{M_1}\mathcal{H}_{M_2}$ to indicate that $\mathcal{H}$ is an M_1-M_2-bimodule. Examples of bimodules include the *trivial* M_1-bimodule ${}_{M_1}L^2(M_1)_{M_1}$ and the *coarse* M_1-M_2-bimodule ${}_{M_1}L^2(M_1) \otimes L^2(M_2)_{M_2}$.

Next, we recall a few notions and constructions involving bimodules (see [9, Appendix B] and [33]). If $\mathcal{H}$ and $\mathcal{K}$ are M_1-M_2-bimodules, we say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ and write $\mathcal{H} \subset_{\mathrm{weak}} \mathcal{K}$ if $\|\pi_{\mathcal{H}}(T)\| \leqslant \|\pi_{\mathcal{K}}(T)\|$, for all $T \in M_1 \otimes M_2^{\mathrm{op}}$. If $\mathcal{H}$ is an M_1-M_2-bimodule and $\mathcal{K}$ is an M_2-M_3-bimodule, then the *Connes fusion tensor product of $\mathcal{H}$ and $\mathcal{K}$* is an M_1-M_3-bimodule denoted by $\mathcal{H} \otimes_{M_2} \mathcal{K}$. If $\Phi : M_1 \to M_2$ is a unital normal completely positive map, then there is a unique M_1-M_2-bimodule, denoted by $\mathcal{H}_\Phi$, with a unit vector $\xi_\Phi \in \mathcal{H}_\Phi$ such that $M_1\xi_\Phi M_2$ is dense in $\mathcal{H}_\Phi$ and $\langle x\xi_\Phi y, \xi_\Phi\rangle = \tau_2(\Phi(x)y)$, for all $x \in M_1$ and $y \in M_2$. The next result analyzes the Connes fusion tensor product of bimodules associated with completely positive maps.

Lemma 2.2. *Let $\Phi : M_1 \to M_2$ and $\Psi : M_2 \to M_3$ be unital normal completely positive maps, where $(M_1, \tau_1), (M_2, \tau_2), (M_3, \tau_3)$ are tracial von Neumann algebras. Then the following hold:*

1. *The M_1-M_3-bimodule $\mathcal{H}_{\Psi\circ Ad(u)\circ\Phi}$ is isomorphic to a sub-bimodule of $\mathcal{H}_\Phi \otimes_{M_2} \mathcal{H}_\Psi$, for every $u \in \mathcal{U}(M_2)$.*
2. *If $\mathcal{U}$ is a set of unitaries in M_2 whose span is $\|.\|_2$-dense in M_2, then the M_1-M_3-bimodule $\mathcal{H}_\Phi \otimes_{M_2} \mathcal{H}_\Psi$ is isomorphic to a sub-bimodule of $\bigoplus_{u\in\mathcal{U}} \mathcal{H}_{\Psi\circ Ad(u)\circ\Phi}$.*

Proof. For $u \in \mathcal{U}(M_2)$, we denote $\eta_u := \xi_\Phi u^* \otimes_{M_2} \xi_\Psi \in \mathcal{H}_\Phi \otimes_{M_2} \mathcal{H}_\Psi$. Following [33, § 1.3.1], for every $x \in M_1$, $y \in M_3$, we have that

$$\langle x\eta_u y, \eta_u\rangle = \langle x\xi_\Phi u^* \otimes_{M_2} \xi_\Psi y, \xi_\Phi u^* \otimes_{M_2} \xi_\Psi\rangle = \langle x\xi_\Phi u^* p, \xi_\Phi u^*\rangle = \tau_2(\Phi(x)u^* pu),$$

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where $p \in M_2$ is such that $\tau_2(zp) = \langle z\xi_\Psi y, \xi_\Psi \rangle = \tau_3(\Psi(z)y)$, for all $z \in M_2$. Thus, for all $x \in M_1, y \in M_3$, we have that $\langle x\eta_u y, \eta_u \rangle = \tau_2(u\Phi(x)u^* p) = \tau_3(\Psi(u\Phi(x)u^*)y)$. This shows that the M_1-M_3-bimodule $\overline{M_1\eta_u M_3}$ is isomorphic to $\mathcal{H}_{\Psi\circ \mathrm{Ad}(u)\circ\Phi}$ and proves the first assertion of the lemma.

Finally, note that if the span of $\mathcal{U} \subset \mathcal{U}(M_2)$ is $\|.\|_2$-dense in M_2, then the span of $\{\overline{M_1\eta_u M_3} \mid u \in \mathcal{U}\}$ is dense in $\mathcal{H}_\Phi \otimes_{M_2} \mathcal{H}_\Psi$. This implies the second assertion. □

2.3. Intertwining by bimodules

We next recall from [35, Theorem 2.1 and Corollary 2.3] the powerful *intertwining-by-bimodules* technique of Popa.

Theorem 2.3 [35]. *Let (M, τ) be a tracial von Neumann algebra and $P \subset pMp$, $Q \subset qMq$ be unital von Neumann subalgebras, for some projections $p, q \in M$. Then the following conditions are equivalent:*

- *There exist projections $p_0 \in P, q_0 \in Q$, a $*$-homomorphism $\theta : p_0Pp_0 \to q_0Qq_0$ and a non-zero partial isometry $v \in q_0Mp_0$ such that $\theta(x)v = vx$, for all $x \in p_0Pp_0$.*
- *There is no net $u_n \in \mathcal{U}(P)$ satisfying $\|E_Q(x^*u_n y)\|_2 \to 0$, for all $x, y \in pMq$.*
- *There exists a non-zero projection $f \in P' \cap \langle M, e_Q\rangle$ with $\hat{\tau}(f) < \infty$.*

If one of these conditions holds true, then we write $P \prec_M Q$ and say that a corner of P embeds into Q inside M. If $Pp' \prec_M Q$ for any non-zero projection $p' \in P' \cap pMp$, then we write $P \prec^s_M Q$.

2.4. Amenability

A tracial von Neumann algebra (M, τ) is called *amenable* if there exists a positive linear functional $\varphi : \mathbb{B}(L^2(M)) \to \mathbb{C}$ such that $\varphi_{|M} = \tau$ and φ is *M-central*, in the following sense: $\varphi(xT) = \varphi(Tx)$, for all $x \in M$ and $T \in \mathbb{B}(L^2(M))$. Equivalently, (M, τ) is amenable if ${}_M L^2(M)_M$ is weakly contained in ${}_M L^2(M) \otimes L^2(M)_M$. By Connes' celebrated classification of amenable factors [8], M is amenable if and only if it is approximately finite dimensional.

Next, we recall the notion of relative amenability introduced by Ozawa and Popa. Let $p \in M$ be a projection and $P \subset pMp$, $Q \subset M$ be von Neumann subalgebras. Following [31, § 2.2], we say that P is *amenable relative to Q inside M* if there exists a positive linear functional $\varphi : p\langle M, e_Q\rangle p \to \mathbb{C}$ such that $\varphi_{|pMp} = \tau$ and φ is P-central.

As shown in [11, Lemma 2.7], relative amenability is closed under inductive limits. Here we establish the following generalization of this result, which we will need later on. Given a set I, we denote by $\lim_n$ a state on $\ell^\infty(I)$ which extends the usual limit.

Lemma 2.4. *Let (M, τ) be a tracial von Neumann algebra and $P, Q \subset M$ be von Neumann subalgebras. Assume that $P_n \subset M$, $n \in I$, is a net of von Neumann subalgebras such that $\|E_{P_n}(x) - x\|_2 \to 0$, for all $x \in P$, and $p_n \in P_n' \cap M$ are projections such that $P_n p_n$ is amenable relative to Q inside M, for every $n \in I$. Then there exists a projection $p \in P' \cap M$ such that Pp is amenable relative to Q inside M and $\tau(p) \geqslant \lim_n \tau(p_n)$.*

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Proof. We may clearly assume that $c := \lim_n \tau(p_n) > 0$ and $\tau(p_n) > 0$, for every n. For every n, let $\varphi_n : p_n\langle M, e_Q\rangle p_n \to \mathbb{C}$ be a $P_n p_n$-central positive linear functional such that $\varphi_{n|p_n M p_n} = \tau$. The Cauchy–Schwarz inequality implies that $|\varphi_n(p_n T x p_n)| \leqslant \sqrt{\varphi_n(p_n T T^* p_n)\varphi_n(p_n x^* x p_n)} \leqslant \|T\|\|x\|_2$ and, similarly, that $|\varphi_n(p_n x T p_n)| \leqslant \|T\|\|x\|_2$, for all $x \in M$, $T \in \langle M, e_Q\rangle$.

We define a state $\varphi : \langle M, e_Q\rangle \to \mathbb{C}$ by letting

$$\varphi(T) = \lim_n \frac{\varphi_n(p_n T p_n)}{\tau(p_n)}, \quad \text{for every } T \in \langle M, e_Q\rangle.$$

We claim that φ is P-central. To this end, let $x \in P$, $T \in \langle M, e_Q\rangle$ and $n \in I$. Since φ_n is $P_n p_n$-central, $\varphi_n(p_n T E_{P_n}(x) p_n) = \varphi_n(p_n E_{P_n}(x) T p_n)$, and, thus,

$$\begin{aligned}|\varphi_n(p_n T x p_n) - \varphi_n(p_n x T p_n)| &\leqslant |\varphi_n(p_n T(x - E_{P_n}(x))p_n)| + |\varphi_n(p_n(x - E_{P_n}(x))T p_n)| \\ &\leqslant 2\|T\|\|x - E_{P_n}(x)\|_2.\end{aligned}$$

Since $\|x - E_{P_n}(x)\|_2 \to 0$ and $\lim_n \tau(p_n) > 0$, we get that $\varphi(Tx) = \varphi(xT)$, and the claim is proven.

Finally, note that $\varphi_{|M} \leqslant \frac{1}{c}\tau$. Thus, we can find $y \in P' \cap M$ such that $0 \leqslant y \leqslant \frac{1}{c}$ and $\varphi(x) = \tau(xy)$, for all $x \in M$. Let $p \in P' \cap M$ be the support projection of y. Then $y \leqslant \frac{1}{c}p$; hence, $\tau(p) \geqslant c\tau(y) = c$. Since the restriction of φ to $p(P' \cap M)p$ is faithful, [31, Theorem 2.1] implies that Pp is amenable relative to Q inside M, which finishes the proof. □

Corollary 2.5. *Let (M, τ) and (N, τ') be tracial von Neumann algebras. Assume that there exists a net of von Neumann subalgebras $P_n \subset M$, $n \in I$, and trace preserving $*$-homomorphisms $\pi_n : N \to M$ such that $\|\pi_n(x) - E_{P_n}(\pi_n(x))\|_2 \to 0$, for every $x \in N$. For $n \in I$, let $p_n \in P_n' \cap M$ be a projection such that $P_n p_n$ is amenable. Then there is a projection $z \in \mathcal{Z}(N)$ such that Nz is amenable and $\tau(z) \geqslant \lim_n \tau(p_n)$. In particular, if P_n is amenable for every n, then N is amenable.*

Proof. For every n, let $M_n = M$ and view P_n and N as subalgebras of M_n via the identity map and π_n, respectively. If we put $\tilde{M} = *_{N,n\in I} M_n$, then we have $\|E_{P_n}(x) - x\|_2 \to 0$, for every $x \in N$. Since $P_n p_n$ is amenable for every n, Lemma 2.4 implies the existence of a projection $p \in N' \cap \tilde{M}$ such that Np is amenable and $\tau(p) \geqslant \lim_n \tau(p_n)$. Thus, if z is the support projection of $E_{\mathcal{Z}(N)}(p)$, then Nz is amenable. Since $z \geqslant p$, we have that $\tau(z) \geqslant \tau(p)$, which finishes the proof. □

The next lemma, which appears to be of independent interest, provides general conditions which guarantee that if P is amenable relative to a decreasing net of subalgebras Q_n, then P is amenable relative to their intersection, $\bigcap_n Q_n$. More generally, we have the following.

