Combinatorics, Probability and Computing (2014) 23, 161–200.
doi:10.1017/S0963548313000576

Large Deviations and Ratio Limit Theorems for Pattern-Avoiding Permutations

MAHSHID ATAPOUR[1] and NEAL MADRAS[2†]

[1]Department of Finance and Management Science, University of Saskatchewan, 25 Campus Drive, Saskatoon, Saskatchewan S7N 5A7, Canada
(e-mail: atapour@math.usask.ca)

[2]Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
(e-mail: madras@mathstat.yorku.ca)

Received 20 July 2011; revised 11 October 2013; first published online 13 December 2013

For a fixed permutation τ, let $\mathcal{S}_N(\tau)$ be the set of permutations on N elements that avoid the pattern τ. Madras and Liu (2010) conjectured that $\lim_{N\to\infty}\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|}$ exists; if it does, it must equal the Stanley–Wilf limit. We prove the conjecture for every permutation τ of length 5 or less, as well as for some longer cases (including 704 of the 720 permutations of length 6). We also consider permutations drawn at random from $\mathcal{S}_N(\tau)$, and we investigate properties of their graphs (viewing permutations as functions on $\{1,\dots,N\}$) scaled down to the unit square $[0,1]^2$. We prove exact large deviation results for these graphs when τ has length 3; it follows, for example, that it is exponentially unlikely for a random 312-avoiding permutation to have points above the diagonal strip $|y-x|<\epsilon$, but not unlikely to have points below the strip. For general τ, we show that some neighbourhood of the upper left corner of $[0,1]^2$ is exponentially unlikely to contain a point of the graph if and only if τ starts with its largest element. For patterns such as $\tau = 4231$ we establish that this neighbourhood can be extended along the sides of $[0,1]^2$ to come arbitrarily close to the corner points $(0,0)$ and $(1,1)$, as simulations had suggested.

2010 *Mathematics subject classification*: Primary 60C05
Secondary 05A05, 05A16, 60F10

1. Introduction

For each natural number N, let $\mathcal{S}_N$ denote the set of all permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_N$ of $\{1,2,\dots,N\}$. Fix an integer k and a permutation τ in $\mathcal{S}_k$. For $N \geqslant k$, a permutation $\sigma \in \mathcal{S}_N$ is said to 'contain the pattern τ' if there is a subsequence of k elements of σ that appears in the same relative order as τ (see Definition 2 for the precise statement). We

† Supported in part by a Discovery Grant from NSERC Canada.

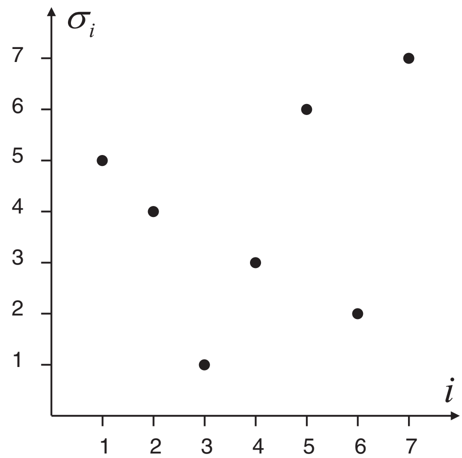

Figure 1. The graph of $\sigma = 5413627 \in \mathcal{S}_7$ viewed as the function $i \mapsto \sigma_i$. It can be seen that σ contains the patterns 2134 and 1234 but avoids the patterns 12345 and 54321. The pattern 21 occurs in σ in several ways (*e.g.*, $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$, $\sigma_4\sigma_6$, and $\sigma_5\sigma_6$), but σ contains a tight occurrence of 21 only once ($\sigma_1\sigma_2$).

say that σ avoids τ (or is τ-avoiding) if it does not contain the pattern τ. For example, $\sigma = 5413627 \in \mathcal{S}_7$ contains the patterns 2134 and 1234 (since σ contains the subsequences $\sigma_1\sigma_3\sigma_5\sigma_7 = 5167$ and $\sigma_3\sigma_4\sigma_5\sigma_7 = 1367$) but avoids the patterns 12345 and 54321 (see Figure 1). Given $\tau \in \mathcal{S}_k$, we denote the set of τ-avoiding permutations in $\mathcal{S}_N$ by $\mathcal{S}_N(\tau)$. Hence 5413627 is in $\mathcal{S}_7(12345)$ as well as $\mathcal{S}_7(54321)$ but is not in $\mathcal{S}_7(2134)$ or $\mathcal{S}_7(1234)$.

The number of elements in $\mathcal{S}_N(\tau)$, denoted $|\mathcal{S}_N(\tau)|$, has been computed for only some cases; in general, this seems to be a challenging computational problem. The case of pattern length $k = 2$ is easy, since the only 12-avoiding permutation (respectively, 21-avoiding permutation) in $\mathcal{S}_N$ is the permutation which has all the entries in decreasing (respectively, increasing) order. Thus $|\mathcal{S}_N(12)| = |\mathcal{S}_N(21)| = 1$ for every N. For $k = 3$, it is known [21] that $|\mathcal{S}_N(\tau)|$ is the same for all $\tau \in \mathcal{S}_3$ and is equal to the Nth Catalan number, that is,

$$|\mathcal{S}_N(\tau)| = \frac{\binom{2N}{N}}{N+1} \quad \text{for every } N \geqslant 1 \text{ and every } \tau \in \mathcal{S}_3. \tag{1.1}$$

This also shows that

$$\lim_{N\to\infty} |\mathcal{S}_N(\tau)|^{1/N} = 4 \quad \text{for every } \tau \in \mathcal{S}_3.$$

Thus, for the case $k = 3$, $|\mathcal{S}_N(\tau)|$ has a finite exponential growth rate, while $|\mathcal{S}_N| = N!$ grows superexponentially (much faster). In 1980, the Stanley–Wilf conjecture predicted that $|\mathcal{S}_N(\tau)|$ has a finite exponential growth rate for every $\tau \in \mathcal{S}_k$ ($k \geqslant 2$). In 1999, Bóna [6] proved the conjecture to be true for a special class of patterns. In 2000, Alon and Friedgut [2] proved a general bound on $|\mathcal{S}_N(\tau)|$ that grew only slightly faster than exponentially. However, a complete proof of the conjecture that would include all the patterns was not found until 2004, when Marcus and Tardos [18] showed that

$$L(\tau) := \lim_{N\to\infty} |\mathcal{S}_N(\tau)|^{1/N} \quad \text{exists and is finite for every } \tau. \tag{1.2}$$

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For $k = 2$ and $k = 3$ we know that the so-called Stanley–Wilf limit $L(\tau)$ is independent of τ and is respectively equal to 1 and 4. For $k > 3$, we know that $L(\tau)$ does depend on τ and we only know the value of $L(\tau)$ for some of the patterns $\tau \in \mathcal{S}_k$. For $k \geqslant 2$, Regev [20] proved that the Stanley–Wilf limit of the special pattern $\tau = 12\cdots k$ (the so-called increasing pattern) is given by

$$L(12\cdots k) = (k-1)^2.$$

For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_N \in \mathcal{S}_N$, the *complement* of σ is $\sigma^c = \sigma_1^c\sigma_2^c\cdots\sigma_N^c$, where $\sigma_i^c = N+1-\sigma_i$ for $1 \leqslant i \leqslant N$. The *reverse* of σ is defined to be $\sigma^r = \sigma_N\sigma_{N-1}\cdots\sigma_2\sigma_1$. It can be easily checked that σ is τ-avoiding if and only if σ^c (respectively, σ^r) is τ^c-avoiding (respectively, τ^r-avoiding). Hence, this gives rise to a bijection between $\mathcal{S}_N(\tau)$ and $\mathcal{S}_N(\tau^c)$ (respectively, $\mathcal{S}_N(\tau^r)$). Therefore,

$$|\mathcal{S}_N(\tau)| = |\mathcal{S}_N(\tau^c)| = |\mathcal{S}_N(\tau^r)| = |\mathcal{S}_N(\tau^{rc})|$$

and hence $L(\tau) = L(\tau^c) = L(\tau^r) = L(\tau^{rc})$. In addition, we have ([7], p. 136)

$$|\mathcal{S}_N(\tau)| = |\mathcal{S}_N(\tau^{-1})| \quad \text{and} \quad L(\tau) = L(\tau^{-1}),$$

where τ^{-1} is the inverse permutation of τ (*i.e.*, for $\tau \in \mathcal{S}_k$, $\tau^{-1} = \rho_1\rho_2\cdots\rho_k$, such that $\tau_{\rho_i} = i$ for every i).

For $k = 4$, $L(\tau)$ is known for all $\tau \in \mathcal{S}_4$ except for $\tau = 4231$ and its complement 1324. We know [3] that

$$L(\tau) = \sup_{N\geqslant 1} |\mathcal{S}_N(\tau)|^{1/N}.$$

Thus, for each $N \geqslant 1$, $|\mathcal{S}_N(\tau)|^{1/N}$ is a lower bound on $L(\tau)$. But exact enumeration of $|\mathcal{S}_N(\tau)|$ is difficult and exact values of $|\mathcal{S}_N(4231)|$ are known only up to $N = 25$ [1]. This gives rise to the (weak) lower bound $|\mathcal{S}_{25}(4231)|^{1/25} \approx 5.64$ for $L(4231)$. In 1999 Arratia [3] conjectured that $L(\tau) \leqslant (k-1)^2$ for every $\tau \in \mathcal{S}_k$. However, this conjecture was shown to be wrong by Albert, Elder, Rechnitzer, Westcott and Zabrocki [1]. In fact, they showed that $L(4231) \geqslant 9.47$ while $(k-1)^2 = 9$ for $k = 4$. For several years, the best published upper bound for $L(4231)$ was 288 [7], until Claesson, Jelínek and Steingrímsson [11] recently proved that $L(4231) \leqslant 16$; a more recent preprint of Bóna [9] lowers the upper bound to $7 + 4\sqrt{3} \approx 13.93$. So it remains an active open problem to find the exact value or an accurate estimate of $L(4231)$.

Madras and Liu [15] proposed an alternative method to approximate $L(4231)$. They used Monte Carlo simulation to estimate $L(4231)$ statistically. To do this, they first made the following natural conjecture.

Conjecture 1.1 (Ratio Limit Conjecture).

$$\text{For every } k \geqslant 2 \text{ and every } \tau \in \mathcal{S}_k, \quad \lim_{N\to\infty} \frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

When k is 2 or 3, known exact formulas verify the Ratio Limit Theorem (as we shall refer to the Ratio Limit Conjecture in those cases for which proofs are known). Madras

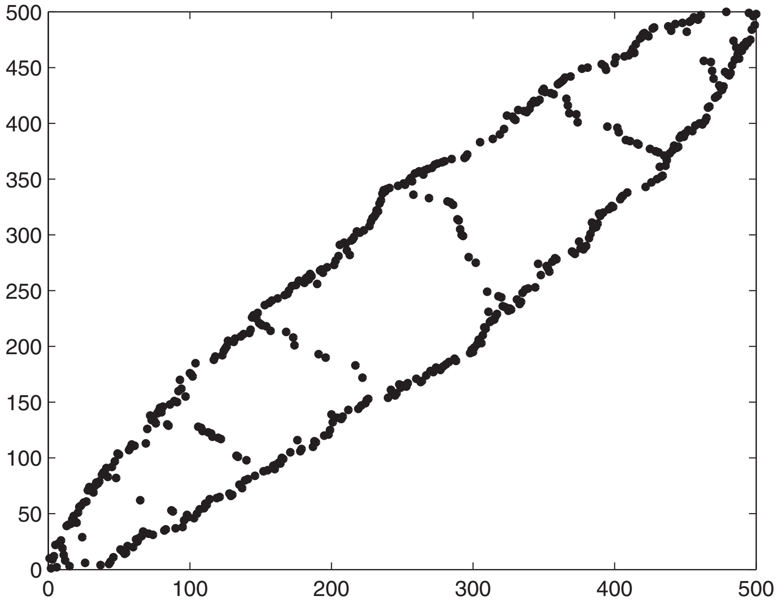

Figure 2. An example of a randomly generated 4231-avoiding permutation with 500 entries.

and Liu [15] used Monte Carlo methods to estimate the ratio

$$\frac{|\mathcal{S}_{N+1}(4231)|}{|\mathcal{S}_N(4231)|}$$

for some moderately large values of N. Then, assuming that Conjecture 1.1 was true for $\tau = 4231$, they extrapolated the results to the $N \to \infty$ limit, obtaining a 95% confidence interval of $[10.71, 11.83]$ for $L(4231)$ (subject to model uncertainty as well as statistical error).

Madras and Liu [15] also addressed the question: 'What does a typical 4231-avoiding permutation look like?' In order to approach this question, they used Monte Carlo methods to produce some random samples of 4231-avoiding permutations for several (relatively large) values of the permutation length N. A permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_N \in \mathcal{S}_N$ can be illustrated by its graph as the set of N points $\{(i, \sigma_i) : i = 1, \ldots, N\}$ in the Cartesian plane (see Figure 1). Based on the graphs of the random permutations generated by the Monte Carlo simulation (*e.g.*, see Figure 2), Madras and Liu [15] made some conjectures about the typical shape of a 4231-avoiding permutation. In particular, they predicted that, roughly speaking, there are some specific regions in the square $[1, N]^2$ such that for large N the probability is exponentially small that a point (i, σ_i) (for some $1 \leqslant i \leqslant N$) belongs to these regions.

The present paper is mainly motivated by the predictions and conjectures based on the Monte Carlo results in [15]. We prove the Ratio Limit Conjecture 1.1 for $k \leqslant 5$ (Theorem 7.1), as well as when $\tau = \tau_1 \cdots \tau_k$ satisfies at least one of the following conditions.

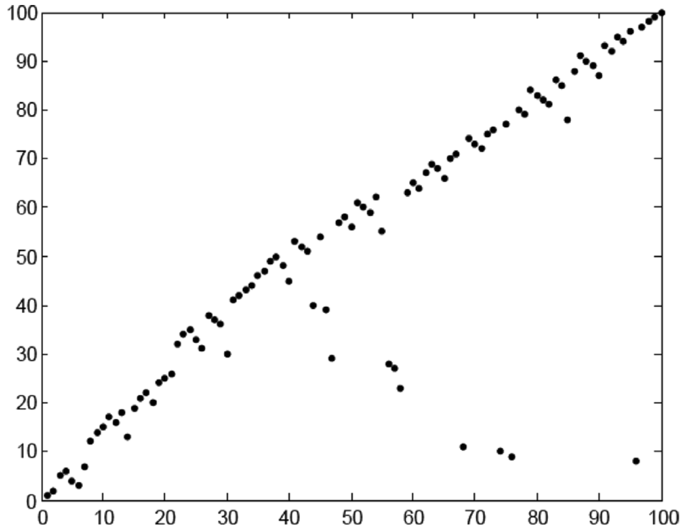

Figure 3. An example of a randomly generated 312-avoiding permutation in $\mathcal{S}_{100}(312)$.

(C1) τ does not contain any tight occurrence of 12, or does not contain any tight occurrence of 21 (Theorem 5.9 and Remark 1). (The notion of 'tight occurrence of patterns' is explained later in this section.)

(C2) τ_1 (or τ_k) is equal to either 1 or k (Theorem 6.4 and Remark 2).

