(G, μ)-DISPLAYS AND RAPOPORT–ZINK SPACES

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Abstract Let (G, μ) be a pair of a reductive group G over the p-adic integers and a minuscule cocharacter μ of G defined over an unramified extension. We introduce and study '(G, μ)-displays' which generalize Zink's Witt vector displays. We use these to define certain Rapoport–Zink formal schemes purely group theoretically, i.e. without p-divisible groups.

Keywords: 11-XX Number Theory; 14-XX Algebraic Geometry; p-divisible groups

Contents

1. Introduction

In the theory of Shimura varieties, as interpreted by Deligne [9], one starts with a 'Shimura datum' (G, X). This is a pair of a (connected) reductive algebraic group G over the field of rational numbers $\mathbb{Q}$, and a symmetric Hermitian domain $X = \{h\}$ for $G(\mathbb{R})$ given as a $G(\mathbb{R})$-conjugacy class of an algebraic group homomorphism $h : \mathbb{C}^* \to G_{\mathbb{R}}$ over the real numbers $\mathbb{R}$, that satisfies certain axioms. For each open compact subgroup K of the finite adeles $G(\mathbb{A}^f)$, one considers the Shimura variety $\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}^f)/K)$; this complex analytic space is actually an algebraic variety with a canonical model over a number field, the so-called reflex field of the pair (G, X).

The proposal that there should exist a similar theory of 'p-adic local Shimura varieties' was recently put forward by Rapoport and Viehmann [21]. The current paper can be viewed as a contribution to this theory. In [21], one starts with the 'local Shimura datum'. This is a triple $(G, \{\mu\}, [b])$ consisting of a connected reductive algebraic group G over $\mathbb{Q}_p$, a conjugacy class $\{\mu\}$ of minuscule cocharacters of $G_{\overline{\mathbb{Q}}_p}$, and a σ-conjugacy class $[b]$ of elements in $G(L)$, satisfying some simple axioms. (Here, L is the completion of the maximal unramified extension of $\mathbb{Q}_p$ and σ the canonical lift of Frobenius.) For each open compact subgroup $K \subset G(\mathbb{Q}_p)$ we should have the local Shimura variety $\mathrm{M}_K(G, \{\mu\}, [b])$; this is expected to be a rigid analytic space with a canonical model over the 'local reflex field' which is a finite extension E of $\mathbb{Q}_p$ that depends only on $(G, \{\mu\})$. See [21] for more details and expected properties of the local Shimura varieties. Examples of such local Shimura varieties have first been constructed in the work of Rapoport and Zink [22] in some special cases. There, they appear as covers of the generic fibers of certain formal schemes over the ring of integers $\mathcal{O}_E$. These formal schemes (which we call Rapoport–Zink formal schemes) are moduli spaces parametrizing p-divisible groups (with additional structure) with a quasi-isogeny to a fixed p-divisible group. They can be viewed as integral models of the desired local Shimura varieties.

In this paper, we consider the case in which the local Shimura datum is *unramified*. In particular, the open compact subgroup K is maximal hyperspecial. Then, starting from $(G, \{\mu\}, [b])$, we give a functor $\mathrm{RZ}_{G,\mu,b}$ on p-nilpotent algebras. As we explain below, this functor has a direct group theoretic definition which uses only G, suitable representatives μ and b, and involves rings of Witt vectors. We conjecture that $\mathrm{RZ}_{G,\mu,b}$ is represented by a formal scheme which should then be an integral model of the sought-after local

Shimura variety for this hyperspecial level. We essentially show this conjecture when the local Shimura datum is of *Hodge type*, i.e. when it embedds into a local Shimura datum for GL_n.

The main tool we use is a variation of the theory of Zink displays. By the work of Zink and Lau, (formal) p-divisible groups over a p-adically complete and separated algebra R are classified by displays. These are projective finitely generated modules P over the ring of Witt vectors $W(R)$ with additional structure given by a suitable filtration $I(R)P \subset Q \subset P$ and a Frobenius semi-linear operator $V^{-1} : Q \to P$ that satisfy certain axioms. Here, $I(R)$ is the kernel of the projection $w_0 : W(R) \to R$ given by $w_0(r_0, r_1, \ldots) = r_0$.

In this paper, we develop a theory of displays 'with (G, μ)-structure': Instead of projective $W(R)$-modules, we use G-torsors over $W(R)$ or, equivalently, L^+G-torsors over R. Here, G is a reductive group scheme over $\mathbb{Z}_p$ and μ is a (minuscule) cocharacter of G defined over a finite unramified extension $W(k_0)$. Then L^+G is the positive 'Witt loop group scheme of G' defined by $L^+G(R) = G(W(R))$. To explain the definition of (G, μ)-displays we need to introduce some more objects: We let H^μ be the subgroup scheme of $L^+G_{W(k_0)}$ with R-valued points $H^\mu(R)$ given by those $g \in G(W(R))$ whose projection $g_0 \in G(R)$ lands in the R-points of the parabolic subgroup $P_\mu \subset G$ associated to μ. We then construct the 'divided Frobenius' which is a group scheme homomorphism

$$\Phi_{G,\mu} : H^\mu \to L^+G_{W(k_0)}$$

such that, for $h \in H^\mu(R)$, we have

$$\Phi_{G,\mu}(h) = \mu^\sigma(p) \cdot F(h) \cdot \mu^\sigma(p)^{-1}$$

in $G(W(R)[1/p])$. Here F is induced by the Frobenius on $W(R)$.

Definition 1.0.1. A (G, μ)-display is a triple $\mathcal{D} := (P, Q, u)$ which we can write

$$P \hookleftarrow Q \xrightarrow{u} P,$$

where P is a L^+G-torsor and Q a H^μ-torsor. We ask that

$$P = Q \times_{H^\mu} L^+G_{W(k_0)}$$

and that u is a morphism compatible with the actions on Q and P and with $\Phi_{G,\mu}$, in the sense that $u(q \cdot h) = u(q) \cdot \Phi_{G,\mu}(h)$, for all $q \in Q(R)$, $h \in H^\mu(R)$.

(This definition first appeared in [3], see also [26] for a similar construction. One can view this structure as formally similar to that of a Drinfeld shtuka with $W(R)$ replacing the affine ring $R[t]$ of a curve.)

Locally the morphism u is given by a point of L^+G and we can see that (G, μ)-displays are objects of the quotient stack

$$[L^+G_{W(k_0)}/_{\Phi_{G,\mu}} H^\mu]$$

where the action of H^μ is by $\Phi_{G,\mu}$-conjugation: $g \cdot h := h^{-1}g\Phi_{G,\mu}(h)$.

Apparently, the notion of a (G, μ)-display is sufficiently well-behaved and we can generalize several of the results of Zink on Witt vector displays, for example, about

deformation theory. We can also define a notion of a G-quasi-isogeny between two (G, μ)-displays and show that a triple (G, μ, b) allows us to give a 'base-point' (G, μ)-display $\mathcal{D}_0$ defined over $k = \bar{k}_0$. With these ingredients, we now describe the 'Rapoport–Zink functor' $\mathrm{RZ}_{G,\mu,b}$:

By definition, $\mathrm{RZ}_{G,\mu,b}$ sends a p-nilpotent $W(k)$-algebra R to the set of isomorphism classes of pairs $(\mathcal{D}, \rho)$, where $\mathcal{D}$ is a (G, μ)-display over R, and $\rho : \mathcal{D} \times_R R/pR \dashrightarrow \mathcal{D}_0 \times_k R/pR$ is a G-quasi-isogeny.

We can also view $\mathrm{RZ}_{G,\mu,b}$ as given by a quotient stack. Set LG for the Witt loop group scheme of G given by $LG(R) = G(W(R)[1/p])$. Then $\mathrm{RZ}_{G,\mu,b}$ is given by the isomorphism classes of objects of the (fpqc, or étale) quotient stack

$$\mathcal{RZ}_{G,\mu,b} := [(L^+G \times_{LG,\mu,b} LG)/H^\mu]$$

for the action of H^μ given by

$$(U, g) \cdot h = (h^{-1} \cdot U \cdot \Phi_{G,\mu}(h), g \cdot h).$$

Here, the fiber product $(L^+G \times_{LG,\mu,b} LG)(R)$ is by definition the set of pairs (U, g) with $U \in L^+G(R)$, $g \in LG(R)$, such that

$$g^{-1}bF(g) = U\mu^\sigma(p)$$

in $LG(R)$. The σ-centralizer group

$$J_b(\mathbb{Q}_p) = \{j \in G(L) \mid j^{-1}b\sigma(j) = b\}$$

acts on $\mathcal{RZ}_{G,\mu,b}$ by

$$j \cdot (U, g) = (U, j \cdot g).$$

It follows from the definition that the k-valued points of $\mathrm{RZ}_{G,\mu,b}$ are given by the affine Deligne–Lusztig set

$$\mathrm{RZ}_{G,\mu,b}(k) = \{g \in G(L) \mid g^{-1}b\sigma(g) \in G(W)\mu^\sigma(p)G(W)\}/G(W).$$

Here, $W = W(k)$, $L = W(k)[1/p]$.

Assuming an additional mild condition on the slopes of b, we conjecture that the functor $\mathrm{RZ}_{G,b,\mu}$ is representable by a formal W-scheme which is formally locally of finite type and W-formally smooth. For $G = \mathrm{GL}_n$, and when b has no zero slopes, this follows from the results of Rapoport–Zink and Zink and Lau. Indeed, in that case, the results of Zink and Lau imply that our functor is equivalent to the functor of isomorphism classes of (formal) p-divisible groups with a quasi-isogeny to a fixed p-divisible group considered in [22]; the representability of that functor is one of the main results of loc. cit. For more general (G, μ), we show that pairs $(\mathcal{D}, \rho)$ over R have no automorphisms when R is Noetherian.

When (G, μ) is of Hodge type, i.e. when there is an embedding $i : G \hookrightarrow \mathrm{GL}_n$ with $i \cdot \mu$ conjugate to one of the standard minuscule cocharacters of GL_n, we show that the restriction of the functor $\mathrm{RZ}_{G,\mu,b}$ to Noetherian algebras is representable as desired. This is one of the main results of the paper. The basic idea of the proof is as follows: We show (Corollary 5.2.6) that when $(G, \mu) \hookrightarrow (G', \mu')$ is an embedding of local Shimura data, the corresponding morphism of stacks $\mathcal{RZ}_{G,\mu,b} \to \mathcal{RZ}_{G',\mu',i(b)}$, when restricted to

Noetherian algebras, is relatively representable by a closed immersion. (Proving this is, maybe surprisingly, involved; the main step is the descent result of Proposition 5.2.5. Part of the difficulty comes from the need to handle the nilradical of various rings that appear in the argument.) In the Hodge type case, we have an embedding $(G, \mu) \hookrightarrow (\mathrm{GL}_n, \mu')$ and $\mathcal{RZ}_{\mathrm{GL}_n, \mu', i(b)}$ is representable by the results of Zink, Lau, and Rapoport–Zink as explained above. The result follows by combining these two statements. (In all of this, we have to assume that the slopes of $i(b)$ do not include 0.)

A construction of Rapoport–Zink formal schemes in the Hodge type case was also given by Kim [15], and, under an additional condition, independently by Howard and the second author [14]. It is easy to see that, in the Hodge type case, the formal scheme representing $\mathrm{RZ}_{G, \mu, b}$ given in this paper coincides with the corresponding formal schemes constructed in [22] (in the PEL and EL cases), in [15], and in [14]. Hence, our results give a unified group theoretic description of these formal schemes as moduli functors and describe their R-valued points for all Noetherian algebras R. In particular, they imply the existence of isomorphisms between 'classical' Rapoport–Zink spaces when the corresponding local Shimura data are isomorphic (for example, because of exceptional isomorphisms between the underlying groups); this answers a question of Rapoport.

Let us also mention here that the restriction of our functors to perfect k-algebras has already appeared in the work of Zhu [27] (see also [2]). In fact, when we consider functors with values in perfect algebras there is a more comprehensive theory that employs the 'Witt vector affine Grassmannian' which does not require the assumption that μ is minuscule. However, the techniques of [27] and [2] cannot handle p-nilpotent algebras and only give information about the perfection of the special fiber. On the other extreme, when one considers only the generic fibers, in the Hodge type case, Scholze and Scholze–Weinstein [25] can give a construction of the inverse limit of the tower of local Shimura varieties as a perfectoid space (see also [7]). There are also related constructions of more general spaces (even for μ not minuscule) that use G-bundles on the Fontaine–Fargues curve and Scholze's theory of diamonds [11, 24]. Again, the more classical integral theory in this paper is in a different direction. Nevertheless, it would be interesting to directly compare these constructions with ours. In another direction, it should also be possible to develop a theory of 'relative' Rapoport–Zink spaces by combining our group theoretic constructions with the theory of relative displays of T. Ahsendorf (see for example [1]). Then one can compare these with the 'absolute' Rapoport–Zink spaces of the current paper when the group is given by Weil restriction of scalars (see [23] for an example of such a comparison).

We now briefly describe the contents of the paper: We start with preliminaries on Witt vectors, various notions of 'Witt loop schemes' (these are variations of the Greenberg transform), and a review of the main definitions of Zink's theory of displays. In §3, we define the group theoretic displays ((G, μ)-displays). We discuss several of their basic properties, study their deformation theory and define a notion of quasi-isogeny. In §4, we give our group theoretic definition of the Rapoport–Zink stacks and state the representability conjecture. In §5, we show representability for local Shimura data of Hodge type over Noetherian rings. At the end of the paper, we include three short appendices: The first reviews certain facts about parabolic subgroups of reductive groups

and the second discusses torsors for Witt loop group schemes. Finally, the third appendix gives some results on nilradicals of certain rings which are used in the proof of the main representability theorem.

2. Preliminaries

Let p be a prime number. Denote by k an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Set $W = W(k)$ for the ring of Witt vectors and $K = W[1/p]$ for its fraction field. Denote by $\bar{K}$ an algebraic closure of K. We use the symbol k_0 to denote a finite field of cardinality $q = p^f$ contained in k.

If $\mathcal{O}$ is a $\mathbb{Z}_p$-algebra, we denote by $\mathrm{ANilp}_{\mathcal{O}}$ the category of $\mathcal{O}$-algebras in which p is nipotent. Similarly, we let $\mathrm{Nilp}_{\mathcal{O}}$ be the category of $\mathrm{Spec}(\mathcal{O})$-schemes S which are such that p is Zariski locally nilpotent on $\mathcal{O}_S$.

In most of the paper, G stands for a connected reductive group scheme over $\mathbb{Z}_p$. Its generic fiber is a connected reductive group over $\mathbb{Q}_p$, and is unramified, *i.e.* quasi-split and split over an unramified extension of $\mathbb{Q}_p$. Conversely, every unramified connected reductive group over $\mathbb{Q}_p$ is isomorphic to the generic fiber of such a G.

2.1. Witt vectors

If R is a commutative ring which is a $\mathbb{Z}_p$-algebra, we denote by $W_n(R)$ the ring of p-typical Witt vectors $(r_0, \ldots, r_{n-1})$ of length n with entries in R. We allow in the notation $n = \infty$; in this case, we simply denote by $W(R)$ the ring of Witt vectors with entries in R. For $r \in R$, we set

$$[r] = (r, 0, 0, \ldots).$$

The ring structure on $W_n(R)$ is functorial in R. Recall the ring homomorphisms ('ghost coordinates')

$$w_k : W_n(R) \to R; \quad (r_0, r_1, \ldots, r_{n-1}) \mapsto r_0^{p^k} + p r_1^{p^{k-1}} + \cdots + p^k r_k.$$

We denote by $I_n(R)$, or simply $I(R)$ if $n = \infty$, the kernel of $w_0 : W_n(R) \to R$. The Frobenius F_n and Verschiebung V_n are maps $W_{n+1}(R) \to W_n(R)$, respectively $W_n(R) \to W_{n+1}(R)$, that satisfy the defining relations

$$w_k(F_n x) = w_{k+1}(x), \quad w_k(V_n x) = p w_{k-1}(x), \quad w_0(V_n x) = 0.$$

The Frobenius F_n is a ring homomorphism. The Verschiebung V_n is additive, is given by

$$V_n(r_0, r_1, \ldots, r_{n-1}) = (0, r_0, r_1, \ldots, r_{n-1}),$$

and we have $V_n W_n(R) = I_{n+1}(R)$. Again, we usually omit the subscript n if $n = \infty$. We have the identities

$$F \circ V = p, \quad V(Fx \cdot y) = x \cdot V(y).$$

The following will be used later in the paper. We denote by $J(R)$ the Jacobson radical of R.

Lemma 2.1.1. *Let $\mathcal{A} \subset W_n(R)$ be an ideal, and suppose that one of the following two assertions holds:*

(i) $p \in J(R)$ *and* $Rw_0(\mathcal{A}) = R$.

(ii) $n < \infty$ *and* $Rw_k(\mathcal{A}) = R$ *holds for all* $0 \leqslant k < n$.

Then $\mathcal{A} = W_n(R)$.

Proof. We begin with case (ii), which we handle by induction on n: It suffices to check $V^{n-1}([x]) = (0, 0, \ldots, x) \in \mathcal{A}$, for any $x \in R$. (Here, for simplicity, we denote the composition $V_{n-1} \circ V_{n-2} \circ \cdots \circ V_1$ as V^{n-1}.) The assumption allows us to write the element x as a sum $\sum_{i=1}^m a_i w_{n-1}(f_i)$ where $a_i \in R$ and $f_i \in \mathcal{A}$. It follows that

$$V^{n-1}([x]) = \sum_{i=1}^m V^{n-1}([a_i w_{n-1}(f_i)]) = \sum_{i=1}^m f_i V^{n-1}([a_i]),$$

as elements of $W_n(R)$, which gives what we wanted. It remains to consider (i). Notice that w_0 is surjective, so that $w_0(\mathcal{A}) = Rw_0(\mathcal{A})$. It follows that we can assume $\mathcal{A}$ is principal, i.e. $\mathcal{A} = W_n(R)f$, and we only need to check that f is a unit in every quotient $W_n(R)/(V_{n-1} \circ \cdots \circ V_{n-k})W_{n-k}(R) = W_k(R)$ of Witt vectors of finite length. (This then also implies the case $n = \infty$.) However, this follows from (ii) together with $p \in J(R)$ and $w_k(f) \equiv w_0(f)^{p^k} \bmod p$ for every k. $\square$

If X is a scheme over $W(R)$, we denote by $^F X$ the scheme over $W(R)$ obtained by pulling back via the Frobenius, i.e.

$$^F X = X \times_{\mathrm{Spec}(W(R)), F} \mathrm{Spec}(W(R)).$$

2.2. Greenberg transforms and Witt loop schemes

Suppose that X is an affine scheme which is of finite type, respectively of finite presentation, over $W_n(R)$. By [12, §4] (see also [18, Proposition 29]), the functor $R' \to X(W_n(R'))$ is represented by an affine scheme $F_n X$ over R; this is of finite type, respectively of finite presentation, if $n < \infty$. The scheme $F_n X$ is sometimes called the Greenberg transform of X. (Again, for $n = \infty$, we simply write FX instead of $F_\infty X$. Also, if X is a scheme over $W(R)$, we write $F_n X$ instead of $F_n(X \otimes_{W(R)} W_n(R))$.)

We can also consider the functor $R' \to X(W(R')[1/p])$. By [18, Proposition 32], we see that this functor is represented by an Ind-scheme $F_{(p)}X$ over R which we might call the Witt loop scheme of X. In [18], $F_{(p)}X$ is called the 'localized' Greenberg transform of X.

We collect a few useful properties of the Greenberg transforms F_n.

Proposition 2.2.1.

(a) *If X and Y are two affine finite type schemes over $W_n(R)$, then there is a natural isomorphism*

$$F_n(X \times_{\mathrm{Spec}(W_n(R))} Y) \cong F_n X \times_{\mathrm{Spec}(R)} F_n Y. \qquad (2.2.1.1)$$

(b) *If $f : X \to Y$ is a formally smooth, respectively formally étale, morphism of affine schemes over $W_n(R)$, then $F_n f : F_n X \to F_n Y$ is formally smooth, respectively formally étale, morphism of affine schemes over R.*

(c) *If X is affine and smooth over $W_n(R)$, and $n < \infty$, then $F_n X$ is smooth over R.*

(d) *If X is affine and smooth over $W(R)$ ($n = \infty$), then FX is flat and formally smooth over R.*

Proof. Part (a) follows quickly from the defining property of the Greenberg transforms $F_n X(R') = X(W_n(R'))$, $F_n Y(R') = X(W_n(R'))$. Let us show part (b). Consider an R-algebra B with a nilpotent ideal $I \subset B$ and set $\bar{B} = B/I$. Then there is a natural map of sets

$$a : F_n X(B) \to F_n Y(B) \times_{F_n Y(\bar{B})} F_n X(\bar{B}).$$

By definition, $F_n f$ is formally smooth, respectively formally étale, if a is surjective, respectively bijective, for all such pairs $I \subset B$. By definition, the map a is the natural map

$$a : X(W_n(B)) \to Y(W_n(B)) \times_{Y(W_n(\bar{B}))} X(W_n(\bar{B})).$$

However, we can easily see that the kernel $W_n(I)$ of $W_n(B) \to W_n(\bar{B})$ is still nilpotent (also for $n = \infty$). In fact, if $I^r = 0$, then $W_n(I)^r = 0$. Hence, the surjectivity, respectively bijectivity, of a follows since f is assumed to be formally smooth, respectively formally étale. Part (c) follows from (b) and the above since then $F_n X \to \operatorname{Spec}(R)$ is of finite presentation, and so smooth amounts to formally smooth. Finally, to show (d) observe that, under our assumptions, the natural morphism $F_{n+1} X \to F_n X$ is formally smooth, for all n. (This is obtained using the fact that if $\bar{B} = B/I$ with I nilpotent, then the natural homomorphism $W_{n+1}(B) \to W_n(B) \times_{W_n(\bar{B})} W_{n+1}(\bar{B})$ is surjective with nilpotent kernel.) If we write A_n for the R-algebra with $F_n(X \otimes_{W(R)} W_n(R)) = \operatorname{Spec}(A_n)$, we have $FX = \operatorname{Spec}(\varinjlim_n A_n)$. Formal smoothness of FX over R follows as above. Flatness also follows from (c) since smooth implies flat, using also that a direct limit of flat R-algebras is R-flat. $\square$

2.2.2. If X is an affine finite type scheme over $W(k_0)$, we write $L_n^+ X$ and LX for the Greenberg and localized Greenberg transforms of the base changes of X to $W_n(W(k_0))$ by the natural Cartier ring homomorphism $\Delta_n : W(k_0) \to W_n(W(k_0))$ characterized by $w_k(\Delta_n(x)) = F^k(x)$. These Greenberg transforms are schemes, respectively an Ind-scheme, over $W(k_0)$.