Lemma 2.6. *Let (M, τ) be a tracial von Neumann algebra and $Q \subset M$ a von Neumann subalgebra. Assume that there exist nets of von Neumann subalgebras $Q_n, M_n \subset M$ such that*

(1) *$Q \subset M_n \cap Q_n$ and ${}_{Q_n}L^2(M)_{M_n} \subset_{weak} {}_{Q_n}L^2(Q_n) \otimes_Q L^2(M_n)_{M_n}$, for every n;*
(2) *$\|x - E_{M_n}(x)\|_2 \to 0$, for every $x \in M$.*

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If $P \subset M$ is a von Neumann subalgebra which is amenable relative to Q_n inside M, for every n, then P is amenable relative to Q inside M.

Lemma 2.6 applies, in particular, if there exists $u_n \in \mathcal{U}(M)$ such that $u_n P u_n^* \subset Q_n$ or, more generally, if $P \prec^s_M Q_n$, for every n. Indeed, by [11, Lemma 2.6(3)], the latter condition implies that P is amenable relative to Q_n inside M.

Proof. Assume that P is amenable relative to Q_n, for every n. Then [31, Theorem 2.1] gives that ${}_P L^2(M)_M \subset_{\text{weak}} {}_P L^2(M) \otimes_{Q_n} L^2(M)_M$, and, thus, ${}_P L^2(M)_{M_n} \subset_{\text{weak}} {}_P L^2(M) \otimes_{Q_n} L^2(M)_{M_n}$, for every n. Since ${}_{Q_n} L^2(M)_{M_n} \subset_{\text{weak}} {}_{Q_n} L^2(Q_n) \otimes_Q L^2(M_n)_{M_n}$, we further get that ${}_P L^2(M)_{M_n} \subset_{\text{weak}} {}_P L^2(M) \otimes_Q L^2(M_n)_{M_n}$, and, thus,

$$\begin{aligned} {}_P L^2(M) \otimes_{M_n} L^2(M)_M &\subset_{\text{weak}} {}_P L^2(M) \otimes_Q L^2(M_n) \otimes_{M_n} L^2(M)_M \\ &= {}_P L^2(M) \otimes_Q L^2(M)_M, \quad \text{for every } n. \end{aligned}$$

On the other hand, since $\|x - E_{M_n}(x)\|_2 \to 0$, for every $x \in M$, we have

$${}_P L^2(M)_M \subset_{\text{weak}} \bigoplus_n {}_P L^2(M) \otimes_{M_n} L^2(M)_M.$$

By combining the last two displayed inclusions, we get that ${}_P L^2(M)_M \subset_{\text{weak}} {}_P L^2(M) \otimes_Q L^2(M)_M$, and, therefore, P is amenable relative to Q inside M. $\square$

Remark 2.7. Several weaker versions of particular cases of Lemma 2.6 have been observed before. Indeed, conditions (1) and (2) from Lemma 2.6 are satisfied in the following two cases:

(a) $M = *_{Q,k\in\mathbb{N}} M_k$ is an amalgamated free product of tracial von Neumann algebras $(M_k)_{k\in\mathbb{N}}$ over a common subalgebra Q, $Q_n = *_{Q,k\geqslant n} M_k$ and $M_n = *_{Q,k<n} M_k$.

(b) $M = (\bar{\bigotimes}_{k\in\mathbb{N}} M_k)\bar{\otimes} Q$ is an infinite tensor product of tracial von Neumann algebras $(M_k)_{k\in\mathbb{N}}$ and Q, $Q_n = (\bar{\bigotimes}_{k\geqslant n} M_k)\bar{\otimes} Q$ and $M_n = (\bar{\bigotimes}_{k<n} M_k)\bar{\otimes} Q$.

Lemma 2.6 was first noticed by the first author in case (a) under the assumption that P can be unitarily conjugated into Q_n and extended in [19, Proposition 4.2] to cover the more general assumption that $P \prec^s_M Q_n$. When $Q = \mathbb{C}1$, the latter result was also noticed by Boutonnet and Vaes (personal communication), whose proof inspired our Lemma 2.6. In case (b), weaker versions of Lemma 2.6 were obtained in [22, Lemma 4.4] and [5, Proposition 2.7].

2.5. Malleable deformations

In [34, 35], Popa introduced the notion of an *s-malleable deformation* of a von Neumann algebra. In combination with his powerful *deformation/rigidity* techniques, this notion has led to remarkable progress in the theory of von Neumann algebras (see, e.g., [20, 38, 45]). S-malleable deformations will also play an important role in this paper.

Definition 2.8. Let (M, τ) be a tracial von Neumann algebra. We say that a triple $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}}, \beta)$ is an *s-malleable deformation* of M if the following conditions hold:

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(1) $(\tilde{M}, \tilde{\tau})$ is a tracial von Neumann algebra such that $\tilde{M} \supset M$ and $\tilde{\tau}_{|M} = \tau$;

(2) $(\alpha_t)_{t\in\mathbb{R}} \subset \mathrm{Aut}(\tilde{M}, \tilde{\tau})$ is a one-parameter group with $\lim_{t\to 0} \|\alpha_t(x) - x\|_2 = 0$, for all $x \in \tilde{M}$.

(3) $\beta \in \mathrm{Aut}(\tilde{M}, \tilde{\tau})$ satisfies $\beta^2 = \mathrm{Id}_{\tilde{M}}$, $\beta\alpha_t\beta^{-1} = \alpha_{-t}$ for all $t \in \mathbb{R}$, and $\beta(x) = x$, for all $x \in M$.

As established in [36], s-malleable deformations have the following 'transversality' property.

Lemma 2.9 [36, Lemma 2.1]. *For any $x \in M$ and $t \in \mathbb{R}$, we have*

$$\|x - \alpha_{2t}(x)\|_2 \leqslant 2\,\|\alpha_t(x) - E_M(\alpha_t(x))\|_2\,.$$

2.6. Gaussian and free Bogoljubov actions

We next discuss two kinds of actions that will play a crucial role in this paper, Gaussian and free Bogoljubov actions. Below we describe one possible construction of these actions, following [32] and [47]. For further properties of Gaussian and free Bogoljubov actions, we refer the reader to [3] and [18], respectively.

For the remainder of the preliminaries, we fix an orthogonal representation $\pi : \Gamma \to \mathcal{O}(H_{\mathbb{R}})$ of a countable group Γ on a real Hilbert space $H_{\mathbb{R}}$. Let $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Hilbert space, $H^{\otimes n}$ its nth tensor power and $H^{\odot n}$ its symmetric nth tensor power. The latter is the closed subspace of $H^{\otimes n}$ spanned by vectors of the form

$$\xi_1 \odot \cdots \odot \xi_n := \frac{1}{n!} \sum_{\sigma\in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)},$$

with the inner product normalized such that $\|\xi\|^2_{H^{\odot n}} = n!\,\|\xi\|^2_{H^{\otimes n}}$. We then consider the *symmetric Fock space*

$$\mathcal{S}(H) := \mathbb{C}\Omega \oplus \bigoplus_{n\geqslant 1} H^{\odot n},$$

where the unit vector Ω is the so-called *vacuum vector*. Any vector $\xi \in H$ gives rise to an unbounded operator ℓ_ξ on $\mathcal{S}(H)$, the so-called *left creation operator*, defined by

$$\ell_\xi(\Omega) = \xi, \quad \text{and} \quad \ell_\xi(\xi_1 \odot \cdots \odot \xi_n) = \xi \odot \xi_1 \odot \cdots \odot \xi_n.$$

Denoting $s(\xi) = \ell_\xi + \ell_\xi^*$, one checks that the operators $\{s(\xi)\}_{\xi\in H}$ commute. Moreover, one can show [32] that with respect to the vacuum state $\langle \cdot\Omega, \Omega\rangle$, they can be regarded as independent random variables with Gaussian distribution $\mathcal{N}(0, \|\xi\|^2)$.

Consider the abelian von Neumann algebra $A_\pi \subset B(\mathcal{S}(H))$ generated by all operators of the form

$$\omega(\xi_1, \ldots, \xi_n) := \exp(i\pi s(\xi_1) \ldots s(\xi_n)),$$

together with the trace $\tau = \langle \cdot\Omega, \Omega\rangle$. Any orthogonal operator $T \in \mathcal{O}(H_{\mathbb{R}})$ can also be viewed as a unitary operator on its complexification H and gives rise to a unitary operator on $\mathcal{S}(H)$, which we will still denote by T, defined by

$$T(\Omega) = \Omega, \quad \text{and} \quad T(\xi_1 \odot \cdots \odot \xi_n) = (T\xi_1) \odot \cdots \odot (T\xi_n).$$

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One then checks that $T\omega(\xi_1,\ldots,\xi_n)T^* = \omega(T\xi_1,\ldots,T\xi_n)$; hence, T normalizes A_π. Since $T(\Omega)=\Omega$, $\mathrm{Ad}(T)$ is a trace preserving automorphism of A_π.

Definition 2.10. The *Gaussian action* associated with π is the action $\sigma=\sigma_\pi:\Gamma \curvearrowright (A_\pi,\tau)$ defined by $\sigma_g = \mathrm{Ad}(\pi(g))$, for every $g\in\Gamma$.

One can easily check that the unitaries $\omega(\xi)$ satisfy the properties $\omega(0)=1$, $\omega(\xi+\eta)=\omega(\xi)\omega(\eta)$, $\tau(\omega(\xi))=\exp(-\|\xi\|^2)$ and $\sigma_g(\omega(\xi))=\omega(\pi(g)\xi)$ for all $\xi,\eta\in H$, $g\in\Gamma$. This, in fact, gives an equivalent description of the Gaussian action (see [46]).