(C3) $\tau = \tau_1\tau_2\cdots\tau_k \in \mathcal{S}_k$ $(k \geqslant 6)$ satisfies $\tau_1, \tau_2, \tau_{k-1}, \tau_k \in \{q+1, q+2, q+3, q+4\}$ (for some $q \geqslant 0$) and τ contains exactly one tight occurrence of 12 and one tight occurrence of 21, which are at positions $i = 1$ and $i = k-1$, in either order (Theorem 7.4).

(In the terminology of Atkinson and Stitt [4], τ satisfies (C1) if and only if τ or τ^r is 'irreducible'.) Theorem 7.5 observes that the Ratio Limit Theorem is now known to hold for all but 16 of the 720 patterns in $\mathcal{S}_6$.

Our next class of results concerns permutations drawn at random from a set $\mathcal{S}_N(\tau)$. Accordingly, we make the following definition.

Definition 1. Consider a pattern τ. For each $N \geqslant 1$, let P_N^τ be the uniform probability distribution on the set $\mathcal{S}_N(\tau)$, that is, $P_N^\tau(A) = |A|/|\mathcal{S}_N(\tau)|$ for every $A \subset \mathcal{S}_N(\tau)$. We shall write $\pi = \pi_1\pi_2\cdots\pi_N$ to denote a random permutation from the distribution P_N^τ.

Figure 3 is an example of a randomly generated 312-avoiding permutation with length 100. It (and others similar to it) suggest that 312-avoiding permutations are likely to stay near or below the diagonal. This is confirmed by Theorem 1.2 below, which gives explicit large deviation results for P_N^τ when τ has length three. It also shows that, with high probability, 321-avoiding permutations stay close to the diagonal $x = y$. To describe the result formally, define the function

$$\mathcal{K}(s,t) = \frac{1}{4}\frac{(2-s-t)^{2-s-t}(s+t)^{s+t}}{(1-s)^{1-s}(1-t)^{1-t}t^t s^s} \quad \text{if } (s,t) \in [0,1]^2, \tag{1.3}$$

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where we interpret 0^0 to be 1. Observe that $\mathcal{K}(s,s)=1$ for every $s \in [0,1]$. Also note that

$$\mathcal{K}(s,t) = 2^{\mathcal{H}(s)+\mathcal{H}(t)-2\mathcal{H}((s+t)/2)},$$

where $\mathcal{H}(s) = -s\log_2 s - (1-s)\log_2(1-s)$ is the standard binary entropy function. Since $\mathcal{H}$ is strictly concave, we see that $\mathcal{K}(s,t) < 1$ whenever $s \neq t$. Also, define

$$\mathcal{K}^*(s,t) = \begin{cases} \mathcal{K}(s,t) & \text{if } 0 \leqslant s \leqslant t \leqslant 1, \\ 1 & \text{if } 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Theorem 1.2. *Let D be a relatively open subset of the unit square $[0,1]^2$. Then*

$$\lim_{N\to\infty}\left[P_N^{321}\left\{\left(\frac{i}{N},\frac{\pi_i}{N}\right) \in D \text{ for some } i \in \{1,\ldots,N\}\right\}\right]^{1/N} = \sup\{\mathcal{K}(s,t) : (s,t) \in D\} \quad (1.4)$$

and

$$\lim_{N\to\infty}\left[P_N^{312}\left\{\left(\frac{i}{N},\frac{\pi_i}{N}\right) \in D \text{ for some } i \in \{1,\ldots,N\}\right\}\right]^{1/N} = \sup\{\mathcal{K}^*(s,t) : (s,t) \in D\}. \quad (1.5)$$

For a random 312-avoiding permutation, equation (1.5) tells us that it is not unusual to have points of the graph well below the $x=y$ diagonal. Complementing this result, Proposition 4.2 shows that the number of points well below the diagonal is $o(N)$ with high probability. Madras and Pehlivan [16] consider the properties of random 312-avoiding permutations below the diagonal in more detail. Theorem 1.2 also follows directly from recent independent work of Miner and Pak [19], who investigate fine asymptotics of $P_N^\tau(\pi_i = j)$ for $\tau \in \mathcal{S}_3$.

The exact calculations in Theorem 1.2 are due in large part to the tractability of Catalan numbers (1.1) and the many things associated with them. For longer patterns, our results are considerably less precise and more qualitative. For our next results, consider the graph of a random permutation from $\mathcal{S}_N(\tau)$, scaled down to the unit square $[0,1]^2$. Under what condition is some neighbourhood of the upper left corner of $[0,1]^2$ very likely to contain no points of the graph (as seems to be the case in Figures 2 and 3)? The answer is: *if and only if* $\tau_1 = k$. The next theorem presents a more precise formulation of this assertion. It assumes that $\tau_1 > \tau_k$, since otherwise we have $P_N^\tau(\pi_1 = N) = |\mathcal{S}_{N-1}(\tau)|/|\mathcal{S}_N(\tau)|$, which is not small (this equation holds because if $\tau_1 < \tau_k$, then $N\sigma_2\cdots\sigma_N \in \mathcal{S}_N(\tau)$ if and only if $\sigma_2\cdots\sigma_N \in \mathcal{S}_{N-1}(\tau)$).

Theorem 1.3. *Assume $\tau \in \mathcal{S}_k$ and $\tau_1 > \tau_k$.*

(a) *Assume $\tau_1 = k$. Then there exists an open neighbourhood D of the point* $(0,1)$ *such that*

$$\limsup_{N\to\infty}\left[P_N^{\tau}\left\{\left(\frac{i}{N},\frac{\pi_i}{N}\right) \in D \text{ for some } i \in \{1,\ldots,N\}\right\}\right]^{1/N} < 1.$$

(b) *Assume* $\tau_1 < k$. *Then for every non-empty relatively open subset* D *of* $\{(s,t) : 0 \leqslant s \leqslant t \leqslant 1\}$

$$\lim_{N\to\infty}\left[P_N^\tau\left\{\left(\frac{i}{N},\frac{\pi_i}{N}\right) \in D \text{ for some } i \in \{1,\ldots,N\}\right\}\right]^{1/N} = 1.$$

This theorem is a consequence of the stronger results Theorem 8.1 and Proposition 3.1. One might wonder whether the conclusion of Theorem 1.3(b) holds whenever $\tau_1<\tau_k$, but Theorem 1.2 implies that $\tau = 123$ is a counterexample to this conjecture.

Monte Carlo samples from $\mathcal{S}_N(4231)$ (see [15] and Figure 2) suggest the conjecture that for the pattern $\tau = 4231$, the set D of Theorem 1.3(a) can be extended along the sides of the square $[0,1]^2$ to come arbitrarily close to the corner points $(0,0)$ and $(1,1)$. More generally, we have the following result. Its statement refers to the Tight Pattern Insertion Property, which is a technical condition that we shall introduce in Definition 10 (and is satisfied in the case $\tau = 4231$). We do not believe that this condition is the best possible, but we cannot prove a significantly more general result.

Theorem 1.4. *Let* $\tau \in \mathcal{S}_k$ *be a pattern such that* $\tau_1 = k$ *and* $\tau_2\tau_3\cdots\tau_k$ *satisfies the Tight Pattern Insertion Property. Then, for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\limsup_{N\to\infty}\left[P_N^\tau\left\{\left(\frac{i}{N},\frac{\pi_i}{N}\right) \in [0,\delta]\times[\epsilon,1] \text{ for some } i \in \{1,\ldots,N\}\right\}\right]^{1/N} < 1.$$

The symmetry of the pattern 4231 implies that the set of graphs of members of $\mathcal{S}_N(4231)$ is invariant upon reflection through the lines $x = y$ and $x + y = N + 1$. This observation, together with Theorem 1.4, proves the conjecture for the pattern 4231 that was mentioned in the paragraph preceding this theorem.

The key points in our proofs of the Ratio Limit Theorems, as well as Theorems 1.3(a) and 1.4, are (i) the so-called 'pattern theorem' argument first introduced by Kesten in [14]† and (ii) the notion of 'tight occurrence of patterns'.

Roughly speaking, a pattern $\tau = \tau_1\tau_2\cdots\tau_k \in \mathcal{S}_k$ occurs tightly in $\sigma = \sigma_1\sigma_2\cdots\sigma_N \in \mathcal{S}_N$ (or equivalently, σ contains a tight occurrence of τ) if it can be covered by a closed $(k-1)\times(k-1)$ box in the graph of σ. (More formally, there must exist a contiguous subsequence $\sigma_i\sigma_{i+1}\cdots\sigma_{i+k-1}$ that forms the pattern τ, with the additional property that, for some integer h, we have $\sigma_m \in [h, h+k-1]$ for every $m = i,\ldots,i+k-1$. See Definition 4. In the terminology of Bousquet-Mélou, Claesson, Dukes and Kitaev [10], a tight occurrence of $\tau = \tau_1\cdots\tau_k$ is equivalent to the occurrence of the 'bivincular' pattern $(\tau,[1,k-1],[1,k-1])$.) See Figure 1 for an example.

† Historically, the word 'pattern' has more than one meaning. Kesten [14] focused on self-avoiding walks, and other authors in that field have since referred to his result, and others like it, as 'pattern theorems' (*e.g.*, [13] and Chapter 7 of [17]), even though Kesten did not use this term. These results are in the spirit of our Proposition 5.5 below, and their use of 'pattern' is more akin to what we call 'tight occurrence of a pattern'.

We prove a Kesten-type 'pattern theorem' for $\mathcal{S}_N(\tau)$ when $\tau \in \mathcal{S}_k$ has some specific properties that will be explained in detail later. We shall prove that there exists an $\epsilon > 0$ such that there is an exponentially small probability that a random permutation in $\mathcal{S}_N(\tau)$ has fewer than ϵN tight occurrences of specified (other) patterns, as N approaches infinity. To see why this helps with the Ratio Limit Theorems, suppose that almost every long τ-avoiding permutation contains many tight occurrences of 12s and of 123s. If τ contains no tight 123, then we can change any of σ's tight 12s into tight 123s, and *vice versa*, without affecting τ-avoidance. The corresponding relation between $\mathcal{S}_N(\tau)$ and $\mathcal{S}_{N+1}(\tau)$ is many-to-many, and is very similar to the corresponding relation between $\mathcal{S}_{N-1}(\tau)$ and $\mathcal{S}_N(\tau)$. Essentially, Kesten used this idea to show that $|\mathcal{S}_{N+1}(\tau)|/|\mathcal{S}_N(\tau)|$ cannot be too different from $|\mathcal{S}_N(\tau)|/|\mathcal{S}_{N-1}(\tau)|$. The core of this argument is in Lemma 5.7 and Proposition 5.8, while Corollary 2.3 provides the finishing touch.

This paper is organized as follows. Section 2 states the definitions and the basic terminology needed for the rest of the paper, and presents some background results and useful bounds. Section 3 gives the (relatively easy) proof of Theorem 1.3(b), which tells us when certain regions of the graph of a random τ-avoiding permutation are reasonably likely to be occupied. Section 4 proves the shape results for permutations avoiding patterns of length 3. Section 5 proves the Ratio Limit Theorem for $\mathcal{S}_N(\tau)$ when τ does not contain a tight occurrence of the 12 pattern (or does not contain a tight occurrence of the 21 pattern). Section 6 discusses the Ratio Limit Theorem for the case that $\tau = k\tau_2 \cdots \tau_k \in \mathcal{S}_k$. In Section 7 the Ratio Limit Theorem is proved for $\mathcal{S}_N(\tau)$ when $k \leqslant 5$ and some cases of $k = 6$. Sections 8 and 9 prove the shape results of Theorems 1.3(a) and 1.4 respectively.

2. Basic terminology and useful results

For each positive integer N, let $\mathcal{S}_N$ denote the set of all permutations of the numbers $1, \ldots, N$. We represent a permutation as a string,

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_N,$$

where each σ_i is in $\{1, \ldots, N\}$, and $\sigma_i \neq \sigma_j$ whenever $i \neq j$ $(1 \leqslant i, j \leqslant N)$. Viewed as a function (as in Figure 1), the map $i \mapsto \sigma_i$ is a bijection of $\{1, \ldots, N\}$ to itself.

Definition 2. Let k be a positive integer $(k \geqslant 2)$.

(a) Let $\tau = \tau_1 \tau_2 \cdots \tau_k$ be a permutation in $\mathcal{S}_k$. A string of k distinct integers $a_1 a_2 \cdots a_k$ *forms the pattern* τ if, for each $i = 1, \ldots, k$, a_i is the (τ_i)th smallest element of $\{a_1, \ldots, a_k\}$ (*e.g.*, 7932 forms the pattern 3421 because 7 is the third smallest, 9 is the fourth smallest, *etc.*). In this case, we also write $\tau = \mathrm{Patt}(a_1 a_2 \cdots a_k)$, thus defining a map from strings of distinct integers to permutations.

(b) Let N be an integer with $N \geqslant k$. For $\sigma \in \mathcal{S}_N$ and $\tau \in \mathcal{S}_k$, we say that σ *contains the pattern* τ if some k-element subsequence $\sigma_{i(1)} \sigma_{i(2)} \cdots \sigma_{i(k)}$ of σ (where $1 \leqslant i(1) < i(2) < \cdots < i(k) \leqslant N$) forms the pattern τ. If σ contains τ, we also say that the pattern τ occurs in σ. We say that σ *avoids the pattern* τ if σ does not contain τ. Let $\mathcal{S}_N(\tau)$ be the set of all permutations of $\{1, \ldots, N\}$ that avoid τ.

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Let

$$L(\tau) := \lim_{N\to\infty} |\mathcal{S}_N(\tau)|^{1/N} = \sup_{N\geqslant 1} |\mathcal{S}_N(\tau)|^{1/N}.$$

In 2004, Marcus and Tardos [18] proved that the quantity $L(\tau)$ exists and is finite for every τ. The second equality follows from the fact that $|\mathcal{S}_N(\tau)|$ is supermultiplicative in N (as observed by [3]). In particular,

$$|\mathcal{S}_N(\tau)| \leqslant L(\tau)^N \quad \text{for every } N. \tag{2.1}$$

We shall use the notation that $\lfloor x\rfloor$ is the largest integer less than or equal to x, and $\lceil x\rceil$ is the smallest integer greater than or equal to x.

The following elementary bounds will be used frequently.

Lemma 2.1.

(a) *Let s be a positive real number. There exist constants c_1 and c_2 (depending on s) such that*

$$\frac{c_1}{\sqrt{n}}(sn/e)^{sn} \leqslant \lfloor sn\rfloor ! \leqslant c_2\sqrt{n}(sn/e)^{sn} \quad \textit{for every integer } n \geqslant 1.$$

(b) *For integers $k > m > 0$,*

$$\binom{k}{m} \leqslant \left(\frac{k}{m}\right)^k \left(\frac{k}{k-m}\right)^{k-m}.$$

(c) *Assume $0 < a < b$, and let $t = a/b$. Then there exist constants c_3 and c_4 (depending on a and b) such that*

$$\frac{c_3}{n^{5/2}}(t^t(1-t)^{1-t})^{-bn} \leqslant \binom{\lfloor bn\rfloor}{\lfloor an\rfloor} \leqslant c_4(t^t(1-t)^{1-t})^{-bn}$$

for every integer $n \geqslant 1$.

Proof. **(a)** This follows from Stirling's formula $\lim_{x\to\infty}\Gamma(x+1)(e/x)^x/\sqrt{x} = \sqrt{2\pi}$ for the Gamma function Γ, and the bounds $\Gamma(ns+1) \geqslant \lfloor sn\rfloor ! \geqslant \Gamma(ns) = \Gamma(ns+1)/ns$ for $ns \geqslant 2$.

