

ON PROJECTIONAL RESOLUTION OF IDENTITY  
ON THE DUALS OF CERTAIN BANACH SPACES

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A consequence of the main proposition includes results of Tacon, and John and Zizler and says: If a Banach space  $X$  possesses a continuous Gâteaux differentiable function with bounded nonempty support and with norm-weak continuous derivative, then its dual  $X^*$  admits a projectional resolution of the identity and a continuous linear one-to-one mapping into  $c_0(\Gamma)$ . The proof is easy and selfcontained and does not use any complicated geometrical lemma. If the space  $X$  is in addition weakly countably determined, then  $X^*$  has an equivalent dual locally uniformly rotund norm. It is also shown that  $\ell_\infty$  admits no continuous Gâteaux differentiable function with bounded nonempty support.

1. Notation and introduction.

We start by recalling some concepts and notation. All Banach spaces are assumed to be real and infinite dimensional. If  $(X, \|\cdot\|)$  is a Banach space, then  $X^*$  denotes its (topological) dual and  $X^{**}$  its second dual. If  $V$  is a subspace of  $X$  and  $f$  is a function on  $X$ ,

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then  $f|_V$  stands for the restriction of  $f$  to  $V$ . The topological concepts as "closed", "dense", ..., are always meant in the norm topology, unless otherwise specified. If  $S$  is a subset of  $X$ , we use the symbols  $\overline{S}$ ,  $\overline{\text{sp}} S$ ,  $\text{card } S$ , and  $\text{dens } S$  to denote the closure, closed linear span, cardinality, and density character of  $S$ , respectively;  $\text{dens } S$  being defined as the smallest cardinality of a dense subset of  $S$ .

In this note we propose an easy and selfcontained proof of the well-known theorem of Tacon [10], which asserts that the dual of a Banach space possessing some kind of smoothness admits a projectional resolution of the identity and a continuous linear one-to-one mapping into  $c_0(\Gamma)$ . In fact, we extend this result a little.

**PROPOSITION 1.** *Let  $X$  be a Banach space admitting a norm-weak continuous mapping  $D$  defined on an open subset  $B$  of  $X$  into  $X^*$  such that  $\overline{\text{sp}} \{Dx|_V : x \in V \cap B\} = V^*$  for every subspace  $V$  of  $X$ .*

*Denote by  $\mu$  the first ordinal of cardinality  $\text{dens } X$ .*

*Then there exists a nondecreasing "long sequence"  $\{X_\alpha : 0 \leq \alpha \leq \mu\}$  of subspaces of  $X$  such that  $X_0 = \{0\}$ ,  $X_\mu = X$  and for all  $0 < \alpha \leq \mu$*

- (i)  $\text{dens } X_\alpha = \text{dens } \overline{\text{sp}} D(X_\alpha \cap B) \leq \max(\overline{\alpha}, \aleph_0)$ ,
- (ii)  $\overline{\text{sp}} D(X_\alpha \cap B)$  is linearly isometric with  $X_\alpha^*$  under the restriction mapping  $f \longrightarrow f|_{X_\alpha}$ ,
- (iii)  $\bigcup_{\gamma < \alpha} X_{\gamma+1}$  is a dense subset of  $X_\alpha$ ,
- (iv)  $\bigcup_{\gamma < \alpha} \overline{\text{sp}} D(X_{\alpha+1} \cap B)$  is a dense subset of  $\overline{\text{sp}} D(X_\alpha \cap B)$ .

*There also exist linear projections  $\{P_\alpha : 0 \leq \alpha \leq \mu\}$  on  $X^*$  such that  $P_0 \equiv 0$ ,  $P_\mu = \text{identity}$ , and for all  $0 < \alpha \leq \mu$*

- (v)  $\|P_\alpha\| = 1$ ,
- (vi)  $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$  if  $\beta < \alpha$ ,
- (vii)  $P_\alpha X^* = \overline{\bigcup_{\gamma < \alpha} P_{\gamma+1} X^*} = \overline{\text{sp}} D(X_\alpha \cap B)$ .

Thus one can construct by a standard process (see, for example [6, Lemma 2], [10]) the announced injection from  $X^*$  into  $c_0(\Gamma)$ . Also,

combining this proposition with a recent renorming result due to Zizler [13], we get that  $X^*$  admits an equivalent (not necessarily dual) locally uniformly rotund norm.