The free Bogoljubov action arises in a similar way using the *full Fock space*

$$\mathcal{F}(H) := \mathbb{C}\Omega \oplus \bigoplus_{n\geqslant 1} H^{\otimes n}.$$

We consider the *left creation operator* L_ξ associated with $\xi\in H$ defined by

$$L_\xi(\Omega)=\xi, \quad \text{and} \quad L_\xi(\xi_1\otimes\cdots\otimes\xi_n)=\xi\otimes\xi_1\otimes\cdots\otimes\xi_n.$$

Putting $W(\xi)=L_\xi+L_\xi^*$, one can show [47] that the distribution of the self-adjoint operator $W(\xi)$ with respect to the vacuum state $\langle\cdot\Omega,\Omega\rangle$ is the semicircular law supported on $[-2\|\xi\|, 2\|\xi\|]$ and that for any orthogonal set of vectors from $H_\mathbb{R}$, the associated family of operators is freely independent with respect to $\langle\cdot\Omega,\Omega\rangle$.

Denote by $\Gamma(H_\mathbb{R})''$ the von Neumann algebra generated by $\{W(\xi)\mid \xi\in H_\mathbb{R}\}$. Then $\Gamma(H_\mathbb{R})''$ is isomorphic to the free group factor $L(\mathbb{F}_{\dim(H_\mathbb{R})})$. Moreover, $\tau=\langle\cdot\Omega,\Omega\rangle$ is a normal faithful trace on $\Gamma(H_\mathbb{R})''$. As for the symmetric Fock space, any operator $T\in\mathcal{O}(H_\mathbb{R})$ induces an operator $T\in\mathcal{U}(\mathcal{F}(H))$, satisfying $\mathrm{Ad}(T)(W(\xi))=W(T\xi)$.

Definition 2.11. The *free Bogoljubov action* associated with π is the action $\rho=\rho_\pi:\Gamma\curvearrowright (\Gamma(H_\mathbb{R})'',\tau)$ defined by $\rho_g=\mathrm{Ad}(\pi(g))$, for every $g\in\Gamma$.

Since $\overline{\Gamma(H_\mathbb{R})''\Omega}=\mathcal{F}(H)$, the Koopman representation associated with ρ of Γ on $L^2(\Gamma(H_\mathbb{R})'')$ is isomorphic to the representation of Γ on $\mathcal{F}(H)$. This implies the following fact which will be needed later on.

Lemma 2.12. *Denote by ρ_0 the restriction of the Koopman representation of ρ to $L^2(\Gamma(H_\mathbb{R})'')\ominus\mathbb{C}1$. If $\pi^{\otimes k}$ is weakly contained in the left regular representation of Γ, for some $k\in\mathbb{N}$, then $\rho_0^{\otimes k}$ is weakly contained in the left regular representation of Γ.*

2.7. Deformations associated with Gaussian and free Bogoljubov actions

We will now recall the construction of s-malleable deformations of the crossed product von Neumann algebras associated with the above actions. On $H_\mathbb{R}\oplus H_\mathbb{R}$, consider the orthogonal operators

$$A_t=\begin{pmatrix}\cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t)\\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)\end{pmatrix}, \quad t\in\mathbb{R}, \quad \text{and} \quad B=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

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We note that, canonically, $A_{\pi\oplus\pi} \cong A_\pi \bar{\otimes} A_\pi$ and $\Gamma(H_{\mathbb{R}} \oplus H_{\mathbb{R}})'' \cong \Gamma(H_{\mathbb{R}})'' * \Gamma(H_{\mathbb{R}})''$. Under these identifications, we have that $\sigma_{\pi\oplus\pi} \cong \sigma_\pi \otimes \sigma_\pi$ and $\rho_{\pi\oplus\pi} \cong \rho_\pi * \rho_\pi$, respectively. Associated with the operators A_t and B, we get automorphisms

$$\alpha_t := \mathrm{Ad}(A_t), \quad t \in \mathbb{R}, \quad \text{and} \quad \beta := \mathrm{Ad}(B)$$

of $A_\pi \bar{\otimes} A_\pi$ and $\Gamma(H_{\mathbb{R}})'' * \Gamma(H_{\mathbb{R}})''$, respectively. Since A_t and B commute with $\pi \oplus \pi$, it follows that α_t and β commute with $\sigma_\pi \otimes \sigma_\pi$ and $\rho_\pi * \rho_\pi$, respectively.

- For the Gaussian action, let $M = A_\pi \rtimes \Gamma$, $\tilde{M} = (A_\pi \bar{\otimes} A_\pi) \rtimes \Gamma$, and view M as a subalgebra of $\tilde{M}$ via $M \cong (A_\pi \bar{\otimes} 1) \rtimes \Gamma$. By the discussion above, the automorphisms α_t and β of $A_\pi \bar{\otimes} A_\pi$ extend to automorphisms of $\tilde{M}$ by letting $\alpha_t(u_g) = \beta(u_g) = u_g$, for all $g \in \Gamma$.
- For the free Bogoljubov action, let $M = \Gamma(H_{\mathbb{R}})'' \rtimes \Gamma$, $\tilde{M} = (\Gamma(H_{\mathbb{R}})'' * \Gamma(H_{\mathbb{R}})'') \rtimes \Gamma$, and view M as a subalgebra of $\tilde{M}$ via $M \cong (\Gamma(H_{\mathbb{R}})'' * 1) \rtimes \Gamma$. By the discussion above, the automorphisms α_t and β of $\Gamma(H_{\mathbb{R}})'' * \Gamma(H_{\mathbb{R}})''$ extend to automorphisms of $\tilde{M}$ by letting $\alpha_t(u_g) = \beta(u_g) = u_g$, for all $g \in \Gamma$.

In both cases, it is easy to check that $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}}, \beta)$ is an s-malleable deformation of M.

3. Spectral gap rigidity

This section is devoted to the following rigidity result and its Corollary 3.2.

Theorem 3.1. *Let (M, τ) be a tracial von Neumann algebra and $N, P \subset M$ be von Neumann subalgebras. Assume that there exists an s-malleable deformation $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}}, \beta)$ such that*

1. *The M-bimodule $\mathcal{H} := L^2(\tilde{M}) \ominus L^2(M)$ has the property that $\mathcal{H}^{\otimes_M k}$ is weakly contained in the bimodule $L^2(M) \otimes_N L^2(M)$, for some $k \in \mathbb{N}$.*
2. *The M-bimodule $L^2(\tilde{M})$ with the bimodular structure given by $x \cdot \xi \cdot y = x\xi\alpha_1(y)$, for every $x, y \in M, \xi \in L^2(\tilde{M})$, is contained in a multiple of the bimodule $L^2(M) \otimes_P L^2(M)$.*

Let $Q \subset M$ be a von Neumann subalgebra such that Qp is not amenable relative to N inside M, for any non-zero projection $p \in Q' \cap M$. Then $Q' \cap M \prec^s_M P$.

The proof of Theorem 3.1 relies on Popa's deformation/rigidity theory and notably uses his spectral gap rigidity principle introduced in [36, 37]. Theorem 3.1 and Corollary 3.2 were inspired by Boutonnet's work (see [2] and [3, Chapter II]), whose exposition we follow closely. Finally, we note that condition (1) in Theorem 3.1 was first considered by Sinclair in [43].

Corollary 3.2. *Let Γ be a countable group and $\pi : \Gamma \to \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ be an orthogonal representation. Assume that $\pi^{\otimes k}$ is weakly contained in the left regular representation of Γ, for some $k \in \mathbb{N}$. Let $\Gamma \curvearrowright (C, \tau)$ be either the Gaussian action or the free Bogoljubov action associated with π. Let $\Gamma \curvearrowright (D, \rho)$ be a trace preserving action on a tracial von Neumann algebra D, consider the diagonal product action $\Gamma \curvearrowright (C \bar{\otimes} D, \tau \otimes \rho)$, and denote $M = (C \bar{\otimes} D) \rtimes \Gamma$.*

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Let $Q \subset M$ be a von Neumann subalgebra such that Qp is not amenable relative to D inside M, for any non-zero projection $p \in Q' \cap M$. Then $Q' \cap M \prec^s_M D \rtimes \Gamma$.

The remainder of this section is devoted to the proofs of Theorem 3.1 and Corollary 3.2.

Lemma 3.3 [2]. *Let $(\tilde{M}, \tau)$ be a tracial von Neumann algebra and $N \subset M \subset \tilde{M}$ be von Neumann subalgebras. Assume that the M-bimodule $\mathcal{H} := L^2(\tilde{M}) \ominus L^2(M)$ has the property that $\mathcal{H}^{\otimes_M k}$ is weakly contained in the bimodule $L^2(M) \otimes_N L^2(M)$, for some $k \in \mathbb{N}$.*

Let $Q \subset M$ be a von Neumann subalgebra such that Qp is not amenable relative to N inside M, for any non-zero projection $p \in Q' \cap M$. Then $Q' \cap \tilde{M}^\omega \subset M^\omega$. In particular, $Q' \cap \tilde{M} \subset M$.

Proof. The proof of [2, Lemma 2.3], which applies verbatim for $N = \mathbb{C}1$, works in general. □

The following lemma is a standard application of Popa's spectral gap rigidity principle.

Lemma 3.4. *Let (M, τ) be a tracial von Neumann algebra and $N \subset M$ be a von Neumann subalgebra. Assume that there exists an s-malleable deformation $(\tilde{M}, (\alpha_t)_{t\in\mathbb{R}}, \beta)$ such that the M-bimodule $\mathcal{H} := L^2(\tilde{M}) \ominus L^2(M)$ has the property that $\mathcal{H}^{\otimes_M k}$ is weakly contained in the bimodule $L^2(M) \otimes_N L^2(M)$, for some $k \in \mathbb{N}$.*

Let $Q \subset M$ be a von Neumann subalgebra such that Qp is not amenable relative to N inside M, for any non-zero projection $p \in Q' \cap M$. Then α_t converges uniformly on $(Q' \cap M)_1$.

Proof. Fix $\varepsilon > 0$. Since $Q' \cap \tilde{M}^\omega \subset M^\omega$ by Lemma 3.3, there exist $x_1, \ldots, x_n \in Q$ and $\delta > 0$ such that for all $y \in (\tilde{M})_1$:

$$\forall i \in \{1, \ldots, n\} : \|[y, x_i]\|_2 \leqslant \delta \quad \Longrightarrow \quad \|y - E_M(y)\|_2 \leqslant \varepsilon.$$

Taking $t > 0$ such that $\|\alpha_s(x_i) - x_i\|_2 \leqslant \frac{\delta}{2}$ for all $1 \leqslant i \leqslant n$ and all $s \in [0, t]$, we get for any $x \in (Q' \cap M)_1$

$$\begin{aligned}\|\alpha_s(x)x_i - x_i\alpha_s(x)\|_2 &= \|x\alpha_{-s}(x_i) - \alpha_{-s}(x_i)x\|_2 \\ &\leqslant 2\,\|x\|\,\|\alpha_{-s}(x_i) - x_i\|_2 + \|xx_i - x_ix\|_2 \\ &\leqslant 2\,\|\alpha_s(x_i) - x_i\|_2 \\ &\leqslant \delta.\end{aligned}$$

Hence, for all $s \in [0, t]$ and $x \in (Q' \cap M)_1$, we have $\|\alpha_s(x) - E_M(\alpha_s(x))\|_2 \leqslant \varepsilon$, and, thus, by Lemma 2.9, $\|\alpha_{2s}(x) - x\|_2 \leqslant 2\varepsilon$. It follows that α_t converges uniformly on $(Q' \cap M)_1$. □

Lemma 3.5. *Assume the setting of Lemma 3.4 and let $p \in (Q' \cap M)' \cap M$ be a non-zero projection. Then there is a non-zero element $a_1 \in p\tilde{M}\alpha_1(p)$ such that $xa_1 = a_1\alpha_1(x)$ for all $x \in (Q' \cap M)p$.*

Proof. We follow closely the proof of [35, Theorem 4.1]. Put $D = Q' \cap M$ and fix a projection $p \in D' \cap M$.