(b) Consideration of the binomial probability distribution shows that, for every p in $(0,1)$, we have

$$\binom{k}{m} p^m(1-p)^{k-m} \leqslant 1, \quad \text{and hence} \quad \binom{k}{m} \leqslant \left(\frac{1}{p}\right)^m \left(\frac{1}{1-p}\right)^{k-m}. \tag{2.2}$$

Now take $p = m/k$ to obtain part (b).

(c) The left inequality follows from part (a) and the bound $\lfloor bn\rfloor - \lfloor an\rfloor \leqslant \lfloor (b-a)n\rfloor + 1$. For the right inequality, apply inequality (2.2) with $k = \lfloor bn\rfloor$, $m = \lfloor an\rfloor$, and $p = t$, and use the bound $\lfloor bn\rfloor - \lfloor an\rfloor \leqslant (b-a)n + 1$. □

To prove the Ratio Limit Theorems, we shall use the following lemma and its corollary. It is a modification of a similar result in [14].

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Lemma 2.2 (Lemma 7.3.1 of [17]). *Let $\{a_N\}$ be a sequence of positive numbers. Assume that:*

(i) $\lim_{N\to\infty} a_N^{1/N} = \mu$,
(ii) $\liminf_{N\to\infty}(a_{N+1}/a_N) > 0$, *and*
(iii) *there exists a constant $D > 0$ such that*

$$\frac{a_{N+2}}{a_N} \geqslant \left(\frac{a_{N+1}}{a_N}\right)^2 - \frac{D}{N}$$

for all sufficiently large N. Then

$$\lim_{N\to\infty} \frac{a_{N+1}}{a_N} = \mu.$$

Observe that if D were 0 in Lemma 2.2(iii), then the sequence of ratios $\{a_{N+1}/a_N\}$ would be increasing and hence clearly convergent.

Corollary 2.3. *Let τ be a pattern. Assume there exists a constant $D > 0$ such that*

$$\frac{|\mathcal{S}_{N+2}(\tau)|}{|\mathcal{S}_N(\tau)|} \geqslant \left(\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|}\right)^2 - \frac{D}{N}$$

for all sufficiently large N. Then

$$\lim_{N\to\infty} \frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

Proof. This follows directly from Lemma 2.2 with $a_N = |\mathcal{S}_N(\tau)|$ and $\mu = L(\tau)$. Condition (i) follows from equation (1.2), and condition (ii) follows from the obvious inequality $|\mathcal{S}_{N+1}(\tau)| \geqslant |\mathcal{S}_N(\tau)|$. □

3. Proof of Theorem 1.3(b)

Suppose τ is a pattern in $\mathcal{S}_k$ with $\tau_1 > \tau_k$. This section proves that if $\tau_1 \neq k$, then no region above the diagonal $x = y$ is very unlikely to contain a point of the graph of a randomly chosen τ-avoiding permutation. More precisely, we shall prove the following result, which immediately implies Theorem 1.3(b).

Proposition 3.1. *Assume $\tau \in \mathcal{S}_k$ and $\tau_k < \tau_1 < k$. Choose $\{i(N)\}$, $\{j(N)\}$ such that $1 \leqslant i(N) < j(N) \leqslant N$ for every N. Then*

$$\lim_{N\to\infty} |\{\sigma \in \mathcal{S}_N(\tau) : \sigma_{i(N)} = j(N)\}|^{1/N} = L(\tau).$$

□

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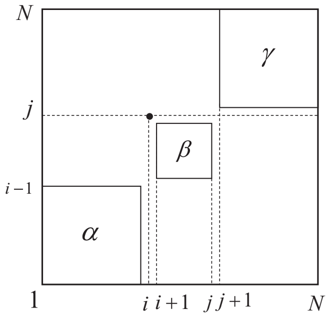

Figure 4. How the function F constructs the permutation σ from the three permutations α, β and γ in the proof of Proposition 3.1.

Proof. Fix $1 \leqslant i < j \leqslant N$. Define $F : \mathcal{S}_{i-1}(\tau) \times \mathcal{S}_{j-i}(\tau) \times \mathcal{S}_{N-j}(\tau) \to \mathcal{S}_N$ as follows. Suppose that

$$\alpha = \alpha_1\alpha_2 \cdots \alpha_{i-1} \in \mathcal{S}_{i-1}(\tau),$$
$$\beta = \beta_1\beta_2 \cdots \beta_{j-i} \in \mathcal{S}_{j-i}(\tau) \quad \text{and}$$
$$\gamma = \gamma_1\gamma_2 \cdots \gamma_{N-j} \in \mathcal{S}_{N-j}(\tau).$$

Let

$$\sigma_k = \begin{cases} \alpha_k & \text{if } 1 \leqslant k \leqslant i-1, \\ j & \text{if } k = i, \\ i-1+\beta_{k-i} & \text{if } i < k \leqslant j, \\ j + \gamma_{k-j} & \text{if } j < k \leqslant N. \end{cases}$$

Then $F(\alpha, \beta, \gamma) = \sigma_1\sigma_2 \cdots \sigma_N$. It is clear from the construction that σ has the following block structure: the graph of σ is obtained by first adding a point to the upper left corner of the graph of β (this point will become (i, j)), then inserting a translate of the result at the upper right corner of α, and finally inserting a translate of the graph of γ at the upper right corner of β. See Figure 4.

We next show that $\sigma \in \mathcal{S}_N(\tau)$. Suppose to the contrary that σ contains the pattern τ so there exists a subsequence $\sigma_{\ell[1]} \cdots \sigma_{\ell[k]}$ which forms the pattern τ. We consider three separate cases for $\ell[1]$:

(1) $1 \leqslant \ell[1] \leqslant i-1$,
(2) $i \leqslant \ell[1] \leqslant j$, or
(3) $j < \ell[1] \leqslant N$.

First assume that $i \leqslant \ell[1] \leqslant j$. Then, because of the block structure of σ and the fact that $\sigma_{\ell[1]} > \sigma_{\ell[k]}$ (since $\tau_1 > \tau_k$), we should also have $\ell[k] \leqslant j$. Also $\tau_1 < k$ implies that $\ell[1] \neq i$. Hence this would result in a τ-pattern in β. This is a contradiction since $\beta \in \mathcal{S}_{j-i}(\tau)$. The other two cases can be treated similarly. Therefore, $\sigma \in \mathcal{S}_N(\tau)$.

It is also clear that F is one-to-one, so

$$|\mathcal{S}_{i-1}(\tau)|\,|\mathcal{S}_{j-i}(\tau)|\,|\mathcal{S}_{N-j}(\tau)| \leqslant |\{\sigma \in \mathcal{S}_N(\tau) : \sigma_i = j\}|.$$

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Let $\epsilon > 0$. There exists $A > 0$ such that $|\mathcal{S}_k(\tau)| > A(L(\tau) - \epsilon)^k$ for all $k \geqslant 0$ ($|\mathcal{S}_0(\tau)| := 1$). Therefore $|\{\sigma \in \mathcal{S}_N(\tau) : \sigma_i = j\}| > A^3(L(\tau) - \epsilon)^{N-1}$, and the proposition follows. □

4. 312-avoiding permutations

The main purpose of this section is to prove Theorem 1.2. Most of the work for this is done in Theorem 4.1. Proposition 4.2 and Theorem 4.3 are consequences of Theorem 4.1. The proof of Theorem 1.2 appears at the end of the section.

Recall $L(312) = L(321) = 4$. Define the function $\mathcal{L}$ on $[0,1]^2$ by $\mathcal{L} = 4\mathcal{K}$, where $\mathcal{K}$ was defined in equation (1.3), that is,

$$\mathcal{L}(s,t) = \frac{(2-s-t)^{2-s-t}(s+t)^{s+t}}{(1-s)^{1-s}(1-t)^{1-t}t^t s^s} \quad \text{if } (s,t) \in [0,1]^2,$$

where we interpret $0^0 = 1$. The following subset of permutations will play an important role.

Definition 3. For a pattern τ and integers $1 \leqslant I \leqslant J \leqslant N$, define

$$\mathcal{S}_{N,I,J}(\tau) = \{\sigma \in \mathcal{S}_N(\tau) : \sigma_I = J \text{ and } \sigma_k < J \text{ for } k = 1, \ldots, I-1\}.$$

Theorem 4.1. *Let Δ be the triangle $\{(s,t) : 0 \leqslant s < t \leqslant 1\}$. For a pattern τ and a relatively open subset D of Δ, let*

$$\mathcal{S}_N[\tau, D] = \left\{\sigma \in \mathcal{S}_N(\tau) : \left(\frac{i}{N}, \frac{\sigma_i}{N}\right) \in D \text{ for some } i \in \{1,\ldots,N\}\right\}.$$

Then

$$\lim_{N\to\infty} |\mathcal{S}_N[312, D]|^{1/N} = \sup\{\mathcal{L}(s,t) : (s,t) \in D\}.$$

Proof. In this proof, for brevity, we shall write $\mathcal{S}_{N,I,J}$ to denote $\mathcal{S}_{N,I,J}(312)$ from Definition 3. The key to the proof is the exact calculation of the size of $\mathcal{S}_{N,I,J}$.

Assume $1 \leqslant I < J \leqslant N$. For $\sigma \in \mathcal{S}_{N,I,J}$, define

$$\begin{aligned}
\mathcal{U} = \mathcal{U}(\sigma) &= [1, J) \setminus \{\sigma_1, \ldots, \sigma_{I-1}\}, \\
\mathcal{W} = \mathcal{W}(\sigma) &= \{i \in (I, N] : \sigma_i < J\}, \\
\xi = \xi(\sigma) &= \text{Patt}(\sigma_1 \cdots \sigma_{I-1}), \\
\psi = \psi(\sigma) &= \text{Patt}(\sigma_i : i > I, \sigma_i > J),
\end{aligned}$$

where in the last line we write $(\sigma_i : i > I, \sigma_i > J)$ to denote the subsequence of σ consisting of elements σ_i satisfying $i > I$ and $\sigma_i > J$. See Figure 5. Also, where there is no chance of confusion, we write intervals to denote contiguous sets of integers (*e.g.*, we write $[1, J)$ to denote $[1, J) \cap \mathbb{Z}$). We interpret $[a, b] = \emptyset$ if $b < a$, and $[a, b) = \emptyset$ if $b \leqslant a$.

Observe the following:

(a) $|\mathcal{U}| = J - I = |\mathcal{W}|$,

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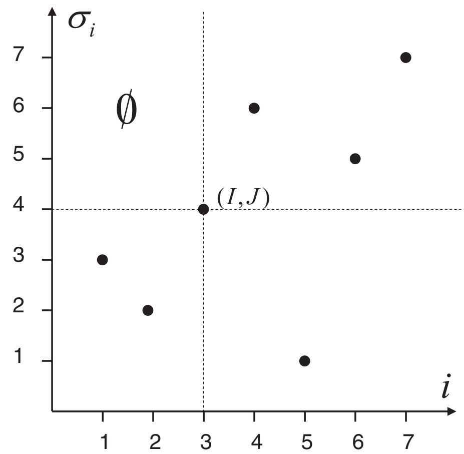

Figure 5. Let $\sigma = 3246157$, $I = 3$ and $J = 4$. Then $\mathcal{U} = [1,4) \setminus \{\sigma_1, \sigma_2\} = \{1\}$, $\mathcal{W} = \{i \in (3,7] : \sigma_i < 4\} = \{5\}$, $\xi = \mathrm{Patt}(\sigma_1, \sigma_2) = \mathrm{Patt}(3,2) = 21$ and $\psi = \mathrm{Patt}(\sigma_i : i > 3, \sigma_i > 4) = \mathrm{Patt}(6,5,7) = 213$. The interior of the rectangle labelled '$\emptyset$' is empty, *i.e.*, it contains no points of the graph of σ.

(b) the subsequence $(\sigma_i : i \in \mathcal{W})$ is decreasing (since σ avoids 312, and $\sigma_I = J > \sigma_i$ for every i in $\mathcal{W}$), and

(c) $\xi \in \mathcal{S}_{I-1}(312)$ and $\psi \in \mathcal{S}_{N-J}(312)$.

For a set A and an integer $m > 0$, let $\mathcal{P}_m(A)$ be the collection of all m-element subsets of A. From the above observations (especially (b)), we see that the map

$$\begin{aligned} \mathcal{S}_{N,I,J} &\to \mathcal{P}_{J-I}([1,J)) \times \mathcal{P}_{J-I}((I,N]) \times \mathcal{S}_{I-1}(312) \times \mathcal{S}_{N-J}(312), \\ \sigma &\mapsto (\mathcal{U}, \mathcal{W}, \xi, \psi) \end{aligned} \tag{4.1}$$

is one-to-one. To help evaluate $|\mathcal{S}_{N,I,J}|$, we shall characterize the image of $\mathcal{S}_{N,I,J}$ under this map.

For $\sigma \in \mathcal{S}_{N,I,J}$, we can express $\mathcal{W}(\sigma)$ as the union of some number of non-contiguous intervals of integers, that is,

$$\mathcal{W}(\sigma) = \left(\bigcup_{l=1}^{m} [B_{l-1}, A_l) \right) \cup [B_m, N],$$

where the B_l and A_l are integers satisfying

$$I + 1 = B_0 \leqslant A_1 < B_1 < A_2 < B_2 < \cdots < B_m \leqslant N + 1,$$

and also

$$(I, N] \setminus \mathcal{W}(\sigma) = \bigcup_{l=1}^{m} [A_l, B_l).$$

Observe that if $s \in [A_l, B_l)$ and $t \in [A_r, B_r)$ with $l < r$, then we must have $\sigma_s < \sigma_t$ (otherwise, $\sigma_s \sigma_{B_l} \sigma_t$ would form the pattern 312: to see this, recall that $\sigma_{B_l} < J < \min\{\sigma_s, \sigma_t\}$). Therefore ψ must be of the form

$$\psi = \psi_1 \circ \psi_2 \circ \cdots \circ \psi_m, \quad \text{where } \psi_l \in \mathcal{S}_{B_l - A_l}(312) \text{ for each } l, \tag{4.2}$$

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and '$\circ$' is the concatenation operator

$$(\theta_1\cdots\theta_u)\circ(\phi_1\cdots\phi_v) := (\theta_1\cdots\theta_u, (\phi_1+u)\cdots(\phi_v+u))$$

for $\theta\in\mathcal{S}_u$ and $\phi\in\mathcal{S}_v$. Let $\Psi[\mathcal{W}]$ be the set of all permutations ψ of the form (4.2) (where m and the B_l–A_l depend on $\mathcal{W}$).

Similarly, for $\sigma\in\mathcal{S}_{N,I,J}$, we can express

$$\mathcal{U}(\sigma) = \left(\bigcup_{l=1}^{\bar{m}}[D_{l-1},E_l)\right)\cup[D_{\bar{m}},J),$$

where the D_l and E_l are integers satisfying

$$1 = D_0 \leqslant E_1 < D_1 < E_2 < D_2 < \cdots < D_{\bar{m}} \leqslant J,$$

and

$$[1,J)\setminus\mathcal{U}(\sigma) = \bigcup_{l=1}^{\bar{m}}[E_l,D_l).$$

Observe that if $s<t<I$ and $\sigma_s\in[E_l,D_l)$, then we must have $\sigma_t\geqslant E_l$ (otherwise, $\sigma_s\sigma_t\sigma_u$ would form the pattern 312, where $\sigma_u = D_{l-1}$). Therefore ξ must be of the form

$$\xi_1\circ\xi_2\circ\cdots\circ\xi_{\bar{m}}, \quad \text{where } \xi_l\in\mathcal{S}_{D_l-E_l}(312) \text{ for each } l. \tag{4.3}$$

Let $\Xi[\mathcal{U}]$ be the set of all permutations ξ of the form (4.3) (where $\bar{m}$ and the D_l–E_l depend on $\mathcal{U}$).