Since $w_0 \cdot \Delta$ is the identity, there is a natural morphism

$$L^+ X \to X$$

induced by $w_0 : W(R) \to R$. We denote by $s_0 \in X(R)$ the image of the point $s \in L^+ X(R) = X(W(R))$ under this map.

The homomorphism Δ_n commutes with the Frobenius on the source and target ([28, Lemma 52]) and we can see that we have natural isomorphisms

$$^F(L^+ X) \simeq L^+(^F X), \quad ^F(LX) \simeq L(^F X),$$

while F induces natural morphisms

$$F : L^+ X \to {}^F(L^+ X), \quad F : LX \to {}^F(LX)$$

which cover the Frobenius isomorphism of $\operatorname{Spec}(W(k_0))$.

If X is in addition a group scheme over $W(k_0)$, then $L_n^+ X$, respectively LX, are group schemes, respectively is a Ind-group scheme, over $W(k_0)$. In this case, the isomorphisms and morphisms above are group scheme, respectively Ind-group scheme, homomorphisms.

Remark 2.2.3. Suppose that R is a k_0-algebra via $\phi : k_0 \to R$ and $X = \mathrm{Spec}(A)$ an affine finite type $W(k_0)$-scheme. We view R as a $W(k_0)$-algebra via the composition $W(k_0) \to k_0 \to R$ where the first map is w_0. By our definition of the $W(k_0)$-scheme $L^+ X$ above, which uses the Cartier homomorphism, the points $L^+ X(R) = X(W(R))$ are given by ring homomorphisms $A \to W(R)$ such that the composition $W(k_0) \to A \to W(R)$ is equal to

$$W(k_0) \overset{\Delta}{\to} W(W(k_0)) \overset{\epsilon}{\to} W(k_0) \overset{W(\phi)}{\longrightarrow} W(R),$$

where the second map ϵ is the result of applying the functor $W(-)$ to $w_0 : W(k_0) \to k_0$. However, $\epsilon \circ \Delta$ is the identity by [28, equation (92)] and so this composition is equal to $W(\phi) : W(k_0) \to W(R)$.

This gives a simpler description of the special fiber $L^+ X \otimes_{W(k_0)} k_0$; its R-valued points for $\phi : k_0 \to R$ are the $W(R)$-valued points of X where $W(R)$ is regarded as a $W(k_0)$-algebra via $W(\phi)$. In particular, we see that

$$FX = L^+ X \otimes_{W(k_0)} k_0.$$

As mentioned above, some authors call the special fiber $FX = L^+ X \otimes_{W(k_0)} k_0$ the Greenberg transform of X.

2.3. Displays

Let us now quickly review the definition of displays as in [28]. Suppose that R is a (commutative) ring which is p-adically complete and separated. A display over R is a quadruple

$$\mathcal{D} = (M, N, F_0, F_1)$$

where M is a finitely generated projective $W(R)$-module, N a submodule such that $I(R)M \subset N$ and M/N a projective R-module, $F_0 : M \to M$ and $F_1 : N \to M$ are F-linear maps such that the image $F_1(Q)$ generates M as a $W(R)$-module, and we have $F_1(Vw \cdot x) = wF_0(x)$ for $w \in W(R)$ and $x \in M$. See [28] and other places for the definition of nilpotence and of a nilpotent display. To avoid confusion, let us note that in [28] displays are called $3n$-displays ($3n =$ not necessarily nilpotent), while nilpotent displays are called displays. The notation for these objects there is (P, Q, F, V^{-1}) in which V^{-1} is just a symbol.

Over a perfect field, a display is the same as a Dieudonné module (M, F, V); then $N = V(M)$ and F_1 is the inverse of V. In that case, the nilpotence condition means that V is p-adically topologically nilpotent.

Displays over R form a category and Zink constructs a functor BT from nilpotent displays over R to formal p-divisible groups over R which he shows to be an equivalence of categories in many cases, for example when R/pR is a finitely generated algebra over $\mathbb{F}_p$. Later, Lau [19] showed that BT gives an equivalence for all R which are p-adically complete and separated.

Let $\mathcal{D} = (M, N, F_0, F_1)$ be a display over R and suppose there exists a 'normal decomposition' of $\mathcal{D}$, i.e. a decomposition $M = T \oplus L$ with T, L two $W(R)$-modules such that also $N = I(R)T \oplus L$. Suppose that both T and L are free $W(R)$-modules of rank d and $h - d$ (we can always find such a decomposition Zariski locally on $\mathrm{Spec}(R)$, see [28]). Assume $e_1, \ldots, e_d$ is a basis of T and $e_{d+1}, \ldots, e_h$ a basis of L. Then there is an invertible matrix $U = (u_{ij})$ in $\mathrm{GL}_h(W(R)) = L^+\,\mathrm{GL}_h(R)$ such that

$$F_0 e_j = \sum_{i=1}^{h} u_{ij} e_i, \quad 1 \leqslant j \leqslant d,$$

$$F_1 e_j = \sum_{i=1}^{h} u_{ij} e_i, \quad d+1 \leqslant j \leqslant h.$$

We can write this as a block matrix

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A of size $d \times d$ and D of size $(h-d) \times (h-d)$. Suppose that $\mathcal{D}'$ is another display over R which is also given by a block matrix U' with blocks of the same sizes. Then a morphism of displays $\mathcal{D} \to \mathcal{D}'$ is given by a block matrix of the form

$$H = \begin{pmatrix} X & V(Y) \\ Z & T \end{pmatrix}$$

with X, Y, Z, T blocks with coefficients in $W(R)$ which satisfies

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} F(X) & Y \\ pF(Z) & F(T) \end{pmatrix} = \begin{pmatrix} X & V(Y) \\ Z & T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.3.0.1}$$

Set

$$\Phi(H) = \Phi\left(\begin{pmatrix} X & V(Y) \\ Z & T \end{pmatrix} \right) = \begin{pmatrix} F(X) & Y \\ pF(Z) & F(T) \end{pmatrix}. \tag{2.3.0.2}$$

The morphism $\mathcal{D} \to \mathcal{D}'$ is an isomorphism if and only if H is invertible. Then the identity above can be written

$$H^{-1} U' \Phi(H) = U. \tag{2.3.0.3}$$

By [28, Theorem 37] displays form a fpqc stack over $\mathrm{Nilp}_{\mathbb{Z}_p}$. The above discussion then implies that displays of rank h and $\mathrm{rank}_R(M/N) = d$ are given by the fpqc quotient stack

$$[L^+\,\mathrm{GL}_h /_\Phi H^{(d, h-d)}]$$

over $\mathrm{Nilp}_{\mathbb{Z}_p}$. Here, $H^{(d, h-d)}(R)$ is the subgroup of $L^+\,\mathrm{GL}_h(R) = \mathrm{GL}_h(W(R))$ of matrices of the form

$$H = \begin{pmatrix} X & V(Y) \\ Z & T \end{pmatrix}$$

as above and the quotient is for the right action by 'Φ-conjugation' as in (2.3.0.3). Of course, here Φ is the F-linear map given by (2.3.0.2).

3. (G, μ)-displays

In this section, we define (G, μ)-displays and show several basic properties.

3.1. The divided Frobenius $\Phi_{G,\mu}$

We start by defining the 'divided Frobenius'. This generalizes the map Φ of the previous section and plays a central role in everything that follows.

3.1.1. Suppose G is a reductive group scheme over $\mathbb{Z}_p$ and that

$$\mu : \mathbb{G}_{m\,W(k_0)} \to G_{W(k_0)}$$

is a minuscule cocharacter. As in Appendix A we denote by $P_\mu \subset G_{W(k_0)}$ the parabolic subgroup scheme defined by μ. This is the parabolic subgroup of G such that $P_\mu \times_{W(k_0)} W$ contains exactly the root groups U_a of the split group G_W, for all roots a with $\langle \mu, a \rangle \geqslant 0$. We denote by U_μ the corresponding unipotent group which is the unipotent radical of P_μ.

We denote by $\mathfrak{g}, \mathfrak{p}, \mathfrak{p}^-, \mathfrak{u}, \mathfrak{u}^-$, the Lie algebras of $G, P_\mu, P_{\mu^{-1}}, U_\mu, U_{\mu^{-1}}$; these are finite free $\mathbb{Z}_p$-, respectively $W(k_0)$-modules, and we have the weight decompositions

$$W(k_0) \otimes_{\mathbb{Z}_p} \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-. \tag{3.1.1.1}$$

We denote by H^μ the group scheme over $\mathrm{Spec}(W(k_0))$ with

$$H^\mu(R) = \{g \in L^+G(R) \mid g_0 \in P_\mu(R)\}.$$

We can see that H^μ is a closed subgroup scheme of L^+G.

Proposition 3.1.2. *There is a group scheme homomorphism*

$$\Phi_{G,\mu} : H^\mu \to L^+G_{W(k_0)}$$

characterized by the following property: We have

$$\Phi_{G,\mu}(h) = F \cdot (\mu(p) \cdot h \cdot \mu(p)^{-1}) \in {}^F G(W(R)[1/p]) = G(W(R)[1/p]). \tag{3.1.2.1}$$

Proof. Consider the group scheme $L^{>0}U_{\mu^{-1}} \subset L^+U_{\mu^{-1}}$ with R-valued points $u \in U_{\mu^{-1}}(W(R))$ such that $u_0 = 1$.

We use the following:

Proposition 3.1.3. *Assume that p is in the Jacobson radical $J(R)$ of R. Then multiplication in $L^+G(R)$ gives a bijection*

$$L^+P_\mu(R) \times L^{>0}U_{\mu^{-1}}(R) \cong H^\mu(R).$$

Proof. Let $h \in H^\mu(R) \subset L^+G(R) = G(W(R))$. We would like to show that the corresponding morphism $h : \mathrm{Spec}(W(R)) \to G$ factors through $G^* = P_\mu \times_{W(k_0)} U_{\mu^{-1}}$. Recall that by A.0.2, G^* is an open subscheme of the affine scheme $G = \mathrm{Spec}(A_0)$ with the open immersion $G^* \to G$ given by multiplication. Suppose $\sqrt{\mathcal{I}} = \mathcal{I} \subset A_0$ is the ideal corresponding to the reduced induced subscheme structure on the complement $G - G^*$. The element h is given by $h^* : A_0 \to W(R)$. Consider the composition

$$h_0^* = w_0 \circ h^* : A_0 \to W(R) \to R.$$

We know that $h_0^*(\mathcal{I})R = R$. By part (i) of Lemma 2.1.1 it follows that $h^*(\mathcal{I})W(R) = W(R)$, and so h factors through G^*. The result now follows from the definition of H^μ. $\square$

We now continue with the construction of $\Phi_{G,\mu}$. By Lemma A.0.5, we have $U_{\mu^{-1}} \simeq \mathbb{G}_a^r \times_{\mathbb{Z}_p} W(k_0)$. We have $L^+\mathbb{G}_a^r(R) = W(R)^r$, $L^{>0}\mathbb{G}_a^r(R) = I(R)^r$ and we can define

$$V^{-1} : L^{>0}\mathbb{G}_a^r \to {}^F L^+\mathbb{G}_a^r$$

as given by the direct sum of r copies of $V^{-1} : I(R) \to W(R)$. This gives

$$V^{-1} : L^{>0}U_{\mu^{-1}} \to {}^F L^+ U_{\mu^{-1}}$$

which is independent of choices. Consider also the composition

$$F \cdot \mathrm{Int}_\mu(p) : L^+ P_\mu(R) \to L^+ P_\mu(R) \to {}^F L^+ G(R) = L^+ G(R).$$

Here we are using the extension of conjugation $\mathrm{Int}_\mu(p) : \mathbb{A}^1 \to \mathrm{End}(P_\mu)$ (see A.0.1) applied to $W(R)$ so that $\mathrm{Int}_\mu(p) \in \mathrm{End}(P_\mu)(W(R))$.

First suppose that $p \in J(R)$ and let $h \in H^\mu(R)$; by Proposition 3.1.3, we can write (uniquely)

$$h = h' \cdot h'', \quad h' \in L^+ P_\mu(R), \quad h'' \in L^{>0}U_{\mu^{-1}}(R),$$

and we set

$$\Phi_1(R)(h) := (F \cdot \mathrm{Int}_\mu(p))(h') \cdot V^{-1}(h'') \in {}^F L^+ G(R) = L^+ G(R).$$

This gives

$$\Phi_1(R) : H^\mu(R) \to L^+ G(R)$$

when $p \in J(R)$.

Now suppose R is arbitrary; consider $R_1 = (1 + pR)^{-1}R$ and $R_2 = R[1/p]$. This gives a faithfully flat cover of $\mathrm{Spec}(R)$

$$\mathrm{Spec}(R) = \mathrm{Spec}(R_1) \cup \mathrm{Spec}(R_2).$$

Notice that p is in every maximal ideal of R_1; indeed if $\mathfrak{m}$ is such an ideal and p is not in $\mathfrak{m}$, we can find $r = b/(1 + pb') \in R_1$, such that $pr - 1 \in \mathfrak{m}$. This gives $1 + p(b' - b) \in \mathfrak{m}$, a contradiction, since this is a unit in R_1. Therefore, p is in the Jacobson radical of R_1.

• Let $h_2 \in H^\mu(R_2) = H^\mu(R[1/p])$. Since

$$p \in W(R[1/p]))^\times = \mathbb{G}_m(W(R[1/p])),$$

we can consider $\mu(p), \mu(p)^{-1} \in L^+ G(R[1/p])$. We define

$$\Phi_2(R_2)(h_2) := F(\mu(p) \cdot h_2 \cdot \mu(p)^{-1}) \in L^+ G(R[1/p]).$$

• Let $h_1 \in H^\mu(R_1) = H^\mu((1 + pR)^{-1}R)$. Since $p \in J(R_1)$, we consider $\Phi_1(R_1)(h_1) \in L^+ G(R_1)$ as above.

Now let us apply this to $R = \mathcal{O}_{H^\mu}$ the affine algebra of H^μ and the universal points $h_i \in H^\mu(R_i)$. We obtain elements $\Phi_i(R_i)(h_i) \in L^+ G(R_i)$. These points agree over $\mathrm{Spec}(R_{12})$ with $R_{12} = (1 + pR)^{-1}R \otimes_R R[1/p] = (1 + pR)^{-1}R[1/p]$. Indeed, when p is invertible,

the map $V^{-1} : L^{>0}U_{\mu^{-1}} \to {}^F L^+ U_{\mu^{-1}}$ agrees with division by p followed by F. On the other hand, the adjoint action of $\mu(p)^{-1}$ on $\mathfrak{u}^- = \mathrm{Lie}(U_{\mu^{-1}})$ agrees with multiplication by p. Therefore, by descent, we obtain a well-defined morphism

$$\Phi_{G,\mu} : H^\mu \to L^+ G_{W(k_0)}. \tag{3.1.3.1}$$

Notice that $R = \mathcal{O}_{H^\mu}$ is p-torsion free and so $R \subset R_2 = R[1/p]$ and $W(R) \subset W(R[1/p])$. Since $\Phi_2(R_2)$ is obviously a group homomorphism we conclude that $\Phi_{G,\mu}$ is a group scheme homomorphism which satisfies the identity (3.1.2.1). In fact, by considering $R = \mathcal{O}_{H^\mu}$ we see that $\Phi_{G,\mu}$ is the unique morphism that satisfies that identity. $\square$

3.2. Definitions and basic properties

Suppose that (G, μ) is a pair of a reductive group scheme over $\mathbb{Z}_p$ and a minuscule cocharacter $\mu : \mathbb{G}_{W(k_0)} \to G_{W(k_0)}$, where k_0 is a finite field of characteristic p. In the previous paragraph, we have constructed H^μ and

$$\Phi_{G,\mu} : H^\mu \to {}^F L^+ G_{W(k_0)} = L^+ G_{W(k_0)}$$

over $\mathrm{Spec}(W(k_0))$. Suppose S is a $W(k_0)$-scheme.

Definition 3.2.1. A (G, μ)-display over S is a triple $\mathcal{D} := (P, Q, u)$ where

- Q is an (fpqc locally trivial) H^μ-torsor over S,
- $P := Q \times_{H^\mu} L^+ G_{W(k_0)}$ is the induced $L^+ G_{W(k_0)}$-torsor,
- $u : Q \to P$ is a morphism which is compatible with $\Phi_{G,\mu}$ in the sense that $u(q \cdot h) = u(q) \cdot \Phi_{G,\mu}(h)$.

Notice that P is determined by Q by $P := Q \times_{H^\mu} L^+ G_{W(k_0)}$ and so we sometimes omit it from the notation. Our convention is that groups act on the right.

A morphism $(P_1, Q_1, u_1) \to (P_2, Q_2, u_2)$ between two (G, μ)-displays is a H^μ-torsor isomorphism $Q_1 \xrightarrow{\sim} Q_2$ which is compatible with u_1 and u_2 in the obvious manner.

In most of the paper, we consider (G, μ)-displays over schemes in $\mathrm{Nilp}_{W(k_0)}$. We can see that (G, μ)-displays form a fpqc stack in groupoids over $\mathrm{Nilp}_{W(k_0)}$ which we denote by $\mathcal{B}(G, \mu)$.

Example 3.2.2. For non-negative integers $d \leqslant h$, we let $\mu_{d,h} : \mathbb{G}_m \to \mathrm{GL}_h$ be the minuscule cocharacter of GL_h over $\mathbb{Z}_p$ given by[1]

$$\mu_{d,h}(z) = \mathrm{diag}(1^{(d)}, z^{(h-d)}).$$

The discussion in 2.3 implies that there is an equivalence between the stack of $(\mathrm{GL}_h, \mu_{d,h})$-displays, and the stack of (Zink, not-necessarily-nilpotent) displays of rank h and dimension d.

---

[1] The notation $z^{(r)}$ means that there are r copies of z.

3.2.3. The fpqc quotient $L^+ G_{W(k_0)}/H^\mu$ can be identified with the homogeneous space $X_\mu = G_{W(k_0)}/P_\mu$. If $\mathcal{D} := (P, Q, u)$ is a (G, μ)-display over S then the quotient P/H^μ is a X_μ-bundle. Since the $L^+ G_{W(k_0)}$-torsor P admits a reduction (given by Q) to the subgroup H^μ we also obtain a section

$$\alpha : S = Q/H^\mu \to P/H^\mu$$

of this X_μ-bundle over S. We call this section the Hodge filtration of the (G, μ)-display $\mathcal{D}$. We also consider the vector bundle

$$T_\mathcal{D} := \alpha^*(\mathcal{N}_{\alpha(S)|P/H^\mu})$$

over S obtained by pulling back via α the normal bundle of the (regular) closed immersion $\alpha(S) \hookrightarrow P/H^\mu$. Under a condition on $\mathcal{D}$, we see that the bundle $T_\mathcal{D}$ controls the deformations of $\mathcal{D}$ (see Theorem 3.5.11).

3.2.4. Suppose $S = \mathrm{Spec}(B)$ with B a p-adically complete and separated $W(k_0)$-algebra. Then using Proposition B.0.2 (see also Remark B.0.5(ii)) we can reinterpret the datum of the H^μ-torsor Q as a pair $(\mathcal{P}, \alpha)$, where $\mathcal{P}$ is a G-torsor over $\mathrm{Spec}(W(B))$ and α a section over $\mathrm{Spec}(B)$ of the X_μ-bundle obtained by first restricting $\mathcal{P}$ along the closed immersion $\mathrm{Spec}(B) \to \mathrm{Spec}(W(B))$ and then taking quotient by the action of P_μ. The G-torsor $\mathcal{P}$ corresponds, via Appendix B.0.2, to $P := Q \times_{H^\mu} L^+ G_{W(k_0)}$ so that $P = F\mathcal{P}$, the X_μ-bundle is P/H^μ and the section is $Q/H^\mu \to P/H^\mu$ as above.

Definition 3.2.5. We say that a (G, μ)-display $\mathcal{D} = (P, Q, u)$ is banal, if the torsor Q is trivial. Banal (G, μ)-displays over S in $\mathrm{Nilp}_{W(k_0)}$ give a full subgroupoid $B(G, \mu)(S)$ of $\mathcal{B}(G, \mu)(S)$.

Remark 3.2.6. By Corollary B.0.3, any (fpqc locally trivial) H^μ-torsor or $L^+ G$-torsor over S in $\mathrm{Nilp}_{W(k_0)}$ is locally trivial for the étale topology on S. Therefore, every (G, μ)-display is banal locally for the étale topology on S.

3.2.7. Suppose that (P, Q, u) is banal. Then, after choosing a trivialization

$$\alpha : H^\mu \xrightarrow{\sim} Q$$

which also induces $\alpha : L^+ G_{W(k_0)} \xrightarrow{\sim} P := Q \times_{H^\mu} L^+ G_{W(k_0)}$, the triple (P, Q, u) is determined by

$$U := \alpha^{-1} u(\alpha(1)) \in {}^F L^+ G(\mathcal{O}_S(S)) = G(W(\mathcal{O}_S(S))).$$

A different trivialization $\alpha' = \alpha \cdot h$ gives $U' = \alpha'^{-1} u(\alpha'(1))$. We can see that

$$U' = h^{-1} \cdot U \cdot \Phi_{G,\mu}(h).$$

Therefore, the groupoid of banal (G, μ)-displays $B(G, \mu)(S)$ can be identified with the quotient groupoid

$$[L^+ G(S)/_{\Phi_{G,\mu}} H^\mu(S)]$$

where the action is by $\Phi_{G,\mu}$-conjugation as above. This implies that the fpqc stack of (G, μ)-displays can be identified with a fpqc quotient stack

$$\mathcal{B}(G, \mu) = [L^+ G_{W(k_0)}/_{\Phi_{G,\mu}} H^\mu].$$

3.2.8. By the definition, there is a natural forgetful morphism

$$q : \mathcal{B}(G, \mu) \to BH^\mu, \quad (P, Q, u) \mapsto Q,$$

where BH^μ is the fpqc stack of H^μ-torsors.