Claim 1. *For any $t > 0$ small enough, there exists a non-zero element $a_t \in p\tilde{M}\alpha_t(p)$ such that $a_t = ua_t\alpha_t(u^*)$ for all $u \in \mathcal{U}(Dp)$.*

Proof of Claim 1. By Lemma 3.4, $\alpha_t \to \mathrm{id}$ uniformly on $(Dp)_1$, as $t \to 0$. Thus, for any $t > 0$ small enough, we have that $\|u - \alpha_t(u)\|_2^2 \leqslant \tau(p)$, and, hence,

$$\Re\tau(u\alpha_t(u^*)) \geqslant \frac{\tau(p)}{2}, \quad \text{for all } u \in \mathcal{U}(Dp). \tag{3.1}$$

Consider the unique element a_t of minimal $\|.\|_2$-norm in the $\|.\|_2$-closure of the convex hull of the set $\{u\alpha_t(u^*) \mid u \in \mathcal{U}(Dp)\}$. By uniqueness, we have $a_t = ua_t\alpha_t(u^*)$ for all $u \in \mathcal{U}(Dp)$. Moreover, by (3.1), we get $\Re\tau(a_t) \geqslant \frac{\tau(p)}{2} > 0$; hence, $a_t \neq 0$. □

Claim 2. *Let $t > 0$ and $a_t \in p\tilde{M}\alpha_t(p)$ be a non-zero element such that $a_t = ua_t\alpha_t(u^*)$ for all $u \in \mathcal{U}(Dp)$. Then there exists $b \in Q$ such that $a_{2t} := \alpha_t(\beta(a_t^*)ba_t) \neq 0$. Moreover, $a_{2t} \in p\tilde{M}\alpha_{2t}(p)$ satisfies $a_{2t} = ua_{2t}\alpha_{2t}(u^*)$ for all $u \in \mathcal{U}(Dp)$.*

Proof of Claim 2. To prove the first part of the claim, assume that $\alpha_t(\beta(a_t^*)ba_t) = 0$, and, thus, $\beta(a_t^*)ba_t = 0$, for all $b \in Q$. Thus, if we let $r = a_ta_t^* \in \tilde{M}$, then since $\beta(u_1^*) = u_1^*$, we get that

$$\beta(u_1ru_1^*)u_2ru_2^* = \beta(u_1a_t)(\beta(a_t^*)u_1^*u_2a_t)(a_t^*u_2^*) = 0, \quad \text{for all } u_1, u_2 \in \mathcal{U}(Q). \tag{3.2}$$

Let s be the element of minimal $\|.\|_2$-norm in the $\|.\|_2$-closure of the convex hull of the set $\{uru^* \mid u \in \mathcal{U}(Q)\}$. Since $\tau(s) = \tau(r) > 0$ and $s \geqslant 0$, we get that $s \neq 0$ and further that $s^2 \neq 0$. By uniqueness, we have that $s \in Q' \cap \tilde{M}$ and since $Q' \cap \tilde{M} \subset M$ by Lemma 3.3, we conclude that $s \in M$. By combining the last two facts, we get that $\beta(s)s = s^2 \neq 0$. This, however, contradicts (3.2) which implies that $\beta(s)s = 0$. The moreover assertion is now a straightforward calculation. □

By Claim 1, its conclusion holds for $t = 2^{-k}$ for some $k \in \mathbb{N}$. Using Claim 2 and induction, we then find $0 \neq a_1 \in p\tilde{M}\alpha_1(p)$ such that $a_1 = ua_1\alpha_1(u^*)$, for all $u \in \mathcal{U}(Dp)$. □

Proof of Theorem 3.1. Let $p \in (Q' \cap M)' \cap M$ be a non-zero projection. We need to show that $(Q' \cap M)p \prec_M P$. By Lemma 3.5, we can find $0 \neq a_1 \in p\tilde{M}\alpha_1(p)$ such that $xa_1 = a_1\alpha_1(x)$ for all $x \in (Q' \cap M)p$. Thus, the pMp-bimodule ${}_{pMp}L^2(\tilde{M})_{\alpha_1(pMp)}$ contains a non-zero $(Q' \cap M)p$-central vector. Since this bimodule is contained in a multiple of $pL^2(M) \otimes_P L^2(M)p$ by assumption (2), we get that $pL^2(M) \otimes_P L^2(M)p$ contains a non-zero $(Q' \cap M)p$-central vector. In other words, the pMp-bimodule $pL^2(\langle M, e_P\rangle)p$ contains a non-zero $(Q' \cap M)p$-central vector ξ. Let $\varepsilon > 0$ such that $f = 1_{[\varepsilon,\infty)}(\xi^*\xi) \neq 0$. Then we have that $f \in ((Q' \cap M)p)' \cap p\langle M, e_P\rangle p$. Since $\hat{\tau}(f) \leqslant \|\xi\|^2/\varepsilon < \infty$, Theorem 2.3 implies that $(Q' \cap M)p \prec_M P$, thus finishing the proof of the theorem. □

Proof of Corollary 3.2. In § 2.7, we defined an s-malleable deformation $(\tilde{C} \rtimes \Gamma, (\alpha_t)_{t\in\mathbb{R}}, \beta)$ of $C \rtimes \Gamma$, where $\tilde{C} = C\bar{\otimes}C$ or $\tilde{C} = C * C$, depending on whether $\Gamma \curvearrowright C$ is the Gaussian action or the free Bogoljubov action associated with π, respectively. By construction, $\alpha_t(\tilde{C}) = \tilde{C}$, $\beta(\tilde{C}) = \tilde{C}$ and $\alpha_t(u_g) = u_g$, for all $t \in \mathbb{R}$ and $g \in \Gamma$. Recall

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that $M=(C\bar{\otimes}D)\rtimes\Gamma$ and put $\tilde{M}=(\tilde{C}\bar{\otimes}D)\rtimes\Gamma$. We extend α_t and β to automorphisms of $\tilde{M}$ by letting $\alpha_t(x)=\beta(x)=x$, for all $t\in\mathbb{R}$ and $x\in D$. Then $(\tilde{M},(\alpha_t)_{t\in\mathbb{R}},\beta)$ is an s-malleable deformation of M. In order to derive the conclusion, it remains to verify that conditions (1) and (2) from Theorem 3.1 are satisfied with $N=D$ and $P=D\rtimes\Gamma$.

As in the proof of [46, Lemma 3.5], given a unitary representation $\eta:\Gamma\to\mathcal{U}(\mathcal{K})$, we define $\mathcal{K}_\eta=\mathcal{K}\otimes L^2(M)$ and endow it with the following M-bimodule structure:

$$(au_g)\cdot(\xi\otimes x)\cdot(bu_h)=\eta_g(\xi)\otimes au_gxbu_h,\quad\text{for all } a,b\in C\bar{\otimes}D,\, g,h\in\Gamma,\, x\in M,\text{ and }\xi\in\mathcal{K}.$$

If $\eta':\Gamma\to\mathcal{U}(\mathcal{K}')$ is another unitary representation of Γ, then $\mathcal{K}_{\eta\otimes\eta'}\cong\mathcal{K}_\eta\otimes_M\mathcal{K}_{\eta'}$, and if η is weakly contained in η', then $\mathcal{K}_\eta\subset_{\text{weak}}\mathcal{K}_{\eta'}$. If $\lambda:\Gamma\to\mathcal{U}(\ell^2(\Gamma))$ is the left regular representation, then it is straightforward to check that $\mathcal{K}_\lambda\cong L^2(M)\bar{\otimes}_{C\bar{\otimes}D}L^2(M)$ as M-bimodules via the isomorphism

$$\mathcal{K}_\lambda=\ell^2(\Gamma)\otimes L^2(M)\to L^2(M)\bar{\otimes}_{C\bar{\otimes}D}L^2(M):\delta_g\otimes x\mapsto u_g\otimes_{C\bar{\otimes}D}u_g^*x.$$

Case 1. $\Gamma\curvearrowright^\sigma(C,\tau)$ is the Gaussian action associated with π.

Let $\sigma_0:\Gamma\to\mathcal{U}(L^2(C)\ominus\mathbb{C}1)$ be the restriction of the Koopman representation of σ to $L^2(C)\ominus\mathbb{C}1$. Since $\pi^{\otimes k}$ is weakly contained in the left regular representation λ of Γ, the same holds for $\sigma_0^{\otimes k}$ by [32, Proposition 2.7] and [3, Proposition II.1.15]. Moreover, we note that as M-bimodules, $\mathcal{K}_{\sigma_0}$ is isomorphic to $L^2(\tilde{M})\ominus L^2(M)$ via the isomorphism

$$\mathcal{K}_{\sigma_0}=(L^2(C)\ominus\mathbb{C}1)\otimes L^2(M)\to L^2(\tilde{M})\ominus L^2(M):c_1\otimes((c_2\otimes d)u_g)\mapsto(c_1\otimes c_2\otimes d)u_g.$$

We conclude that

$$(L^2(\tilde{M})\ominus L^2(M))^{\otimes_M k}\cong\mathcal{K}_{\sigma_0}^{\otimes_M k}\cong\mathcal{K}_{\sigma_0^{\otimes k}}\subset_{\text{weak}}\mathcal{K}_\lambda.$$

Since C is abelian, hence amenable, $\mathcal{K}_\lambda\cong L^2(M)\otimes_{C\bar{\otimes}D}L^2(M)$ is weakly contained in $L^2(M)\bar{\bigotimes}_D L^2(M)$, proving condition (1). Since $L^2(\tilde{M})=\overline{M\alpha_1(M)}^{\|\cdot\|_2}$ and $\tau(x\alpha_1(y))=\tau(xE_{D\rtimes\Gamma}(y))$, for all $x,y\in M$, the M-bimodule ${}_ML^2(\tilde{M})_{\alpha_1(M)}$ is isomorphic to $L^2(M)\otimes_{D\rtimes\Gamma}L^2(M)$. Thus, condition (2) also holds.

Case 2. $\Gamma\curvearrowright^\rho(C,\tau)$ is the free Bogoljubov action associated with π.

We will denote still by ρ the diagonal product action of Γ on $\tilde{C}\bar{\otimes}D$.