From the above analysis, it is not hard to see that the map

$$\mathcal{S}_{N,I,J}\to\left(\bigcup_{\mathcal{U}\in\mathcal{P}_{J-I}([1,J))}\{\mathcal{U}\}\times\Xi[\mathcal{U}]\right)\times\left(\bigcup_{\mathcal{W}\in\mathcal{P}_{J-I}((I,N])}\{\mathcal{W}\}\times\Psi[\mathcal{W}]\right), \tag{4.4}$$

$$\sigma\mapsto(\mathcal{U}(\sigma),\xi(\sigma),\mathcal{W}(\sigma),\psi(\sigma)) \tag{4.5}$$

is a surjection, hence a bijection.

Now we count the right-hand side of (4.4). For a given $\mathcal{W}$, we have

$$|\Psi[\mathcal{W}]| = \prod_{l=1}^{m} C_{B_l-A_l},$$

where $C_1, C_2,\ldots$ are the Catalan numbers:

$$C_k = |\mathcal{S}_k(312)| = \frac{\binom{2k}{k}}{k+1}.$$

It follows that

$$\left|\bigcup_{\mathcal{W}\in\mathcal{P}_{J-I}((I,N])}\{\mathcal{W}\}\times\Psi[\mathcal{W}]\right| = \sum_{m=1}^{N-J}\left(\Sigma_m^*\prod_{l=1}^{m}C_{k_l}\right)W_m^*, \tag{4.6}$$

where:

Σ_m^* is the summation over all $k_1,\ldots,k_m\geqslant 1$ such that $k_1+\cdots+k_m = N-J$ (here we associate k_l with B_l-A_l), and

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W_m^* is the number of choices of integers with $r_0 \geqslant 0$, $r_m \geqslant 0$, $r_l \geqslant 1$ for $l = 1, \ldots, m-1$, and $r_0 + \cdots + r_m = J - I$ (here we associate r_l with $A_{l+1} - B_m$ for $l = 0, \ldots, m-1$, and r_m with $N + 1 - B_m$).

Therefore we have

$$W_m^* = \binom{J - I + 1}{m}. \tag{4.7}$$

For a non-negative integer p and a power series F, we use the notation $[x^p]F(x)$ to denote the coefficient of x^p in $F(x)$.

Recall that the generating function of the Catalan numbers (starting from $k = 1$) is

$$\mathcal{C}(x) := \sum_{k=1}^{\infty} C_k x^k = \frac{1 - \sqrt{1 - 4x}}{2x} - 1.$$

Therefore $\Sigma_m^* \prod_{l=1}^m C_{k_l}$ is the coefficient of x^{N-J} in $\mathcal{C}(x)^m$. By equations (4.6) and (4.7), we have

$$\begin{aligned}\left| \bigcup_{\mathcal{W} \in \mathcal{P}_{J-I}((I,N])} \{\mathcal{W}\} \times \Psi[\mathcal{W}] \right| &= \sum_{m=1}^{N-J} \left([x^{N-J}] \mathcal{C}(x)^m \right) \binom{J - I + 1}{m} \\ &= [x^{N-J}] \sum_{m=1}^{N-J} \mathcal{C}(x)^m \binom{J - I + 1}{m}. \end{aligned} \tag{4.8}$$

Next, we claim that

$$[x^{N-J}] \sum_{m=1}^{N-J} \mathcal{C}(x)^m \binom{J - I + 1}{m} = [x^{N-J}] \sum_{m=1}^{J-I+1} \mathcal{C}(x)^m \binom{J - I + 1}{m}. \tag{4.9}$$

On the one hand, if $J - I + 1 \leqslant N - J$, then (4.9) is true because

$$\binom{J - I + 1}{m} = 0 \quad \text{for } m > J - I + 1.$$

On the other hand, if $J - I + 1 > N - J$, then (4.9) is true because

$$[x^{N-J}] \mathcal{C}(x)^m = 0 \quad \text{for } m > N - J$$

(observe that m is the smallest power of x in $\mathcal{C}(x)^m$). So the claim is true, and together with (4.8) it implies that

$$\begin{aligned}\left| \bigcup_{\mathcal{W} \in \mathcal{P}_{J-I}((I,N])} \{\mathcal{W}\} \times \Psi[\mathcal{W}] \right| &= [x^{N-J}] (1 + \mathcal{C}(x))^{J-I+1} \\ &= [x^{N-J}] \left(\frac{1 - \sqrt{1 - 4x}}{2x} \right)^{J-I+1} \\ &= [x^{N-I+1}] \left(\frac{1 - \sqrt{1 - 4x}}{2} \right)^{J-I+1}. \end{aligned} \tag{4.10}$$

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An analogous argument shows that

$$\left|\bigcup_{\mathcal{U}\in\mathcal{P}_{J-I}([1,J))}\{\mathcal{U}\}\times\Xi[\mathcal{U}]\right| = [x^{I-1}](1+\mathcal{C}(x))^{J-I+1}$$
$$= [x^J]\left(\frac{1-\sqrt{1-4x}}{2}\right)^{J-I+1}. \tag{4.11}$$

We shall also use the identity (see p. 68 of [12])

$$[x^n]\left(\frac{1-\sqrt{1-4x}}{2}\right)^k = \frac{k}{n}\binom{2n-k-1}{n-1}. \tag{4.12}$$

Since the map of equations (4.4)–(4.5) is a bijection, we see from equations (4.10), (4.11) and (4.12) that

$$|\mathcal{S}_{N,I,J}| = [x^{N-I+1}]\left(\frac{1-\sqrt{1-4x}}{2}\right)^{J-I+1}\times[x^J]\left(\frac{1-\sqrt{1-4x}}{2}\right)^{J-I+1}$$
$$= \frac{(J-I+1)^2}{J(N-I+1)}\binom{2N-I-J}{N-I}\binom{I+J-2}{J-1}. \tag{4.13}$$

We now state some consequences of equation (4.13). First, by Lemma 2.1(b) and the simple bound $\binom{I+J-2}{J-1}\leqslant\binom{I+J}{J}$, we have

$$|\mathcal{S}_{N,I,J}|\leqslant\mathcal{L}\left(\frac{I}{N},\frac{J}{N}\right)^N \quad \text{for } I\leqslant J\leqslant N. \tag{4.14}$$

Next, we introduce the following condition:

$$\text{the integer sequences } \{i(N)\} \text{ and } \{j(N)\} \text{ satisfy } 1\leqslant i(N)<j(N)\leqslant N \text{ and } \lim_{N\to\infty}(i(N)/N, j(N)/N)=(s,t)\in[0,1]^2. \tag{4.15}$$

If condition (4.15) holds, then

$$\lim_{N\to\infty}|\mathcal{S}_{N,i(N),j(N)}|^{1/N}=\mathcal{L}(s,t). \tag{4.16}$$

For $I\leqslant J\leqslant N$, define

$$\mathcal{S}^*_{N,I,J}=\{\sigma\in\mathcal{S}_N(312):\sigma_I=J\}.$$

Then $\mathcal{S}^*_{N,I,J}$ is a subset of $\cup_{i\leqslant I,j\geqslant J}\mathcal{S}_{N,i,j}$. Then since $\mathcal{L}(s,t)$ is increasing in s and decreasing in t on Δ, we see that

$$|\mathcal{S}_{N,I,J}|\leqslant|\mathcal{S}^*_{N,I,J}|\leqslant N^2\mathcal{L}\left(\frac{I}{N},\frac{J}{N}\right)^N.$$

From this we conclude that if condition (4.15) holds, then

$$\lim_{N\to\infty}|\mathcal{S}^*_{N,i(N),j(N)}|^{1/N}=\mathcal{L}(s,t). \tag{4.17}$$

Now consider $(s,t)\in D$. Choose sequences $\{i(N)\}$ and $\{j(N)\}$ such that condition (4.15) holds and $(i(N)/N, j(N)/N)\in D$ for every N. Then $\mathcal{S}^*_{N,i(N),j(N)}\subset\mathcal{S}_N[312,D]$, so equation

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(4.17) implies that

$$\liminf_{N\to\infty} |\mathcal{S}_N[312,D]|^{1/N} \geqslant \sup\{\mathcal{L}(s,t) : (s,t) \in D\}. \tag{4.18}$$

Finally, it is not hard to see that

$$|\mathcal{S}_N[312,D]| \leqslant \sum_{(i,j)\in\mathbf{N}^2:(i/N,j/N)\in D} |\mathcal{S}^*_{N,i,j}| \leqslant N^4 \sup\{\mathcal{L}(s,t)^N : (s,t)\in D\}.$$

This provides the counterpart to equation (4.18) that completes the proof of the theorem. □

Next, we consider the probability that the graph of a random permutation $\pi = \pi_1\cdots\pi_N$ in $\mathcal{S}_N(312)$ (with uniform distribution P_N^{312}) has more than tN points (i,π_i) $(i\in\{1,2,\ldots,N\})$ below a neighbourhood of radius δN of the diagonal of $[1,N]\times[1,N]$ (for given $\delta>0$ and $t>0$). The next proposition shows that this probability decays exponentially as $N\to\infty$.

Given $\sigma = \sigma_1,\ldots,\sigma_N \in \mathcal{S}_N$ and $A>0$, let $K_N(\sigma,A) = |\{i : \sigma_i < i - A\}|$.

Proposition 4.2. *Let $\delta > 0$ and $0 < t < 1$. Then*

$$\limsup_{N\to\infty} [P_N^{312}(K_N(\pi,\delta N) > tN)]^{1/N} < 1.$$

Proof. Let $\epsilon = \delta t/(1-t)$. We first claim that if $K_N(\sigma,\delta N) \geqslant tN$ then $\sigma_i > i + \epsilon N$ for some i, that is, the graph of σ has a point (i,σ_i) above an ϵN-neighbourhood of the diagonal. Suppose to the contrary that $\sigma_i \leqslant i+\epsilon N$ for every $1\leqslant i\leqslant N$. Let $A=\delta N$ and $B=\epsilon N$. Then

$$\sigma_i < i - A \quad \text{for } K_N(\sigma,A) \text{ values of } i$$

and

$$\sigma_i < i + B \quad \text{for every (other) } i.$$

So

$$\sum_{j=1}^N j = \sum_{i=1}^N \sigma_i < \sum_{i=1}^N i + B(N - K_N(\sigma,A)) - AK_N(\sigma,A),$$

and hence $BN - BK_N(\sigma,A) - AK_N(\sigma,A) > 0$, that is,

$$K_N(\sigma,A) < \frac{B}{B+A}N = tN,$$

which is a contradiction. Hence

$$P_N^{312}(K_N(\pi,\delta N)\geqslant tN) \leqslant P_N^{312}(\pi_i > i+\epsilon N \text{ for some } i).$$

Therefore, using Theorem 4.1, we conclude that

$$\limsup_{N\to\infty} [P_N^{312}(K_N(\pi,\delta N) > tN)]^{1/N} < 1.$$ □

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We remark that the region below a neighbourhood of the diagonal is not very likely to be *completely* empty. In fact, Proposition 3.1 (applied to the reverse complement of 312) implies that

$$\lim_{N\to\infty} [P_N^{312}(K_N(\pi, \delta N) > 0)]^{1/N} = 1$$

for every $\delta \in (0,1)$.

Finally, equation (1.4) of Theorem 1.2 is a consequence of the following result and symmetry.

Theorem 4.3. *Recall the terminology of Theorem 4.1. Let D be a relatively open subset of Δ. Then*

$$\lim_{N\to\infty} |\mathcal{S}_N[321, D]|^{1/N} = \sup\{\mathcal{L}(s,t) : (s,t) \in D\}.$$

Proof. Recall the definition of $\mathcal{S}_{N,I,J}(\tau)$ from Definition 3. Simion and Schmidt [21] define a bijection from $\mathcal{S}_N(312)$ to $\mathcal{S}_N(321)$ that is also a bijection from $\mathcal{S}_{N,I,J}(312)$ to $\mathcal{S}_{N,I,J}(321)$ whenever $I \leqslant J \leqslant N$ (the details are also given in Section 4.2 of [7]). That is, $|\mathcal{S}_{N,I,J}(321)| = |\mathcal{S}_{N,I,J}(312)|$, and the latter is given by equation (4.13). Therefore the part of the proof of Theorem 4.1 subsequent to equation (4.13) applies to the pattern 321 as well as to 312. □

Proof of Theorem 1.2. Recall that $L(321) = L(312) = 4$. Equation (1.4) follows directly from Theorem 4.3 and the symmetry of 321 under reverse complement. In the case that D lies entirely above the diagonal $y = x$, equation (1.5) is equivalent to Theorem 4.1. For other open sets D, the limit in equation (1.5) is 1, and this is a consequence of Proposition 3.1 applied to $\tau = 231$, which is the reverse complement of 312. □

5. Tight occurrence of 12 and 21 patterns

The goal of this section is to first obtain a pattern theorem for $\mathcal{S}_N(\tau)$ when τ satisfies condition (C1) from the Introduction, and then prove the Ratio Limit Theorem for $\mathcal{S}_N(\tau)$ under this condition. Recall that (C1) says that τ contains no tight occurrence of 12 (or contains no tight occurrence of 21).

Definition 4. Given a positive integer t, let $\gamma \in \mathcal{S}_t$ and $\sigma \in \mathcal{S}_N$.

(a) Let $i \in \{1, \ldots, N-t+1\}$. We say that the pattern γ occurs tightly in σ (or equivalently, σ contains a tight occurrence of γ) at position i if $\sigma_i\sigma_{i+1}\cdots\sigma_{i+t-1}$ forms a γ pattern and satisfies the relation

$$\max\{\sigma_{i+j}\}_{0\leqslant j\leqslant t-1} - \min\{\sigma_{i+j}\}_{0\leqslant j\leqslant t-1} = t-1.$$

(b) Let $h \in \{1, \ldots, N-t+1\}$. We say that the pattern γ occurs tightly in σ (or equivalently, σ contains a tight occurrence of γ) at height h if there exists an $i \in \{1, \ldots, N-t+1\}$ such that $\sigma_i\sigma_{i+1}\cdots\sigma_{i+t-1}$ forms a γ pattern, and $h \leqslant \sigma_{i+j} \leqslant h+t-1$ for every $j \in$

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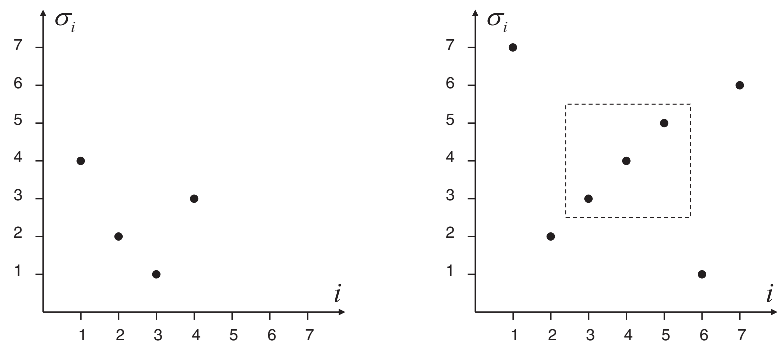

(a) $\sigma = 4213$ (b) $\mathcal{H}_3(\sigma; h^u) = 7234516$

Figure 6. Let $\sigma = 4213$, $t = 3$, $h = 2$ and $\tau = 132$, which contains no tight occurrence of 12. Then $\sigma = 4213 \in \mathcal{S}_4(\tau)$, and by Lemma 5.3, $\mathcal{H}_3(\sigma; h^u) = 7234516 \in \mathcal{S}_7(\tau)$.