Lemma 3.2.9.

(a) *The morphism $q : \mathcal{B}(G, \mu) \to BH^\mu$ is representable and affine.*

(b) *The diagonal morphism $\Delta : \mathcal{B}(G, \mu) \to \mathcal{B}(G, \mu) \times_{W(k_0)} \mathcal{B}(G, \mu)$ is representable and affine.*

Proof. If $Q_i \to S$, $i = 1, 2$, are H^μ-torsors, we can see by descent, that the functor on S-schemes

$$(T \to S) \mapsto \mathrm{Isom}_{H^\mu}(Q_1 \times_S T, Q_2 \times_S T)$$

is represented by an affine S-scheme $\underline{\mathrm{Isom}}_{H^\mu}(Q_1, Q_2)$. This implies that the diagonal of the stack BH^μ is representable and affine. There is a similar statement for isomorphisms between L^+G-torsors.

Now suppose that $S \to BH^\mu$ is the morphism corresponding to an H^μ-torsor Q. Recall that we have $P = Q \times_{H^\mu} L^+G$. Then the fibered product $S \times_{BH^\mu} \mathcal{B}(G, \mu)$ is represented by the affine S-scheme $\underline{\mathrm{Isom}}_{L^+G}(P', P)$, where $P' = Q \times_{H^\mu, \Phi_{G,\mu}} L^+G$. Part (a) now follows easily from the definition of $\mathcal{B}(G, \mu)$; then part (b) follows quickly from part (a) and the above. $\square$

3.2.10. Suppose that A is a Noetherian $W(k_0)$-algebra complete and separated for the I-adic topology for $I \subset A$ an ideal that contains a power of p. Then A is also complete and separated for the p-adic topology.

Proposition 3.2.11. *There is a natural equivalence between the category of (G, μ)-displays $\mathcal{D}$ over A and the category of compatible systems of (G, μ)-displays $\mathcal{D}_n$ over A/I^n, $n \geqslant 1$, given by $\mathcal{D} \mapsto \{\mathcal{D} \times_A A/I^n\}_n$.*

Proof. The full faithfulness of the functor $\mathcal{D} \mapsto \{\mathcal{D} \times_A A/I^n\}_n$ follows easily by using Lemma 3.2.9(b). Let us show essential surjectivity: Consider a compatible sequence of (G, μ)-displays $\mathcal{D}_n = (P_n, Q_n, u_n)$ over A/I^n; we would like to construct $\mathcal{D} = (P, Q, u)$ over A. Using Lemma B.0.4(b) and Remark B.0.5(i) we can construct a L^+G-torsor P over A with compatible isomorphisms $P \times_A A_n \simeq P_n$. To give the H^μ-torsor Q over A we use Remark 3.2.4 and apply Grothendieck's algebraization theorem to the proper morphism $P/H^\mu \to \mathrm{Spec}(A)$. Finally, the homomorphism $u : Q \to P$ is given from $\{u_n\}_n$ using that $\underline{\mathrm{Isom}}_{L^+G}(Q \times_{H^\mu, \Phi_{G,\mu}} L^+G, P)$ is affine (see also the proof of 3.2.9(b) above). $\square$

3.2.12. Suppose that $\mu' : \mathbb{G}_{m\,W(k_0)} \to G_{W(k_0)}$ is conjugate to μ, i.e. $\mu' = \mathrm{Int}(g) \circ \mu$, for $g \in G(W(k_0))$. Then $\mathrm{Int}(g) : H^\mu \xrightarrow{\sim} H^{\mu'}$ and

$$\Phi_{G,\mu'} = \mathrm{Int}(^F g) \cdot \Phi_{G,\mu} \cdot \mathrm{Int}(g)^{-1}.$$

Using this, we see that conjugation by g gives an isomorphism $\mathcal{B}(G, \mu) \xrightarrow{\sim} \mathcal{B}(G, \mu')$, cf. [3, §3.3.2].

3.2.13. Suppose that (G_i, μ_i), $i = 1$, 2, are two pairs as above and that we are given a group scheme homomorphism $\rho : G_1 \to G_2$ such that $\mu_2 = \rho(\mu_1)$. We then write $\rho : (G_1, \mu_1) \to (G_2, \mu_2)$. Then ρ induces group scheme homomorphisms $\rho : H_1^{\mu_1} \to H_2^{\mu_2}$ and $\rho : L^+ G_1 \to L^+ G_2$ and we have $\rho \cdot \Phi_{G_1,\mu_1} = \Phi_{G_2,\mu_2} \cdot \rho$. Using this we obtain a morphism

$$\rho_* : \mathcal{B}(G_1, \mu_1) \to \mathcal{B}(G_2, \mu_2).$$

(cf. [3, § 3.3.1]).

3.2.14. Let us discuss (G, μ)-displays over the algebraically closed field k. Since $I(k) = V W(k) = p W(k)$, we have

$$H^\mu(k) = \mu^{-1}(p) G(W) \mu(p) \cap G(W)$$

and

$$\Phi_{G,\mu}(h) = F(\mu(p) h \mu^{-1}(p)) = \mu^\sigma(p) \sigma(h) \mu^\sigma(p)^{-1}.$$

Proposition 3.2.15. *The set of isomorphism classes of (G, μ)-displays over k is in 1-to-1 correspondence with the quotient*

$$G(W) \mu^\sigma(p) G(W) /_\sigma G(W)$$

by σ-conjugation.

Proof. Since both torsors Q and P are trivial, a (G, μ)-display $\mathcal{D} = (P, Q, u)$ over k is given by $u \in G(W)$. The action by $\Phi_{G,\mu}$-conjugation is given by

$$u \mapsto h^{-1} \cdot u \cdot \Phi_{G,\mu}(h) = h^{-1} u \mu^\sigma(p) \sigma(h) \mu^\sigma(p)^{-1}.$$

If we set $g = u \mu^\sigma(p)$, we see that this action replaces g by $h^{-1} g \sigma(h)$. Therefore, the isomorphism classes of (G, μ)-displays over k are in 1-to-1 correspondence with the quotient

$$G(W) \mu^\sigma(p) /_\sigma (\mu(p)^{-1} G(W) \mu(p) \cap G(W))$$

by σ-conjugation. Consider the natural map

$$G(W) \mu^\sigma(p) /_\sigma (\mu(p)^{-1} G(W) \mu(p) \cap G(W)) \to G(W) \mu^\sigma(p) G(W) /_\sigma G(W).$$

If $g \mu^\sigma(p) = h^{-1} g' \mu^\sigma(p) \sigma(h)$ for $h \in G(W)$, then $g'^{-1} h g = \mu^\sigma(p) \sigma(h) \mu^\sigma(p)^{-1}$. This implies $\sigma(h) \in \mu^\sigma(p)^{-1} G(W) \mu^\sigma(p)$, so $h \in \mu(p)^{-1} G(W) \mu(p) \cap G(W)$. This shows that the above map is injective. The map is also surjective: consider $u_1 \mu^\sigma(p) u_2$ and write $u_2 = \sigma(h)^{-1}$, for $h \in G(W)$. Then we have

$$h^{-1} u_1 \mu^\sigma(p) u_2 \sigma(h) \in G(W) \mu^\sigma(p)$$

which shows $u_1 \mu^\sigma(p) u_2$ is in the image. $\qquad\square$

3.2.16. Notice that the above proof also shows that, if $\mathcal{D}$ is a (G, μ)-display over k given by $u \in G(W)$, then the σ-conjugacy class of $b := u \mu^\sigma(p) \in G(W[1/p])$ only depends on the isomorphism class of $\mathcal{D}$. The pair (μ, b) gives a filtered G-isocrystal over $W[1/p]$ as considered in [22, Chapter 1].

We denote by $\mathbf{D}$ the algebraic torus over $\mathbb{Q}_p$ whose character group is $\mathbb{Q}$. Kottwitz [17] associates to the Frobenius conjugacy class b a morphism of algebraic groups defined over $K := W[1/p]$

$$\nu_b : \mathbf{D}_K \to G_K$$

called the slope morphism.

If $\rho : G \to \mathrm{GL}(V)$ is a $\mathbb{Q}_p$-rational representation of G, the morphism $\rho \cdot \nu_b$ defines a $\mathbb{Q}$-grading on the vector space $V \otimes_{\mathbb{Q}_p} K$. The morphism ν_b is characterized by the property that this grading is the slope decomposition of the isocrystal over K associated to b and V. The rational number λ is called a slope of $\rho \cdot \nu_b$ if the corresponding isotypic component of $V \otimes_{\mathbb{Q}_p} K$ is not equal to zero. In particular, we can consider the adjoint representation $\rho = \mathrm{Ad}^G : G \to \mathrm{GL}(\mathrm{Lie}(G))$ and the slopes of $\mathrm{Ad}^G(b)$.

3.3. Quasi-isogenies

3.3.1. Suppose that S is a scheme over $\mathrm{Spec}(\mathbb{F}_p)$ and denote by $F : S \to S$ the absolute Frobenius. Recall that if $Y \to S$ is an S-scheme (or Ind-scheme) we set

$$^F Y = Y \times_{S,F} S.$$

Since G is defined over $\mathbb{Z}_p$ we have a natural isomorphism $^F LG \xrightarrow{\sim} LG$.

Definition 3.3.2. Suppose that S is a scheme over $\mathrm{Spec}(\mathbb{F}_p)$.

A G-isodisplay over S is a pair (T, ϕ) of a LG-torsor T over S and an isomorphism $\phi : {}^F T \xrightarrow{\sim} T$.

An (iso)morphism $\alpha : (T_1, \phi_1) \xrightarrow{\sim} (T_2, \phi_2)$ of two G-isodisplays over S is an isomorphism of LG-torsors $\alpha : T_1 \xrightarrow{\sim} T_2$ which is compatible with ϕ_1, ϕ_2, in the sense that: $\alpha \cdot \phi_1 = \phi_2 \cdot {}^F \alpha$.

3.3.3. Let us now show how to associate a G-isodisplay

$$\mathcal{D}[1/p] = (P[1/p], \phi)$$

to a (G, μ)-display $\mathcal{D} = (P, Q, u)$ over $S \in \mathrm{Nilp}_{W(k_0)}$.

We first assume $p \cdot \mathcal{O}_S = 0$. If P is a L^+G-torsor over S in $\mathrm{Nilp}_{W(k_0)}$, then we set

$$P[1/p] := P \times_{L^+G} LG$$

for the LG-torsor over S obtained from P using the natural $L^+G \to LG$. Then if (P, Q, u) is a (G, μ)-display over S, we construct an isomorphism

$$\phi : {}^F P[1/p] \xrightarrow{\sim} P[1/p]$$

of LG-torsors over S. We can assume that S is affine $S = \mathrm{Spec}(R)$; the general case can then be handled by descent as below.

Suppose first that Q is the trivial H^μ-torsor and choose a trivialization of Q so that u is given by $U \in G(W(R))$. Both $P[1/p]$ and $^F P[1/p]$ are then identified with the trivial

LG-torsor. Define $\phi : {}^F P[1/p] \xrightarrow{\sim} P[1/p]$ to be the morphism given by left multiplication by $U\mu^\sigma(p) \in G(W(R)[1/p]) = LG(R)$:

$$\phi(x) = U\mu^\sigma(p) \cdot x.$$

After changing the trivialization of Q by right multiplication by $h \in H^\mu(R)$, U changes to $U' = h^{-1}U\Phi_{G,\mu}(h)$, and the morphism ϕ now is given by multiplication by

$$
\begin{aligned}
U'\mu^\sigma(p) &= h^{-1}U\Phi_{G,\mu}(h)\mu^\sigma(p) \\
&= h^{-1}UF(\mu(p)h\mu(p)^{-1})\mu^\sigma(p) = h^{-1}U\mu^\sigma(p)F(h).
\end{aligned}
$$

This shows that $\phi : {}^F P[1/p] \xrightarrow{\sim} P[1/p]$ is independent of our choice of trivialization of the H-torsor Q. We can now easily see using descent that this construction also produces $\phi : {}^F P[1/p] \xrightarrow{\sim} P[1/p]$ in general.

For S in $\mathrm{Nilp}_{W(k_0)}$, set $\bar{S} = S \otimes_{W(k_0)} k_0$ for its special fiber.

Definition 3.3.4. If $\mathcal{D}$ is a (G, μ)-display over S in $\mathrm{Nilp}_{W(k_0)}$ we set $\mathcal{D}[1/p] = (P[1/p], \phi)$ to be the G-isodisplay which is associated to its special fiber $\mathcal{D} \times_S \bar{S}$, by the above construction.

Notice that for S in $\mathrm{Nilp}_{W(k_0)}$, we have

$$(P \times_{L^+G} LG) \times_S \bar{S} = (P \times_S \bar{S})[1/p].$$

This is because, if $S = \mathrm{Spec}(R)$, we have $W(R)[1/p] = W(R/pR)[1/p]$, since $p^m R = 0$ implies $p^m W(pR) = 0$.

Definition 3.3.5. A G-quasi-isogeny

$$\alpha : \mathcal{D}_1 = (P_1, Q_1, u_1) \dashrightarrow \mathcal{D}_2 = (P_2, Q_2, u_2)$$

between two (G, μ)-displays over S in $\mathrm{Nilp}_{W(k_0)}$ is an isomorphism

$$\alpha : (P_1[1/p], \phi_1) \xrightarrow{\sim} (P_2[1/p], \phi_2)$$

between their corresponding G-isodisplays, i.e. an isomorphism of LG-torsors $\alpha : P_1[1/p] \xrightarrow{\sim} P_2[1/p]$ which is compatible with ϕ_1, ϕ_2, in the sense that $\alpha \cdot \phi_1 = \phi_2 \cdot {}^F\alpha$.

Remark 3.3.6. Recall (Example 3.2.2) that there is an equivalence between the stack of $(\mathrm{GL}_h, \mu_{d,h})$-displays, and the stack of Zink (not-necessarily-nilpotent) displays of rank h and dimension d, over $\mathrm{Nilp}_{W(k_0)}$. It follows from [28, Proposition 66] that, under this equivalence, GL_n-quasi-isogenies as defined here, bijectively correspond to quasi-isogenies between displays as considered in loc. cit.

3.4. The adjoint nilpotent condition

3.4.1. Suppose that $\mathcal{D}$ is a (G, μ)-display over a $W(k_0)$-scheme S. Let x be a point of $\bar{S} = S \otimes_{W(k_0)} k_0$ and let $k(x)^{\mathrm{ac}}$ be an algebraic closure of the residue field $k(x)$. Set $L(x) =$

$W(k(x)^{\mathrm{ac}})[1/p]$. By 3.2.14 applied to $k = k(x)^{\mathrm{ac}}$, we see that $\mathcal{D} \times_S k(x)^{\mathrm{ac}}$ is described by an element $u := u(x) \in G(W(k(x)^{\mathrm{ac}}))$ and that the Frobenius conjugacy class of $b(x) = u(x)\mu^\sigma(p)$ in $G(L(x))$ is well-defined.

Definition 3.4.2. We say that the (G, μ)-display $\mathcal{D}$ over S is adjoint nilpotent if, for all $x \in \bar{S}$, all the slopes of $\mathrm{Ad}^G(b(x)) \in \mathrm{GL}(\mathrm{Lie}(G))(L(x))$ are > -1. We denote by $\mathcal{B}^{\mathrm{nil}}(G, \mu)$ the full substack of $\mathcal{B}(G, \mu)$ given by adjoint nilpotent (G, μ)-displays.

3.4.3. We continue with the above assumptions and notations. For every $x \in \bar{S}$, consider the isocrystal given by the Frobenius semi-linear $\phi(x) := \mathrm{Ad}^G(b(x)) \circ F_{k(x)^{\mathrm{ac}}}$ acting on $\mathrm{GL}(\mathrm{Lie}(G))(L(x))$. The Hodge weights of this isocrystal with respect to the lattice $\mathcal{L}_x = \mathfrak{g}_{W(k(x)^{\mathrm{ac}})} = \mathrm{Lie}(G)_{W(k(x)^{\mathrm{ac}})} \subset \mathrm{Lie}(G)_{L(x)}$ are just the weights of μ^{-1} on $\mathfrak{g}_{W(k(x)^{\mathrm{ac}})}$. Since μ is minuscule these weights lie in the set $\{-1, 0, 1\}$. Hence, the Newton slopes of the isocrystal are all greater than or equal to -1, by the Hodge–Newton inequality. We then obtain that $\mathcal{D}$ is adjoint nilpotent if and only if for all $x \in \bar{S}$, we have $\phi(x)^r \mathcal{L}_x \subset p^{1-r} \mathcal{L}_x$, with $r = \dim_{\mathbb{Z}_p}(\mathfrak{g})$. Using this, we see that we also have (cf. [3, Definition 3.23]):

Lemma 3.4.4. *Suppose π is the projector on $W(k_0) \otimes_{\mathbb{Z}_p} \mathrm{Lie}(G)$ with $\mathrm{im}(\pi) = \mathrm{Lie}\, U_{\mu^{-1}}$ and $\ker(\pi) = \mathrm{Lie}\, P_\mu$. Let $\mathcal{D}$ be a banal (G, μ)-display over a k_0-algebra R and let $U \in G(W(R))$ be a representative for it. Then $\mathcal{D}$ is adjoint nilpotent if and only if the endomorphism on $R \otimes_{\mathbb{Z}_p} \mathrm{Lie}(G)$ defined by*

$$\mathrm{Ad}^G(w_0(U)) \circ (F_R \otimes \mathrm{id}_{\mathrm{Lie}\, G}) \circ (\mathrm{id}_R \otimes \pi)$$

is nilpotent.

Proof. When $R = k$ is an algebraically closed field the condition that the endomorphism in the statement is nilpotent is equivalent to

$$(\phi \cdot p)^r (\mathrm{Lie}(G)_{W(k)}) \subset p(\mathrm{Lie}(G)_{W(k)}),$$

where $\phi := \mathrm{Ad}^G(U\mu^\sigma(p)) \circ F_k$; given the above discussion this gives the result in this case. The case of general R follows from this. $\qquad\square$

Remark 3.4.5. Suppose that $(G, \mu) = (\mathrm{GL}_h, \mu_{d,h})$. Recall that, by Example 3.2.2, there is an equivalence between $\mathcal{B}(\mathrm{GL}_h, \mu_{d,h})$, i.e. the stack of $(\mathrm{GL}_h, \mu_{d,h})$-displays, and the stack of (Zink, not-necessarily-nilpotent) displays of rank h and dimension d.

Suppose that R is in $\mathrm{ANilp}_{W(k_0)}$ and that $\mathcal{D}$ is a $(\mathrm{GL}_h, \mu_{d,h})$-display over R. The adjoint nilpotence condition on $\mathcal{D}$ (as defined above) is related to Zink's nilpotence condition [28, Definition 11/Definition 13] on the corresponding Zink display as follows:

First assume that $R = k$ is an algebraically closed field and that the display $\mathcal{D}$ is given by $b = u\mu_{d,h}(p) \in \mathrm{GL}_h(W(k)[1/p])$: Then we observe that the isocrystal on $\mathrm{Lie}(\mathrm{GL}_h)_{W(k)[1/p]}$ given by $\mathrm{Ad}^G(b)$ has a non-trivial isotypic component of slope -1 if and only if the isocrystal given by b has non-trivial isotypical components for both slopes 0 and 1. (This is obtained by a simple consideration of roots and weights for GL_h.) For a general R as above, this now implies that $\mathcal{D}$ is adjoint nilpotent if and only if there

exist radical R-ideals I_{nil} and I_{uni} with $I_{\mathrm{nil}} \cap I_{\mathrm{uni}} = \sqrt{pR}$, such that both the Zink display $\mathcal{D}_{R/I_{\mathrm{nil}}}$ and the dual $(\mathcal{D}_{R/I_{\mathrm{uni}}})^t$ satisfy Zink's nilpotence condition. In particular, if either $\mathcal{D}_{R/pR}$ or $(\mathcal{D}_{R/pR})^t$ satisfy Zink's nilpotence condition then $\mathcal{D}$ is adjoint nilpotent.

3.5. Liftings and deformation theory

In this subsection, we describe the deformation theory of (G, μ)-displays which satisfy the adjoint nilpotence condition. Many of our results are group theoretic versions of corresponding results of Zink [28] about deformations of nilpotent displays.

3.5.1. Let A be a p-adically separated and complete $W(k_0)$-algebra and fix a p-adically closed ideal $\mathfrak{a} \subset A$ with divided powers which are compatible with the natural divided powers on $pW(k_0)$. In addition, let us assume that $\mathfrak{a}$ is p-adically topologically nilpotent. Then A is also complete for the $\mathfrak{a}$-adic topology. (The most useful case is when A is in $\mathrm{ANilp}_{W(k_0)}$ and $\mathfrak{a} \subset A$ is a nilpotent pd-ideal.)

We set $I_{\mathfrak{a}}(A) := W(\mathfrak{a}) + I(A)$, which is a pd-ideal of $W(A)$. In this situation, Zink's logarithmic ghost coordinates [28, §1.4] establish an isomorphism

$$\log : W(\mathfrak{a}) \xrightarrow{\cong} \prod_{i=0}^{\infty} \mathfrak{a}$$

leading to splittings $W(\mathfrak{a}) = \mathfrak{a} \oplus I(\mathfrak{a})$ and $I_{\mathfrak{a}}(A) = \mathfrak{a} \oplus I(A)$. We write the elements of $\prod_{i=0}^{\infty} \mathfrak{a}$ using brackets $[a_0, a_1, \ldots]$. We have $F([a_0, a_1, \ldots]) = [pa_1, pa_2, \ldots]$, so $F(\mathfrak{a}) = 0$. The splittings allow one to define the important F-linear extension $V_{\mathfrak{a}}^{-1} : I_{\mathfrak{a}}(A) \to W(A)$ of $V^{-1} = V_{\mathfrak{a}}^{-1}|_{I(A)}$ by setting $0 = V_{\mathfrak{a}}^{-1}|_{\mathfrak{a}}$ (see [28, Lemma 38]).