Claim. *Let $\xi=\xi_1\xi_2\ldots\xi_n\in\tilde{C}=C*C$, where $\xi_1\in 1*(C\ominus\mathbb{C}1),\xi_2\in(C\ominus\mathbb{C}1)*1,\ldots,\xi_n\in 1*(C\ominus\mathbb{C}1)$. Then the M-bimodule $\mathcal{L}_\xi:=\overline{M\xi M}$ satisfies $\mathcal{L}_\xi^{\otimes_M k}\subset_{weak}L^2(M)\otimes_D L^2(M)$.*

Proof of the claim. Define $\varphi:\Gamma\to\mathbb{C}$ and the completely positive map $\Phi:M\to M$ by letting $\varphi(g)=\langle\rho_g(\xi),\xi\rangle$ and $\Phi((c\otimes d)u_g)=\tau(c)\varphi(g)(1\otimes d)u_g$, for all $c\in C,d\in D$ and $g\in\Gamma$.

If $c,c'\in C*1$, $d,d'\in D$ and $g,g'\in\Gamma$, then $\langle c\rho_g(\xi)c',\xi\rangle=\tau(\xi^*c\rho_g(\xi)c')=\tau(c)\tau(c')\varphi(g)$, and, thus,

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$$\begin{aligned}\langle (c\otimes d)u_g\xi u_{g'}(c'\otimes d'),\xi\rangle &= \delta_{gg',e}\langle c\rho_g(\xi)c',\xi\rangle\langle dd',1\rangle\\ &= \delta_{gg',e}\varphi(g)\tau(c)\tau(c')\tau(dd')\\ &= \tau(\Phi((c\otimes d)u_g)u_{g'}(c'\otimes d')).\end{aligned}$$

In other words, using the notation from § 2.2, this means that $\mathcal{L}_\xi\cong\mathcal{H}_\Phi$, as M-bimodules. Note that if $v\in\mathcal{U}(C)$, $w\in\mathcal{U}(D)$, $h\in\Gamma$, then for all $d\in D$ and $g\in\Gamma$, we have that

$$[\Phi\circ\mathrm{Ad}((v\otimes w)u_h)]((1\otimes d)u_g)=\tau(v\rho_{hgh^{-1}}(v)^*)\varphi(hgh^{-1})\mathrm{Ad}((1\otimes w)u_h)((1\otimes d)u_g). \tag{3.3}$$

Let $\mathcal{U}$ be the set of unitaries $u\in M$ of the form $u=(v\otimes w)u_h$, with $v\in\mathcal{U}(C)$, $w\in\mathcal{U}(D)$, $h\in\Gamma$. Since the span of $\mathcal{U}$ is $\|.\|_2$-dense in M, Lemma 2.2(2) implies that the M-bimodule $\mathcal{L}_\xi^{\otimes_M k}\cong\mathcal{H}_\Phi^{\otimes_M k}$ is isomorphic to a sub-bimodule of

$$\bigoplus_{u_1,\ldots,u_{k-1}\in\mathcal{U}}\mathcal{H}_{\Phi\circ\mathrm{Ad}(u_{k-1})\circ\Phi\circ\cdots\circ\mathrm{Ad}(u_1)\circ\Phi}.$$

We fix $u_1,\ldots,u_{k-1}\in\mathcal{U}$ and denote $\Psi:=\Phi\circ\mathrm{Ad}(u_{k-1})\circ\Phi\circ\cdots\circ\mathrm{Ad}(u_1)\circ\Phi:M\to M$. Thus, in order to prove the claim, it suffices to argue that $\mathcal{H}_\Psi\subset_{\mathrm{weak}}L^2(M)\otimes_D L^2(M)$. To this end, for $i\in\{1,\ldots,k-1\}$, write $u_i=(v_i\otimes w_i)u_{h_i}$, where $v_i\in\mathcal{U}(C)$, $w_i\in\mathcal{U}(D)$ and $h_i\in\Gamma$. We define $U=(1\otimes w_{k-1})u_{h_{k-1}}\ldots(1\otimes w_1)u_{h_1}\in\mathcal{U}(D\rtimes\Gamma)$ and a positive definite function $\psi:\Gamma\to\mathbb{C}$ by letting

$$\psi(g)=\prod_{i=1}^{k-1}\tau(v_i\rho_{h_i\ldots h_1gh_1^{-1}\ldots h_i^{-1}}(v_i)^*),\quad \text{for all } g\in\Gamma.$$

By using (3.3) and induction, it follows that for all $c\in C$, $d\in D$ and $g\in\Gamma$, we have that

$$\Psi((c\otimes d)u_g)=\tau(c)\psi(g)\varphi(g)\prod_{i=1}^{k-1}\varphi(h_i\ldots h_1gh_1^{-1}\ldots h_i^{-1})\mathrm{Ad}(U)((1\otimes d)u_g). \tag{3.4}$$

Let $\Theta:M\to M$ and $\Omega:M\to M$ be the completely positive maps given by $\Theta(xu_g)=\psi(g)xu_g$ and $\Omega(xu_g)=\varphi(g)\prod_{i=1}^{k-1}\varphi(h_i\ldots h_1gh_1^{-1}\ldots h_i^{-1})xu_g$, for all $x\in C\bar{\otimes}D$ and $g\in\Gamma$. Then (3.4) rewrites as $\Psi=\mathrm{Ad}(U)\circ\Theta\circ\Omega\circ E_{D\rtimes\Gamma}$. By Lemma 2.2(1), we get that

$$\text{the } M\text{-bimodule } \mathcal{H}_\Psi \text{ is isomorphic to a sub-bimodule of } \mathcal{H}_{E_{D\rtimes\Gamma}}\otimes_M\mathcal{H}_\Omega\otimes_M\mathcal{H}_\Theta. \tag{3.5}$$

Let $\rho_0:\Gamma\to\mathcal{U}(L^2(C)\ominus\mathbb{C}1)$ be the restriction of the Koopman representation of ρ to $L^2(C)\ominus\mathbb{C}1$. Since $\varphi(g)=\langle\rho_g(\xi),\xi\rangle=\prod_{i=1}^n\langle\rho_g(\xi_i),\xi_i\rangle$ and $\xi_i\in C\ominus\mathbb{C}1$, for all $g\in\Gamma$ and $i\in\{1,\ldots,n\}$, it follows that the M-bimodule $\mathcal{H}_\Omega$ is isomorphic to a sub-bimodule of $\mathcal{K}_{\rho_0^{\otimes kn}}$. Since $\pi^{\otimes k}$ is weakly contained in the left regular representation λ, so is $\rho_0^{\otimes k}$ by Lemma 2.12. Thus, $\rho_0^{\otimes kn}$ is weakly contained in λ. Hence, $\mathcal{K}_{\rho_0^{\otimes kn}}\subset_{\mathrm{weak}}\mathcal{K}_\lambda\cong\mathcal{H}_{E_{C\bar{\otimes}D}}$. Altogether, we conclude that $\mathcal{H}_\Omega\subset_{\mathrm{weak}}\mathcal{H}_{E_{C\bar{\otimes}D}}$. In combination with (3.5), we derive that

$$\mathcal{H}_\Psi\subset_{\mathrm{weak}}\mathcal{H}_{E_{D\rtimes\Gamma}}\otimes_M\mathcal{H}_{E_{C\bar{\otimes}D}}\otimes_M\mathcal{H}_\Theta. \tag{3.6}$$

Since $\mathcal{H}_{E_N}\cong L^2(M)\otimes_N L^2(M)$, for any von Neumann subalgebra $N\subset M$, and the $(D\rtimes\Gamma)$-$(C\bar{\otimes}D)$-bimodule $L^2(M)$ is isomorphic to $L^2(D\rtimes\Gamma)\otimes_D L^2(C\bar{\otimes}D)$, it follows that

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$\mathcal{H}_\Psi \subset_{\text{weak}} L^2(M) \otimes_D \mathcal{H}_\Theta$. Using that D is regular in M and $\Theta_{|D} = \text{id}_D$, it is easy to see that $L^2(M) \otimes_D \mathcal{H}_\Theta$ is isomorphic to a sub-bimodule of a multiple of $L^2(M) \otimes_D L^2(M)$. Thus, $\mathcal{H}_\Psi \subset_{\text{weak}} L^2(M) \otimes_D L^2(M)$, which finishes the proof of the claim. $\square$

Since $L^2(\tilde{M}) \ominus L^2(M)$ decomposes as a direct sum of M-bimodules of the form $\mathcal{L}_\xi$ as in the claim, condition (1) follows. To verify condition (2), let $\xi \in \tilde{C}$ be a non-zero element of the form $\xi = \xi_1\xi_2 \dots \xi_n$, where $\xi_1 \in 1 * (C \ominus \mathbb{C}1), \xi_2 \in (C \ominus \mathbb{C}1) * 1, \dots, \xi_n \in (C \ominus \mathbb{C}1) * 1$. Using a calculation similar to the one in the claim, it follows that the M-bimodule ${}_M\overline{M\xi\alpha_1(M)}_{\alpha_1(M)}$ is isomorphic to a submodule of a multiple of $L^2(M) \otimes_{D \rtimes \Gamma} L^2(M)$. This implies that condition (2) holds in case (2) and finishes the proof of Corollary 3.2. $\square$

4. Proofs of Theorems A and B

The proofs of Theorems A and B rely on the following consequence of Corollary 3.2.

Lemma 4.1. *Let Γ be a non-amenable group. For $k \in \mathbb{N}$, let $\pi_k : \Gamma \to \mathcal{O}(\mathcal{H}_k)$ be an orthogonal representation such that $\pi_k^{\otimes l(k)}$ is weakly contained in the left regular representation of Γ, for some $l(k) \in \mathbb{N}$. Let $\Gamma \curvearrowright (B_k, \tau_k)$ be either the Gaussian or the free Bogoljubov action associated with π_k. Let $\Gamma \curvearrowright (B, \tau) := \bar{\bigotimes}_k (B_k, \tau_k)$ be the diagonal product action and denote $M = B \rtimes \Gamma$. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of von Neumann subalgebras of M such that $\|x - E_{M_n}(x)\|_2 \to 0$, for every $x \in M$.*

Then there exist projections $p_n \in \mathcal{Z}(M_n' \cap M)$, for $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \tau(p_n) = 1$ and $(M_n' \cap M)p_n \prec_M^s (\bar{\bigotimes}_{k>N} B_k) \rtimes \Gamma$, for every $n, N \in \mathbb{N}$.

Moreover, if Γ is not inner amenable, then there exist projections $r_n \in \mathcal{Z}(M_n' \cap M)$, for $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \tau(r_n) = 1$ and $(M_n' \cap M)r_n$ is amenable, for every $n \in \mathbb{N}$.

Proof. Let $q_n \in \mathcal{Z}(M_n' \cap M)$ be the largest projection such that $M_n q_n$ is amenable relative to B. We claim that $\tau(q_n) \to 0$. Otherwise, after replacing $(M_n)_{n\in\mathbb{N}}$ with a subsequence, we may assume that $\tau(q_n) \to c > 0$. By Lemma 2.4, this implies that there is a non-zero projection $q \in \mathcal{Z}(M)$ such that Mq is amenable relative to B. Since M is a factor, this would give that M is amenable relative to B and hence that Γ is amenable by [31, Proposition 2.4], which is a contradiction.