$\{0, 1, \ldots, t-1\}$. Note that the subsequence $\sigma_i\sigma_{i+1}\cdots\sigma_{i+t-1}$ can be covered by the $(t-1)\times(t-1)$ box $[i, i+t-1]\times[h, h+t-1]$ in the graph of σ.

Note that we sometimes refer to a tight occurrence of γ in σ at position i (height h) as σ containing γ at position i (height h). Given a positive integer t, let $\alpha_t := 12\cdots t$ and $\beta_t := t(t-1)\cdots 1$. In particular, if $\gamma = \alpha_t$ ($\gamma = \beta_t$) then Definition 4 can be restated as Example 5.1 (Example 5.2) below.

Example 5.1. *Let t be a positive integer and $\sigma \in \mathcal{S}_N$.*

(a) *Let $i \in \{1, \ldots, N-t+1\}$. An α_t occurs tightly in σ at position i if and only if $\sigma_{i+j} = \sigma_i + j$ for $j \in \{0, 1, \ldots, t-1\}$.*
(b) *Let $h \in \{1, \ldots, N-t+1\}$. An α_t occurs tightly in σ at height h if and only if there exists an $i \in \{1, \ldots, N-t+1\}$ such that $\sigma_{i+j} = h + j$ for $j \in \{0, 1, \ldots, t-1\}$.*

Example 5.2. *Let t be a positive integer and $\sigma \in \mathcal{S}_N$.*

(a) *Let $i \in \{1, \ldots, N-t+1\}$. A β_t occurs tightly in σ at position i if and only if $\sigma_{i+j} = \sigma_i + t - j - 1$ for $j \in \{0, 1, \ldots, t-1\}$.*
(b) *Let $h \in \{1, \ldots, N-t+1\}$. A β_t occurs tightly in σ at height h if and only if there exists an $i \in \{1, \ldots, N-t+1\}$ such that $\sigma_{i+j} = h + t - j - 1$ for every $j \in \{0, 1, \ldots, t-1\}$.*

The following definition introduces an operation which, roughly speaking, inserts an α_t tightly and immediately above and to the right of the point at a given height h in the graph of σ (for an example, see Figure 6).

Definition 5. Let t be a positive integer and $\sigma \in \mathcal{S}_N$.

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(a) Let $h \in \{1,\dots,N\}$. Choose J such that $\sigma_J = h$. Define the permutation θ in $\mathcal{S}_{N+t}$ as follows:

$$\theta_i = \begin{cases} \sigma_i & \text{if } i \leqslant J \text{ and } \sigma_i \leqslant h,\\ \sigma_i + t & \text{if } i < J \text{ and } \sigma_i > h,\\ h+j & \text{if } i = J+j \text{ for } j \in \{1,\dots,t\},\\ \sigma_{i-t} & \text{if } i > J+t \text{ and } \sigma_{i-t} < h,\\ \sigma_{i-t}+t & \text{if } i > J+t \text{ and } \sigma_{i-t} > h. \end{cases}$$

Then we denote θ by $\mathcal{H}_t(\sigma;h^u)$.

(b) Assume $N \geqslant h_1 > h_2 > \cdots > h_r \geqslant 1$. Let $\sigma^{(0)} = \sigma$ and let $\sigma^{(w)} = \mathcal{H}_t(\sigma^{(w-1)};h_w^u)$ for $w = 1,\dots,r$. Then we denote $\sigma^{(r)}$ by $\mathcal{H}_t(\sigma;h_1^u,\dots,h_r^u)$.

Similarly, the following definition introduces an operation which, roughly speaking, inserts a β_t tightly and immediately below and to the right of the point at a given height h.

Definition 6. Let t be a positive integer and $\sigma \in \mathcal{S}_N$.

(a) Let $h \in \{1,\dots,N\}$. Choose J such that $\sigma_J = h$. Define the permutation γ in $\mathcal{S}_{N+t}$ as follows:

$$\gamma_i = \begin{cases} \sigma_i & \text{if } i < J \text{ and } \sigma_i < h,\\ \sigma_i + t & \text{if } i \leqslant J \text{ and } \sigma_i \geqslant h,\\ h+t-j & \text{if } i = J+j \text{ for } j \in \{1,\dots,t\},\\ \sigma_{i-t} & \text{if } i > J+t \text{ and } \sigma_{i-t} < h,\\ \sigma_{i-t}+t & \text{if } i > J+t \text{ and } \sigma_{i-t} > h. \end{cases}$$

Then we denote γ by $\mathcal{H}_t(\sigma;h^d)$.

(b) Assume $N \geqslant h_1 > h_2 > \cdots > h_r \geqslant 1$. Let $\sigma^{(0)} = \sigma$ and let $\sigma^{(w)} = \mathcal{H}_t(\sigma^{(w-1)};h_w^d)$ for $w = 1,\dots,r$. Then we denote $\sigma^{(r)}$ by $\mathcal{H}_t(\sigma;h_1^d,\dots,h_r^d)$.

The following lemma describes a situation where a permutation $\sigma \in \mathcal{S}_N(\tau)$ remains τ-avoiding after the tight insertion of an α_t immediately above and to the right of the point at a given height h, that is, $\mathcal{H}_t(\sigma;h^u) \in \mathcal{S}_{N+t}(\tau)$ (see Figure 6).

Lemma 5.3. *Let $\tau \in \mathcal{S}_k$ be a k-permutation that contains no tight occurrence of* 12. *Then, for any $\sigma := \sigma_1\cdots\sigma_N \in \mathcal{S}_N(\tau)$, $h \in \{1,\dots,N\}$ and $t \in \mathbb{N}$, we have $\mathcal{H}_t(\sigma;h^u) \in \mathcal{S}_{N+t}(\tau)$.*

Proof. First note that for $t \geqslant 2$, $\mathcal{H}_t(\sigma;h^u) = \mathcal{H}_1(\mathcal{H}_{t-1}(\sigma;h^u);(h+t-1)^u)$. So it suffices to prove the statement for $t = 1$ and the complete proof will follow by induction on t. Suppose to the contrary that $\mathcal{H}_1(\sigma;h^u) := \theta_1\cdots\theta_{N+1}$ contains the pattern τ. So there exists a subsequence $\theta_{i[1]}\cdots\theta_{i[k]}$ which forms the pattern τ. Since σ avoids τ, the α_1 inserted tightly at height h of σ has to contribute to the formation of τ in $\mathcal{H}_1(\sigma;h^u)$, that is, there exists some $1 \leqslant K \leqslant k$ such that $\theta_{i[K]} = h+1$. It

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follows from Definition 5 that $i[K] = J+1$ and $\theta_{i[K]-1} = \theta_J = \sigma_J = h$. By assumption, τ does not contain any tight occurrence of 12, and hence $i[K-1] < i[K]-1 = J$. So $\theta_{i[1]} \cdots \theta_{i[K-1]} \theta_{i[K]-1} \theta_{i[K+1]} \cdots \theta_{i[k]}$ is also a τ pattern. Using Definition 5, this pattern gives rise to the τ pattern $\sigma_{i[1]} \cdots \sigma_{i[K-1]} \sigma_{i[K]-1} \sigma_{i[K+1]-1} \cdots \sigma_{i[k]-1}$ in σ, which is a contradiction. Hence the proof is complete. □

Similarly, the following lemma gives a condition under which a permutation $\sigma \in \mathcal{S}_N(\tau)$ remains τ-avoiding after the tight insertion of β_t at the bottom right corner of the point at a given height h, that is, $\mathcal{H}_t(\sigma; h^d) \in \mathcal{S}_{N+t}(\tau)$. We omit the proof, since it is essentially the same as for Lemma 5.3.

Lemma 5.4. *Let $\tau \in \mathcal{S}_k$ be a k-permutation that contains no tight occurrence of* 21. *Then, for any $\sigma \in \mathcal{S}_N(\tau)$, $h \in \{1,\ldots,N\}$ and any $t \in \mathbb{N}$, we have $\mathcal{H}_t(\sigma; h^d) \in \mathcal{S}_{N+t}(\tau)$.*

Given $\tau \in \mathcal{S}_k$, an integer $t > 1$ and real $c > 0$, let $\mathcal{B} = \{\gamma_1, \ldots, \gamma_q\}$ be a finite set of patterns of length t. Let $\mathcal{S}_N(\tau; \mathcal{B} < c)$ denote the set of all the permutations in $\mathcal{S}_N(\tau)$ which contain fewer than c tight occurrences of the patterns in $\mathcal{B}$, that is, $\#\gamma_1 + \cdots + \#\gamma_q < c$, where $\#\gamma_i$ denotes the number of tight occurrences of γ_i in the permutation ($1 \leqslant i \leqslant q$).

Definition 7. Let t be a positive integer and $\sigma \in \mathcal{S}_N$. Assume $(x_1, \ldots, x_r) \in \{u, d\}^r$ and $N \geqslant h_1 > h_2 > \cdots > h_r \geqslant 1$. Let $\sigma^{(0)} = \sigma$ and let $\sigma^{(w)} = \mathcal{H}_t(\sigma^{(w-1)}; h_w^{x_w})$ for $w = 1, \ldots, r$. Then we denote $\sigma^{(r)}$ by $\mathcal{H}_t(\sigma; h_1^{x_1}, \ldots, h_r^{x_r})$.

In other words, $\mathcal{H}_t(\sigma; h_1^{x_1}, h_2^{x_2}, \ldots, h_r^{x_r})$ ($x_i \in \{u, d\}$ for $1 \leqslant i \leqslant r$) denotes the permutation obtained by (i) inserting an α_t tightly at the top right corner of the point at height h_i ($1 \leqslant i \leqslant r$), as described in Definition 5, if $x_i = u$, and (ii) inserting β_t tightly at the bottom right corner of the point at height h_i ($1 \leqslant i \leqslant r$), as described in Definition 6, if $x_i = d$.

The following proposition gives a condition under which a pattern theorem holds for $\mathcal{S}_N(\tau)$. It will be used to prove Corollary 5.6 and Theorem 7.4.

Proposition 5.5. *Fix $t \in \mathbb{N}$ and $\tau \in \mathcal{S}_k$. Suppose there exists an $\epsilon > 0$ such that for every $\sigma \in \mathcal{S}_N(\tau)$ there are heights*

$$N \geqslant h_1(\sigma) > h_2(\sigma) > \cdots > h_{\lfloor \epsilon N \rfloor}(\sigma) \geqslant 1 \quad \textit{and} \quad (x_1(\sigma), \ldots, x_{\lfloor \epsilon N \rfloor}(\sigma)) \in \{u, d\}^{\lfloor \epsilon N \rfloor}$$

such that

$$\mathcal{H}_t\big(\sigma; h_1(\sigma)^{x_1(\sigma)}, \ldots, h_{\lfloor \epsilon N \rfloor}(\sigma)^{x_{\lfloor \epsilon N \rfloor}(\sigma)}\big) \in \mathcal{S}_{N+t\lfloor \epsilon N \rfloor}(\tau).$$

Then there exists a $\delta > 0$ such that the following strict inequality holds:

$$\limsup_{N \to \infty} |\mathcal{S}_N(\tau; \{\alpha_{t+1}, \beta_{t+1}\} < \delta N)|^{1/N} < L(\tau).$$

Proof. Take $0 < \delta < \epsilon$ and let r be a positive integer. Let V be the set of all $(2r+1)$-tuples $\vec{v} = (\sigma, \tilde{h}_1, \ldots, \tilde{h}_r, z_1, \ldots, z_r)$, where $\sigma \in \mathcal{S}_N(\tau)$, $N \geqslant \tilde{h}_1 > \tilde{h}_2 > \cdots > \tilde{h}_r \geqslant 1$, and $z_1, \ldots, z_r \in$

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$\{u,d\}$. We shall write $\mathcal{H}_t(\vec{v})$ to denote $\mathcal{H}_t(\sigma;\tilde{h}_1^{z_1},\ldots,\tilde{h}_r^{z_r})$. Also, let V_δ be the set of $\vec{v}$ in V such that $\sigma \in \mathcal{S}_N(\tau;\{\alpha_{t+1},\beta_{t+1}\} < \delta N)$.

For given $\vec{v} = (\sigma,\tilde{h}_1,\ldots,\tilde{h}_r,z_1,\ldots,z_r) \in V_\delta$, we want an upper bound on the number of $\vec{w} = (\hat{\sigma},\hat{h}_1,\ldots,\hat{h},\hat{z}_1,\ldots,\hat{z}_r)$ in V such that $\mathcal{H}_t(\vec{w}) = \mathcal{H}_t(\vec{v})$. Let j denote the number of tight occurrences of the patterns in $\{\alpha_{t+1},\beta_{t+1}\}$ in σ. Then $0 \leqslant j \leqslant \lfloor \delta N \rfloor$ and $\mathcal{H}_t(\vec{v})$ contains at most $tr+j$ tight occurrences of the patterns in $\{\alpha_{t+1},\beta_{t+1}\}$, so there are at most $\binom{tr+j}{r}$ ways to remove r of the tight occurrences of the patterns in $\{\alpha_t,\beta_t\}$ in order to construct $\hat{\sigma}$ (and $\vec{w}$). Hence the number of choices for $\vec{w}$ is at most $\binom{tr+\lfloor \delta N\rfloor}{r}$.