3.5.2. Now let (G, μ) be as above. We set $H^{\mu}(A, \mathfrak{a})$ to be the inverse image of $H^{\mu}(A/\mathfrak{a})$ in $L^+G(A)$. Using an argument as in the proof of Proposition 3.1.3 we deduce that

$$H^{\mu}(A, \mathfrak{a}) = U_{\mu^{-1}}(W(\mathfrak{a})) H^{\mu}(A).$$

We obtain further decompositions

$$H^{\mu}(A, \mathfrak{a}) = U_{\mu^{-1}}(\mathfrak{a}) H^{\mu}(A) = G(\mathfrak{a}) H^{\mu}(A),$$

where $G(\mathfrak{a}) = \ker(G(A) \to G(A/\mathfrak{a}))$ and similarly for $U_{\mu^{-1}}(\mathfrak{a})$. We can now see that $\Phi_{G,\mu}(A) : H^{\mu}(A) \to L^+G(A)$ can be extended to a map

$$\Psi_{\mathfrak{a}} : H^{\mu}(A, \mathfrak{a}) \to L^+G(A)$$

vanishing on $G(\mathfrak{a})$. (Notice that $\Phi_{G,\mu}$ takes the identity value on $P_{\mu}(\mathfrak{a}) = H^{\mu}(A) \cap G(\mathfrak{a})$, cf. [3, Corollary 3.11].) We similarly get

$$\Psi_{\mathfrak{a}}(G(W(\mathfrak{a})) \subset G(W(\mathfrak{a})).$$

3.5.3. Recall the weight decomposition

$$W(k_0) \otimes_{\mathbb{Z}_p} \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-$$

induced by our minuscule cocharacter μ. We can also construct Lie-theoretic analogs of the group $H^{\mu}(A, \mathfrak{a})$ and the homomorphism $\Psi_{\mathfrak{a}}$. Namely, we set

$$\mathfrak{h}_{\mathfrak{a}}^{\mu}(A) = (I_{\mathfrak{a}}(A) \otimes_{W(k_0)} \mathfrak{u}^-) \oplus (W(A) \otimes_{W(k_0)} \mathfrak{p}), \tag{3.5.3.1}$$

and we define

$$\psi_{\mathfrak{a}} : \mathfrak{h}_{\mathfrak{a}}^{\mu}(A) \to W(A) \otimes_{\mathbb{Z}_p} \mathfrak{g} \tag{3.5.3.2}$$

to be $V_{\mathfrak{a}}^{-1}$ on $I_{\mathfrak{a}}(A) \otimes_{W(k_0)} \mathfrak{u}^-$, and $p^m F$ on all summands of μ-weight $m \geqslant 0$.

Theorem 3.5.4. *Under the above assumptions on $\mathfrak{a} \subset A$, and with the above notations, suppose in addition that we have U, $U' \in L^+G(A) = G(W(A))$ that satisfy the adjoint nilpotence condition and are such that*

$$U' \equiv U \mod W(\mathfrak{a}).$$

Then there is a unique element $h \in G(W(\mathfrak{a}))$ such that $U' = h^{-1}U\Psi_{\mathfrak{a}}(h)$.

Proof. We endow $G(W(\mathfrak{a})) = \ker(G(W(A)) \to G(W(A/\mathfrak{a})))$ with the topology coming from the restriction of the p-adic topology of A, with respect to which it is separated and complete. Consider the chain of ideals $\mathfrak{a}_n = \mathfrak{a}(pA + \mathfrak{a})^n$ which are pd-subideals of $\mathfrak{a}$. Then the ideal $\mathfrak{a}$ is complete and separated for the topology given by $(\mathfrak{a}_n)$. Note that $\Psi_{\mathfrak{a}}(G(W(\mathfrak{a}_n)) \subset G(W(\mathfrak{a}_n))$, because $\mathfrak{a}_n$ is a pd-subideal of $\mathfrak{a}$. By an inductive procedure we construct elements $h_n \in G(W(\mathfrak{a}))$ such that

$$h_n \equiv h_{n-1} \mod G(W(\mathfrak{a}_{n-1}))$$

and

$$h_n^{-1} U \Psi_{\mathfrak{a}}(h_n) \equiv U' \mod G(W(\mathfrak{a}_n)):$$

Then $h = \varprojlim_n h_n$ is the required element. Set $h_0 = 1$ and for $n \geqslant 1$ consider the element

$$U'' := h_{n-1}^{-1} U \Psi_{\mathfrak{a}}(h_{n-1}).$$

By the induction hypothesis we have

$$U'' \equiv U' \mod G(W(\mathfrak{a}_{n-1})).$$

We can define a function K from

$$W(\mathfrak{a}_{n-1}/\mathfrak{a}_n) \otimes_{\mathbb{Z}_p} \mathfrak{g} \overset{\exp}{\cong} G(W(\mathfrak{a}_{n-1}/\mathfrak{a}_n)) \cong G(W(\mathfrak{a}_{n-1}))/G(W(\mathfrak{a}_n)).$$

to itself by setting

$$K(X) := \exp^{-1}(U'' \Psi_{\mathfrak{a}}(\widetilde{\exp(X)}) U'^{-1} \mod G(W(\mathfrak{a}_n))),$$

where the tilde denotes a lift of $\exp(X) \in G(W(\mathfrak{a}_{n-1}/\mathfrak{a}_n))$ to $G(W(\mathfrak{a}_{n-1}))$. For $Y, X \in W(\mathfrak{a}_{n-1}/\mathfrak{a}_n) \otimes_{\mathbb{Z}_p} \mathfrak{g}$, we have

$$K(Y + X) = K(Y) + (\mathrm{Ad}^G(U) \circ \psi_{\mathfrak{a}_{n-1}/\mathfrak{a}_n})(X). \tag{3.5.4.1}$$

(Here $\psi_{\mathfrak{a}_{n-1}/\mathfrak{a}_n}$ is the map of our construction above applied to the ring $A/\mathfrak{a}_n$ and its pd-ideal $\mathfrak{a}_{n-1}/\mathfrak{a}_n$. Note also that since U, U' and U'' are congruent modulo $W(\mathfrak{a}_{n-1})$, the maps $\mathrm{Ad}^G(U)$, $\mathrm{Ad}^G(U')$ and $\mathrm{Ad}^G(U'')$ induce the same operator on $W(\mathfrak{a}_{n-1}/\mathfrak{a}_n) \otimes_{\mathbb{Z}_p} \mathfrak{g}$.)

Now observe that $W(\mathfrak{a}_{n-1}/\mathfrak{a}_n) \simeq \prod_{i=0}^{\infty} \mathfrak{a}_{n-1}/\mathfrak{a}_n$ can be regarded as a $A/(pA + \mathfrak{a})$-module; the module structure on the i-th component of the right hand side is given by $s \cdot a = F^i(s)a$, where F is the absolute Frobenius on $A/(pA + \mathfrak{a})$. Let

$$\psi^* : (A/(pA + \mathfrak{a})) \otimes_{\mathbb{Z}_p} \mathfrak{g} \to (A/(pA + \mathfrak{a})) \otimes_{\mathbb{Z}_p} \mathfrak{g}$$

be the composition of the absolute Frobenius on $(A/(pA + \mathfrak{a})) \otimes_{\mathbb{Z}_p} \mathfrak{g}$ with the projection π of $W(k_0) \otimes_{\mathbb{Z}_p} \mathfrak{g}$ onto $\mathfrak{u}^-$ killing $\mathfrak{p}$. Since $V_{\mathfrak{a}}^{-1}([x_0, x_1, x_2, \ldots]) = [x_1, x_2, \ldots]$, we can deduce from the definition of $\psi_{\mathfrak{a}_{n-1}/\mathfrak{a}_n}$ that

$$\psi_{\mathfrak{a}_{n-1}/\mathfrak{a}_n} = \mathrm{id}_{W(\mathfrak{a}_{n-1}/\mathfrak{a}_n)} \otimes_{A/(pA+\mathfrak{a})} \psi^*.$$

This together with Lemma 3.4.4 shows that the adjoint nilpotence of U implies that $\mathrm{Ad}^G(U) \circ \psi_{\mathfrak{a}_{n-1}/\mathfrak{a}_n}$ is nilpotent on $W(\mathfrak{a}_{n-1}/\mathfrak{a}_n) \otimes_{\mathbb{Z}_p} \mathfrak{g}$. In turn, this nilpotence together with (3.5.4.1) implies that K has a unique fixed point in $G(W(\mathfrak{a}_{n-1}/\mathfrak{a}_n)) \simeq W(\mathfrak{a}_{n-1}/\mathfrak{a}_n) \otimes_{\mathbb{Z}_p} \mathfrak{g}$. Finally, let us choose a lift h_* to $G(W(\mathfrak{a}_{n-1}))$ of the fixed point of K, and let us write $h_n := h_{n-1}h_*$. We can now see that

$$h_n^{-1} U \Psi_{\mathfrak{a}}(h_n) = h_*^{-1} h_{n-1}^{-1} U \Psi_{\mathfrak{a}}(h_{n-1}) \Psi_{\mathfrak{a}}(h_*) = h_*^{-1} U'' \Psi_{\mathfrak{a}}(h_*) = U'$$

modulo $G(W(\mathfrak{a}_n))$. This completes the inductive step that shows the existence of h. (Notice that this is a group theoretic generalization of the proof of [28, Theorem 44].)

Now let us show the uniqueness of h: We can see that it is enough to show that $h^{-1} U \Psi_{\mathfrak{a}}(h) = U = U'$ and $h \in G(W(\mathfrak{a}))$ imply $h = 1$. In fact, it is enough by induction to assume that $h \in G(W(\mathfrak{a}_{n-1}))$. Apply the argument above to h instead of h_{n-1}; then we have $U'' \equiv U \mod G(W(\mathfrak{a}_{n-1}))$. Since obviously $K(0) = 0$, the uniqueness of the fixed point of the map K implies that $h \in G(W(\mathfrak{a}_n))$. Since $\bigcap_n \mathfrak{a}_n = (0)$, we conclude that $h = 1$. $\square$

3.5.5. Recall that by Lemma 3.2.9 the diagonal of $\mathcal{B}(G, \mu)$ is affine. We now show that the diagonal of $\mathcal{B}^{\mathrm{nil}}(G, \mu)$ is p-adically formally unramified, cf. [28, Proposition 40], [3, Corollary 3.26(ii)]:

Corollary 3.5.6. *Let A be a $\mathfrak{a}$-adically separated algebra in $\mathrm{ANilp}_{W(k_0)}$, where $\mathfrak{a}$ is an ideal that contains some power of p. Let ϕ be an automorphism of an adjoint nilpotent (G, μ)-display $\mathcal{D}$ over $\mathrm{Spec}\, A$. Then $\phi_{A/\mathfrak{a}}$ is the identity if and only if ϕ is the identity.*

Proof. It is enough to prove, that if $\phi_{A/\mathfrak{a}}$ is the identity, then $\phi_{A/\mathfrak{a}^n}$ is the identity for every n, since by Lemma 3.2.9(b), the diagonal of the stack (G, μ)-displays is affine, hence separated. By a straightforward induction argument it suffices to do this under the additional assumption that $\mathfrak{a}^2 = 0$. After passing to some affine étale covering we can also assume that $\mathcal{D}$ is banal. Then since ideals of vanishing square have a natural pd-structure we can apply Theorem 3.5.4 and the result follows immediately from the uniqueness in the statement there. $\square$

3.5.7. Suppose that $\mathcal{D}_0$ is a adjoint nilpotent (G, μ)-display over $A_0 = A/\mathfrak{a}$ which is banal, and so given by a $U_0 \in G(W(A_0))$. Using Theorem 3.5.4 we classify lifts ('deformations') of $\mathcal{D}_0$ to displays over A, up to isomorphism. By definition, a lift

of $\mathcal{D}_0$ is a pair $(\mathcal{D}, \delta)$ of a (G, μ) display $\mathcal{D}$ over A together with an isomorphism $\delta : \mathcal{D}_0 \xrightarrow{\sim} \mathcal{D} \otimes_A A/\mathfrak{a}$. Since $\mathcal{D}$ is also banal by B.0.3, such pairs are given by pairs (U, h_0) with $U \in G(W(A))$, $h_0 \in H^\mu(A_0)$, such that

$$U \bmod \mathfrak{a} = h_0^{-1} U_0 \Phi_{G,\mu}(h_0);$$

two such pairs (U, h_0), (U', h_0'), are isomorphic if there is $h \in H^\mu(A)$ with

$$h \bmod \mathfrak{a} = h_0^{-1} h_0', \quad U' = h^{-1} U \Phi_{G,\mu}(h).$$

Suppose we are given a pair (U, h_0) up to isomorphism. Under our assumptions, since also H^μ is formally smooth, we can lift $h_0 \in H^\mu(A_0)$ to $\tilde{h}_0 \in H^\mu(A)$; this gives an isomorphism of (U, h_0) to a pair of the form $(U', 1)$, i.e. with $U' \bmod \mathfrak{a} = U_0$; hence, in classifying pairs (U, h_0) up to isomorphism, we can always assume that the second component is trivial, i.e. $h_0 = 1$; then $U \bmod \mathfrak{a} = U_0$. Suppose that $(U, 1)$, $(U', 1)$, are two such pairs. By Theorem 3.5.4 there is unique $g \in G(W(\mathfrak{a}))$ such that

$$U' = g^{-1} U \Psi_{\mathfrak{a}}(g).$$

Using this and the fact that the restriction of $\Psi_{\mathfrak{a}}$ to $H^\mu(\mathfrak{a})$ coincides with $\Phi_{G,\mu}$, we see that the following is true:

After choosing a lift U of U_0, we can identify the set of lifts of the (G, μ)-display $\mathcal{D}_0$ up to isomorphism with the set of right cosets $G(W(\mathfrak{a}))/H^\mu(\mathfrak{a})$; the bijection is given by

$$g H^\mu(\mathfrak{a}) \mapsto [(g^{-1} U \Psi_{\mathfrak{a}}(g), 1)].$$

We can see that the natural map

$$G(W(\mathfrak{a}))/H^\mu(\mathfrak{a}) \hookrightarrow G(W(A))/H^\mu(A) \to (G/P_\mu)(A)$$

identifies $G(W(\mathfrak{a}))/H^\mu(\mathfrak{a})$ with the set of A-valued points of the homogeneous space G/P_μ that reduce to the identity coset modulo $\mathfrak{a}$. (As usual, we can then also see that this set is in bijection with the set of liftings of the Hodge filtration of $\mathcal{D}_0$.)

3.5.8. Continue with the above assumptions and notations but suppose that in addition we have $\mathfrak{a}^2 = (0)$. Then we have as above

$$\mathfrak{a} \otimes_{W(k_0)} \mathfrak{u}^- \xrightarrow{\sim} G(W(\mathfrak{a}))/H^\mu(\mathfrak{a}).$$

Hence, if $\mathfrak{a}^2 = (0)$, the choice of a lift U of U_0 gives a bijection between the set of all lifts of $\mathcal{D}_0$ up to isomorphism and the set $\mathfrak{a} \otimes_{W(k_0)} \mathfrak{u}^-$. In fact, the above discussion shows that there is an action of the group $\mathfrak{a} \otimes_{W(k_0)} \mathfrak{u}^-$ on the set of all lifts of $\mathcal{D}_0$ up to isomorphism, and this set is a (trivial) principal homogeneous space for the group $\mathfrak{a} \otimes_{W(k_0)} \mathfrak{u}^-$.

3.5.9. Suppose that k is a separably closed field of characteristic p and let $\mathcal{D}_0$ be an adjoint nilpotent (G, μ) display over $S_0 = \mathrm{Spec}(k)$. We can then consider the functor $\mathrm{Def}_{\mathcal{D}_0}$ of formal deformations of $\mathcal{D}_0$. This is a functor on the category of augmented local Artinian $W(k)$-algebras, i.e. local Artinian $W(k)$-algebras $(A, \mathfrak{m})$ with an isomorphism $A/\mathfrak{m} \xrightarrow{\sim} k$ with $\mathrm{Def}_{\mathcal{D}_0}(A)$ the set of isomorphism classes of lifts of $\mathcal{D}_0$ to a (G, μ)-display

over A. Here, as above, a lift of $\mathcal{D}_0$ is by definition a pair $(\mathcal{D}, \delta)$ of a (G, μ)-display $\mathcal{D}$ over A together with an isomorphism $\delta : \mathcal{D}_0 \xrightarrow{\sim} \mathcal{D} \otimes_A k$.

The display $\mathcal{D}_0$ is banal and given by a matrix $U_0 \in G(W(k))$. Choose a basis $e_1, \dots, e_r$ of the $W(k_0)$-module $\mathfrak{u}^- = \mathrm{Lie}(U_{\mu^{-1}})$. By A.0.5 this induces an isomorphism $\mathbb{G}_a^r \xrightarrow{\sim} U_{\mu^{-1}}$; we write by $\exp(a_1 e_1 + \cdots + a_r e_r)$ the point of $U_{\mu^{-1}}$ which is the image of $(a_1, \dots, a_r)$. Set $A = W(k)[[t_1, \dots, t_r]]$ for the power series ring with r variables. (More canonically, we can take A to be the formal completion of $U_{\mu^{-1}} \otimes_{W(k_0)} W(k)$ at the origin.) Let us set

$$g_{\mathrm{uni}} = \exp([t_1]e_1 + \cdots + [t_r]e_r) \in U_{\mu^{-1}}(W(A)) \subset G(W(A))$$

where $[t_i] = (t_i, 0, 0, \dots) \in W(A)$ is the Teichmuller lift. Set

$$U_{\mathrm{uni}} = g_{\mathrm{uni}}^{-1} U_0 \in G(W(A)).$$

The element U_{uni} defines a (G, μ)-display $\mathcal{D}_{\mathrm{uni}}$ over A.

We claim that $\mathcal{D}_{\mathrm{uni}}$ prorepresents the functor $\mathrm{Def}_{\mathcal{D}_0}$ of formal deformations of $\mathcal{D}_0$:

Given the above work, the proof of this is very similar to the proof of the corresponding statement for Zink displays given in [28, pp. 173–176]. We just sketch the argument here: We first observe that, by the discussion in the above paragraph, $\mathrm{Def}_{\mathcal{D}_0}$ is formally smooth. Next, we notice that Theorem 3.5.4 (or the discussion above) quickly implies that the reduction of $\mathcal{D}_{\mathrm{uni}}$ over $A_2 := A/(t_1, \dots, t_r)^2$ is universal for deformations over augmented local Artinian $W(k)$-algebras with maximal ideal of square zero. (Notice that over A_2, we have $\Psi_{(t_1, \dots, t_r)}(g_{\mathrm{uni}}) = 1$.) This implies that the morphism of functors on augmented local Artinian $W(k)$-algebras $\epsilon : \mathrm{Spf}(A) \to \mathrm{Def}_{\mathcal{D}_0}$ given by $\mathcal{D}_{\mathrm{uni}}$ induces an isomorphism on tangent spaces. It now follows that ϵ is an isomorphism and this concludes the proof. The same argument works even if the field k is not separably closed provided $\mathcal{D}_0$ is banal.

3.5.10. We can similarly obtain a global version of some deformation theory statements in which we consider adjoint nilpotent (G, μ)-displays over general schemes in $\mathrm{Nilp}_{W(k_0)}$ (cf. [3, § 3.5] and especially [3, Corollary 3.27]).

Let S be a scheme in $\mathrm{Nilp}_{W(k_0)}$ and recall that if $\mathcal{D} = (P, Q, u)$ is a (G, μ)-display over S, then the Hodge filtration of $\mathcal{D}$ is a section

$$\alpha : S \to P/H^\mu$$

of the $X_\mu = G/P_\mu$-bundle $P/H^\mu \to S$. Recall also the vector bundle $T_{\mathcal{D}}$ over S obtained by pulling back via α the normal bundle of the (regular) closed immersion $\alpha(S) \hookrightarrow \mathcal{P}/H^\mu$.

Suppose $\mathcal{I} \subset \mathcal{O}_S$ is an ideal sheaf with $\mathcal{I}^2 = 0$. For any S-scheme X we denote $S_0 \times_S X = X_0$, where $S_0 \hookrightarrow S$ is the nilimmersion defined by $\mathcal{I}$. We can view

$$0 \to \mathcal{I} \hookrightarrow \mathcal{O}_S \twoheadrightarrow \mathcal{O}_{S_0} \to 0$$

as an exact sequence on abelian sheaves on the fpqc site of S: More specifically let $\mathcal{O}_{S_0}(X)$, $\mathcal{O}_S(X)$, and $\mathcal{I}(X)$ be the global sections of X_0, X, and the kernel of $\mathcal{O}_S(X) \to \mathcal{O}_{S_0}(X)$.

Now let $\mathcal{D}_0$ be an adjoint nilpotent (G, μ)-display over S_0. Again, by a lift of $\mathcal{D}_0$ over an S-scheme X we mean a pair $(\mathcal{D}, \delta)$ where $\mathcal{D}$ is a (G, μ)-display over X, and $\delta : \mathcal{D}_0 \times_{S_0} X_0 \xrightarrow{\sim} \mathcal{D} \times_X X_0$ is an isomorphism. By Corollary 3.5.6 no lift has any automorphism other than the identity. Let $\mathbf{D}_{\mathcal{D}_0, S}(X)$ be the set of isomorphism classes of lifts over

X. By pull-back of lifts this is a presheaf on S. Let us check that $D_{\mathcal{D}_0, S}$ satisfies the sheaf axiom for a faithfully flat map $Y \to X$: If the pull-backs $\mathrm{pr}_1^*(\mathcal{D})$ and $\mathrm{pr}_2^*(\mathcal{D})$ of some $(\mathcal{D}, \delta) \in D_{\mathcal{D}_0, S}(Y)$ agree for the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : Y \times_X Y \to Y$, then this means that there exists $\beta : \mathrm{pr}_1^*(\mathcal{D}) \overset{\simeq}{\to} \mathrm{pr}_2^*(\mathcal{D})$ which restricts to the identity on $Y_0 \times_{X_0} Y_0$. We easily deduce the cocycle condition for β, because any equality of isomorphisms of displays over $Y \times_X Y \times_X Y$ can be checked over $Y_0 \times_{X_0} Y_0 \times_{X_0} Y_0$, by Corollary 3.5.6. This shows that $\mathcal{D}$ (respectively δ) descends to X (respectively X_0). It follows that $D_{\mathcal{D}_0, S}$ is a fpqc-sheaf on S.