Next, fix $n \in \mathbb{N}$ and put $p_n = 1 - q_n$. Then $M_n p'$ is not amenable relative to B, for any non-zero projection $p' \in (M_n' \cap M)p_n$. Otherwise, [11, Lemma 2.6(2)] would provide a non-zero projection $z \in \mathcal{Z}(M_n' \cap M)p_n$ such that $M_n z$ is amenable relative to B, contradicting the maximality of q_n. Let $i \in \mathbb{N}$ and denote $C_i = \bar{\bigotimes}_{k\neq i} B_k$. Since $C_i \subset B$, $M_n p'$ is not amenable relative to C_i, for any non-zero projection $p' \in (M_n' \cap M)p_n$. Since $\Gamma \curvearrowright B_i$ is either the Gaussian or the free Bogoljubov action associated with π_i and a multiple of π_i is weakly contained in the left regular representation of Γ, we can apply Corollary 3.2 to the inclusion $M_n p_n \subset M = (B_i \bar{\otimes} C_i) \rtimes \Gamma$ to deduce that

$$(M_n' \cap M)p_n \prec_M^s C_i \rtimes \Gamma, \quad \text{for all } i \in \mathbb{N}. \tag{4.1}$$

Let $N \in \mathbb{N}$. Since the subalgebras $\{C_i\}_{i=1}^N$ of M are regular and any two form a commuting square, (4.1) and [11, Lemma 2.8(2)] imply that $(M_n' \cap M)p_n \prec_M^s \bigcap_{i=1}^N (C_i \rtimes \Gamma) = (\bar{\bigotimes}_{k>N} B_k) \rtimes \Gamma$. Since $\tau(p_n) \to 1$, this proves the main assertion.

For the moreover assertion, assume that Γ is not inner amenable. Then by [7], we get that $M'\cap M^\omega \subset B^\omega$ and, hence, $\prod_\omega (M_n'\cap M)\subset M'\cap M^\omega\subset B^\omega$. By combining this with [21, Lemmas 2.2 and 2.3], we can find projections $f_n\in\mathcal{Z}(M_n'\cap M)$ such that $\tau(f_n)\to 1$ and

$$(M_n'\cap M)f_n \prec^s_M B,\quad \text{for every } n\in\mathbb{N}. \tag{4.2}$$

Put $r_n = p_n\wedge f_n\in\mathcal{Z}(M_n'\cap M)$. Then $(M_n'\cap M)r_n\prec^s_M B$ and $(M_n'\cap M)r_n\prec^s_M (\bar{\bigotimes}_{k>N}B_k)\rtimes\Gamma$, for every $n,N\in\mathbb{N}$. Since B is regular in M, B and $(\bar{\bigotimes}_{k>N}B_k)\rtimes\Gamma$ form a commuting square and $B\cap((\bar{\bigotimes}_{k>N}B_k)\rtimes\Gamma)=\bar{\bigotimes}_{k>N}B_k$, [11, Lemma 2.8(2)] implies that

$$(M_n'\cap M)r_n\prec^s_M \bar{\bigotimes}_{k>N}B_k,\quad \text{for every } n,N\in\mathbb{N}. \tag{4.3}$$

For $N\in\mathbb{N}$, put $Q_N=\bar{\bigotimes}_{k>N}B_k$ and $R_N=(\bar{\bigotimes}_{k\leqslant N}B_k)\rtimes\Gamma$. Then $\|x-E_{R_N}(x)\|_2\to 0$, for any $x\in M$, and ${}_{Q_N}L^2(M)_{R_N}\cong {}_{Q_N}L^2(Q_N)\otimes L^2(R_N)_{R_N}$, for any $N\in\mathbb{N}$. These facts and (4.3) imply that we can apply Lemma 2.6 to deduce that $(M_n'\cap M)r_n$ is amenable, for every $n\in\mathbb{N}$. □

4.1. Proof of Theorem A

Assume, by contradiction, that M admits a residual sequence $(A_n)_n$. For $n\in\mathbb{N}$, let $M_n=A_n'\cap M$. Since $\prod_\omega A_n\subset\bigcap_n A_n^\omega\subset M'\cap M^\omega$, Lemma 2.1 implies that $\|x-E_{M_n}(x)\|_2\to 0$, for every $x\in M$. By Lemma 4.1, we can find projections $p_n\in\mathcal{Z}(M_n'\cap M)$ such that $\tau(p_n)\to 1$ and $(M_n'\cap M)p_n\prec^s_M(\bar{\bigotimes}_{l>N}B_l)\rtimes\Gamma$, for every $n,N\in\mathbb{N}$. Since $A_n\subset M_n'\cap M$, we thus get that

$$A_np_n\prec^s_M\left(\bar{\bigotimes}_{l>N}B_l\right)\rtimes\Gamma,\quad \text{for every } n,N\in\mathbb{N}. \tag{4.4}$$

Let $n\in\mathbb{N}$ be fixed such that $\tau(p_n)>15/16$. Recall that $\Gamma\curvearrowright B_k$ is the Gaussian action associated with π_k and denote $U_k^m=\omega(\xi_k^m)\in\mathcal{U}(B_k)$, for every $k,m\in\mathbb{N}$.

Claim. *There exists $k\in\mathbb{N}$ such that $\|U_k^m-E_{A_n}(U_k^m)\|_2\leqslant 1/16$, for every $m\in\mathbb{N}$.*

Proof of the claim. Assuming the claim is false, for every $k\in\mathbb{N}$, we can find $m(k)\in\mathbb{N}$ such that $U_k:=U_k^{m(k)}\in\mathcal{U}(B_k)$ satisfies $\|U_k-E_{A_n}(U_k)\|_2>1/16$. Since $1-e^{-t}\leqslant t$, for any $t\geqslant 0$, we get

$$\begin{aligned}\|u_gU_ku_g^*-U_k\|_2&=\|\omega(\pi_k(g)(\xi_k^{m(k)}))-\omega(\xi_k^{m(k)})\|_2\\&=\sqrt{2(1-\exp(-\|\pi_k(g)(\xi_k^{m(k)})-\xi_k^{m(k)}\|^2))}\\&\leqslant\sqrt{2}\|\pi_k(g)(\xi_k^{m(k)})-\xi_k^{m(k)}\|,\quad\text{for every } g\in\Gamma.\end{aligned}$$

Since $\sup_{m\in\mathbb{N}}\|\pi_k(g)(\xi_k^m)-\xi_k^m\|\to 0$, we deduce that $\|u_gU_ku_g^*-U_k\|_2\to 0$, for every $g\in\Gamma$. Since $U_k\in\mathcal{U}(B_k)$, we also have that $U_kx=xU_k$, for every $x\in B$. By combining the last two facts, we get that $U:=(U_k)\in M'\cap M^\omega$. However, since $\|U-E_{A_n^\omega}(U)\|_2=\lim_{k\to\omega}\|U_k-E_{A_n}(U_k)\|_2\geqslant 1/16$, this contradicts that $M'\cap M^\omega\subset A_n^\omega$. Altogether, this proves the claim. □

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Let $k \in \mathbb{N}$ be as in the claim and put $V_m = U_k^m - \tau(U_k^m)$. Then we have $V_m \in B_k$, $\|V_m\| \leqslant 2$, $\|V_m\|_2 = \sqrt{1-\exp(-1)}$ and $\|V_m - E_{A_n}(V_m)\|_2 \leqslant 1/16$, for every $m \in \mathbb{N}$. Since $\tau(V_{m'}^* V_m) = 0$, for all $m \neq m'$, we also have that $V_m \to 0$ weakly.

By specializing (4.4) to $N = k$, we get that $A_n p_n \prec_M^s (\bar{\bigotimes}_{l>k} B_l) \rtimes \Gamma$. This implies that we can find a finite dimensional subspace $\mathcal{K} \subset \bar{\bigotimes}_{l\leqslant k} B_l$ such that if e denotes the orthogonal projection from $L^2(M)$ onto the $\|.\|_2$-closed linear span of $\{(y \otimes z)u_g \mid y \in \mathcal{K}, z \in \bar{\bigotimes}_{l>k} B_l, g \in \Gamma\}$, then

$$\|x - e(x)\|_2 \leqslant 1/16, \quad \text{for all } x \in (A_n p_n)_1. \tag{4.5}$$

Next, if $m \in \mathbb{N}$, then $\|V_m - E_{A_n}(V_m)\| \leqslant 1/16$ and hence $\|V_m p_n - E_{A_n}(V_m) p_n\|_2 \leqslant 1/16$. Since $E_{A_n}(V_m) p_n \in A_n p_n$ and $\|E_{A_n}(V_m) p_n\| \leqslant 2$, (4.5) gives $\|E_{A_n}(V_m) p_n - e(E_{A_n}(V_m) p_n)\|_2 \leqslant 1/8$. Combining the last two inequalities further implies that

$$\|V_m p_n - e(V_m p_n)\|_2 \leqslant 1/4, \quad \text{for every } m \in \mathbb{N}. \tag{4.6}$$

Now, we claim that

$$\lim_{m\to\infty} \|E_{(\bar{\bigotimes}_{l>k} B_l)\rtimes\Gamma}(x V_m y)\|_2 = 0, \quad \text{for all } x, y \in M. \tag{4.7}$$

Indeed, it is enough to check this when $x = u_g(a \otimes b)$ and $y = (c \otimes d)u_h$, for $a, c \in \bar{\bigotimes}_{l\leqslant k} B_l$, $b, d \in \bar{\bigotimes}_{l>k} B_l$ and $g, h \in \Gamma$. Then, since $V_m \in B_k$, we have $E_{(\bar{\bigotimes}_{l>k} B_l)\rtimes\Gamma}(x V_m y) = \tau(a V_m b) u_g b d u_h$ and the conclusion follows since $V_m \to 0$ weakly. This proves (4.7).

Let $\{\xi_j\}_{j=1}^r$ be an orthonormal basis for $\mathcal{K}$. Since $E_{(\bar{\bigotimes}_{l>k} B_l)\rtimes\Gamma}(\xi_i^* \xi_j) = \delta_{i,j}$, for all $i, j \in \{1, \ldots, r\}$, we get that $e(x) = \sum_{j=1}^r \xi_j E_{(\bar{\bigotimes}_{l>k} B_l)\rtimes\Gamma}(\xi_j^* x)$, for every $x \in M$. In combination with (4.7), it follows that $\|e(V_m p_n)\|_2 \to 0$. On the other hand, since $\|V_m\| \leqslant 2$ and $\tau(p_n) > 15/16$, we have that

$$\begin{aligned}\|V_m p_n\|_2 &\geqslant \|V_m\|_2 - \|V_m(1-p_n)\|_2 \geqslant \|V_m\|_2 - 2\|1-p_n\|_2 \\ &= \sqrt{1-\exp(-1)} - 2\sqrt{1-\tau(p_n)} > 1/4, \quad \text{for every } m \in \mathbb{N}.\end{aligned}$$

Altogether, we get that $\liminf_{m\to\infty} \|V_m p_n - e(V_m p_n)\|_2 > 1/4$, which contradicts (4.6). So M cannot have a residual sequence. □

Remark 4.2. The proof of Theorem A shows that there is no sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M such that $\prod_\omega A_n \subset M' \cap M^\omega \subset \bigcap_{n\in\mathbb{N}} A_n^\omega$. In particular, there is no sequence $(A_n)_{n\in\mathbb{N}}$ of von Neumann subalgebras of M which satisfies conditions (2) and (3) of Definition 1.1.