Let $a = \epsilon/((t^t+2)L(\tau)^t)$ and $r = \lfloor aN \rfloor$. Let

$$\mathcal{P} = \{\vec{w} \in V : \mathcal{H}_t(\vec{w}) \in \mathcal{S}_{N+tr}(\tau) \cap \mathcal{H}_t(V_\delta)\},$$

where $\mathcal{H}_t(V_\delta) = \{\mathcal{H}_t(\vec{v}) : \vec{v} \in V_\delta\}$. Then

$$|\mathcal{S}_N(\tau;\{\alpha_{t+1},\beta_{t+1}\} < \delta N)|\binom{\lfloor \epsilon N\rfloor}{r} \leqslant |\mathcal{P}| \leqslant |\mathcal{S}_{N+tr}(\tau)|\binom{tr+\lfloor \delta N\rfloor}{r}$$

(the upper bound follows from the bound of the previous paragraph, and the lower bound follows by considering only $\vec{v}$ in V_δ for which every $\tilde{h}_i$ is in $\{h_1(\sigma),\ldots,h_{\lfloor \epsilon N\rfloor}(\sigma)\}$). Therefore, using Lemma 2.1(c), we have

$$\begin{aligned}
&|\mathcal{S}_N(\tau;\{\alpha_{t+1},\beta_{t+1}\} < \delta N)| \\
&\quad\leqslant |\mathcal{S}_{N+t\lfloor aN\rfloor}(\tau)|\binom{\lfloor (at+\delta)N\rfloor}{\lfloor aN\rfloor}\Big/\binom{\lfloor \epsilon N\rfloor}{\lfloor aN\rfloor} \\
&\quad\leqslant c\,[L(\tau)]^{(N+t\lfloor aN\rfloor)}N^{5/2}\left(\left(\frac{a}{\epsilon}\right)^{a/\epsilon}\left(1-\frac{a}{\epsilon}\right)^{(1-a/\epsilon)}\right)^{\epsilon N} \\
&\qquad\times\left(\left(\frac{a}{at+\delta}\right)^{(a/(at+\delta))}\left(1-\frac{a}{at+\delta}\right)^{(1-a/(at+\delta))}\right)^{-(at+\delta)N} \\
&\quad\leqslant \frac{c\,[L(\tau)]^N N^{5/2}}{(t^t+2)^{aN}}\left(\frac{at+\delta}{a}\right)^{aN}\left(\frac{at+\delta}{\delta+a(t-1)}\right)^{(\delta+a(t-1))N},
\end{aligned}$$

where c is a constant independent of N. Thus

$$\limsup_{N\to\infty}|\mathcal{S}_N(\tau;\{\alpha_{t+1},\beta_{t+1}\} < \delta N)|^{1/N} \leqslant \frac{L(\tau)}{(t^t+2)^a}\left(\frac{at+\delta}{a}\right)^a\left(\frac{at+\delta}{\delta+a(t-1)}\right)^{\delta+a(t-1)}.$$

We know that

$$\lim_{\delta\to 0}\left(\frac{at+\delta}{a}\right)^a\left(\frac{at+\delta}{\delta+a(t-1)}\right)^{\delta+a(t-1)} = \begin{cases} \left(\frac{t^t}{(t-1)^{t-1}}\right)^a & \text{if } t>1, \\ 1 & \text{if } t=1. \end{cases}$$

Therefore, we can choose δ sufficiently small that

$$\left(\frac{at+\delta}{a}\right)^a\left(\frac{at+\delta}{\delta+a(t-1)}\right)^{\delta+a(t-1)} < (t^t+1)^a,$$

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which implies

$$\begin{aligned}
&\limsup_{N\to\infty} |\mathcal{S}_N(\tau;\{\alpha_{t+1},\beta_{t+1}\} < \delta N)|^{1/N} \\
&\leqslant \frac{L(\tau)}{(t^t+2)^a}\left(\frac{at+\delta}{a}\right)^a\left(\frac{at+\delta}{\delta+a(t-1)}\right)^{\delta+a(t-1)} \\
&\leqslant L(\tau)\left(\frac{t^t+1}{t^t+2}\right)^a \\
&< L(\tau).
\end{aligned}$$

□

Corollary 5.6. *Let $t > 1$ be an integer. Let $\tau \in \mathcal{S}_k$ be a k-permutation that contains no tight occurrence of* 12 (*or contains no tight occurrence of* 21). *Then there exists a $\delta > 0$ such that*

$$\limsup_{N\to\infty} |\mathcal{S}_N(\tau;\{\alpha_t,\beta_t\} < \delta N)|^{1/N} < L(\tau).$$

Proof. By Lemma 5.3 (Lemma 5.4), the conditions of Proposition 5.5 are satisfied when τ does not contain any tight occurrence of 12 (21). So, by Proposition 5.5, there exists a $\delta > 0$ such that

$$\limsup_{N\to\infty} |\mathcal{S}_N(\tau;\{\alpha_t,\beta_t\} < \delta N)|^{1/N} < L(\tau).$$

□

Next, we introduce some new notation and prove two important results which, together with Corollary 2.3, immediately lead to the proof of the Ratio Limit Theorem for $\mathcal{S}_N(\tau)$ provided that τ does not contain any tight occurrence of 12 (or does not contain any tight occurrence of 21). Fix two sets of patterns, U and V. For non-negative integers a and b, let $\mathcal{S}_N(\tau;a,b)$ denote the set of all the permutations of $\mathcal{S}_N(\tau)$ that contain exactly a tight occurrences of patterns in U and b tight occurrences of patterns in V. (To be clear: if $\sigma \in \mathcal{S}_N(\tau)$ and σ has a_l tight occurrences of the lth pattern in U, then $a = \sum_l a_l$.) Also, let $\mathcal{S}_N(\tau;\geqslant a,\geqslant b)$ denote the set of all the permutations of $\mathcal{S}_N(\tau)$ that contain at least a (respectively, b) tight occurrences of patterns in U (respectively, V). Then

$$|\mathcal{S}_N(\tau;\geqslant a,\geqslant b)| := \sum_{i\geqslant a, j\geqslant b} |\mathcal{S}_N(\tau;i,j)|.$$

In particular,

$$|\mathcal{S}_N(\tau;\geqslant 0,\geqslant 0)| = |\mathcal{S}_N(\tau)|.$$

Then

$$\mathcal{S}_N(\tau;\geqslant a,> b),\quad \mathcal{S}_N(\tau;> a,> b)\quad \text{and}\quad \mathcal{S}_N(\tau;\geqslant a,< b)$$

are defined similarly.

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Lemma 5.7. *Given* $\tau := \tau_1\tau_2\cdots\tau_k \in \mathcal{S}_k$, *let* $t \geqslant k$. *Let* $U := \{\alpha_t, \beta_t\}$ *and* $V := \{\alpha_{t+1}, \beta_{t+1}\}$. *Then the following identity holds:*

$$a|\mathcal{S}_N(\tau; a, b)| = (b+1)|\mathcal{S}_{N+1}(\tau; a+1, b+1)|,$$

for all $a > 0$ *and* $b \geqslant 0$.

Proof. In this proof, we shall write $\mathcal{H}$ to denote $\mathcal{H}_1$. To prove the lemma, we shall define a bijection

$$\Phi : \{1, \dots, a\} \times \mathcal{S}_N(\tau; a, b) \to \{1, \dots, b+1\} \times \mathcal{S}_{N+1}(\tau; a+1, b+1).$$

For $i \in \{1, \dots, a\}$ and $\sigma \in \mathcal{S}_N(\tau; a, b)$, we define $\Phi(i, \sigma)$ to be $(j, \mathcal{H}(\sigma; h^x))$, where x, h and j are obtained as follows. If the ith tight pattern from U occurring in σ (*i.e.*, the ith occurrence in order from left to right in σ) is α_t (respectively, β_t) then $x = u$ (respectively, $x = d$); J is such that $\sigma_{J-t+1}\sigma_{J-t+2}\cdots\sigma_J$ is the ith tight pattern in U occurring in σ, $h = \sigma_J$, and j is such that $\theta_{J-t+1}\theta_{J-t+2}\cdots\theta_{J+1}$ is the jth tight pattern from V occurring in $\theta = \mathcal{H}(\sigma; h^x)$.

We must first show that

$$\Phi(i, \sigma) = (j, \mathcal{H}(\sigma; h^x)) \in \{1, \dots, b+1\} \times \mathcal{S}_{N+1}(\tau; a+1, b+1).$$

It is clear from the construction that $\mathcal{H}(\sigma; h^x)$ contains $a+1$ occurrences of the tight patterns in U and $b+1$ occurrences of the tight patterns in V. It only remains to prove that it avoids τ. Suppose to the contrary that $\mathcal{H}(\sigma; h^x)$ contains τ. Assume first that $x = u$. Then there exists a subsequence $\theta_{i[1]}\cdots\theta_{i[k]}$ forming the pattern τ. Since σ avoids τ, the tight α_1 inserted at height h of σ has to contribute to the formation of τ in $\mathcal{H}(\sigma; h^u)$. So there exists $1 \leqslant K \leqslant k$ such that $\theta_{i[K]} = h+1$. By Definition 5, we know that $i[K] = J+1$. Let w be the smallest number in $\{1, \dots, K\}$ such that $i[w], i[w+1], \dots, i[K]$ is a sequence of consecutive integers. Since $w \leqslant K \leqslant k < t+1$, it follows that $\theta_{i[w]}\theta_{i[w+1]}\cdots\theta_{i[K]}$ and $\theta_{i[w]-1}\theta_{i[w]}\theta_{i[w+1]}\cdots\theta_{i[K-1]}$ are both tight α_{K+1-w} patterns. Since $i[w]-1 \neq i[w-1]$, we know that

$$\theta_{i[1]}\cdots\theta_{i[w-1]}\theta_{i[w]-1}\theta_{i[w]}\theta_{i[w+1]}\cdots\theta_{i[K-1]}\theta_{i[K+1]}\cdots\theta_{i[k]} \tag{5.1}$$

forms the same pattern as $\theta_{i[1]}\cdots\theta_{i[k]}$ (which is τ). Since $\theta_{i[K]}$ does not appear in (5.1), it follows that (5.1) translates back into a τ pattern in σ, which is a contradiction. Therefore $\mathcal{H}(\sigma; h^x)$ avoids τ. A similar argument applies when $x = d$.

Now we show that Φ is a one-to-one correspondence. First, let

$$\big(j_1, \mathcal{H}\big(\sigma; h_1^{x[1]}\big)\big), \big(j_2, \mathcal{H}\big(\hat{\sigma}; h_2^{x[2]}\big)\big) \in \{1, \dots, b+1\} \times \mathcal{S}_{N+1}(\tau; a+1, b+1)$$

satisfy

$$\big(j_1, \mathcal{H}\big(\sigma; h_1^{x[1]}\big)\big) = \big(j_2, \mathcal{H}\big(\hat{\sigma}; h_2^{x[2]}\big)\big).$$

So $j_1 = j_2$ and

$$\mathcal{H}\big(\sigma; h_1^{x[1]}\big) = \mathcal{H}\big(\hat{\sigma}; h_2^{x[2]}\big),$$

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and thus $x[1] = x[2]$. Suppose first that $x[1] = x[2] = u$. By the definition of Φ, this together with $j_1 = j_2$ implies that $h_1 = h_2$. Hence $i_1 = i_2$ and $\sigma = \hat{\sigma}$, which leads to $(i_1, \sigma) = (i_2, \hat{\sigma})$ (a similar argument shows that $(i_1, \sigma) = (i_2, \hat{\sigma})$ for the case of $x[1] = x[2] = d$). Therefore, Φ is one-to-one.

Let

$$(j, \theta) \in \{1, \ldots, b+1\} \times \mathcal{S}_{N+1}(\tau; a+1, b+1).$$

Then there exists some J such that $\theta_{J-t+1}\theta_{J-t+2}\cdots\theta_{J+1}$ is the jth pattern from V occurring in θ. Suppose that this pattern is α_{t+1}. A similar argument works if the pattern is β_{t+1}. Let $h = \theta_J$, and then $\theta = \mathcal{H}(\sigma; h^u)$, where $\sigma \in \mathcal{S}_N(\tau; a, b)$ is obtained by removing θ_{J+1} from θ and shifting the entries; $\sigma_{J-t+1}\sigma_{J-t+2}\cdots\sigma_J$ is a tight occurrence of α_t so there exists an $i \in \{1, \ldots, a\}$ such that $\sigma_{J-t+1}\sigma_{J-t+2}\cdots\sigma_J$ is the ith α_t in σ. Therefore, $(j, \theta) = \Phi\big((i, \mathcal{H}(\sigma; h^u))\big)$, which proves that Φ is onto. □

The next result is essentially due to Kesten [14], in the context of self-avoiding walks rather than permutations.

Proposition 5.8. *Let $\tau \in \mathcal{S}_k$ be a permutation that contains no tight occurrence of* 12. *There exists a positive constant Γ such that*

$$\frac{|\mathcal{S}_{N+2}(\tau)|}{|\mathcal{S}_N(\tau)|} \geqslant \left(\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|}\right)^2 - \frac{\Gamma}{N}$$

for all sufficiently large N.

Proof. By Lemma 5.7, we have

$$\begin{aligned} |\mathcal{S}_{N+1}(\tau; \geqslant 0, > 1)| &= |\mathcal{S}_{N+1}(\tau; > 1, > 1)| \\ &= \sum_{i>0, j>0} |\mathcal{S}_{N+1}(\tau; i+1, j+1)| \\ &= \sum_{i>0, j>0} |\mathcal{S}_N(\tau; i, j)| \frac{i}{j+1}, \end{aligned}$$

and

$$\begin{aligned} |\mathcal{S}_{N+2}(\tau; \geqslant 0, > 2)| &= |\mathcal{S}_{N+2}(\tau; > 2, > 2)| \\ &= \sum_{i>0, j>0} |\mathcal{S}_{N+2}(\tau; i+2, j+2)| \\ &= \sum_{i>0, j>0} |\mathcal{S}_N(\tau; i, j)| \frac{i(i+1)}{(j+1)(j+2)}. \end{aligned}$$

The Schwarz inequality implies

$$\left(\sum_{i>0, j>0} |\mathcal{S}_N(\tau; i, j)| \frac{i}{j+1}\right)^2 \leqslant \left(\sum_{i>0, j>0} |\mathcal{S}_N(\tau; i, j)|\right)\left(\sum_{i>0, j>0} |\mathcal{S}_N(\tau; i, j)| \frac{i^2}{(j+1)^2}\right).$$

We get

$$\left[|\mathcal{S}_{N+1}(\tau;\geqslant 0,>1)|\right]^2 \leqslant \left(\sum_{i>0,j>0} |\mathcal{S}_N(\tau;i,j)|\right)\left(\sum_{i>0,j>0} |\mathcal{S}_N(\tau;i,j)|\frac{i^2}{(j+1)^2}\right)$$
$$\leqslant |\mathcal{S}_N(\tau)|\left(\sum_{i>0,j>0} |\mathcal{S}_N(\tau;i,j)|\frac{i^2}{(j+1)^2}\right).$$

For $N \geqslant 1$, let

$$\Xi_N = \frac{|\mathcal{S}_{N+2}(\tau;\geqslant 0,>2)|}{|\mathcal{S}_N(\tau)|} - \left(\frac{|\mathcal{S}_{N+1}(\tau;\geqslant 0,>1)|}{|\mathcal{S}_N(\tau)|}\right)^2$$

and

$$\Delta_N = \frac{|\mathcal{S}_{N+2}(\tau)|}{|\mathcal{S}_N(\tau)|} - \left(\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|}\right)^2 - \Xi_N$$
$$= \frac{|\mathcal{S}_{N+2}(\tau)| - |\mathcal{S}_{N+2}(\tau;\geqslant 0,>2)|}{|\mathcal{S}_N(\tau)|} - \frac{|\mathcal{S}_{N+1}(\tau)|^2 - |\mathcal{S}_{N+1}(\tau;\geqslant 0,>1)|^2}{|\mathcal{S}_N(\tau)|^2}.$$

Corollary 5.6 shows that Δ_N decays to zero exponentially rapidly as N increases. Now, we show that $\Xi_N \geqslant -A/N$ for some constant A. Then

$$\Xi_N \geqslant \left(\sum_{i>0,j>0} |\mathcal{S}_N(\tau;i,j)|\frac{i(i+1)}{(j+1)(j+2)} - \sum_{i>0,j>0} |\mathcal{S}_N(\tau;i,j)|\frac{i^2}{(j+1)^2}\right)\frac{1}{|\mathcal{S}_N(\tau)|}$$
$$= \frac{1}{|\mathcal{S}_N(\tau)|}\sum_{i>0,j>0} |\mathcal{S}_N(\tau;i,j)|\frac{(-i^2+ij+i)}{(j+1)^2(j+2)}.$$

For every N, no permutation in $\mathcal{S}_N(\tau)$ contains more than N tight occurrences of the patterns in U or V. Hence, the term $-i^2+ij+i$ is greater than $-N^2$. By Corollary 5.6, there exists $\epsilon > 0$ such that

$$\limsup_{N\to\infty}\left(1 - \frac{|\mathcal{S}_N(\tau;\geqslant 0,\geqslant \epsilon N)|}{|\mathcal{S}_N(\tau)|}\right)^{1/N} < 1.$$

Splitting the sum over j into $\epsilon N \leqslant j \leqslant N$ and $0 < j < \epsilon N$, we obtain

$$\Xi_N \geqslant \frac{-N^2|\mathcal{S}_N(\tau;\geqslant 0,\geqslant \epsilon N)|}{(\epsilon N)^3|\mathcal{S}_N(\tau)|} + (-N^2)\left(\frac{\sum_{i\geqslant 0,0<j<\epsilon N}|\mathcal{S}_N(\tau;i,j)|}{|\mathcal{S}_N(\tau)|}\right)$$
$$\geqslant \frac{-N^2|\mathcal{S}_N(\tau;\geqslant 0,\geqslant \epsilon N)|}{(\epsilon N)^3|\mathcal{S}_N(\tau)|} + (-N^2)\left(1 - \frac{|\mathcal{S}_N(\tau;\geqslant 0,\geqslant \epsilon N)|}{|\mathcal{S}_N(\tau)|}\right).$$

As $N \to \infty$, the first term on the right-hand side is asymptotic to $-\epsilon^{-3}/N$, and the second term decays to zero exponentially. Thus, letting $\Gamma = 2\epsilon^{-3}$, the proof of the theorem is completed. □

Theorem 5.9. *Let $\tau \in \mathcal{S}_k$. Assume that τ contains no tight occurrence of* 12. *Then*

$$\lim_{N\to\infty}\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

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Proof. The proof follows from Proposition 5.8 and Corollary 2.3. □

Remark 1. By symmetry, clearly the above result also holds if we assume that τ does not contain any tight occurrence of 21.