Theorem 3.5.11. *Suppose that $\mathcal{D}_0$ is an adjoint nilpotent (G, μ)-display over S_0 and that $S_0 \hookrightarrow S$ is a closed immersion defined by an ideal sheaf $\mathcal{I}$ of square zero. Then the above functor $D_{\mathcal{D}_0, S}$ has the structure of a locally trivial principal homogeneous space for the sheaf $\mathcal{I} \otimes_{\mathcal{O}_S} T_{\mathcal{D}_0}$. In particular, the set $D_{\mathcal{D}_0, S}(S)$ of isomorphism classes of lifts of $\mathcal{D}_0$ over S is either empty or is a principal homogeneous space for $\Gamma(S, \mathcal{I} \otimes_{\mathcal{O}_S} T_{\mathcal{D}_0})$.*

Proof. The case that S is affine and the display is banal is given by 3.5.8. (To see this one uses the fact that the Lie algebra $\mathfrak{u}^-$ gives the tangent space of the homogeneous space X_μ at the identity $1 \cdot P_\mu$.) The general case follows from this and descent. □

Corollary 3.5.12. *If S_0 is affine, then $D_{\mathcal{D}_0, S}$ is globally trivial, i.e. $D_{\mathcal{D}_0, S}(S) \neq \emptyset$.* □

3.6. Faithfulness up to isogeny

If B is a ring such that both pB and $\sqrt{0_B}$ are nilpotent the forgetful functor from (Zink-)displays to isodisplays is faithful (see [28, p. 186]). Here, we describe an extension of this which needs the notion of Frobenius separatedness defined in Appendix C. We write $\mathrm{ANilp}_W^{\mathrm{aFs}}$ for the full subcategory of ANilp_W consisting of W-algebras which are almost Frobenius separated.

Proposition 3.6.1. *Suppose that B is in $\mathrm{ANilp}_W^{\mathrm{aFs}}$. Then the functor of 3.3.3 from adjoint nilpotent (G, μ)–displays over B to G-isodisplays over B/pB is faithful.*

Proof. Let ϕ be an automorphism of a display $\mathcal{D} = (P, Q, u)$ with (G, μ)-structure over B, such that ϕ and $\mathrm{id}_{\mathcal{D}}$ give rise to the same self-G-quasi-isogeny. We would like to prove $\phi = \mathrm{id}_{\mathcal{D}}$. Using Corollary 3.5.6 and since p is nilpotent in B, we can easily reduce to the case $pB = 0$. Let $\mathfrak{a}$ be the smallest ideal of B such that $\phi \equiv \mathrm{id}_{\mathcal{D}} \mod \mathfrak{a}$; this ideal exists since by Lemma 3.2.9(b) the diagonal of $\mathcal{B}(G, \mu)$ is representable and affine. In fact, the B-scheme representing the automorphisms of the display $\mathcal{D}$ is a closed subscheme of the Greenberg transform $F\mathcal{I}$ of $\mathcal{I} := \underline{\mathrm{Aut}}_G(\mathcal{P})$, where $\mathcal{P}$ is the G-torsor over $W(B)$ that corresponds to the L^+G-torsor P by Proposition B.0.2. Corollary 3.5.6 on the rigidity of automorphisms implies that $\mathfrak{a} = \mathfrak{a}^2$. For simplicity, set $J_n = \ker(F_B^n) = \{x \in B \mid x^{p^n} = 0\}$. Note that $W(J_n) = W(B)[p^n]$, and

$$\bigcup_n W(J_n) = \ker(W(B) \to W(B)_{\mathbb{Q}} = W(B)[1/p]).$$

Since G is of finite type over $\mathbb{Z}_p$, by descent we see that the G-torsor $\mathcal{P}$ and also the scheme $\mathcal{I}$, are of finite presentation over $W(B)$. Since ϕ becomes the identity in $F_{(p)}\mathcal{I}(B) =$

$\mathcal{I}(W(B)[1/p])$, there must exist some integer N such that ϕ is equal to the identity in $\mathcal{I}(W(B/J_N))$. We deduce that $\mathfrak{a} \subset J_N$ which is bounded nilpotent, while $\mathfrak{a} = \mathfrak{a}^2$. Hence, by Lemma C.0.4, $\mathfrak{a} = 0$ which completes the proof. □

3.7. Display blocks

In this subsection, we define and discuss the notion of a display block. This is a technical construction which is useful for handling a certain type of descent that appears in the proof of Proposition 5.2.5 and of the main result Theorem 5.1.3. The subsection can be omitted at first reading.

3.7.1. We consider two pairs (G, μ) and (G', μ') where both G and G' are reductive group schemes over $\mathbb{Z}_p$, and μ and μ' are, respectively, minuscule cocharacters of G and G' defined over $W(k_0)$. We suppose that there is a group scheme homomorphism $i : G \to G'$ which is a closed immersion and is such that $\mu' = i \cdot \mu$ as in 3.2.13. We denote this set-up by writing $i : (G, \mu) \hookrightarrow (G', \mu')$.

Note that if for a (G, μ)-display $\mathcal{D}$ the (G', μ')-display $i(\mathcal{D})$ is adjoint nilpotent then $\mathcal{D}$ is also adjoint nilpotent (since the Frobenius isocrystal given by $\mathrm{Ad}^G(\mathcal{D})[1/p]$ is a sub-isocrystal of the one given by $\mathrm{Ad}^{G'}(i(\mathcal{D}))[1/p]$).

Definition 3.7.2. Fix $i : (G, \mu) \hookrightarrow (G', \mu')$ as above. Consider an injective homomorphism $A \hookrightarrow B$ in $\mathrm{ANilp}_{W(k_0)}$. An (A, B)-display block is a triple $(\mathcal{D}', \mathcal{D}, \psi)$ where

- $\mathcal{D}'$ is a (G', μ')-display over A,
- $\mathcal{D}$ is a (G, μ)-display over B, and
- $\psi : i(\mathcal{D}) \xrightarrow{\sim} \mathcal{D}' \times_A B$ is an isomorphism of (G', μ')-displays.

An isomorphism of two (A, B)-display blocks

$$(\mathcal{C}', \mathcal{C}, \phi) \xrightarrow{\sim} (\mathcal{D}', \mathcal{D}, \psi).$$

is a pair of isomorphisms $\epsilon' : \mathcal{C}' \xrightarrow{\sim} \mathcal{D}'$ and $\epsilon : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ satisfying

$$\psi \circ i(\epsilon) \circ \phi^{-1} = \epsilon' \times_A B.$$

Definition 3.7.3. The (A, B)-display block $(\mathcal{D}', \mathcal{D}, \psi)$ is called *effective* if there exists a (G, μ)-display $\mathcal{D}^\diamond$ over A such that

$$(\mathcal{D}', \mathcal{D}, \psi) \cong (i(\mathcal{D}^\diamond), \mathcal{D}^\diamond \times_A B, i(\mathrm{id}_{\mathcal{D}^\diamond}) \times_A B).$$

Then the (G, μ)-display $\mathcal{D}^\diamond$ is unique up to a unique isomorphism.

3.7.4. Consider an ideal $\mathfrak{b} \subset B$ and write $\mathfrak{a} := \mathfrak{b} \cap A \subset A$. By the reduction of $(\mathcal{D}', \mathcal{D}, \psi)$ modulo $\mathfrak{b}$ we mean the $(A/\mathfrak{a}, B/\mathfrak{b})$-display block given by the triple $(\mathcal{D}' \times_A A/\mathfrak{a}, \mathcal{D} \times_B B/\mathfrak{b}, \psi \times_B B/\mathfrak{b})$.

3.7.5. We first need the following result, which is a slight generalization of [3, Corollary 3.28]:

Lemma 3.7.6. *Suppose* $i : (G, \mu) \hookrightarrow (G', \mu')$ *is as above, and consider homomorphisms* $A \hookrightarrow B \twoheadrightarrow B/\mathfrak{b}$ *in* $\mathrm{ANilp}_{W(k_0)}$. *We assume that*

(i) B *is* $\mathfrak{b}$*-adically separated, i.e.* $(0) = \bigcap_n \mathfrak{b}^n$, *and*

(ii) A *is* $\mathfrak{b}$*-adically closed as a subset of* B, *i.e.* $A = \bigcap_n (A + \mathfrak{b}^n)$.

Let $\mathfrak{a} := A \cap \mathfrak{b}$ *so that we have* $A/\mathfrak{a} \hookrightarrow B/\mathfrak{b}$.

Suppose that $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are two* (G, μ)*-displays over* A. *We assume that* $i(\mathcal{D}_j \times_A A/\mathfrak{a})$, $j = 1, 2$ *are adjoint nilpotent* (G', μ')*-displays (so that then* $\mathcal{D}_j \times_A A/\mathfrak{a}$ *are also adjoint nilpotent* (G, μ)*-displays). Let*

$$\psi : i(\mathcal{D}_1) \times_A B \xrightarrow{\sim} i(D_2) \times_A B$$

be a (G', μ')*-isomorphism over* B, *and*

$$\phi_0 : \mathcal{D}_1 \times_A A/\mathfrak{a} \xrightarrow{\sim} \mathcal{D}_2 \times_A A/\mathfrak{a}$$

a (G, μ)*-isomorphism over* $A/\mathfrak{a}$. *Assume the base changes of* ϕ_0 *and* ψ *to* $B/\mathfrak{b}$ *are compatible in the obvious sense. Then there is a unique* (G, μ)*-isomorphism* $\phi : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$ *over* A *that induces both* ϕ_0 *and* ψ.

Proof. We first deal with the case that $\mathfrak{b}$ is nilpotent. Then, by induction we can assume that $\mathfrak{b}^2 = 0$. Now we can use the map ϕ_0 in order to view $\mathcal{D}_1$ as a lift of $\bar{\mathcal{D}}_2 := \mathcal{D}_2 \times_A A/\mathfrak{a}$, so that there exists a well-defined element $N \in T_{\bar{\mathcal{D}}_2} \otimes_A \mathfrak{a}$ that measures the difference between the elements $\mathcal{D}_1$ and $\mathcal{D}_2$ of the deformation $\mathbf{D}_{\bar{\mathcal{D}}_2, \mathrm{Spec}\, A}$. The existence of ψ implies that the image of N in $T_{i(\bar{\mathcal{D}}_2)} \otimes_A \mathfrak{b}$ has to vanish. Since $T_{\bar{\mathcal{D}}_2} \otimes_A \mathfrak{a} \to T_{i(\bar{\mathcal{D}}_2)} \otimes_A \mathfrak{b}$ is injective, this yields $N = 0$ which implies the existence of ϕ over A.

We now deal with the general case: The argument above gives that, for every $n \geqslant 1$, there is a (G, μ)-display isomorphism $\phi_n : \mathcal{D}_1 \times_A A/(A \cap \mathfrak{b}^n) \xrightarrow{\sim} \mathcal{D}_2 \times_A A/(A \cap \mathfrak{b}^n)$ which lifts ϕ_0 and is compatible with $\psi \times_B B/\mathfrak{b}^n$. This, since the diagonal of $\mathcal{B}(G', \mu')$ is affine by Lemma 3.2.9, implies that ψ, as a (G', μ')-display isomorphism, is actually defined over the subring $A + \mathfrak{b}^n \subset B$ and therefore over $A = \bigcap_n (A + \mathfrak{b}^n)$. Now, by the above, $\psi \times_A A/(A \cap \mathfrak{b}^n) = \phi_n$ which is a (G, μ)-isomorphism for all $n \geqslant 1$. Since $\bigcap_n (A \cap \mathfrak{b}^n) = 0$ we obtain that ψ is also a (G, μ)-isomorphism. $\square$

3.7.7. We end this paragraph with two lemmas that, roughly speaking, show that under some assumptions, certain deformations and liftings of effective display blocks are still effective. In both of these, we fix $i : (G, \mu) \hookrightarrow (G', \mu')$ as above.

Lemma 3.7.8. *Consider homomorphisms* $A \hookrightarrow B \twoheadrightarrow B/\mathfrak{b}$ *in* $\mathrm{ANilp}_{W(k_0)}$. *Suppose that* $\mathfrak{b}$ *is nilpotent and set* $\mathfrak{a} := \mathfrak{b} \cap A$. *Let* $\mathcal{D}_0^{\diamond}$ *be a* (G, μ)*-display over* $A/\mathfrak{a}$ *such that* $i(\mathcal{D}_0^{\diamond})$ *is an adjoint nilpotent* (G', μ')*-display. Then the assignment*

$$\mathcal{D}^{\diamond} \mapsto (i(\mathcal{D}^{\diamond}), \mathcal{D}^{\diamond} \times_A B, i(\mathrm{id}_{\mathcal{D}^{\diamond}}) \times_A B)$$

induces a canonical bijection from the set of isomorphism classes of deformations of the (G, μ)-display $\mathcal{D}_0^\diamond$ over A to the set of isomorphism classes of deformations of the $(A/\mathfrak{a}, B/\mathfrak{b})$-display block

$$(i(\mathcal{D}_0^\diamond), \mathcal{D}_0^\diamond \times_{A/\mathfrak{a}} B/\mathfrak{b}, i(\mathrm{id}_{\mathcal{D}_0^\diamond}) \times_{A/\mathfrak{a}} B/\mathfrak{b})$$

over (A, B).

Proof. We can assume $\mathfrak{b}^2 = 0$, by induction. In this case, Corollary 3.5.12 shows that both of the two sets in question are principal homogeneous spaces, the former one under $T_{\mathcal{D}_0^\diamond} \otimes_A \mathfrak{a}$ and the latter one under

$$(T_{i(\mathcal{D}_0^\diamond)} \otimes_A \mathfrak{a}) \cap (T_{\mathcal{D}_0^\diamond} \otimes_A \mathfrak{b}),$$

the intersection taking place in $T_{i(\mathcal{D}_0^\diamond)} \otimes_A \mathfrak{b}$. We get the result as these two groups are equal. $\qquad\square$

Lemma 3.7.9. *Consider homomorphisms $A \hookrightarrow B \twoheadrightarrow B/\mathfrak{b}$ in $\mathrm{ANilp}_{W(k_0)}$. Assume that $(A, \mathfrak{m})$ is a complete local Noetherian ring, and that one of the following two assertions holds:*

(i) *There exists a regular sequence (f, g) in $\mathfrak{m}$ such that $B = A[g^{-1}]$ and $\mathfrak{b}$ is generated by f.*

(ii) *B is a finitely generated A-module and $\mathfrak{b} = J(B)$.*

Let $(\mathcal{D}', \mathcal{D}, \psi)$ be a (A, B)-display block with $\mathcal{D}'$ adjoint nilpotent whose reduction modulo $\mathfrak{b}$ is effective and given by a (G, μ)-display over $A/A \cap \mathfrak{b}$ which is banal over $A/\mathfrak{m}$. Then $(\mathcal{D}', \mathcal{D}, \psi)$ is effective.

Proof. Let us write $\mathfrak{a}_n$ for the intersection $A \cap \mathfrak{b}^n$. In case (ii) the Artin–Rees lemma implies that the ideals $\mathfrak{a}_n$ form a basis of neighborhoods for the $\mathfrak{m}$-adic topology on A. In case (i) each $\mathfrak{a}_n$ is generated by f^n. Thus, in both cases B is $\mathfrak{b}$-adically separated and A is $\mathfrak{b}$-adically closed as a subset of B, as $\mathfrak{a}_n$ tends $\mathfrak{m}$-adically toward 0_A. By Lemma 3.7.8 we can choose a sequence of suitable adjoint nilpotent (G, μ)-displays $\mathcal{D}_n^\diamond$ over $A/\mathfrak{a}_n$ and isomorphisms:

$$(i(\mathcal{D}_n^\diamond), \mathcal{D}_n^\diamond \times_{A/\mathfrak{a}_n} B/\mathfrak{b}^n, i(\mathrm{id}_{\mathcal{D}_n^\diamond}) \times_{A/\mathfrak{a}_n} B/\mathfrak{b}^n) \cong (\mathcal{D}', \mathcal{D}, \psi) \bmod \mathfrak{b}^n.$$

Using the existence of the universal deformation over the formal deformation space of the (banal) (G, μ)-display $\mathcal{D}_1^\diamond \times_{A/A \cap \mathfrak{b}} A/\mathfrak{m}$ over $A/\mathfrak{m}$ we can now construct a 'limit' (G, μ)-display $\varprojlim_n \mathcal{D}_n^\diamond = \mathcal{D}^\diamond$ over $A = \varprojlim_n A/\mathfrak{a}_n$ (cf. Proposition 3.2.11). Passing to the limit we get:

$$(i(\mathcal{D}^\diamond), \mathcal{D}^\diamond \times_A \hat{B}, i(\mathrm{id}_{\mathcal{D}^\diamond}) \times_A \hat{B}) \cong (\mathcal{D}', \mathcal{D} \times_B \hat{B}, \psi \times_B \hat{B}),$$

where $\hat{B} = \varprojlim_n B/\mathfrak{b}^n$. Since B is $\mathfrak{b}$-adically separated $B \hookrightarrow \hat{B}$. The above isomorphisms allows us to identify the L^+G-torsors corresponding to the two displays $\mathcal{D}$ and $\mathcal{D}^\diamond \times_A B$ over B as two L^+G-equivariant subschemes of the L^+G'-torsor corresponding $\mathcal{D}' \times_A B$ which are the same after base changing by $B \to \hat{B}$. Using that G'/G is represented by

an affine scheme [8, 6.12] we can see that this implies that these subschemes are equal. This gives an isomorphism

$$(i(\mathcal{D}^\diamond), \mathcal{D}^\diamond \times_A B, i(\mathrm{id}_{\mathcal{D}^\diamond}) \times_A B) \cong (\mathcal{D}', \mathcal{D}, \psi),$$

which shows that $(\mathcal{D}', \mathcal{D}, \psi)$ is effective. □

4. Rapoport–Zink spaces

We now give our definition of the Rapoport–Zink stack and functor, state the representability conjecture and prove our main result on representability in the Hodge type case.

4.1. Local Shimura data

Let G be a connected reductive group scheme over $\mathbb{Z}_p$. We follow [21] and [14].

4.1.1. Let $([b], \{\mu\})$ be a pair consisting of:

- a $G(\bar{K})$-conjugacy class $\{\mu\}$ of cocharacters $\mu : \mathbb{G}_{m\bar{K}} \to G_{\bar{K}}$,
- a σ-conjugacy class $[b]$ of elements $b \in G(K)$; here, as usual, b and b' are σ-conjugate if there is $g \in G(K)$ with $b' = gb\sigma(g)^{-1}$.

We let $E \subset \bar{K}$ be the field of definition of the conjugacy class $\{\mu\}$. This is the *local reflex field.* Denote by $\mathcal{O}_E$ its valuation ring and by k_E its (finite) residue field. In fact, under our assumption on G, the field $E \subset \bar{K}$ is contained in K and there is a cocharacter $\mu : \mathbb{G}_{mE} \to G_E$ in the conjugacy class $\{\mu\}$ that is defined over E; see [16, Lemma (1.1.3)]. In fact, we can find a representative μ that extends to an integral cocharacter

$$\mu : \mathbb{G}_{m\mathcal{O}_E} \to G_{\mathcal{O}_E}, \tag{4.1.1.1}$$

and the $G(\mathcal{O}_E)$-conjugacy class of such an μ is well-defined. In what follows, we usually assume that μ is such a representative. We can identify $\mathcal{O}_E$ with the ring of Witt vectors $W(k_E)$ and we have $E = W(k_E)[1/p]$.

We write $\mu^\sigma = \sigma(\mu)$ for the Frobenius conjugate of (4.1.1.1).

Definition 4.1.2 (cf. [21, Definition 5.1]). A *local unramified Shimura datum* is a triple $(G, [b], \{\mu\})$, in which G is a connected reductive group over $\mathbb{Z}_p$, the pair $([b], \{\mu\})$ is as above, and we assume

  (i) $\{\mu\}$ is minuscule,

  (ii) for some (equivalently, any) integral representative (4.1.1.1) of $\{\mu\}$, the σ-conjugacy class $[b]$ has a representative

$$b \in G(W)\mu^\sigma(p)G(W). \tag{4.1.2.1}$$

By [20, Theorem 4.2], assumptions (i) and (ii) imply that $[b]$ lies in the set $B(G_{\mathbb{Q}_p}, \{\mu\})$ of neutral acceptable elements for $\{\mu\}$; see [21, Definition 2.3]. In particular, $(G_{\mathbb{Q}_p}, [b], \{\mu\})$ is a local Shimura datum in the sense of [21, Definition 5.1].

4.2. Definitions and a representability conjecture

Fix a local unramified Shimura datum $(G, [b], \{\mu\})$ and choose an integral representative μ of the conjugacy class $\{\mu\}$ as in (4.1.1.1). Choose a representative b of the σ-conjugacy class $[b]$ such that $b = u\mu^\sigma(p)$, with $u \in G(W)$. Consider the (G, μ)-display $\mathcal{D}_0 = (P_0, Q_0, u_0)$ over k given by $P_0 = L^+ G_W$, $Q_0 = H^\mu$, and $u_0 : H^\mu \to L^+ G_W$ given as the composition of the inclusion followed by left multiplication by u.

Suppose that S is a scheme in Nilp_W and consider pairs $(\mathcal{D}, \rho)$ with

- $\mathcal{D}$ a (G, μ)-display over S,
- $\rho : \bar{\mathcal{D}} \dashrightarrow \mathcal{D}_0 \times_k \bar{S}$ a G-quasi-isogeny over $\bar{S}$.

(In the above, $\bar{S}$ and $\bar{\mathcal{D}}$ denote the reductions of S and $\mathcal{D}$ modulo p.) Consider the natural notion of isomorphism between two such pairs.

We denote by $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}$ the stack of groupoids over Nilp_W that classifies pairs $(\mathcal{D}, \rho)$ as above. We can see that this is a fpqc stack. Denote by $\mathrm{RZ}_{G,\mu,b}$ the corresponding functor $\mathrm{Nilp}_W \to \mathbf{Sets}$ which sends S to the isomorphism classes of pairs $(\mathcal{D}, \rho)$ over S as above.