Let us recall that Tacon [10] assumed  $X$  to have  $\hat{G}$ âteaux differentiable norm whose derivative is norm-weak continuous; while John and Zizler [7] constructed the above projections if  $X$  admitted a continuously Fréchet differentiable function with bounded nonempty support. Both these results are easily seen to be included in our proposition when one takes account of the following

**LEMMA 0.** *Let a Banach space  $X$  possess a continuous and  $\hat{G}$ âteaux differentiable function  $\phi : X \rightarrow \mathbb{R}$  such that the set  $B = \{x \in X : \phi(x) \neq 0\}$  is nonempty and bounded. Then there exists a mapping  $D : B \rightarrow X^*$  such that the set  $\{Dx|_V : x \in V \cap B\}$  is dense in  $V^*$  for every subspace  $V$  of  $X$ . The mapping  $D$  can be chosen to be norm-weak continuous if  $\phi'$  (the derivative of  $\phi$ ) is.*

*If, in addition, either  $\phi'$  is norm-weak continuous or  $\text{dens } X = \text{card } X$ , then  $\text{dens } X^* = \text{dens } X$ .*

**Proof.** By means of a shift we can ensure that  $0 \in B$ . Let us note that  $B$  is open. We define the function  $\psi : X \rightarrow (0, +\infty]$  by  $\psi(x) = \phi(x)^{-2}$ ,  $x \in X$ . Then  $\psi$  is continuous on  $X$ , and from differential calculus we know that it is  $\hat{G}$ âteaux differentiable on  $B$ . Moreover  $\psi'$  is norm-weak continuous if  $\phi'$  is. We define the mapping  $D$  by  $Dx = \psi'(x)$ ,  $x \in B$ .

Let  $V$  be a subspace of  $X$ . We must show that the set  $\{Dx|_V : x \in V \cap B\}$  is dense in  $V^*$ . We fix  $g \in V^*$  and  $\varepsilon > 0$ . We shall find  $x \in V \cap B$  such that  $\|g - Dx|_V\| \leq \varepsilon$ . Since  $B$  is bounded and contains  $0$ , the function  $\psi|_V - g$  is continuous, bounded from below, and not identically equal to  $+\infty$ . Thus, by Ekeland's variational principle [3, Theorem 1 bis], there exists  $x \in V$  such that  $(\psi|_V)(x+h) - g(x+h) \geq (\psi|_V)(x) - g(x) - \varepsilon\|h\|$  for all  $h \in V$ . It follows that  $x \in V \cap B$  and

$$Dx(h) \geq g(h) - \varepsilon\|h\| \quad \text{for all } h \in V.$$

Hence  $\|g - Dx|_V\| \leq \epsilon$ , which shows that the set  $\{Dx|_V : x \in V \cap B\}$  is dense in  $V^*$ .

It remains to prove the rest of Lemma 0. Let  $S$  be a dense subset of  $B$  with  $\text{card } S = \text{dens } B$ . If  $D$  is norm-weak continuous, then  $D(S)$  is weakly dense in  $D(B)$ ; hence  $\overline{\text{sp}} D(S)$  contains  $D(B)$  and so  $\overline{\text{sp}} D(S) = X^*$  by the first part of Lemma 0. Thus in this case  $\text{dens } X \leq \text{dens } X^* = \text{dens } D(S) \leq \text{card } D(S) \leq \text{card } S = \text{dens } B = \text{dens } X$ . On the other hand, if  $\text{card } X = \text{dens } X$ , then  $\text{dens } X \leq \text{dens } X^* = \text{dens } D(B) \leq \text{card } B = \text{card } X = \text{dens } X$ .

The lemma just proved can be compared with a result of Leduc [9, Proposition 3.6], which asserts that, if the above  $\phi$  has norm-norm continuous derivative, then the unit sphere of  $X$  has a continuous dense image in the dual unit sphere, and hence  $\text{dens } X^* = \text{dens } X$ .