6. Ratio Limit Theorem for $\mathcal{S}_N(k\tau_2\cdots\tau_k)$

In this section we obtain a pattern theorem for $\mathcal{S}_N(\tau)$ when $\tau_1 = \max\{\tau_1,\ldots,\tau_k\}$ (*i.e.*, $\tau_1 = k$), and use it to prove the Ratio Limit Theorem for this τ. Note that condition (C2) from the Introduction is satisfied when $\tau_1 = k$.

This section requires an 'insertion' operation on permutations similar to but different from the one given in Definition 6.

Definition 8. Let $\sigma \in \mathcal{S}_N$ and $t \in \mathbb{N}$.

(a) Let $h \in \{1,\ldots,N\}$. Let $J = \min\{j : \sigma_j \geqslant h\}$. Define the permutation θ in $\mathcal{S}_{N+t}$ as follows:

$$\theta_i = \begin{cases} \sigma_i & \text{if } i < J, \\ h+j & \text{if } i = J+j \text{ for } j \in \{0,\ldots,t-1\}, \\ \sigma_{i-t} & \text{if } i \geqslant J+t \text{ and } \sigma_{i-t} < h, \\ \sigma_{i-t}+t & \text{if } i \geqslant J+t \text{ and } \sigma_{i-t} \geqslant h. \end{cases}$$

Then we denote θ by $\mathcal{I}_t(\sigma;h)$ (see Figure 7 for an example).

(b) Assume $N \geqslant h_1 > h_2 > \cdots > h_r \geqslant 1$. Let $\sigma^{(0)} = \sigma$ and let $\sigma^{(w)} = \mathcal{I}_t(\sigma^{(w-1)};h_w)$ for $w = 1,\ldots,r$. Then we denote $\sigma^{(r)}$ by $\mathcal{I}_t(\sigma;h_1,\ldots,h_r)$.

The following lemma shows that given $\tau := k\tau_2\cdots\tau_k \in \mathcal{S}_N$, a permutation $\sigma \in \mathcal{S}_N(\tau)$ remains τ-avoiding after the operation of $\mathcal{I}_t$ (see Figure 7 for an example).

Lemma 6.1. *Let $\tau = k\tau_2\cdots\tau_k \in \mathcal{S}_k$ (i.e., $\tau_1 = k$). Then, for any $\sigma \in \mathcal{S}_N(\tau)$, $h \in \{1,\ldots,N\}$ and $t \in \mathbb{N}$, we have $\mathcal{I}_t(\sigma;h) \in \mathcal{S}_{N+t}(\tau)$.*

Proof. First note that for $t \geqslant 2$, $\mathcal{I}_t(\sigma;h) = \mathcal{I}_1(\mathcal{I}_{t-1}(\sigma;h);(h+t-1))$. So it suffices to prove the statement for $t = 1$, and the complete proof will follow by induction on t.

Suppose to the contrary that $\mathcal{I}_1(\sigma;h) := \theta_1\cdots\theta_{N+1}$ contains the pattern τ. So there exists a subsequence $\theta_{i[1]}\cdots\theta_{i[k]}$ which forms the pattern τ. Since σ avoids τ, the new point inserted tightly at height h of σ has to contribute to the formation of τ in $\mathcal{I}_1(\sigma;h)$, that is, there exists some $1 \leqslant K \leqslant k$ such that $\theta_{i[K]} = h$. Then K must be 1 (since $\tau_1 = k = \max\{\tau_i\}_{1\leqslant i\leqslant k}$ and J was chosen such that $\sigma_j < h$ for $1 \leqslant j < J$, and consequently $\theta_j < h = \theta_{i[K]}$ for $1 \leqslant j < J$). Moreover, $i[2] \neq J+1$ since $\theta_{i[2]} < \theta_{i[1]} = h$ but $\theta_{J+1} = \sigma_J + 1 > h$. Therefore, replacing $\theta_{i[1]}$ by θ_{J+1}, $\theta_{J+1}\theta_{i[2]}\cdots\theta_{i[k]}$ is also a τ pattern in θ which implies that $\sigma_J\sigma_{i[2]-1}\cdots\sigma_{i[k]-1}$ forms the pattern τ in σ. This is a contradiction, hence the proof is complete. □

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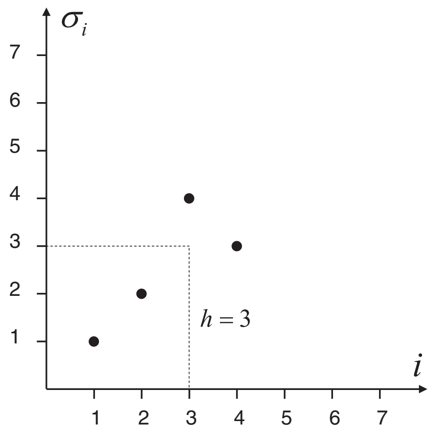
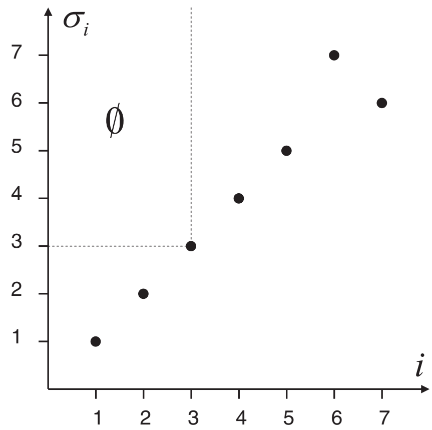


(a) $\sigma = 1243$



(b) $\mathcal{I}_3(\sigma; h) = 1234576$

Figure 7. Let $\sigma = 1243$, $t = 3$, $h = 3$ and $\tau = 321$ which starts with the largest number. Then $\sigma = 1243 \in \mathcal{S}_4(\tau)$ and by Lemma 6.1 $\mathcal{I}_3(\sigma; h) = 1234576 \in \mathcal{S}_7(\tau)$. The interior of the rectangle labelled '$\emptyset$' contains no points of the graph of σ.

Proposition 6.2. *Let $\tau := k\tau_2 \cdots \tau_k \in \mathcal{S}_k$ (i.e., $\tau_1 = k$) and let $t > 1$ be an integer. Then there exists a $\delta > 0$ such that the following strict inequality holds:*

$$\limsup_{N\to\infty} \left(|\mathcal{S}_N(\tau; \{\alpha_t\} < \delta N)|\right)^{1/N} < L(\tau).$$

Proof. Let $\delta > 0$ and $T = \lceil 2tL(\tau)^t \rceil$. For large N, let $r = \lfloor N/2T \rfloor$. For each $i = 1, \ldots, r$, let $\jmath_i := (2(i-1)T, (2i-1)T] \cap \mathbb{Z}$. (We shall only use the interval $\jmath_i$ with reference to values (or heights) σ_k of a permutation, rather than to the indices k of a permutation; to emphasize this, we shall refer to such intervals as 'intervals of heights'.)

Define the function Φ as

$$\Phi : \mathcal{S}_N(\tau; \{\alpha_t\} < \delta N) \times \left((0, T] \cap \mathbb{Z}\right)^r \to \mathcal{S}_{N+tr}(\tau),$$

such that

$$\Phi(\sigma, (\hat{h}_1, \ldots, \hat{h}_r)) = \mathcal{I}_t(\sigma; h_1, \ldots, h_r),$$

where $h_i = \hat{h}_i + 2(r-i)T$ for $1 \leqslant i \leqslant r$. Observe that $h_i \in \jmath_{r+1-i}$ for each i.

Corresponding to the set of intervals of heights for $\sigma \in \mathcal{S}_N$, we define a new set of intervals of heights for $\mathcal{I}_t(\sigma; h_1, \ldots, h_r) \in \mathcal{S}_{N+tr}$ as follows:

$$\Phi^\dagger(\jmath_i) := (2(i-1)T + (i-1)t, (2i-1)T + it], \quad i = 1, \ldots, r.$$

Observe that the r α_t inserted tightly into σ by Φ occur at heights in the intervals $\Phi^\dagger(\jmath_i)$, $i = 1, \ldots, r$.

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Given σ and $\Phi(\sigma,(\hat{h}_1,\ldots,\hat{h}_r)) \in \mathcal{S}_{N+tr}(\tau)$, we now want to find an upper bound on the number of $(\tilde{\sigma},(\tilde{h}_1,\ldots,\tilde{h}_r))$ in the domain of Φ such that

$$\Phi(\tilde{\sigma},(\tilde{h}_1,\ldots,\tilde{h}_r)) = \Phi(\sigma,(\hat{h}_1,\ldots,\hat{h}_r)) = \mathcal{I}_t(\sigma;h_1,\ldots,h_r). \tag{6.1}$$

Let p be the number of intervals $J \in \{J_1,\ldots,J_r\}$ such that an α_t occurs tightly in σ at a height in J. Denote the collection of such intervals by $\{J_{i[1]},\ldots,J_{i[p]}\}$, and let $\{J_{j[1]},\ldots,J_{j[q]}\}$ denote the remaining intervals of $\{J_1,\ldots,J_r\}$ (where $q = r-p$). Then $p < \delta N$ since

$$\sigma \in \mathcal{S}_N(\tau;\{\alpha_t\} < \delta N).$$

Evidently there are at most $T^{\delta N}$ ways to choose the heights $\{\tilde{h}_{i[1]},\ldots,\tilde{h}_{i[p]}\}$.

For every $1 \leqslant \ell \leqslant q$, the length of the largest increasing pattern occurring tightly in σ at a height in $J_{j[\ell]}$ is at most $t-1$. Hence, after the insertion of the α_t (using the function Φ), the length of the largest increasing pattern occurring tightly in $\mathcal{I}_t(\sigma;h_1,\ldots,h_r)$ at a height in $\Phi^\dagger(J_{j[\ell]})$ is at least t and at most $2t-1$; and any other α_R pattern with $R \geqslant t$ occurring tightly in $\mathcal{I}_t(\sigma;h_1,\ldots,h_r)$ at a height in $\Phi^\dagger(J_{j[\ell]})$ must be contained in this largest tight occurring increasing pattern. There are at most t different ways to remove exactly one α_t from $\mathcal{I}_t(\sigma;h_1,\ldots,h_r)$ at a height in each $\Phi^\dagger(J_{j[\ell]})$. Thus, in total there are at most t^r ways to choose the heights $\{\tilde{h}_{j[1]},\ldots,\tilde{h}_{j[q]}\}$ from this collection of intervals.

Therefore, in total, there are at most $T^{\delta N}t^r$ different ways to construct the $(\tilde{\sigma},(\tilde{h}_1,\ldots,\tilde{h}_r))$ in the domain of Φ such that equation (6.1) holds. Therefore

$$|\mathcal{S}_N(\tau;\{\alpha_t\} < \delta N)|\, T^r \leqslant |\mathcal{S}_{N+tr}(\tau)|\, T^{\delta N} t^r,$$

and hence

$$|\mathcal{S}_N(\tau;\{\alpha_t\} < \delta N)| \leqslant \frac{L(\tau)^{N+tr}}{T^r} T^{\delta N} t^r.$$

It follows that

$$\limsup_{N\to\infty} \left(|\mathcal{S}_N(\tau;\{\alpha_t\} < \delta N)|\right)^{1/N} \leqslant L(\tau)\left(\frac{L(\tau)^t\, t}{T}\right)^{1/2T} T^\delta \leqslant L(\tau)\frac{T^\delta}{2^{1/2T}}.$$

Since $\lim_{\delta\to 0} T^\delta = 1$, we can choose δ small enough so that $T^\delta < 2^{1/2T}$. The proposition follows. □

The pattern theorems proved for $\mathcal{S}_N(k\tau_2\cdots\tau_k)$ in Proposition 6.2 and Lemma 5.7 are the main results that lead to the Ratio Limit Theorem for $\mathcal{S}_N(k\tau_2\cdots\tau_k)$.

Proposition 6.3. *Let $\tau := k\tau_2\cdots\tau_k \in \mathcal{S}_k$ (i.e., $\tau_1 = k$). There exists a positive constant Γ such that*

$$\frac{|\mathcal{S}_{N+2}(\tau)|}{|\mathcal{S}_N(\tau)|} \geqslant \left(\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|}\right)^2 - \frac{\Gamma}{N}$$

for all sufficiently large N.

Proof. Using Proposition 6.2 and Lemma 5.7, a similar argument to that of Proposition 5.8 gives the proof. □

Theorem 6.4. *Let* $\tau := k\tau_2 \cdots \tau_k \in \mathcal{S}_k$ *(i.e.,* $\tau_1 = k$*). Then*

$$\lim_{N\to\infty} \frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

Proof. The proof follows from Proposition 6.3 and Corollary 2.3. □

Remark 2. The Ratio Limit Theorem also holds for $\tau := \tau_1\tau_2 \cdots \tau_k \in \mathcal{S}_k$ when $\tau_1 = 1$ or $\tau_k \in \{1, k\}$, that is, when condition (C2) from the Introduction holds. This is obvious by looking at the complement, the reverse or the reverse complement of τ.

7. Ratio Limit Theorem for $\mathcal{S}_N(\tau_1 \cdots \tau_k)$, $k \leqslant 5$

In this section we prove the Ratio Limit Theorem for $\mathcal{S}_N(\tau_1 \cdots \tau_k)$ when $k \leqslant 5$. We also discuss progress towards the case $k = 6$. We prove the Ratio Limit Theorem for $k \geqslant 6$ when condition (C3) from the Introduction is satisfied. Recall that (C3) says that τ contains exactly one tight occurrence of 12 and one tight occurrence of 21, so that they form four consecutive numbers and they occur at the beginning and end of the permutation.