Consider the group of automorphisms $\mathrm{Aut}(\mathcal{D}_0[1/p])$ of the G-isodisplay given by $\mathcal{D}_0$. We can see that

$$\mathrm{Aut}(\mathcal{D}_0[1/p]) \simeq J_b(\mathbb{Q}_p) := \{j \in G(L) \mid j^{-1}b\sigma(j) = b\}.$$

This group acts $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}$ and on the functor $\mathrm{RZ}_{G,\mu,b}$ on the left by

$$j \cdot (\mathcal{D}, \rho) = (\mathcal{D}, j \cdot \rho).$$

We can now state:

Conjecture 4.2.1. *Assume that -1 is not a slope of $\mathrm{Ad}^G(b)$. The functor $\mathrm{RZ}_{G,\mu,b}$ is representable by a formal scheme which is formally smooth and formally locally of finite type over W.*

Here, representability by a formal scheme is in the sense explained in [22]. For $G = \mathrm{GL}_n$ and for b with no slopes equal to 0, the conjecture follows by a combination of the results of Rapoport–Zink and Zink and Lau: By Lau and Zink and the discussion of 2.3 (see also 3.2.2, 3.3.6 and 3.4.5) the functor for GL_n is equivalent to the Rapoport–Zink functor of deformations up to quasi-isogeny of the p-divisible group over k that corresponds to $\mathcal{D}_0$. For local unramified Shimura data of Hodge type, we prove this conjecture for the restriction of the functor $\mathrm{RZ}_{G,\mu,b}$ to locally Noetherian schemes in Nilp_W, see Theorem 5.1.3.

For general (G, μ) and b as above, it follows easily from Proposition 3.6.1 that, if B is Noetherian, then the objects of the groupoid $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}(B)$ have no automorphisms.

Remark 4.2.2. We refer to [22, Definition 3.45] for the notion of a Weil descent datum relative to the extension $W/\mathcal{O}_E$ for a functor on the category Nilp_W. Set $f = [E : \mathbb{Q}_p]$. (Recall that $E/\mathbb{Q}_p$ is finite and unramified.) Denote by $\tau : \mathrm{Spec}(\mathcal{O}_E) \to \mathrm{Spec}(\mathcal{O}_E)$ the morphism induced by $\sigma^f : \mathcal{O}_E \to \mathcal{O}_E$ and by $\bar{\tau}$ its reduction $\bar{\tau} : \mathrm{Spec}(k_E) \to \mathrm{Spec}(k_E)$. Using the construction of loc. cit. 3.48, we can define a Weil descent datum on $\mathrm{RZ}_{G,\mu,b}$

(relative to τ). This datum is given by the isomorphism of functors $\alpha : \mathrm{RZ}_{G,\mu,b} \to \mathrm{RZ}_{G,\mu,b}^{\tau}$ (see loc. cit.) obtained by sending the pair $(\mathcal{D}, \rho)$ to the pair of $\tau^*\mathcal{D}$ together with the G-quasi-isogeny

$$\bar{\tau}^*\bar{\mathcal{D}} \dashrightarrow \bar{\tau}^*(\mathcal{D}_0 \times_k \bar{S}) \dashrightarrow \mathcal{D}_0 \times_k \bar{S}$$

where the first arrow is $\bar{\tau}^*(\rho)$ and the second arrow is given by the relative Frobenius of $\mathcal{D}_0$.

4.2.3. Suppose that $(\mathcal{D}, \rho)$ is as above with $S = \mathrm{Spec}(R)$ affine and $\mathcal{D}$ banal; then $\mathcal{D}$ is determined by $U \in L^+G(R)$; the corresponding G-isocrystal over $\bar{R}$ is given by $U\mu^\sigma(p)$. The G-quasi-isogeny ρ is given by left multiplication by $g \in LG(\bar{R})$ which satisfies the identity

$$bF(g) = u\mu^\sigma(p)F(g) = gU\mu^\sigma(p) \tag{4.2.3.1}$$

in $LG(\bar{R}) = G(W(\bar{R})[1/p])$. Note that since p is nilpotent in R, the ideal $W((p)) \subset W(R)$ is p-power torsion and so

$$W(R)[1/p] = W(\bar{R})[1/p].$$

We conclude that in the banal case, the pair $(\mathcal{D}, \rho)$ is given by a pair $(U, g) \in L^+G(R) \times LG(R)$ which satisfies

$$g^{-1}bF(g) = U\mu^\sigma(p). \tag{4.2.3.2}$$

By the definitions, two pairs (U', g'), (U, g) give isomorphic pairs $(\mathcal{D}, \rho)$, $(\mathcal{D}', \rho')$, when there exists $h \in H^\mu(R)$ such that

$$(U', g') = (h^{-1} \cdot U \cdot \Phi_{G,\mu}(h), g \cdot h). \tag{4.2.3.3}$$

This implies that $\mathrm{RZ}_{G,\mu,b}$ is given by the isomorphism classes of objects of the (fpqc, or étale) quotient stack

$$[(L^+G \times_{LG,\mu,b} LG)/H^\mu].$$

Here the fiber product is

$$
\begin{array}{ccc}
L^+G \times_{LG,\mu,b} LG & \longrightarrow & LG \\
\downarrow & & \downarrow{\scriptstyle c_b} \\
L^+G & \longrightarrow & LG
\end{array}
$$

with $c_b(g) := g^{-1}bF(g)$ and the bottom horizontal map is the natural map followed by right multiplication by $\mu^\sigma(p) \in G(E) \subset LG(R)$. The quotient is for the action of H^μ given by

$$(U, g) \cdot h = (h^{-1} \cdot U \cdot \Phi_{G,\mu}(h), g \cdot h).$$

4.2.4. Continue with the set-up above and assume that $I \subset R$ is a nilpotent ideal. Set $R_0 = R/I$. Then, since $W(R)[1/p] = W(R_0)[1/p]$, the pair (U, g) is determined by (U, g_0). We can use this to deduce that for any $(\mathcal{D}_0, \rho_0)$ over R_0, the forgetful functor $\mathcal{RZ}_{G,\mu,\mathcal{D}_0} \to \mathcal{B}(G, \mu)$, $(\mathcal{D}, \rho) \mapsto \mathcal{D}$, induces an equivalence of deformation functors $\mathrm{D}_{(\mathcal{D}_0,\rho_0),S} \xrightarrow{\sim} \mathrm{D}_{\mathcal{D}_0,S}$. In particular, our results in § 3.5 apply to the deformation theory of $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}$.

4.2.5. Let k' be an algebraically closed field extension of k and set $W' = W(k')$, $K' = W'[1/p]$. In this case, since $W(k')$ is torsion free, the equation $g^{-1}bF(g) = U\mu^\sigma(p)$ shows that U is determined from g. By the above discussion, we have

$$\mathrm{RZ}_{G,\mu,b}(k') = \{g \in G(K') \mid g^{-1}bF(g) \in G(W')\mu^\sigma(p)\}/H^\mu(k'), \qquad (4.2.5.1)$$

where $H^\mu(k') \subset G(K')$ acts on $G(K')$ on the right. Since k' is perfect, we have $I(k') = pW(k')$. This gives

$$H^\mu(k') = G(W') \cap \mu(p)^{-1}G(W')\mu(p).$$

(Hence, the group $H^\mu(k')$ is equal to group of W'-valued points of a parahoric subgroup scheme defined over $W(k_0)$.)

Proposition 4.2.6. *We have a bijection*

$$\mathrm{RZ}_{G,b,\mu}(k') \cong \{g \in G(K') \mid g^{-1}bF(g) \in G(W')\mu^\sigma(p)G(W')\}/G(W')$$

where the quotient is for the natural right action of $G(W')$ on $G(K')$.

The right hand side of the identity in this statement is, by definition, the affine Deligne–Lusztig set

$$X_{G,\mu^\sigma,b}(k') \subset G(K')/G(W')$$

for the data (G, μ^σ, b) and the field k'.

Proof. This follows from 4.2.5.1 and an argument as in the proof of Proposition 3.2.15 (or [14, Proposition 2.4.3(i)]) which shows that the map

$$G(K')/H^\mu(k') \to G(K')/G(W'); \quad gH^\mu(k') \mapsto gG(W'),$$

restricts to a bijection between the set in the right side of 4.2.5.1 and $X_{G,\mu^\sigma,b}(k')$. $\square$

Remark 4.2.7. By [14, Proposition 2.4.3(iii)], the map $g \mapsto \sigma^{-1}(b^{-1}g)$ gives a bijection $X_{G,\mu^\sigma,b}(k') \xrightarrow{\sim} X_{G,\mu,b}(k')$ and so we also have

$$\mathrm{RZ}_{G,\mu,b}(k') \cong X_{G,\mu,b}(k').$$

5. Rapoport–Zink spaces of Hodge type

5.1. Hodge type local Shimura data

The two definitions below are slight variants of definitions in [14].

Definition 5.1.1. The local unramified Shimura datum $(G, [b], \{\mu\})$ is of *Hodge type* if there exists a closed group scheme embedding $\iota : G \hookrightarrow \mathrm{GL}(\Lambda)$, for a free $\mathbb{Z}_p$-module Λ of finite rank, with the following property: After a choice of basis $\Lambda_{\mathcal{O}_E} \xrightarrow{\sim} \mathcal{O}_E^n$, the composite

$$\iota \circ \mu : \mathbb{G}_{m\mathcal{O}_E} \to \mathrm{GL}_{n,\mathcal{O}_E}$$

is the minuscule cocharacter $\mu_{r,n}(a) = \mathrm{diag}(1^{(r)}, a^{(n-r)})$, for some $1 \leqslant r < n$.

Definition 5.1.2. Let $(G, [b], \{\mu\})$ be a local unramified Shimura datum of Hodge type. A *local Hodge embedding datum* for $(G, [b], \{\mu\})$ consists of

• a group scheme embedding $\iota : G \hookrightarrow \mathrm{GL}(\Lambda)$ as above,

• the $G(W)$–σ-conjugacy class $\{gb\sigma(g)^{-1} : g \in G(W)\}$ of a representative $b \in G(W)\mu^\sigma(p)G(W)$ of $[b]$ as in (4.1.2.1).

Notice that the $G(W)$-conjugacy class of an integral representative μ as in (4.1.1.1) is determined from $\{\mu\}$.

We refer to the quadruple (G, b, μ, Λ), where μ is given up to $G(W)$-conjugation, and b up to $G(W)$-σ-conjugation, as a *local unramified Shimura–Hodge datum*.

The following is the main result of the paper:

Theorem 5.1.3. *Assume that $(G, [b], \{\mu\})$ is a local unramified Shimura datum of Hodge type with a local Hodge embedding datum such that $\iota(b)$ has no slope 0. Then the restriction of the functor $\mathrm{RZ}_{G,\mu,b}$ to locally Noetherian schemes in Nilp_W is representable by a formal scheme which is formally smooth and formally locally of finite type over W.*

5.1.4. Fix a local unramified Shimura–Hodge datum (G, b, μ, Λ). The following is obtained similarly to [14, Lemma 2.2.5]. Note however that, here, we are using the covariant Dieudonné module.

Lemma 5.1.5 [14]. *There is a unique, up to isomorphism, p-divisible group*

$$X_0 = X_0(G, b, \mu, \Lambda)$$

over k whose (covariant) Dieudonné module is $\mathbb{D}(X_0)(W) = \Lambda \otimes_{\mathbb{Z}_p} W$ with Frobenius $F = b \circ \sigma$. Moreover, the Hodge filtration

$$V D_k \subset D_k = \mathbb{D}(X_0)(k)$$

is induced by a conjugate of the reduction $\mu_k : \mathbb{G}_{mk} \to G_k$ of μ modulo (p).

In what follows, we show Theorem 5.1.3. We use a natural morphism from $\mathrm{RZ}_{G,\mu,b}$ to the functor represented by the 'classical' Rapoport–Zink formal scheme RZ_{X_0}, where $X_0 = X_0(G, b, \mu, \Lambda)$ is as above.

5.2. The proof of the representability theorem

5.2.1. In this subsection we show Theorem 5.1.3. We are going to use the notion of Frobenius separatedness defined in Appendix C. We write $\mathrm{ANilp}_{\mathcal{O}}^{\mathrm{noeth}}$, $\mathrm{ANilp}_{\mathcal{O}}^{\mathrm{ared}}$, $\mathrm{ANilp}_{\mathcal{O}}^{\mathrm{aFs}}$ be the full subcategories of $\mathrm{ANilp}_{\mathcal{O}}$ consisting of $\mathcal{O}$-algebras which are respectively Noetherian, respectively with nilpotent nilradical, respectively which are almost Frobenius separated.

5.2.2. In what follows we fix a closed group scheme immersion

$$i : (G, \mu) \hookrightarrow (G', \mu')$$

as in 3.7.1. We eventually apply the following statements to the case that $G' = \mathrm{GL}_h$, $\mu' = \mu_{d,h}$, and i is given by a Hodge embedding datum.

Proposition 5.2.3. *Suppose that B is in $\mathrm{ANilp}_W^{\mathrm{aFs}}$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be banal displays with (G,μ)-structure over B such that $i(\mathcal{D}_1)$ and $i(\mathcal{D}_2)$ are adjoint nilpotent. Then a pair (ϕ,ψ) with*

   (i) *$\phi : \mathcal{D}_1 \dashrightarrow \mathcal{D}_2$ a G-quasi-isogeny,*

   (ii) *$\psi : i(\mathcal{D}_1) \to i(\mathcal{D}_2)$ a G'-isomorphism,*

is induced from a G-isomorphism $\epsilon : \mathcal{D}_1 \to \mathcal{D}_2$ if and only if

$$i(\phi) = \psi[1/p],$$

i.e. if and only if $i(\phi)$ and ψ give the same G'-quasi-isogeny $i(\mathcal{D}_1) \dashrightarrow i(\mathcal{D}_2)$. (By Proposition 3.6.1, the isomorphism ϵ is then uniquely determined.)

Proof. Using Lemma 3.7.6 we can reduce to the case $pB = 0$. Again set $J_n = \ker(F_B^n) = \{x \in B | x^{p^n} = 0\}$. Since the displays $\mathcal{D}_i$ are banal, they are represented by elements $U_i \in G(W(B))$. The quasi-isogeny ϕ gives us an element $k \in G(W(B)_{\mathbb{Q}})$ such that $U_2 = k^{-1}U_1\Phi^\mu(k)$ holds in $G(W(B)_{\mathbb{Q}})$. The isomorphism ψ gives us an element $l \in H^{\mu'}(B)$ such that $i(U_2) = l^{-1}i(U_1)\Phi^{\mu'}(l)$ holds in $G'(W(B))$. By our assumption, the images of $i(k)$ and l agree in the group $G'(W(B)_{\mathbb{Q}})$.

Since i is a closed immersion there exists a smallest ideal I_0 of $W(B)$ such that the restriction of the $W(B)$-valued point l to $\mathrm{Spec}(W(B)/I_0)$ factors through, say $h \in G(W(B)/I_0)$. Since $i : G \to G'$ is of finite presentation, I_0 is a finitely generated ideal of $W(B)$.

The existence of k shows that I_0 is contained in $W(B)[p^\infty] = \bigcup_n W(J_n)$. By the finite generation of I_0, we can choose a large enough n such that $I_0 \subset W(J_n)$ and we can consider $h_n \in G(W(B)/W(J_n)) = G(W(B/J_n)) = L^+G(B/J_n)$. Since $H^\mu = L^+G \cap H^{\mu'}$, we see that h_n is in $H^\mu(B/J_n)$. Let $U_{i,n}$ be the images of U_i in the group $G(W(B/J_n))$. Note that the elements $U_{2,n}$ and $h_n^{-1}U_{1,n}\Phi^\mu(h_n)$ are well-defined in $G(W(B/J_n))$ and have the same image in $G'(W(B)_{\mathbb{Q}})$. Since G' is of finite type, there exists another integer $n' \geqslant n$ such that these elements agree already in $G'(W(B/J_{n'}))$ and so also in $G(W(B/J_{n'}))$. Therefore, we have constructed an isomorphism

$$\eta : \mathcal{D}_1 \times_B B/J_{n'} \xrightarrow{\sim} \mathcal{D}_2 \times_B B/J_{n'}$$

with $\eta[1/p] = \phi$ and $i(\eta) \equiv \psi \mod J_{n'}$. Now use deformation theory (as in the proof of Lemma 3.7.6) to extend η to a compatible system of isomorphisms η_N over $B/J_{n'}^N$, for all N, which are given by $h_N \in H^\mu(W(B/J_{n'}^N))$ and which, when viewed in $H^{\mu'}(W(B/J_{n'}^N))$, are all the reduction of a single element $h' \in H^{\mu'}(W(B))$. Set $I = \bigcap_N J_{n'}^N$; by our assumption that B is almost Frobenius separated, I is nilpotent. It follows that $h' \mod I$ is in $H^\mu(W(B/I))$ and so it gives a lift of η to an isomorphism $\tilde{\eta} : \mathcal{D}_1 \times_B B/I \xrightarrow{\sim} \mathcal{D}_2 \times_B B/I$ with $\tilde{\eta}[1/p] = \phi$ and $i(\tilde{\eta}) \equiv \psi \mod I$. We can now conclude by a similar deformation theory argument as before. $\square$

5.2.4. Now let $\mathcal{D}_0$ be a (G,μ)-display defined over an algebraically closed field extension k of k_0. Suppose that the (G',μ')-display $i(\mathcal{D}_0)$ is adjoint nilpotent, then the same holds true for $\mathcal{D}_0$. We can consider the Rapoport–Zink stacks of groupoids $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}$, respectively $\mathcal{RZ}_{G',\mu',i(\mathcal{D}_0)}$, of pairs of (G,μ)-displays, respectively (G',μ')-displays,

together with a G-quasi-isogeny to $\mathcal{D}_0$, respectively a G'-quasi-isogeny to $i(\mathcal{D}_0)$. There is a natural transformation

$$i : \mathcal{RZ}_{G,\mu,\mathcal{D}_0} \to \mathcal{RZ}_{G',\mu',i(\mathcal{D}_0)}.$$

(Note that by Lemma 3.6.1, for B in $\mathrm{ANilp}_{W(k)}^{\mathrm{aFs}}$, the objects of $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}(B)$, $\mathcal{RZ}_{G',\mu',i(\mathcal{D}_0)}(B)$, have no automorphisms.)

Proposition 5.2.5. *Suppose that B is an object of $\mathrm{ANilp}_{W(k)}^{\mathrm{ared}}$ and let $A \subset B$ a Noetherian $W(k)$-subalgebra. Suppose we are given:*

- *$(\mathcal{D}, \delta)$ is an object of $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}(B)$,*
- *$(\mathcal{D}', \delta')$ is an object of $\mathcal{RZ}_{G',\mu',i(\mathcal{D}_0)}(A)$,*
- *ψ an isomorphism $\psi : i((\mathcal{D}, \delta)) \to (\mathcal{D}', \delta') \times_A B$.*

Then there is an object $(\mathcal{D}^{\diamond}, \delta^{\diamond})$ of $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}(A)$ together with isomorphisms

$$(\mathcal{D}^{\diamond}, \delta^{\diamond}) \times_A B \xrightarrow{\sim} (\mathcal{D}, \delta), \quad i((\mathcal{D}^{\diamond}, \delta^{\diamond})) \xrightarrow{\sim} (\mathcal{D}', \delta'),$$

which are compatible with ψ in the appropriate manner.

Proof. Given the above, the data $(\mathcal{D}', \mathcal{D}, \psi)$ give an (A, B)-display block as in Definition 3.7.2. Part of the desired conclusion is that $(\mathcal{D}', \mathcal{D}, \psi)$ is effective. In fact, we first show that it is enough to show this effectivity: Indeed, assume we have that, i.e. suppose that we have constructed a (G, μ)-display $\mathcal{D}^{\diamond}$ over A together with isomorphisms $i(\mathcal{D}^{\diamond}) \xrightarrow{\sim} \mathcal{D}'$, $\mathcal{D}^{\diamond} \times_A B \xrightarrow{\sim} \mathcal{D}$, which are compatible with ψ. We can then give the G-quasi-isogeny $\delta^{\diamond}$ over $\bar{A}$ as follows: The data δ, δ' together with the above isomorphisms give a G'-quasi-isogeny $\tilde{\delta}' : i(\bar{\mathcal{D}}^{\diamond}) \dashrightarrow i(\mathcal{D}_0) \times_k \bar{A}$ and a G-quasi-isogeny $\tilde{\delta} : \bar{\mathcal{D}}^{\diamond} \times_{\bar{A}} \bar{B} \dashrightarrow \mathcal{D}_0 \times_k \bar{B}$. These two are compatible in the sense that

$$i(\tilde{\delta}) = \tilde{\delta}' \times_{\bar{A}} \bar{B}.$$

We claim that this implies that $\tilde{\delta}$ is defined over $\bar{A}$ and so it gives the desired $\delta^{\diamond}$. To see this we can assume that the display $\mathcal{D}^{\diamond}$ is banal (by using étale descent on A and Corollary B.0.3). Then $\tilde{\delta}$, $\tilde{\delta}'$ are given by elements in $g \in G(W(B)[1/p])$ and $g' \in G'(W(A)[1/p])$ respectively and the condition is that these elements coincide in $G'(W(B)[1/p])$. Now observe that our conditions on $A \subset B$ imply

$$W(A)[1/p] \subset W(B)[1/p].$$

(Indeed, since A, B are in $\mathrm{ANilp}_W^{\mathrm{ared}}$ there is an integer n such that p^n annihilates both $W(\mathrm{rad}(A))$ and $W(\mathrm{rad}(B))$. Hence, $W(A)[1/p] = W(A_{\mathrm{red}})[1/p]$, $W(B)[1/p] = W(B_{\mathrm{red}})[1/p]$. Since $W(A_{\mathrm{red}})$, $W(B_{\mathrm{red}})$ are p-torsion free and $A \subset B$ induces $A_{\mathrm{red}} \subset B_{\mathrm{red}}$, the inclusion follows.) Now g is given by $\mathcal{O}_G \to W(B)[1/p]$ and g' by $\mathcal{O}_{G'} \to W(A)[1/p]$. The compatibility condition is that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{G'} & \to & \mathcal{O}_G \\ g' \downarrow & & \downarrow g \\ W(A)[1/p] & \hookrightarrow & W(B)[1/p]. \end{array}$$

Since $\mathcal{O}_{G'} \to \mathcal{O}_G$ is surjective this implies that g factors through $W(A)[1/p]$; this provides the G-quasi-isogeny $\delta^{\diamond}$.