We cannot help mentioning the following consequence of Lemma 0, though it departs from our main direction:  $\ell_\infty$  does not admit a continuous Gâteaux differentiable function with bounded nonempty support. In fact, it is easy to check that  $\text{card } \ell_\infty = \text{dens } \ell_\infty = 2^{\aleph_0}$ . On the other hand  $\ell_\infty$  is isometrical with  $C(\beta\mathbb{N})$  and the Dirac measures  $\{\delta_x : x \in \beta\mathbb{N}\}$  form a discrete subset of  $C(\beta\mathbb{N})^*$ . Hence  $\text{dens } \ell_\infty^* = \text{dens } C(\beta\mathbb{N})^* \geq \text{card } \beta\mathbb{N}$ . But, according to the theorem of Pospíšil [12, p. 71],  $\text{card } \beta\mathbb{N} = 2^{2^{\aleph_0}}$ . Hence  $\text{dens } \ell_\infty^* > \text{dens } \ell_\infty$  and our lemma applies. It should be noted that the non-existence of an equivalent Gâteaux differentiable norm on  $\ell_\infty$  was shown by Day [2, pp. 120-126, 229-230].

Now we return to Proposition 1 and say a few words on the method used in its proof. The rough scheme we follow is, of course, standard. However, the proof of the starting point (Lemma 1) is simple and elementary. Unlike [1], [10], [7] we avoid any complicated geometrical lemma. Also, no argument about compactness is needed.

It should also be noted that our method is different from that of Gul'ko [4] (see also [8]), who has proved the well-known result of Amir and Lindenstrauss [1] in a simple topological way. In fact we do not know if his approach can be adapted to the proof of Proposition 1.

Finally the following result should be stated.

**PROPOSITION 2.** *If the space  $X$  from Proposition 1 is weakly countably determined, then it has an equivalent norm such that the corresponding dual norm is locally uniformly rotund.*

Thus, taking into account Lemma 0, this proposition covers results of John and Zizler [6, Theorem 1], Gutman [5, Theorem 2] and Vařák [11, Theorem 2]. The proof can proceed almost word for word as in [6], taking account of [11]. In fact only slight changes in the proofs of [6, Lemma 5, Lemma 6] need to be made.

## 2. Proof of Proposition 1.

**LEMMA 1.** *Let  $X, B, D$  be as in Proposition 1. Let  $\aleph$  be an infinite cardinal number and let  $Z$  be a subset of  $X$  with  $\text{dens } Z \leq \aleph$ .*

*Then there exists a subspace  $C$  of  $X$  such that  $\text{dens } C = \text{dens } C^* \leq \aleph$ ,  $Z \subset C$ , and that the restriction mapping  $Q: f \rightarrow f|_C$  maps  $\overline{\text{sp}} D(C \cap B)$  onto  $C^*$  isometrically.*

**Proof.** Without loss of generality we may assume that  $B$  contains the unit ball of  $X$ . By induction, we shall construct a nondecreasing sequence  $\{S_n: 0 \leq n < \omega\}$  of subsets of  $X$  such that  $\text{card } S_n \leq \aleph$ ,  $\overline{S_n}$  is linear,  $Z \subset \overline{S_n}$ , and

$$\|f\| = \sup\{f(x): x \in S_{n+1}, \|x\| < 1\} \quad \text{for all } f \in \overline{\text{sp}} D(S_n \cap B)$$

for all  $n < \omega$ . Let  $S_0$  be any subset of  $X$

such that  $\text{card } S_0 \leq \aleph$ ,  $\overline{S_0}$  is linear, and contains  $Z$ . Such an  $S_0$

exists as  $\text{dens } Z \leq \aleph$ . Let us assume that we have constructed  $S_n$  for

some  $n \geq 0$ . Let  $\gamma$  denote the first ordinal of cardinality  $\aleph$ . Let

$\{f_\alpha: \alpha < \gamma\}$  be a dense subset of  $\overline{\text{sp}} D(S_n \cap B)$ . For each  $\alpha < \gamma$  we find

a sequence  $\{x_m^\alpha: m < \omega\}$  in the unit ball of  $X$  (lying in  $B$ ) such that

$$\|f_\alpha\| = \sup\{f_\alpha(x_m^\alpha): m < \omega\}. \quad \text{Let now } S_{n+1} \subset X \text{ be a set of cardinality}$$

at most  $\aleph$  which contains  $S_n$  and is such that  $\overline{S_{n+1}} = \overline{\text{sp}} [\{x_m^\alpha: m < \omega,$

$\alpha < \gamma\} \cup S_n]$ . Then  $S_{n+1}$  has all the prescribed properties.