4.2. Proof of Theorem B

Recall that $\Gamma \curvearrowright B_k$ is the free Bogoljubov action associated with π_k and denote $W_{k,m} = W(\xi_k^m) \in B_k$, for $k \in \mathbb{N}$ and $m \in \{1, 2\}$. Then for any $k \in \mathbb{N}$, $\{W_{k,1}, W_{k,2}\}$ are freely independent semicircular operators with $\|W_{k,1}\| = \|W_{k,2}\| = 2$. Moreover, if $m \in \{1, 2\}$, then for any $g \in \Gamma$, we have that $\|u_g W_{k,m} u_g^* - W_{k,m}\|_2 = \|W(\pi_k(g)(\xi_k^m)) - W(\xi_k^m)\|_2 = \|\pi_k(g)(\xi_k^m) - \xi_k^m\| \to 0$. Since $W_{k,m} \in B_k$, we also have that $\|W_{k,m} x - x W_{k,m}\|_2 \to 0$, for every $x \in B$. By combining the last two facts, we get that $W_m = (W_{k,m})_k \in M' \cap M^\omega$.

Let us first prove the moreover assertion. To this end, let $P \subset M' \cap M^\omega$ be the von Neumann subalgebra generated by W_1 and W_2. Assume, by contradiction, that there is a sequence $(A_n)_n$ of von Neumann subalgebras of M such that

$$P \subset \prod_\omega A_n \subset M' \cap M^\omega.$$

For $n \in \mathbb{N}$, let $M_n = A_n' \cap M$. Lemma 2.1 implies that $\lim_{n\to\omega} \|x - E_{M_n}(x)\|_2 \to 0$, for every $x \in M$. The moreover assertion of Lemma 4.1 implies the existence of projections $r_n \in \mathcal{Z}(M_n' \cap M)$ such that $\lim_{n\to\omega} \tau(r_n) \to 1$ and $(M_n' \cap M)r_n$ is amenable, for every $n \in \mathbb{N}$. Thus, $A_n r_n$ is amenable, for every $n \in \mathbb{N}$.

If $n \in \mathbb{N}$, then since $W_m = (W_{k,m})_k \in P \subset \prod_\omega A_k$, there is $k_n \in \mathbb{N}$ satisfying $\tau(r_{k_n}) \geqslant 1 - 1/n^2$ and $\|W_{k_n,m} - E_{A_{k_n}}(W_{k_n,m})\|_2 \leqslant 1/n$, for every $m \in \{1, 2\}$. Thus, if $B_n = A_{k_n} r_{k_n} \oplus \mathbb{C}(1 - r_{k_n})$, then

$$\|W_{k_n,m} - E_{B_n}(W_{k_n,m})\|_2 \leqslant 1/n + \|1 - r_{k_n}\|_2 \leqslant 2/n, \quad \text{for every } n \in \mathbb{N} \text{ and } m \in \{1, 2\}. \tag{4.8}$$

Let N be the II$_1$ factor generated by two freely independent semicircular operators S_1, S_2 with $\|S_1\| = \|S_2\| = 2$. For $n \in \mathbb{N}$, let $\pi_n : N \to M$ be the unique trace preserving $*$-homomorphism such that $\pi_n(S_m) = W_{k_n,m}$, for all $m \in \{1, 2\}$. Then (4.8) gives that $\|\pi_n(x) - E_{B_n}(\pi_n(x))\|_2 \to 0$, for every $x \in N$. Since B_n is amenable, for every $n \in \mathbb{N}$, Corollary 2.5 implies that N is amenable. Since $N \cong L(\mathbb{F}_2)$ is not amenable, this gives a contradiction and thus proves the moreover assertion.

To prove the main assertion, assume, by contradiction, that M admits a residual sequence $(A_n)_n$. Then $P \subset M' \cap M^\omega = \bigcap_n A_n^\omega$ and since P is separable, we can find an increasing sequence of positive integers (k_n) such that $P \subset \prod_\omega A_{k_n}$. Since $\prod_\omega A_{n_k} \subset \bigcap_n A_n^\omega = M' \cap M^\omega$, this contradicts the moreover assertion. □

5. Stability

5.1. Proof of Proposition C

Since P is amenable, it is approximately finite dimensional by Connes' theorem [8]. Thus, we can find an increasing sequence $(B_k)_k$ of finite dimensional von Neumann subalgebras such that $P = (\bigcup_k B_k)''$. If $k \in \mathbb{N}$, then since B_k is finite dimensional, there exists $S_k \in \omega$ such that for every $n \in S_k$, we have an embedding $B_k \subset M_n$ in such a way that the embedding $B_k \subset \prod_\omega M_n$ is the diagonal embedding. Put $S_0 = \mathbb{N}$.

Claim. *There exists a sequence $(k_n) \subset \mathbb{N}$ such that $n \in S_{k_n}$, for all $n \in \mathbb{N}$, $\lim_{n\to\omega} k_n = +\infty$, and*

$$Q \subset \prod_\omega (B_{k_n}' \cap M_n).$$

Proof of the claim. Since B_k is finite dimensional, $Q \subset P' \cap \prod_\omega M_n \subset B_k' \cap \prod_\omega M_n = \prod_\omega (B_k' \cap M_n)$, for every $k \in \mathbb{N}$. Hence, $Q \subset \bigcap_{k\in\mathbb{N}} \prod_\omega (B_k' \cap M_n)$, i.e.,

$$\lim_{n\to\omega} \left\| q_n - E_{B_k' \cap M_n}(q_n) \right\|_2 = 0, \quad \text{for all } k \in \mathbb{N} \text{ and } q = (q_n) \in Q. \tag{5.1}$$

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Now, let $\{q^{(m)}\}_{m\in\mathbb{N}}$ be a $\|.\|_2$-dense sequence in $(Q)_1$. Let $X_0=\mathbb{N}$ and

$$X_k=\left\{n\in S_k \mid \left\|q_n^{(i)}-E_{B_k'\cap M_n}(q_n^{(i)})\right\|_2\leqslant\frac{1}{k},\ \text{for all } 1\leqslant i\leqslant k\right\}.$$

For $n\in\mathbb{N}$, define k_n to be the largest $k\leqslant n$ such that $n\in X_k$. We claim that $\lim_{n\to\omega}k_n=+\infty$. Otherwise, there exists $k\in\mathbb{N}$ such that $\{n\in\mathbb{N}\mid k_n=k\}\in\omega$. Then $\{n\in\mathbb{N}\mid n\notin X_{k+1}\}\in\omega$. Since $S_{k+1}\in\omega$, this would imply the existence of $i\in\{1,\ldots,k+1\}$ such that we have

$$\left\{n\in\mathbb{N}\mid \left\|q_n^{(i)}-E_{B_{k+1}'\cap M_n}(q_n^{(i)})\right\|_2>\frac{1}{k+1}\right\}\in\omega,$$

and, thus,

$$\lim_{n\to\omega}\left\|q_n^{(i)}-E_{B_{k+1}'\cap M_n}(q_n^{(i)})\right\|_2\geqslant\frac{1}{k+1},$$

contradicting (5.1). By construction, $Q\subset\prod_\omega(B_{k_n}'\cap M_n)$, which finishes the proof of the claim. □

Taking (k_n) as in the claim, we also have that $P\subset\prod_\omega B_{k_n}$. Thus, $P_n=B_{k_n}$ and $Q_n=B_{k_n}'\cap M_n$ verify the conclusion of Proposition C. □

5.2. Proof of Theorem E

In the proof of Theorem E, we will need the following consequence of Corollary 3.2. Recall that a tracial von Neumann algebra (M,τ) is called *solid* [30] if the relative commutant $P'\cap M$ is amenable, for any diffuse von Neumann subalgebra $P\subset M$.

Lemma 5.1. *Let Γ be a countable group and $\pi:\Gamma\to\mathcal{O}(\mathcal{H}_\mathbb{R})$ be a mixing orthogonal representation. Assume that $\pi^{\otimes k}$ is weakly contained in the left regular representation of Γ, for some $k\in\mathbb{N}$. Let $\Gamma\curvearrowright(C,\tau)$ be the free Bogoljubov action associated with π. If $L(\Gamma)$ is solid, then $C\rtimes\Gamma$ is solid.*

Proof. Assume that $L(\Gamma)$ is solid. In order to prove that $M=C\rtimes\Gamma$ is solid, it suffices to show that if $P\subset M$ is a diffuse von Neumann subalgebra, then $P'\cap M$ has an amenable direct summand. Suppose, by contradiction, that $P'\cap M$ has no amenable direct summand. By applying Corollary 3.2, we get that $P\prec_M L(\Gamma)$. Hence, there exist projections $p\in P, q\in L(\Gamma)$, a $*$-homomorphism $\theta:pPp\to qL(\Gamma)q$ and a non-zero partial isometry $v\in qMp$ such that $\theta(x)v=vx$ for all $x\in pPp$. Since π is mixing, the action $\Gamma\curvearrowright C$ is mixing by [18, Proposition 2.6]. Since $\theta(pPp)\subset qL(\Gamma)q$ is a diffuse subalgebra and $vv^*\in\theta(pPp)'\cap qMq$, [35, Theorem 3.1] implies that $q_0:=vv^*\in L(\Gamma)$. Thus, $P_0:=vPv^*$ is a diffuse subalgebra of $q_0L(\Gamma)q_0$. Since $v(P'\cap M)v^*\subset q_0Mq_0$ is a subalgebra which commutes with P_0, [35, Theorem 3.1] gives that $v(P'\cap M)v^*\subset P_0'\cap q_0L(\Gamma)q_0$. Since $L(\Gamma)$ is solid, we get that $v(P'\cap M)v^*$ is amenable and thus $P'\cap M$ has an amenable direct summand. This finishes the proof of the lemma. □

Proof of Theorem E. First, note that if W is a self-adjoint operator in a tracial von Neumann algebra whose distribution with respect to the trace is the semicircular law supported on $[-2,2]$, then $\{W\}''$ is a diffuse abelian von Neumann algebra. Hence, we can find a Borel function $f:[-2,2]\to\mathbb{T}$ such that $U=f(W)\in\{W\}''$ is a Haar unitary,

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i.e., $\tau(U^n) = 0$, for all $n \in \mathbb{Z} \setminus \{0\}$. From now on, fix two freely independent self-adjoint operators W_1, W_2 in a tracial von Neumann algebra whose distribution is the semicircular law supported on $[-2, 2]$. Define $U_1 = f(W_1)$ and $U_2 = f(W_2)$. Then U_1 and U_2 are freely independent Haar unitaries, and, thus, $N = \{U_1, U_2\}''$ satisfies $N = \{U_1\}'' * \{U_2\}'' \cong L(\mathbb{F}_2)$.