To show $(\mathcal{D}', \mathcal{D}, \psi)$ is effective we now proceed in several steps, some of which are similar to [3, proof of Proposition 5.4.(ii)]. As it turns out, the two most important cases that we need to handle are when B is faithfully flat over A (Step 1), and when $A = K[[t]]$ and $B = K((t))$, with K a field (Step 3).

Observe that the statement holds for a finite product $A = \prod_{i=1}^{n} A_i$ if and only if it holds for each factor A_i, so that it does no harm to assume the connectedness of $\mathrm{Spec}\, A$ in each of the steps below.

Step 1. B is faithfully flat over A (e.g. if A is a field).

By Lemma B.0.3, there is an étale faithfully flat $B \to B'$ such that $\mathcal{D} \times_B B'$ is banal. Notice that B' has also nilpotent radical and so we can replace B by B' and assume that $\mathcal{D}$ is banal. Then the conclusion will be a consequence of fpqc descent for $A \to B$ and $\mathcal{RZ}_{G, \mu, \mathcal{D}_0}$. We can use Corollary C.0.6 and Proposition 5.2.3 applied to $B \otimes_A B$ to construct the descent datum as follows:

Let $d_1 : B \to B \otimes_A B$, $d_2 : B \to B \otimes_A B$, be the coprojections and $d : A \to B \otimes_A B$ their common restriction to A. Consider

$$d_i^*(\mathcal{D}, \delta) := (\mathcal{D}, \delta) \times_{B, d_i} (B \times_A B),$$

for $i = 1, 2$, in $\mathcal{RZ}_{G, \mu, \mathcal{D}_0}(B \otimes_A B)$. Define a G-quasi-isogeny $\phi : d_1^* \mathcal{D} \dashrightarrow d_2^* \mathcal{D}$ by setting

$$\phi = d_2^*(\delta)^{-1} \cdot d_1^*(\delta) : d_1^* \mathcal{D} \dashrightarrow d^* \mathcal{D}_0 \dashrightarrow d_2^* \mathcal{D}.$$

There is also a G'-isomorphism $\psi' : d_1^* i(\mathcal{D}) \to d_2^* i(\mathcal{D})$ given by the composition

$$d_1^* i(\mathcal{D}) \xrightarrow{d_1^*(\psi)} d_1^*(\mathcal{D}' \times_A B) = d^* \mathcal{D}' = d_2^*(\mathcal{D}' \times_A B) \xrightarrow{d_2^*(\psi)^{-1}} d_2^* i(\mathcal{D}).$$

We can now see that ψ' gives the G'-quasi-isogeny $i(\phi)$. By Corollary C.0.6, $B \otimes_A B$ is in $\mathrm{ANilp}_W^{\mathrm{aFs}}$ and so we can apply Proposition 5.2.3 to construct the descent datum which is given by a G-isomorphism $\epsilon : d_1^* \mathcal{D} \xrightarrow{\sim} d_2^* \mathcal{D}$ over $B \otimes_A B$. The G-isomorphism ϵ is compatible with the G-quasi-isogenies $d_1^* \delta$ and $d_2^* \delta$.

Step 2. A is a complete local Noetherian ring and B is finite over A.

First let us make the following observation: Let $A \to A'$ be an étale extension of local rings with a finite separable extension of residue fields k' over the residue field $k = A/\mathfrak{m}$ (then A' is also complete Noetherian). Assume we have the result for $A' \hookrightarrow B' := B \otimes_A A'$, and for the base changes of our displays to A' (B' is then also in ANilp_W). Then we can apply descent (or Step 1) to the faithfully flat $A \to A'$ to further descent to A and obtain the result for $A \to B$.

Now apply Step 1 to the inclusion $A/\mathfrak{m} \hookrightarrow B/J(B)$. This shows that the display block $(\mathcal{D}' \times_A A/\mathfrak{m}, \mathcal{D} \times_B B/J(B), \psi \times_B B/J(B))$ is effective and is given by a (G, μ)-display over $k = A/\mathfrak{m}$. There is a finite separable extension k' of k such that this display is banal and we can find an étale extension A' as above with k' as residue field. Now apply part (ii) of Lemma 3.7.9 to the base changes by A' to descent to A' and conclude by using the observation above.

Step 3. $A = K[[t]]$ and $B = K((t))$, where K is a field extension of k.

By using Step 1 we may assume that K is algebraically closed. By base changing to a finite separable extension L of $K((t))$ we can arrange that the (G, μ)-display $\mathcal{D}$ is banal. Notice that if A is the integral closure of $K[[t]]$ in L, then $A = K[[u]]$ for some variable u and $L = K((u))$. Since $A = K[[u]]$ is finite over $K[[t]]$, an application of Step 2 shows that we can always descend along $K[[t]] \hookrightarrow K[[u]]$; hence, we can reduce to the situation $A = K[[t]] \hookrightarrow B = K((t))$ with K algebraically closed and all displays banal. For simplicity of notation, set $R = K[[t]]$, $E = K((t))$. Hence we can find $U' \in G'(W(R))$ for $\mathcal{D}'$, and $U \in G(W(E))$ for $\mathcal{D}$. The isomorphism $\psi : i(\mathcal{D}) \to \mathcal{D}' \times_R E$ is given by $l \in H^{\mu'}(E) \subset G'(W(E))$ with $U = l^{-1}U'\Phi_{G,\mu}(l)$. Using the two isogenies δ, δ' we obtain a factorization

$$l = ab^{-1}$$

in $G'(W(E)[1/p])$. In this, $b \in G(W(E)[1/p])$ gives $\delta : \mathcal{D} \dashrightarrow \mathcal{D}_0 \times_k K((t))$ and $a \in G'(W(R)[1/p])$ gives $\delta' : \mathcal{D}' \dashrightarrow i(\mathcal{D}_0) \times_k R$. Consider the quotient G'/G which is represented by an affine scheme Z [8, 6.12]. The above identity implies that the $W(E)$-valued point of Z which is given by l is equal to the $W(R)[1/p]$-valued point which is given by a with the equality of points considered in $Z(W(E)[1/p])$. Now use

$$W(E) \cap W(R)[1/p] = W(R).$$

(The intersection takes place in $W(E)[1/p]$.) Using that Z is affine, we see that there is a $W(R)$-valued point z of Z which gives both l and a. Since $G' \to Z = G'/G$ is a G-torsor and $W(R) = W(K[[t]])$ is henselian, there is $c \in G'(W(R))$ that lifts z. Then $c = a \cdot d$ with $d \in G(W(R))$. Thus we can adjust b by multiplying by d^{-1} and assume now that $l = ab^{-1}$ with $a \in G'(W(R))$ and $b \in G(W(E))$. Now use that, by the Iwasawa decomposition, we have

$$G(E) = P_\mu(E) \cdot G(R).$$

This and the surjectivity of $G(W(E)) \to G(E)$, $G(W(R)) \to G(R)$ (which holds by Hensel's lemma, since $W(E)$, respectively $W(R)$, is $I(E)$-adically, respectively $I(R)$-adically, complete and G is smooth) gives

$$G(W(E)) = H^\mu(E) \cdot G(W(R)).$$

Write $b = h \cdot g$ with $h \in H^\mu(E)$, $g \in G(W(R))$. Then $l = ab^{-1} = ag^{-1} \cdot h^{-1}$. Hence, we may write $l = ab^{-1}$ with $a \in G'(W(R))$ and $b \in H^\mu(E)$. Observe that then $a \in H^\mu(R)$. Now set

$$U^\diamond = a^{-1}U'\Phi_{G',\mu'}(a) = b^{-1}U\Phi_{G,\mu}(b).$$

This element is in both $G'(W(R))$ and $G(W(E))$, hence it belongs to the intersection $G(W(R))$. We can now see that it defines the desired display $\mathcal{D}^\diamond$; the elements a and b give the isomorphisms of $i(\mathcal{D}^\diamond)$ to $\mathcal{D}$ and of $\mathcal{D}^\diamond \times_R E$ to $\mathcal{D}$ respectively.

In what follows, we denote by $Q(R)$ the total quotient ring of R, *i.e.* the localization $N^{-1}R$ of R at the set of non-zero divisors $N \subset R$.

Step 4. A is a reduced complete local one-dimensional Noetherian ring.

Consider $A \subset Q(A) \subset B \otimes_A Q(A)$; we can apply Step 1 to the base change $Q(A) \subset B \otimes_A Q(A)$. This allows us to reduce to considering $A \subset Q(A)$, i.e. we can assume $B =$

$Q(A)$. Consider the normalization A' of A in $Q(A)$. By [13, Théorème (23.1.5)] this is a finite extension of A. By the Cohen structure theorem we have $A' \simeq K[[t]]$, $Q(A) \simeq K((t))$. The result follows by applying sucessively Step 3 and Step 2.

The sequence of the following four steps settles the case of Noetherian rings of finite Krull dimension, we argue by induction on $\dim(A)$:

Step 5. A is an integrally closed complete local Noetherian ring.

By Serre's condition S_2, there exists a regular sequence of length two, i.e. elements f and g such that f is neither a unit nor a zero-divisor of A and g is neither a unit nor a zero-divisor of A/fA. Notice that $A[1/g]$ is a Noetherian ring of dimension strictly less than $\dim A$, so by induction we can apply Step 8 (for a ring of smaller dimension) to the inclusion $A[1/g] \subset B[1/g]$ and we obtain the desired display $\mathcal{D}^\diamond$ over the ring $A[1/g]$. Thus we may replace B by $A[1/g]$, and so do consider the inclusion $A \subset A[1/g]$, which satisfies the assumption of part (i) of Lemma 3.7.9. (Here, to make sure that the (G, μ)-display over the residue field $A/\mathfrak{m}$ is banal, we might need to base change by a finite étale local extension $A \to A'$ as in Step 2.)

Step 6. A is a complete local Noetherian ring.

Using Lemma 3.7.8 we see that we can assume that A is reduced. Then by an argument as in Step 4, we can reduce to the case $B = Q(A)$. Just as in the Step 4 we consider the normalization A' of A in $Q(A)$. Using the previous step we can replace B by the ring A'. However, by [13, Théorème (23.1.5)], A' is a finite extension of A, so that we can conclude by applying Step 2.

Step 7. A is a local Noetherian ring.

Just as in the step above we can assume A is reduced, putting us into a position where B may be replaced by $Q(A)$. Apply Step 6 to the ring extension $\hat{A} \subset Q(A) \otimes_A \hat{A}$, where $\hat{A}$ denote the completion of the local ring A. This allows us to reduce to the case $A \to B = \hat{A}$. Since $A \to \hat{A}$ is faithfully flat we can conclude by applying Step 1.

Step 8. A is a Noetherian ring of finite Krull dimension.

As before we can assume $A_{\mathrm{red}} = A$ and $B = Q(A)$. By base change, for every maximal ideal $\mathfrak{m}$ of A we obtain a $(A_\mathfrak{m}, Q(A_\mathfrak{m}))$-display block

$$(\mathcal{D}' \times_A A_\mathfrak{m}, \mathcal{D} \times_{Q(A)} Q(A_\mathfrak{m}), \psi \times_{Q(A)} Q(A_\mathfrak{m})),$$

where $A_\mathfrak{m}$ stands for the localization of A at $\mathfrak{m}$.

Set $A^\sharp := \prod_\mathfrak{m} A_\mathfrak{m}$, which is reduced ($\mathfrak{m}$ runs through the set of maximal ideals). Observe that the fact that $A_\mathfrak{m}$ are all reduced implies that the union $\bigcup_\mathfrak{m} \mathrm{Spec}(A_\mathfrak{m}) \hookrightarrow \mathrm{Spec}(A^\sharp)$ of the closed immersions $\mathrm{Spec}(A_\mathfrak{m}) \subset \mathrm{Spec}(A^\sharp)$ is dense in $\mathrm{Spec}(A^\sharp)$. (In fact, in general, if the radical of $\prod_i A_i$ is nilpotent, then $\cup_i \mathrm{Spec}(A_i)$ is dense in $\mathrm{Spec}(\prod_i A_i)$.)

Now let us write $(\mathcal{D}_\mathfrak{m}^\diamond, \delta_\mathfrak{m}^\diamond)$ for its descent to $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}(A_\mathfrak{m})$, of which the existence is granted by the previous step. Now let us construct a product display $\mathcal{D}^\sharp := (P^\sharp, Q^\sharp, u^\sharp) = \prod_\mathfrak{m} \mathcal{D}_\mathfrak{m}^\diamond$ over the ring $A^\sharp$:

(i) We obtain the L^+G-torsor $P^\sharp$ by applying Remark B.0.5. To obtain the descent $Q^\sharp$ to an H^μ-torsor use the observation 3.2.4: We need a section over $\mathrm{Spec}(A^\sharp)$ of the corresponding X_μ-bundle $P^\sharp/H^\mu$ for $X_\mu = G/P_\mu$. By assumption, we have such a section

over $\mathrm{Spec}(A_{\mathfrak{m}})$, for all $\mathfrak{m}$, while the $X_{\mu'}$-bundle $P'/H^{\mu'}$ for $X_{\mu'} = G'/P_{\mu'}$ has a section over $\mathrm{Spec}(A)$ and therefore over $\mathrm{Spec}(A^{\sharp})$. These agree as sections of $P'/H^{\mu'}$ over $\mathrm{Spec}(A_{\mathfrak{m}})$, for all $\mathfrak{m}$. Now notice that, by descent, $P^{\sharp}/H^{\mu} \subset P'^{\sharp}/H^{\mu'}$ is a closed immersion and hence, by the above density of $\cup_{\mathfrak{m}} \mathrm{Spec}(A_{\mathfrak{m}}) \hookrightarrow \mathrm{Spec}(A^{\sharp})$, we see that these give a section of $P^{\sharp}/H^{\mu}$ over $A^{\sharp}$.

(ii) The construction of $u^{\sharp}$ from $(u_{\mathfrak{m}})$ is obtained by an argument as in the proof of essential surjectivity in Lemma B.0.4.

In addition, we need to construct a 'compatible' G-quasi-isogeny $\delta^{\sharp} : \mathcal{D}^{\sharp} \dashrightarrow \mathcal{D}_0 \times_k A^{\sharp}/(p)$: Let us first assume that $\mathcal{D}'$, $\mathcal{D}_{\mathfrak{m}}^{\Diamond}$ are banal and fix trivializations of the torsors P', $P_{\mathfrak{m}}^{\Diamond}$. Then $(\delta_{\mathfrak{m}}^{\Diamond})$ is given by

$$(g_{\mathfrak{m}}) \in \prod_{\mathfrak{m}} G(W(A_{\mathfrak{m}})[1/p]) = G\left(\prod_{\mathfrak{m}}(W(A_{\mathfrak{m}})[1/p])\right);$$

this also lies in $G'(W(A^{\sharp})[1/p])$ and therefore in $G(W(A^{\sharp})[1/p])$. The non-banal case is treated in a similar way by working with, instead of points of affine group schemes, points of the affine schemes of suitable torsor isomorphisms.

Notice that the morphism $A \to A^{\sharp}$ is faithfully flat. By construction, $\mathcal{D}' \times_A A^{\sharp} \cong i(\mathcal{D}^{\sharp})$, and so $(\mathcal{D}^{\sharp}, \delta^{\sharp})$, $(\mathcal{D}', \delta')$, provide data to which we can apply the special case of Step 1.

Step 9. A is an arbitrary Noetherian ring.

Since every local Noetherian ring has finite Krull dimension, we can apply the previous step one more time, because meanwhile we know the local result without a restriction on the dimension. $\square$

Corollary 5.2.6. *Fix a Noetherian algebra A in ANilp_W. Let $(\mathcal{D}', \delta')$ be an object of the stack $\mathcal{RZ}_{G',\mu',i(\mathcal{D}_0)}$ over A. Then there exists an ideal $I \subset A$ such that the following statement is true: For each A-algebra $f : A \to B$ in $\mathrm{ANilp}_W^{\mathrm{ared}}$, we have $f(I) = 0$ if and only if there exists an object $(\mathcal{D}, \delta)$ of $\mathcal{RZ}_{G,\mu,\mathcal{D}_0}$ over B together with an isomorphism $\phi : (\mathcal{D}', \delta') \times_A B \xrightarrow{\sim} i((\mathcal{D}, \delta))$.*

This implies that the morphism $\mathcal{RZ}_{G,\mu,\mathcal{D}_0} \to \mathcal{RZ}_{G',\mu',i(\mathcal{D}_0)}$ of stacks, when restricted over $\mathrm{ANilp}_W^{\mathrm{noeth}}$, is represented by a closed immersion.

Proof. This is similar to [3, proof of Theorem 5.5]: Let us say $(\mathcal{D}', \delta')$ has G-structure over $f : A \to B$ if there is an isomorphism $\phi : (\mathcal{D}', \delta') \times_A B \xrightarrow{\sim} (\mathcal{D}, \delta)$. Then Proposition 5.2.5 implies that $(\mathcal{D}', \delta')$ has a G-structure over the quotient $A/\ker(f)$. Denote by $S(\mathcal{D}', \delta')$ the set of ideals of A for which $(\mathcal{D}', \delta')$ has a G-structure over A/I. If I and J are in $S(\mathcal{D}', \delta')$ we can apply Proposition 5.2.5 to $A/(I \cap J) \hookrightarrow A/I \times A/J$ and deduce that $I \cap J$ is also in $S(\mathcal{D}', \delta')$. In general, finite intersections of ideals in $S(\mathcal{D}', \delta')$ are also in $S(\mathcal{D}', \delta')$. Now consider the ideal

$$\mathfrak{I} := \mathfrak{I}_{(\mathcal{D}', \delta')} = \bigcap_{I \in S(\mathcal{D}', \delta')} I.$$

Consider the reduced product $B^{\flat} = \prod_{I \in S(\mathcal{D}', \delta')} A/\mathrm{rad}(I)$ of the reduced rings $A/\mathrm{rad}(I)$. We can construct a product (G, μ)-display $\mathcal{D}$ and a G-quasi-isogeny of $\mathcal{D}$ to $\mathcal{D}_0$ over this product ring $B^{\flat}$ by an argument as in the proof of Step 8 above. Now apply

Proposition 5.2.5 to

$$A/\operatorname{rad}(\mathfrak{J}) \hookrightarrow B^\flat = \prod_{I \in S(\mathcal{D}', \delta')} A/\operatorname{rad}(I).$$

We obtain that $\operatorname{rad}(\mathfrak{J})$ also belongs to $S(\mathcal{D}', \delta')$. Now repeat the argument and apply Proposition 5.2.5 to

$$A/\mathfrak{J} \hookrightarrow \prod_{I \in S(\mathcal{D}', \delta'), \mathfrak{J} \subset I \subset \operatorname{rad}(\mathfrak{J})} A/I.$$

(Notice that the radical of this product is also a nilpotent ideal.) This implies that $\mathfrak{J} \in S(\mathcal{D}', \delta')$ which is enough to deduce the result. $\square$

Apply the above to the case that $G' = \operatorname{GL}_h$, $\mu' = \mu_{d,h}$, and i is given by a Hodge embedding datum. By the results of Rapoport–Zink, Zink, and Lau, the functor $\operatorname{RZ}_{\operatorname{GL}_h, \mu_{d,h}, i(b)}$ is representable by a W-formal scheme which is locally formally of finite type over W (see 4.2; notice that the base-point p-divisible group is X_0, given in Lemma 5.1.5). By Corollary 5.2.6, we obtain that the restriction of $\operatorname{RZ}_{G, \mu, b}$ to $\operatorname{ANilp}_W^{\operatorname{noeth}}$ is represented by a W-formal closed subscheme of $\operatorname{RZ}_{\operatorname{GL}_h, \mu_{d,h}, i(b)}$ which is then also formally locally of finite type over W. Formal smoothness over W follows from our deformation theory results. This concludes the proof of Theorem 5.1.3. $\square$

Remark 5.2.7. We now easily see that, in the Hodge type case of Theorem 5.1.3, the W-formal scheme representing the restriction of $\operatorname{RZ}_{G, \mu, b}$ above to locally Noetherian schemes is isomorphic to formal schemes constructed in [15] and [14] (when these are defined, for example, when the local Hodge embedding is globally realizable, see loc. cit.) Indeed, all of these are W-formal closed subschemes of the classical Rapoport–Zink W-formal scheme $\operatorname{RZ}_{\operatorname{GL}_h, \mu_{d,h}, i(b)}$ with the same k-valued points (given by the affine Deligne–Lusztig set, Proposition 4.2.6), and the same formal completions at these points (by deformation theory, see 4.2.4 and 3.5) so they agree by flat descent. Let us note, however, a slight difference in notation: The Rapoport–Zink formal scheme for (G, μ, b) in this paper agrees with the one for $(G, \delta \cdot \mu^{-1}, pb^{-1})$ in [14], with $\delta : \mathbb{G}_m \to G \subset \operatorname{GL}(\Lambda)$ the central diagonal torus. (The existence of δ is part of the assumption of being of Hodge type in loc. cit.). The reason for this discrepancy is that [14] uses the contravariant Dieudonné functor for the construction of the base-point p-divisible group X_0 corresponding to $i(\mathcal{D}_0)$.

Remark 5.2.8. In the Hodge type case of Theorem 5.1.3, we can use the descent datum α given in Remark 4.2.2 to descend $\operatorname{RZ}_{G, \mu, b}$ over $\mathcal{O}_E$ as in [22, 3.49, 3.51]. This is done by an argument as in loc. cit., see for example loc. cit. Lemma 3.50.

Appendix A. Minuscule cocharacters and parabolics

Here we collect some notations and standard results on (minuscule) cocharacters and corresponding parabolic and unipotent subgroups of reductive group schemes. We refer the reader to [10] or [6] for more details.

A.0.1. We consider a perfect field k_0 and a smooth affine group scheme G over $W(k_0)$ with connected fibers. Let $\mu : \mathbb{G}_{m, W(k_0)} \to G$ be a group scheme homomorphism (a 'cocharacter' of G). We consider the closed subgroup schemes U_μ and P_μ of G which are defined by the following subfunctors of G on $W(k_0)$-algebras

$$P_\mu(R) = \left\{ g \in G(R) \mid \lim_{t \mapsto 0} \mu(t) g \mu(t)^{-1} \text{ exists} \right\}$$

$$U_\mu(R) = \left\{ g \in G(R) \mid \lim_{t \mapsto 0} \mu(t) g \mu(t)^{-1} = 1 \right\}$$

(See [4, 2.1], or [6, Theorem 4.1.17].)