Continuing in this fashion we can construct the sets  $S_n$  for all

$n < \omega$ . We put  $S = \bigcup_{n < \omega} S_n$ ,  $C = \overline{S}$ ,  $Y = \overline{\text{sp}} D(S \cap B)$ . Then  $Z \subset C$ .

As  $\text{card } S_n \leq \aleph$  for all  $n < \omega$ , we have  $\text{dens } C \leq \text{card } S \leq \aleph$ . Also,  $C$  is linear since  $S_n \subset S_{n+1}$  and the  $\overline{S_n}$  are linear.

Further fix  $f \in Y$  and let  $\epsilon > 0$  be arbitrary. As  $S_n \subset S_{n+1}$ , it follows easily that the set  $\bigcup_{n < \omega} \overline{\text{sp}} D(S_n \cap B)$  is dense in  $Y$ . Hence

there exist  $n < \omega$  and  $g \in \overline{\text{sp}} D(S_n \cap B)$  such that  $\|f - g\| < \epsilon/2$ . Now

$$\begin{aligned} \|f\| &< \|g\| + \epsilon/2 = \sup\{g(x) : x \in S_{n+1}, \|x\| < 1\} + \epsilon/2 < \\ &< \sup\{f(x) : x \in S_{n+1}, \|x\| < 1\} + \epsilon \leq \\ &\leq \sup\{f(x) : x \in C, \|x\| < 1\} + \epsilon = \|Qf\| + \epsilon \leq \|f\| + \epsilon. \end{aligned}$$

Consequently, letting  $\epsilon$  tend to zero, and using the linearity of  $Y$ , we conclude that  $Q$  is an isometry from  $Y$  into  $C^*$ .

It remains to show that  $Q$  is surjective and that  $\text{dens } C^* = \text{dens } C$ . To prove it we shall use the norm-weak continuity of the mapping  $D$ . Thus  $D(S \cap B)$  is weakly dense in  $D(C \cap B)$  and so  $\overline{\text{sp}} D(C \cap B) = \overline{\text{sp}} D(S \cap B) = Y$ . Now

$$Q(Y) = Q[\overline{\text{sp}} D(C \cap B)] = \overline{\text{sp}} Q[D(C \cap B)] = C^*$$

by the other property of  $D$ . Thus the surjectivity of  $Q$  is verified.

Finally, let  $C_0$  be a dense subset of  $C$  with  $\text{card } C_0 = \text{dens } C$ . Then

$$\overline{\text{sp}} D(C \cap B) = \overline{\text{sp}} D(C_0 \cap B),$$

$$C^* = Q[\overline{\text{sp}} D(C \cap B)] = Q[\overline{\text{sp}} D(C_0 \cap B)],$$

and consequently

$$\text{dens } C \leq \text{dens } C^* = \text{dens } D(C_0 \cap B) \leq \text{card } C_0 = \text{dens } C.$$

LEMMA 2. Let  $X$  be a Banach space and  $C \subset X$ ,  $Y \subset X^*$  two subspaces such that the restriction mapping  $Q: f \rightarrow f|_C$  maps  $Y$  onto  $C^*$  isometrically.

Then the mapping  $P: X^* \rightarrow X^*$  defined by  $Pf = Q^{-1}(f|_C)$ ,  $f \in X^*$ , is a norm one projection onto  $Y$  and the canonical image of  $C$  in  $X^{**}$  is weakly dense in  $P^*X^{**}$ .

Moreover, if  $C' \subset C$  and  $Y' \subset Y$  have the same properties as

$C$ ,  $Y$ , and  $P'$  is the corresponding projection, then  $PP' = P'P = P'$ .

Proof. Clearly  $Pf = f$  for all  $f \in Y$ , and  $PX^* = Y$ . Also, since  $Q$  is an isometry,

$$\|Pf\| = \|Q^{-1}(f|_C)\| = \|f|_C\| \leq \|f\|$$

for all  $f \in X^*$ . Hence  $P$  is a norm one projection onto  $Y$ .

Next we shall prove that  $C$  considered as a subset of  $X^{**}$  is weakly\* dense in  $P^*X^{**}$ . Let it be false. Then, by the Hahn Banach theorem, there are  $x^{**} \in X^{**}$  and  $f \in X^*$  such that  $f|_C \equiv 0$  and  $P^*x^{**}(f) \neq 0$ . But  $P^*x^{**}(f) = x^{**}(Pf) = x^{**}(Q^{-1}(f|_C)) = 0$ , a contradiction. Thus  $C$  is weakly\* dense in  $P^*X^{**}$ .