Let $\Gamma = \mathbb{F}_2$ and $a_1, a_2 \in \Gamma$ be free generators. Let $\pi_k : \Gamma \to \mathcal{O}(\mathcal{H}_k)$, $k \in \mathbb{N}$, be a sequence of mixing representations such that a tensor multiple of π_k is weakly contained in the left regular representation of Γ, and there exist unit vectors $\xi_{k,m} \in \mathcal{H}_k$ such that $\|\pi_k(g)(\xi_k^m) - \xi_k^m\| \to 0$, for every $m \in \{1, 2\}$ and $g \in \Gamma$. For instance, let $(\pi_k)_{k\in\mathbb{N}}$ be as in Example 1.5 and note that, by construction, π_k is indeed mixing, for every $k \in \mathbb{N}$. Let $\Gamma \curvearrowright B_k$ be the free Bogoljubov action associated with π_k and denote $M_k = B_k \rtimes \Gamma$, for every $k \in \mathbb{N}$.

Then $W_{k,m} = W(\xi_k^m) \in B_k$ is a self-adjoint operator whose distribution is the semicircular law supported on $[-2, 2]$. Moreover, $\|u_g W_{k,m} - W_{k,m} u_g\|_2 = \|\pi_k(g)(\xi_k^m) - \xi_k^m\| \to 0$, for every $m \in \{1, 2\}$ and $g \in \Gamma$. Thus, if we put $U_{k,m} = f(W_{k,m}) \in \mathcal{U}(B_k)$, then

$$\|u_g U_{k,m} - U_{k,m} u_g\|_2 \to 0, \quad \text{for every } m \in \{1, 2\} \text{ and } g \in \Gamma. \tag{5.2}$$

Let $\rho_k : N \to M_k$ be the unique trace preserving $*$-homomorphism given by $\rho_k(U_1) = U_{k,1}$ and $\rho_k(U_2) = U_{k,2}$. Then (5.2) rewrites as

$$\|u_g \rho_k(U_m) - \rho_k(U_m) u_g\|_2 \to 0, \quad \text{for every } m \in \{1, 2\} \text{ and } g \in \Gamma. \tag{5.3}$$

In the rest of the proof, we treat the two assertions of Theorem E separately.

Part 1. We first prove that $\Gamma \times \Gamma$ is not W*-tracially stable. This readily implies that $\mathbb{F}_l \times \mathbb{F}_m$ is not W*-tracially stable, for every $2 \leqslant l, m \leqslant +\infty$. Assume, by contradiction, that $\Gamma \times \Gamma$ is W*-tracially stable. Using (5.3), we can define a homomorphism $\varphi : \Gamma \times \Gamma \to \mathcal{U}(\prod_\omega M_k)$ by letting

$$\varphi(a_m, e) = (\rho_k(U_m))_k \text{ and } \varphi(e, g) = u_g, \quad \text{for all } m \in \{1, 2\} \text{ and } g \in \Gamma. \tag{5.4}$$

Since $\Gamma \times \Gamma$ is assumed W*-tracially stable, there must be homomorphisms $\varphi_k : \Gamma \times \Gamma \to \mathcal{U}(M_k)$ such that $\varphi = (\varphi_k)_k$. Let $C_k = \varphi_k(\Gamma \times \{e\})''$ and $D_k = \varphi_k(\{e\} \times \Gamma)''$. Then C_k and D_k are commuting von Neumann subalgebras of M_k and we have that

$$\lim_{k\to\omega} \|\rho_k(U_m) - E_{C_k}(\rho_k(U_m))\|_2 = 0, \quad \text{for every } m \in \{1, 2\}, \text{ and} \tag{5.5}$$

$$\lim_{k\to\omega} \|u_g - E_{D_k}(u_g)\|_2 = 0, \quad \text{for every } g \in \Gamma. \tag{5.6}$$

Then (5.5) implies that $\lim\limits_{k\to\omega} \|\rho_k(x) - E_{C_k}(\rho_k(x))\|_2 \to 0$, for every $x \in N$. Since N is a non-amenable II$_1$ factor, Corollary 2.5 implies that if $p_k \in \mathcal{Z}(C_k)$ is the largest projection such that $C_k p_k$ is amenable, then $\lim_{k\to\omega} \tau(p_k) = 0$. Since $L(\Gamma)$ is also a non-amenable II$_1$ factor, by repeating this argument using (5.6), it follows that $\lim_{k\to\omega} \tau(q_k) = 0$, where $q_k \in \mathcal{Z}(D_k)$ denotes the largest projection such that $D_k q_k$ is amenable. Thus, for every $k \in \mathbb{N}$, $r_k := (1 - p_k)(1 - q_k) \in \{C_k, D_k\}''$ is a projection such that $C_k r_k$ and $D_k r_k$ have no amenable direct summands, and $\lim_{k\to\omega} \tau(r_k) = 1$. In particular, we can find k such that $r_k \neq 0$. This implies that $r_k M r_k$, and thus M, is not solid, which is a contradiction by Lemma 5.1. This finishes the proof of the first assertion of Theorem E.

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Part 2. For the moreover assertion, put $B = \bar{\bigotimes}_{k\in\mathbb{N}} B_k$ and $M = B \rtimes \Gamma$. Using the natural embeddings $M_k \subset M$, for every $k \in \mathbb{N}$, we can view $\prod_\omega M_k$ as a subalgebra of M^ω. Thus, we may view φ as a homomorphism $\varphi : \Gamma \times \Gamma \to \mathcal{U}(M^\omega)$. Since by the definition (5.4) of φ we have $\varphi(a, e) \in B^\omega$, $\tau(\varphi(a, e)) = \delta_{a,e}$ and $\varphi(e, g) = u_g$, it follows that $\tau(\varphi(a, g)) = \tau(\varphi(a, e)u_g) = \delta_{(a,g),(e,e)}$, for all $a, g \in \Gamma$. Thus, φ extends to a $*$-homomorphism $\varphi : L(\mathbb{F}_2 \times \mathbb{F}_2) \to M^\omega$.

We claim that there are no homomorphisms $\varphi_k : \Gamma \times \Gamma \to \mathcal{U}(M)$ such that $\varphi = (\varphi_k)_k$. Assume, by contradiction, that such homomorphisms (φ_k) exist. Then $C_k = \varphi_k(\Gamma \times \{e\})''$ and $D_k = \varphi_k(\{e\} \times \Gamma)''$ are commuting von Neumann subalgebras of M such that (5.5) and (5.6) hold.

Since Γ is non-amenable, [31, Proposition 2.4] implies that $L(\Gamma)$ is not amenable relative to B inside M. Thus, since $L(\Gamma)' \cap M = \mathbb{C}1$, there is no non-zero projection $q \in L(\Gamma)' \cap M$ such that $L(\Gamma)q$ is amenable relative to B inside M. Let $q_k \in D_k' \cap M$ be the largest projection such that $D_k q_k$ is amenable relative to B inside M. Then by [11, Lemma 2.6], we have that $q_k \in \mathcal{Z}(D_k' \cap M)$. Since by (5.6) we have that $\lim_\omega \|x - E_{D_k}(x)\|_2 = 0$, for every $x \in L(\Gamma)$, we can apply Lemma 2.4 to conclude that $\lim_\omega \tau(q_k) = 0$.

Next, fix $k \in \mathbb{N}$. Then $D_k p'$ is not amenable relative to B inside M, for any non-zero projection $p' \in (D_k' \cap M)(1 - q_k)$. For $i \in \mathbb{N}$, let $R_i = \bar{\bigotimes}_{l\neq i} B_l$. Then by applying Corollary 3.2 to the decomposition $M = (B_i \bar{\otimes} R_i) \rtimes \Gamma$, it follows that $C_k(1 - q_k) \prec^s_M R_i \rtimes \Gamma$, for every $i \in \mathbb{N}$. If $n_0 \in \mathbb{N}$, then the subalgebras $\{R_i \rtimes \Gamma\}_{i=1}^{n_0}$ of M are regular and any two form a commuting square. Since $\bigcap_{i=1}^{n_0} (R_i \rtimes \Gamma) = (\bar{\bigotimes}_{l>n_0} B_l) \rtimes \Gamma$, [11, Lemma 2.8(2)] implies that

$$C_k(1 - q_k) \prec^s_M \left(\bar{\bigotimes}_{l>n_0} B_l\right) \rtimes \Gamma, \quad \text{for every } k, n_0 \in \mathbb{N}. \tag{5.7}$$

Since $\Gamma = \langle a_1, a_2\rangle$ is not inner amenable, we can find a constant $c > 0$ such that

$$\|x - E_B(x)\|_2 \leqslant c(\|[x, u_{a_1}]\|_2 + \|[x, u_{a_2}]\|_2), \quad \text{for every } x \in M. \tag{5.8}$$

For $k \in \mathbb{N}$, denote $\varepsilon_k = \|u_{a_1} - E_{D_k}(u_{a_1})\|_2 + \|u_{a_2} - E_{D_k}(u_{a_2})\|_2$. Then (5.6) implies that $\lim_\omega \varepsilon_k = 0$. Since C_k and D_k commute, we have that $\|[x, u_{a_1}]\|_2 + \|[x, u_{a_2}]\|_2 \leqslant 2\varepsilon_k$, for all $x \in (C_k)_1$. In combination with (5.8), we get that $\|x - E_B(x)\|_2 \leqslant 2c\varepsilon_k$, for all $x \in (C_k)_1$. By applying [21, Lemma 2.2], we derive the existence of a projection $r_k \in \mathcal{Z}(C_k' \cap M)$ such that $\tau(r_k) \geqslant 1 - 2c\varepsilon_k$ and

$$C_k r_k \prec^s_M B, \quad \text{for every } k \in \mathbb{N}. \tag{5.9}$$

Since B and $(\bar{\bigotimes}_{l>n_0} B_l) \rtimes \Gamma$ are regular subalgebras of M which form a commuting square, if $p_k = (1 - q_k)r_k \in C_k' \cap M$, by combining (5.7), (5.9) and [11, Lemma 2.8(2)], we get that

$$C_k p_k \prec_M \bar{\bigotimes}_{l>n_0} B_l, \quad \text{for every } k, n_0 \in \mathbb{N}. \tag{5.10}$$

Using (5.10) and reasoning as at the end of the proof of Lemma 4.1, it follows that $C_k p_k$ is amenable, for every $k \in \mathbb{N}$. Since $\lim_\omega \tau(q_k) = 0$ and $\lim_\omega \tau(r_k) = 1$, we get that $\lim_\omega \tau(p_k) = 1$. On the other hand, (5.5) implies that $\lim_\omega \|\rho_k(x) - E_{C_k}(\rho_k(x))\|_2 = 0$, for every $x \in N$. By applying Corollary 2.5, we derive that N is amenable, which is a contradiction. □

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