Here '$\lim_{t \to 0}$ exists', by definition, implies that the conjugation action $\mathbb{G}_{W(k_0)} \times_{W(k_0)} P_\mu \to P_\mu$ extends to a morphism

$$\operatorname{Int}_\mu : \mathbb{A}^1_{W(k_0)} \times_{W(k_0)} P_\mu \to P_\mu.$$

This gives an action of the monoid scheme $\mathbb{A}^1_{W(k_0)}$ on P_μ by group scheme endomorphisms. Under this the zero section of $\mathbb{A}^1_{W(k_0)}$ maps U_μ to the neutral section.

A.0.2. By [6, Theorem 4.1.17] we have:

- Under our assumption, U_μ and P_μ are smooth group schemes over $W(k_0)$ with connected fibers.

- The subgroup scheme U_μ is unipotent and the multiplication

$$m : P_\mu \times_{W(k_0)} U_{\mu^{-1}} \to G$$

is an open immersion. We denote by G_μ^* or G^*, if μ is clear from the context, the open subscheme of G given as the image of this morphism.

- If $\mathfrak{g} = \operatorname{Lie}(G) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ is the weight space decomposition of the Lie algebra under the adjoint action (i.e. $\mathfrak{g}_n = \{ v \in \mathfrak{g} \mid \mu(t) v \mu(t)^{-1} = t^n v \}$), then we have

$$\mathfrak{p}_\mu = \operatorname{Lie}(P_\mu) = \bigoplus_{n \geqslant 0} \mathfrak{g}_n,$$

$$\mathfrak{u}_{\mu^{-1}} = \operatorname{Lie}(U_{\mu^{-1}}) = \bigoplus_{n < 0} \mathfrak{g}_n.$$

A.0.3. Suppose that G is connected split reductive over $W(k_0)$ and $T \subset G$ is a split maximal torus such that μ factors through T. Denote by Φ the roots of G and for $a \in \Phi$ by $\mathfrak{g}_a \subset \mathfrak{g}$, respectively U_a, the root $W(k_0)$-subspace of $\mathfrak{g}$, respectively root subgroup scheme of G. Then $\mathfrak{g}_n = \bigoplus_{a | \langle \mu, a \rangle = n} \mathfrak{g}_a$. Denote by $\Phi(\mu) \subset \Phi$ the set of roots a such that $\langle \mu, a \rangle > 0$. Then the multiplication (with the factors in the product taken in any order)

$$\prod_{a \in \Phi(\mu)} U_a \to U_\mu$$

gives an isomorphism of $W(k_0)$-schemes [6, 5]. The subgroup scheme $P_\mu \subset G$ is a parabolic subgroup. The group scheme U_μ is the unipotent radical of P_μ. It contains a finite filtration

$$U_\mu = U_{\Phi(\mu) \geqslant 1} \supseteq U_{\Phi(\mu) \geqslant 2} \supseteq \cdots$$

of normal subgroup schemes such that the multiplication map

$$\prod_{a \mid \langle \mu, a \rangle = n} U_a \to U_{\Phi(\mu) \geqslant n} / U_{\Phi(\mu) \geqslant n+1}$$

(with the factors in the product taken in any order) is an isomorphism of *group* schemes [6, Proposition 5.1.16].

Recall that μ defines a decreasing filtration $\mathrm{Fil}^\bullet(V)$ on each representation $G \to \mathrm{GL}(V)$. We can view $P_\mu \subset G$ as the subgroup scheme that respects the filtration $\mathrm{Fil}^\bullet(V)$. Then $U_\mu \subset P_\mu$ is the subgroup scheme of P_μ that acts trivially on the graded $\mathrm{gr}^\bullet(V)$.

A.0.4. We assume that G is a connected reductive group scheme over $W(k_0)$ with k_0 a finite field. Then there is a finite field extension k_0'/k_0 such that $G \times_{W(k_0)} W(k_0')$ is split. We now assume that the cocharacter μ is minuscule, i.e. that $\langle \mu, a \rangle \in \{-1, 0, 1\}$ for all absolute roots $a \in \Phi$.

Lemma A.0.5. *The unipotent group scheme U_μ is commmutative and is isomorphic to $\mathbb{G}_a^r \times_{\mathbb{Z}_p} W(k_0)$, where $r \geqslant 0$ and $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}_p[T])$ is the additive group scheme over $\mathbb{Z}_p$.*

Proof. First we see that $U_\mu \times_{W(k_0)} W(k_0') \simeq \mathbb{G}_a^r \times_{\mathbb{Z}_p} W(k_0')$: This follows from [6, Proposition 5.1.16] and étale descent (see also loc. cit. Theorem 5.4.3). Indeed, since there are no absolute roots a with $\langle \mu, a \rangle \geqslant 2$, the subgroup scheme $U_{\Phi(\mu) \geqslant 2}$ of loc. cit. is trivial. We can now conclude using [10, XVII.4.1.5] and the fact that all projective finitely generated $W(k_0)$-modules are free. $\qquad \square$

Appendix B. Loop group torsors

Suppose that R is a $W(k_0)$-algebra. As usual, G is a connected reductive group scheme over $\mathbb{Z}_p$. We compare between L^+G-torsors over R and G-torsors over $W(R)$. Here, we view $W(R)$ as a $W(k_0)$-algebra via $W(k_0) \to W(W(k_0)) \to W(R)$. The torsors are, by definition, locally trivial for the fpqc topology. Our convention is that the group acts on the right.

Lemma B.0.1. *If P is a G-torsor over $W(R)$, then the Greenberg transform FP is a L^+G-torsor over R.*

Proof. Observe that P is affine of finite presentation and smooth by descent, and so by Proposition 2.2.1 the Greenberg transform FP is affine flat and formally smooth over R. We can also easily see that $FP \to \mathrm{Spec}(R)$ is surjective, hence faithfully flat. By Proposition 2.2.1 the action morphism $P \times_{W(R)} (G \times_{W(k_0)} W(R)) \to P$ gives an action

$$FP \times_R L^+G \to FP.$$

Since P is a G-torsor the morphism $P \times_{W(R)} (G \times_{W(k_0)} W(R)) \xrightarrow{\sim} P \times_{W(R)} P$ given by $(x, g) \mapsto (x, x \cdot g)$ is an isomorphism. By Proposition 2.2.1 again, the morphism $FP \times_R L^+G \simeq FP \times_R FP$ given by the above action is an isomorphism and the result follows since $FP \to \mathrm{Spec}(R)$ is fpqc. $\qquad \square$

Proposition B.0.2. *Let R be in* $\mathrm{ANilp}_{W(k_0)}$.

(a) *If P is a G-torsor over $W(R)$, then there is an étale faithfully flat ring homomorphism $R \to R'$ such that $P \times_{W(R)} W(R')$ is trivial (i.e. has a section).*

(b) *The functor $P \mapsto FP$ provides an equivalence from the category of G-torsors over $W(R)$ to the category of L^+G-torsors over R.*

Proof. Let us first show part (a): Since G is smooth, the base change $P_0 = P \times_{W(R),w_0} R$ of P by $w_0 : W(R) \to R$ splits locally for the étale topology on R, i.e. there is $R \to R'$ étale faithfully flat such that $P_0 \times_R R'$ has a section s_0'. Now since p is nilpotent in R', the Witt ring $W(R')$ is separated and complete for the topology defined by the powers of $I(R') = \ker(W(R') \to R')$ (see [28, Proposition 3]). By Hensel's lemma, the section s_0' lifts to a section s' of $P \times_{W(R)} W(R')$.

Now let us prove part (b). We first show that the functor is fully faithful. Let $\phi, \psi : P \to P'$ be (iso)morphisms of G-torsors over $W(R)$ such that $F(\phi) = F(\psi)$. By part (a), there is an étale faithfully flat $R \hookrightarrow R'$ such that $P \times_{W(R)} W(R')$, $P' \times_{W(R)} W(R')$, are both trivial G-torsors. Hence, ϕ and ψ are given by multiplication by $g_\phi, g_\psi \in G(W(R'))$. Our assumption $F(\phi) = F(\psi)$ now quickly implies that $g_\phi = g_\psi$ and so $\phi = \psi$. It now remains to show the essential surjectivity of $P \mapsto FP$. First notice that in the case $G = \mathrm{GL}_h$ this is provided by Zink's Witt descent [28, Proposition 33, Corollary 34]). Indeed, a (fpqc locally trivial) $L^+\mathrm{GL}_h$-torsor over R gives by definition 'Witt descent data' on $W(R')^h$ with respect to the faithfully flat $R \to R'$; by loc. cit. these determine a projective finitely generated $W(R)$-module; this is locally free of rank h on $W(R)$ and its scheme of linear automorphisms produce the desired GL_h-torsor over $W(R)$. Let us now handle the case of a general reductive group G. There is a closed group scheme immersion $i : G \hookrightarrow \mathrm{GL}_h$ and the fpqc quotient $\mathrm{GL}_h/G = \mathrm{Spec}(A)$ is represented by an affine scheme [8, Proposition 6.11, Corollary 6.12]. Suppose now that Q is a L^+G-torsor over R and consider the induced $L^+\mathrm{GL}_h$-torsor

$$i(Q) = Q \times_{L^+G,i} L^+\mathrm{GL}_h .$$

By the above discussion, there is a GL_h-torsor P over $W(R)$ such that $FP \cong i(Q)$; this gives a closed immersion $j : Q \hookrightarrow FP$ of schemes over R which is L^+G-equivariant. By descent, the quotient P/G is represented by an affine $W(R)$-scheme Z and $P \to Z = P/G$ is a G-torsor. Then by a similar argument as in the proof of Lemma B.0.1, $FP \to FZ$ is a L^+G-torsor. Now by applying fpqc descent (i.e. 'taking the quotient of j' by the action of L^+G) we obtain a morphism

$$j/L^+G : \mathrm{Spec}(R) \to FZ$$

which amounts to a $W(R)$-valued point of Z. The pull-back of the G-torsor $P \to Z$ along this point gives the desired G-torsor over $W(R)$. $\square$

Corollary B.0.3. *Suppose S is a scheme in* $\mathrm{Nilp}_{W(k_0)}$. *Then all (fpqc locally trivial) L^+G-torsors over S split locally for the étale topology of S.* $\square$

Lemma B.0.4.

(a) *Let A_n, $n \geqslant 1$, be a sequence of local $W(k_0)$-algebras. Then the functor from the category of G-torsors over $A = \prod_n A_n$ to the product of the categories of G-torsors over A_n is an equivalence of categories.*

(b) *Let A be a $W(k_0)$-algebra with a descending chain of ideals $\mathfrak{a}_n$, $n \geqslant 1$, with $\mathfrak{a}_i \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$ and $\mathfrak{a}_1/\mathfrak{a}_n$ nilpotent in $A_n := A/\mathfrak{a}_n$, for all n, and such that $A \cong \varprojlim_n A_n$. Then the functor from the category of G-torsors over A to the category of compatible systems of G-torsors over A_n is an equivalence of categories.*

Proof. In both cases (a) and (b), the full faithfulness is clear and follows from the fact that the scheme of isomorphisms between two G-torsors over A is represented by an affine A-scheme. Also, when $G = \mathrm{GL}_h$, both functors are essentially surjective (for (b) this follows from lifting of projective modules, see for example [28, p. 146–148]). In general, pick a closed group scheme embedding $i : G \hookrightarrow \mathrm{GL}_h$. Consider a (compatible, for part (b)) sequence of G-torsors Q_n over A_n and the corresponding GL_h-torsors $Q_n \times_G \mathrm{GL}_h$. (In case (a), since A_n is local, the GL_h-torsor $Q_n \times_G \mathrm{GL}_h$ is trivial: $\mathrm{GL}_h \times_A A_n \cong Q_n \times_G \mathrm{GL}_h$.) By essential surjectivity for GL_h, there is a GL_h-torsor P over A with (compatible) isomorphisms $P \times_A A_n \cong Q_n \times_G \mathrm{GL}_h$. As in the proof of Proposition B.0.2, the fpqc quotient P/G is representable by an affine A-scheme Z and $P \to Z$ is a G-torsor. The Q_n's give rise to a (compatible) sequence of elements of $Z(A_n)$, which yields a point $Z(A)$; this gives the desired G-torsor over A. $\qquad\square$

Remark B.0.5.

(i) Lemma B.0.4 remains true if G is replaced by L^+G provided that, in addition, all A_n are in $\mathrm{ANilp}_{W(k_0)}$. Indeed, we see that the proof of part (a) goes through after the observation that all $L^+ \mathrm{GL}_h$-torsors over A_n are trivial, given by free $W(A_n)$-modules, of rank h. This is a corollary of Zink's Witt descent (also obtained by combining Proposition B.0.2 with [28, Proposition 35]). For part (b), we can again apply Proposition B.0.2 and observe that we have $W(A) \cong \varprojlim_n W(A_n)$ with $W(\mathfrak{a}_1/\mathfrak{a}_n)$ nilpotent in $W(A/\mathfrak{a}_n)$.

(ii) As a corollary of the above, it also follows that the equivalence of Proposition B.0.2 (b) extends to the case that R is a p-adically complete and separated $W(k_0)$-algebra.

Appendix C. On certain nilradicals

Suppose that R is a (commutative) $\mathbb{F}_p$-algebra. Denote by $F_R : R \to R$ the Frobenius $F_R(x) = x^p$ so that $\ker(F_R^n) = \{x \in R \mid x^{p^n} = 0\}$.

Definition C.0.1.

(a) We say that a $\mathbb{F}_p$-algebra R is *Frobenius separated (Fs)* when, for all $n \geqslant 1$, R is $\ker(F_R^n)$-separated, i.e. when for all $n \geqslant 1$, we have

$$\bigcap_m \ker(F_R^n)^m = 0. \tag{C.0.1.1}$$

(b) We say that a $\mathbb{Z}_{(p)}$-algebra R is *almost Frobenius separated (aFs)* when there exists a nilpotent ideal $\mathfrak{a}$ such that $p \in \mathfrak{a}$ and $R/\mathfrak{a}$ is Frobenius separated.

Lemma C.0.2. *We have:*

(1) *Noetherian rings with $pR = 0$ (respectively $p \in \sqrt{0_R}$) are (almost) Fs.*

(2) *If $R \subset R'$ and R' is (almost) Fs, then so is R.*

(3) *If R_i, $i \in I$, are Fs then the product $\prod_{i \in I} R_i$ is Fs.*

Proof. These properties are easy consequences of the definition. $\qquad\square$

The following is a slight generalization of [3, lemma 5.1]:

Lemma C.0.3. *Let k be a field of characteristic p, and let B and B' be reduced k-algebras. The tensor product $R = B \otimes_k B'$ is Frobenius separated.*

Proof. Notice that we can write $B \otimes_k B' \hookrightarrow \prod_i L_i \otimes_k \prod_j L'_j$ where L_i, L'_j are field extensions of k; then $B \otimes_k B' \subset (\prod_i L_i) \otimes_k (\prod_j L'_j) \hookrightarrow \prod_{i,j} L_i \otimes_k L'_j$. Hence, as Frobenius separatedness is inherited by products and subrings (Lemma C.0.2), without loss of generality, we can assume that B and B' are algebraically closed fields. We may clearly also assume $B = B'$. Zorn's lemma allows us to pick a maximal separable subextension $k \subset S \subset B$, so that $B = S^{1/p^\infty}$. Notice that $T := B \otimes_k S$ is a reduced ring. Let $\{x_i | i \in I\}$ be a p-basis of S (i.e. a subset such that every element of S has a unique representation as a sum $x = \sum_{\underline{n}} a_{\underline{n}}^p \underline{x}^{\underline{n}}$ where $\underline{n} = (n_i)_{i \in I}$ runs through the set of multiindices with $p - 1 \geqslant n_i \geqslant 0$ and $n_i = 0$ for almost all i). It is easy to see that B is the quotient of the polynomial algebra $S[\{b_{i,e} | i \in I, e \geqslant 1\}]$ by the ideal which is generated by $b_{i,e+1}^p - b_{i,e}$ and $b_{i,1}^p - x_i$ for $i \in I$, $e \geqslant 1$. Now consider

$$r_{i,e} := b_{i,e} \otimes 1 - 1 \otimes b_{i,e} \in R.$$

It follows that R is the quotient of the polynomial algebra

$$T[\{r_{i,e} | i \in I, e \geqslant 1\}]$$

by the ideal which is generated by $r_{i,e+1}^p - r_{i,e}$ and $r_{i,1}^p$, for $i \in I$, $e \geqslant 1$. We can deduce that, for each subset $I_0 \subset I$, there exists a ring endomorphism $\theta_{I_0} : R \to R$, defined by $\theta_{I_0}(r_{i,e}) = r_{i,e}$ if $i \in I_0$, $\theta_{I_0}(r_{i,e}) = 0$, if $i \notin I_0$. Notice that each θ_{I_0} preserves $\mathfrak{a}_n := \ker(F_R^n)$, and that $\theta_{I_0} = \theta_{I_0} \circ \theta_{I_0}$.

Now we can see that $\mathfrak{a}_n$ is generated by the elements $r_{i,n}$, so that $\mathfrak{a}_n^\nu$ is generated by the set $\{\prod_i r_{i,n}^{\nu_i} | \sum_i \nu_i = \nu\}$. The only multiindices which give rise to non-zero products are bounded by $\nu_i \leqslant p^n - 1$, and these are seen to involve factors indexed by at least $\nu/(p^n - 1)$ many elements of I. Consequently one has $\theta_{I_0}(\mathfrak{a}_n^\nu) = 0$ provided that $\#(I_0) < \nu/(p^n - 1)$. Now consider some $x \in \bigcap_\nu \mathfrak{a}_n^\nu$, and choose a large finite set I_0 with $\theta_{I_0}(x) = x$. We deduce $x \in \theta_{I_0}(\mathfrak{a}^{1+(p^n-1)\#(I_0)}) = 0$. $\qquad\square$

Lemma C.0.4. *Suppose that A is an almost Frobenius separated $\mathbb{Z}_{(p)}$-algebra. Let $\mathfrak{a} \subset A$ be an idempotent ideal, i.e. $\mathfrak{a} = \mathfrak{a}^2$, which is in addition bounded nilpotent, i.e. there exists $m \geqslant 1$ such that for all $x \in \mathfrak{a}$, $x^m = 0$. Then $\mathfrak{a}$ is the zero ideal.*

Proof. Consider the image $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ in A/pA. It is enough to show $\bar{\mathfrak{a}} = 0$; then $\mathfrak{a} \subset pA$ and so $\mathfrak{a} = \mathfrak{a}^2 \subset p^2 A$ which gives $\mathfrak{a} \subset p^m A = 0$ for $m \gg 0$. Therefore, it is enough to show the result when $pA = 0$. Our assumption implies that there is $n \geqslant 1$ such that $\mathfrak{a} = \mathfrak{a}^2 \subset \ker(F_A^n)$. The result follows since then $\mathfrak{a} = \mathfrak{a}^m \subset \ker(F_A^n)^m$ which gives $\mathfrak{a} \subset \bigcap_m (\ker(F_A^n))^m$. Therefore $\mathfrak{a}$ is nilpotent and $\mathfrak{a} = \mathfrak{a}^2$ gives $\mathfrak{a} = 0$. $\qquad\square$

Remark C.0.5. In fact, the proof only uses that $\cap_m \ker(F_{A/pA})^m$ is nilpotent. Indeed, suppose that $pA = 0$ and that we have $\mathfrak{a} = \mathfrak{a}^2 \subset \ker(F_A^n)$ with $n \geqslant 1$. We can induct on n. Consider $\mathfrak{a}' = \{\sum_i x_i y_i^p \mid x_i \in A, \ y_i \in \mathfrak{a}\}$. This satisfies $\mathfrak{a}' = \mathfrak{a}'^2 \subset \ker(F_A^{n-1})$ and by induction $\mathfrak{a}' = 0$. This gives $\mathfrak{a} \subset \ker(F_A)$ and we can conclude the proof.

Corollary C.0.6. *Let R be a Noetherian ring. If p is nilpotent in R and if B is a flat R-algebra for which $\sqrt{0_B}$ is nilpotent, then $B \otimes_R B$ is almost Frobenius separated (aFs).*

Proof. Recall that we denote by $Q(A)$ the total quotient ring of A, i.e. $Q(A)$ is the localization $N^{-1}A$ on the set of non-zero divisors $N \subset A$. Lemma C.0.3 implies that $Q(B_{\mathrm{red}}) \otimes_{Q(R_{\mathrm{red}})} Q(B_{\mathrm{red}})$ is almost Frobenius separated, and we claim that the kernel of the canonical homomorphism

$$B \otimes_R B \to Q(B_{\mathrm{red}}) \otimes_{Q(R_{\mathrm{red}})} Q(B_{\mathrm{red}})$$

is nilpotent. We first argue that we can quickly reduce to the case that R is reduced. Indeed, since R is Noetherian, $\sqrt{0_R}$ is a nilpotent ideal we can replace R and B by R_{red} and $B/\sqrt{0_R}B$. Now assuming that $R = R_{\mathrm{red}}$, let us write S for the set of non-zero divisors of R. Observe that since B is R-flat the images of the elements of S in B are also non-zero divisors of B, yielding natural injections $B \hookrightarrow S^{-1}B$ and also $B \otimes_R B \hookrightarrow S^{-1}B \otimes_R S^{-1}B$. However, since the kernels of both maps

$$S^{-1}B \otimes_{Q(R)} Q(B_{\mathrm{red}}) \to Q(B_{\mathrm{red}}) \otimes_{Q(R)} Q(B_{\mathrm{red}}),$$

and

$$S^{-1}B \otimes_R S^{-1}B \to S^{-1}B \otimes_R Q(B_{\mathrm{red}})$$

are nilpotent, the statement easily follows. $\qquad\square$

In particular Corollary C.0.6 applies when

- $B = \prod_{\mathfrak{m}} R_{\mathfrak{m}}$, where $\mathfrak{m}$ runs through the set of maximal ideals of the Noetherian ring R and $R_{\mathfrak{m}}$ stands for the localization at $\mathfrak{m}$, and
- R is a local Noetherian ring and $B = \hat{R}$ is its completion.

(In the first situation, the flatness of $B = \prod_{\mathfrak{m}} R_{\mathfrak{m}}$ over the Noetherian ring R follows from [5, Theorem 2.1].)

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