It remains to prove the statement concerning  $C'$  and  $P'$ . Since  $Y' \subset Y$ , we have  $PP' = P'$ . Let now fix  $f \in X$  and  $x \in X$ . As  $C'$  is weakly\* dense in  $P^*X^{**}$ , there is a net  $\{c_\tau\} \subset C'$  converging weakly\* to  $P'^*x$ . Consequently

$$\begin{aligned} P'Pf(x) &= P'^*x(Pf) = \lim_\tau Pf(c_\tau) = \lim_\tau Q^{-1}(f|_C)(c_\tau) = \\ &= \lim_\tau (f|_C)(c_\tau) = \lim_\tau f(c_\tau) = P'^*x(f) = P'f(x) \end{aligned}$$

because  $C' \subset C$ . It follows that  $P'P = P'$ .

Proof of Proposition 1. Let  $\{x_\alpha: 0 \leq \alpha < \mu\}$  be a dense subset of  $X$ . The subspaces  $X_\alpha$  will be constructed by transfinite induction. Put  $X_0 = \{0\}$ . Fix  $\alpha$ ,  $0 < \alpha \leq \mu$  and let us assume that we have found  $\{X_\gamma: \gamma < \alpha\}$  with the properties announced in Proposition 1 and such that  $x_\beta \in X_\gamma$  whenever  $0 \leq \beta < \gamma < \alpha$ . If  $\alpha = \gamma + 1$  for some  $\gamma$ , take as  $X_\alpha$  the  $C$  from Lemma 1 corresponding to  $Z = X_\gamma \cup \{x_\gamma\}$ . By the induction assumption we know that  $\text{dens } Z \leq \max(\bar{\alpha}, \aleph_0)$ . Hence, by Lemma 1, (i) and (ii) are satisfied for our  $\alpha$ . The statements (iii), (iv) are satisfied trivially. Secondly, let  $\alpha$  be a limit ordinal. Then putting  $X_\alpha = \overline{\bigcup_{\gamma < \alpha} X_\gamma}$ , (iii) is satisfied. Since the  $X_\gamma$  are linear and  $X_\beta \subset X_\gamma$  for  $\beta < \gamma < \alpha$ , it follows that  $X_\alpha$  is linear, too. The

norm-weak continuity of  $D$  then ensures that  $\bigcup_{\gamma < \alpha} D(X_\gamma \cap B)$  is weakly dense in  $D(X_\alpha \cap B)$ . Hence (iv) holds. The remaining properties (i) and (ii) can be easily verified in a way similar to that from the last two paragraphs of the proof of Lemma 1 (or otherwise, when carefully reading the proof of Lemma 1 for  $Z = \bigcup_{\gamma < \alpha} X_\gamma$ , we see that we can take

$S_0 = S_1 = \dots =$  any dense subset of  $Z$  with cardinality at most  $\bar{\alpha}$ ; so then  $C = \overline{\bigcup_{n < \omega} S_n} = \overline{\bigcup_{\gamma < \alpha} X_\gamma}$ )

We can thus find subspaces  $X_\alpha$  for all  $0 \leq \alpha \leq \mu$  with the properties (i) - (iv). And applying Lemma 2 we can immediately construct the projections  $P_\alpha$  satisfying (v) - (vii). Finally  $P_0 \equiv 0$  as  $X_0 = \{0\}$ , and we know that  $X_\mu$  contains the set  $\{x_\alpha : 0 \leq \alpha < \mu\}$ , which is dense in  $X$ . Hence  $X_\mu = X$ ,  $\overline{sp} D(X_\mu \cap B) = X^*$ , and  $P_\mu =$  identity.

**Remark.** Propositions 1 and 2 remain valid when the mapping  $D$  is replaced by a norm-weak lower semicontinuous multivalued mapping with separable values at each  $x \in B$ . Thus, especially, the results hold also for those singlevalued mappings  $D$  which are a pointwise limit of a sequence of norm-weak continuous mappings  $D_n : B \longrightarrow X^*$ . A similar remark applies to Lemma 0.

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