Theorem 7.1. *Let* $k \leqslant 5$ *and* $\tau \in \mathcal{S}_k$*. Then*

$$\lim_{N\to\infty} \frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

Proof. It can be easily checked that for $k \leqslant 5$, any $\tau = \tau_1\tau_2 \cdots \tau_k \in \mathcal{S}_k$ satisfies at least one of the following conditions:

(i) $\tau_1 \in \{1, k\}$ or $\tau_k \in \{1, k\}$,
(ii) τ does not contain any tight occurrence of 12,
(iii) τ does not contain any tight occurrence of 21,
(iv) $|\mathcal{S}_N(\tau)| = |\mathcal{S}_N(\hat{\tau})|$ such that $\hat{\tau}$ satisfies (i), (ii) or (iii). In particular, we use the relation ([5])

$$|\mathcal{S}_N(t(t-1)\cdots 1\tau_{t+1}\cdots\tau_k)| = |\mathcal{S}_N(12\cdots t\tau_{t+1}\cdots\tau_k)| \quad (t \in \mathbb{N}).$$

Therefore, Theorem 5.9 and Theorem 6.4 imply that the Ratio Limit Theorem holds for $k \leqslant 5$. □

Given $k = 6$, let $\tau = \tau_1 \cdots \tau_6$. If either $\tau_1 \in \{1, 6\}$ or $\tau_6 \in \{1, 6\}$ then τ satisfies (i) above. Otherwise, there are four cases:

(1) $\tau_2, \tau_5 \in \{1, 6\}$,
(2) $\tau_3, \tau_4 \in \{1, 6\}$,
(3) $\tau_4, \tau_5 \in \{1, 6\}$ or $\tau_2, \tau_3 \in \{1, 6\}$,
(4) $\tau_2, \tau_4 \in \{1, 6\}$ or $\tau_3, \tau_5 \in \{1, 6\}$.

We will investigate each case separately.

The following lemma shows that any permutation from case (1) satisfies one of the conditions (i)–(iv).

Lemma 7.2. *Given $k = 6$, let $\tau = \tau_1 \cdots \tau_6$. If $\tau_2, \tau_5 \in \{1, 6\}$ then one of the conditions* (i)–(iv) *is satisfied.*

Proof. Suppose first that $\tau_2 = 1$ and $\tau_5 = 6$. If $\tau_1 = 2$ or $\tau_6 = 5$ then τ satisfies (iv). Also, if $\tau_1 = 5$ or $\tau_6 = 2$ then τ cannot contain tight occurrences of increasing and decreasing patterns simultaneously, so either (ii) or (iii) holds.

So suppose that $\tau_1, \tau_6 \notin \{2, 5\}$. Thus $\tau_3, \tau_4 \in \{2, 5\}$ and $\tau_1, \tau_6 \in \{3, 4\}$, and such permutations cannot contain any tight occurrence of a decreasing pattern. Therefore, τ satisfies (iii).

Finally, if $\tau_5 = 6$ and $\tau_2 = 1$, then the above argument will be true for τ^c and hence for τ. □

For case (2), all the permutations satisfy one of the conditions (i)–(iv) except when τ, τ^c, τ^r or τ^{rc} belongs to the set

$$C_2 = \{236154, 256143\}.$$

The following lemma was motivated by the pattern 236154. It shows that if condition (C3) holds for τ and if $\sigma \in \mathcal{S}_N(\tau)$, then, roughly speaking, at every point (i, σ_i) an α_1 (or β_1) can be inserted tightly and immediately to the right of and above (or below) (i, σ_i), so that the resulting permutation still avoids τ.

Lemma 7.3. *Let $\tau \in \mathcal{S}_k$ with $k \geqslant 6$. Assume that condition* (C3) *from the Introduction holds for τ, that is, $\tau_1, \tau_2, \tau_{k-1}, \tau_k \in \{q+1, q+2, q+3, q+4\}$ (for some $q \geqslant 0$) and τ contains exactly one occurrence of 12 and one occurrence of 21, with one at position $i = 1$ and the other at position $i = k-1$ (e.g., $\tau = 236154$).*

Then, for any $\sigma \in \mathcal{S}_N(\tau)$ and $h \in \{1, \ldots, N\}$, either $\mathcal{H}_1(\sigma; h^u) \in \mathcal{S}_{N+1}(\tau)$ or $\mathcal{H}_1(\sigma; h^d) \in \mathcal{S}_{N+1}(\tau)$.

Proof. Suppose that $\tau_1 = q+1$, $\tau_2 = q+2$, $\tau_{k-1} = q+4$ and $\tau_k = q+3$. The other three cases follow by symmetry.

Suppose to the contrary that both $\mathcal{H}_1(\sigma; h^u) := \theta_1 \cdots \theta_{N+1}$ and $\mathcal{H}_1(\sigma; h^d) := \gamma_1 \cdots \gamma_{N+1}$ contain the pattern τ. Then there exist subsequences $\theta_{i[1]} \cdots \theta_{i[k]}$ and $\gamma_{l[1]} \cdots \gamma_{l[k]}$, each forming the pattern τ.

Since σ avoids τ, the α_1 inserted tightly at height h of σ must contribute to the formation of τ in $\mathcal{H}_1(\sigma; h^u)$. So there exists $K \in [1, k]$ such that $\theta_{i[K]} = h+1$. By Definition 5, we know that $i[K] = J+1$, where $\sigma_J = h$. Now if $\theta_{i[K-1]} \neq h$ (*i.e.*, if $i[K-1] \neq J$), then $\theta_{i[1]} \cdots \theta_{i[K-1]} \theta_J \theta_{i[K+1]} \cdots \theta_{i[k]}$ is a τ pattern which by Definition 5 translates back to a τ pattern in σ. So $\theta_{i[K-1]} = h$. Since the only tight occurrence of 12 in τ is at position $i = 1$, it follows that $K = 2$ and hence $\theta_{i[1]} = h$ and $\theta_{i[2]} = h+1$ (*i.e.*, $i[1] = J$ and $i[2] = J+1$).

Similarly, the β_1 inserted tightly at height h of σ has to contribute to the formation of τ in $\mathcal{H}_1(\sigma; h^d)$, and since the only tight occurrence of 21 in τ is at position $k-1$, it

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follows that $\gamma_{l[k-1]} = h+1$ and $\gamma_{l[k]} = h$ (*i.e.*, $l[k-1] = J$ and $l[k] = J+1$). Also, since $\tau_1 < \tau_2 < \tau_k$,

$$\sigma_{l[1]} = \gamma_{l[1]} < \gamma_{l[2]} = \sigma_{l[2]} < \gamma_{l[k]} = h. \tag{7.1}$$

On the one hand, if $q = 0$, then for every $j \geqslant 3$ we have $i[j] > i[2] = J+1$ and $\theta_{i[j]} > \theta_{i[2]} = h+1$ (since $\tau_j > \tau_2 = 2$), and hence $\sigma_{i[j]-1} = \theta_{i[j]} - 1 \geqslant h+1$. By equation (7.1), $\sigma_{l[1]}\sigma_{l[2]}\sigma_{i[3]-1}\cdots\sigma_{i[k]-1}$ is a τ pattern in σ, which is a contradiction.

On the other hand, if $q > 0$ then choose $M \in [1,k]$ such that $\tau_M = q$. Then $2 < M < k-1$. Therefore $i[M] > i[2] = J+1$. Since $\tau_1 > \tau_M$ and $\theta_{i[1]} = h$, we see that $\theta_{i[M]} < h$. Therefore, by Definition 5, $\sigma_{i[M]-1} = \theta_{i[M]}$.

Recalling equation (7.1) and the fact that $l[2] < J < i[M]-1$, we now consider two separate cases.

(I) If $\sigma_{i[M]-1} < \sigma_{l[2]}$ then $\sigma_{l[2]}\sigma_J\sigma_{i[3]-1}\cdots\sigma_{i[k]-1}$ is a τ pattern in σ, which is a contradiction.

(II) If $\sigma_{i[M]-1} > \sigma_{l[2]}$ then $\sigma_{l[1]}\cdots\sigma_{l[k-2]}\sigma_J\sigma_{i[M]-1}$ is a τ pattern in σ, which is a contradiction.

Therefore, the proof is complete. □

A corresponding Ratio Limit Theorem now follows as before.

Theorem 7.4. *Let $\tau \in \mathcal{S}_k$ with $k \geqslant 6$. If condition* (C3) *from the Introduction holds, then*

$$\lim_{N\to\infty} \frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

Proof. Lemma 7.3 implies that the conditions of Proposition 5.5 are satisfied. Using Proposition 5.5 and Lemma 5.7, a similar argument to that of Proposition 5.8 shows that there exists a positive constant Γ such that

$$\frac{|\mathcal{S}_{N+2}(\tau)|}{|\mathcal{S}_N(\tau)|} \geqslant \left(\frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|}\right)^2 - \frac{\Gamma}{N}$$

for all sufficiently large N. The theorem now follows from Corollary 2.3. □

The status of Conjecture 1.1 for the case $k = 6$ can now be summarized in the following result. It says that the Ratio Limit Theorem is known (so far) to hold for 704 of the 720 patterns in $\mathcal{S}_6$. The proof follows directly from the results and discussion of this section, together with some case-by-case checking for cases (3) and (4) as listed above. In particular, since 412653 is the inverse of 236154, Theorem 7.4 implies that the Ratio Limit Theorem holds for 412653.

Theorem 7.5. *Let $\tau \in \mathcal{S}_6$. Assume that none of τ, τ^c, τ^r, or τ^{rc} is in*

$$\{256143, 452163, 415632, 512643\}.$$

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Then

$$\lim_{N\to\infty} \frac{|\mathcal{S}_{N+1}(\tau)|}{|\mathcal{S}_N(\tau)|} = L(\tau).$$

We note that $(256143)^{-1} = (452163)^{rc}$ and $(415632)^{-1} = (512643)^{c}$.

8. Upper left corner of the graph of $k\tau_2\cdots\tau_k$-avoiding permutations

Consider the graph of a random permutation from $\mathcal{S}_N(\tau)$. In this section we establish that if $\tau_1 = k$ then there exists some neighbourhood of the upper left corner of $[1,N]^2$ that, with high probability, contains no points of the graph.

Theorem 8.1. *Let $\tau = k\tau_2\cdots\tau_k \in \mathcal{S}_k$, and define $\bar\tau = \tau_2\cdots\tau_k$. Let $A = L(\bar\tau)/L(\tau)$. Assume α and β are numbers in $(0,1)$ such that*

$$\beta - \alpha > \frac{\sqrt{A}}{2} \tag{8.1}$$

and

$$A^{\beta-\alpha} < \frac{(1-\beta)^{1-\beta}\alpha^{\alpha}(\beta-\alpha)^{2(\beta-\alpha)}}{(1-\alpha)^{1-\alpha}\beta^{\beta}}. \tag{8.2}$$

Then

$$\limsup_{N\to\infty} \left[P_N^{\tau}\{\max\{\pi_1,\dots,\pi_{\lfloor\alpha N\rfloor}\} > \beta N\}\right]^{1/N} < 1. \tag{8.3}$$

By Proposition A.1 of Bóna [8], $L(\tau) \geqslant L(\bar\tau) + 1$, and hence $A < 1$. (We note that the strict inequality is important, so the trivial inequality $L(\tau) \geqslant L(\bar\tau)$ is inadequate here.) Since $A < 1$, the assumptions (8.1) and (8.2) hold when α is close to 0 and β is close to 1. Thus Theorem 8.1 implies Theorem 1.3(a), that is, it tells us that when $\tau_1 = k$, a region near the upper left corner of the graph of σ is empty for the vast majority of τ-avoiding permutations σ. This is consistent with results of simulations such as Figure 2.

Proof of Theorem 8.1. Assume that inequalities (8.1) and (8.2) hold. Recall the definition of $\mathcal{S}_{N,i,j}(\tau)$ from Definition 3. First we claim that, for $1 \leqslant i < j \leqslant N$,

$$|\mathcal{S}_{N,i,j}(\tau)| \leqslant \binom{N-i}{N-j}\binom{j-1}{i-1}|\mathcal{S}_{N-j}(\tau)|\,|\mathcal{S}_{i-1}(\tau)|\,|\mathcal{S}_{j-i}(\bar\tau)|. \tag{8.4}$$

This is proved as follows. Consider the graph of an arbitrary $\sigma \in \mathcal{S}_{N,i,j}(\tau)$, viewed as a set of N points in $[1,N]^2$ in the xy-plane. We know (i,j) is one point, and we partition the other $N-1$ points into the following three groups of points in disjoint rectangles.

Group 1: the points that satisfy $y > j$ and $x > i$.
Group 2: the points that satisfy $y < j$ and $x < i$.
Group 3: the points that satisfy $y < j$ and $x > i$.

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(Observe that no points satisfy $y > j$ and $x < i$, by the definition of $\mathcal{S}_{N,i,j}(\tau)$.) Note that there are exactly $N - j$ points in group 1, and the set of their x coordinates can be chosen in $\binom{N-i}{N-j}$ ways. The set of their y coordinates must be $\{j+1, \ldots, N\}$. Given the sets of x and y coordinates, the points of group 1 can be chosen in (at most) $|\mathcal{S}_{N-j}(\tau)|$ ways. Similarly, the $i-1$ points of group 2 can have the set of their y coordinates chosen in $\binom{j-1}{i-1}$ ways and then their precise locations can be chosen in at most $|\mathcal{S}_{i-1}(\tau)|$ ways. This leaves $j - i$ points for group 3. Observe that the points of group 3 cannot contain the pattern $\bar{\tau}$, since such a pattern together with (i, j) would form the pattern τ. The set of x coordinates and the set of y coordinates for group 3 have been determined by choices made for group 1 and group 2 respectively. Given these two sets, there are at most $|\mathcal{S}_{j-i}(\bar{\tau})|$ ways to arrange the points of group 3. The claim (8.4) follows.

By equation (8.4) and Lemma 2.1(b), we have

$$\begin{aligned}
|\mathcal{S}_{N,i,j}(\tau)| &\leqslant \binom{N-i}{N-j}\binom{j-1}{i-1} L(\tau)^{N-j+i-1} L(\bar{\tau})^{j-i} \\
&\leqslant \binom{N-i}{N-j}\binom{j}{i} L(\tau)^{N-1} A^{j-i} \\
&\leqslant \frac{(N-i)^{N-i}}{(N-j)^{N-j}(j-i)^{j-i}} \, \frac{j^j}{i^i (j-i)^{j-i}} \, L(\tau)^N A^{j-i} \\
&= \mathcal{F}\left(\frac{i}{N}, \frac{j}{N}\right)^N L(\tau)^N, \qquad (8.5)
\end{aligned}$$

where

$$\mathcal{F}(s,t) = \frac{(1-s)^{1-s} t^t A^{t-s}}{(1-t)^{1-t}(t-s)^{2(t-s)} s^s}$$

for $0 \leqslant s < t \leqslant 1$, and we interpret 0^0 to be 1. Since

$$\frac{\partial \ln \mathcal{F}}{\partial s} = \ln\left(\frac{(t-s)^2}{s(1-s)A}\right) \quad \text{and} \quad \frac{\partial \ln \mathcal{F}}{\partial t} = \ln\left(\frac{t(1-t)A}{(t-s)^2}\right),$$

we see that $\mathcal{F}$ is increasing in s and decreasing in t whenever $(s,t) \in (0,1)^2$ and $(t-s)^2 > A/4$ (using the fact that $1/4 \geqslant x(1-x)$ for every x). Now, condition (8.1) implies that $(t-s)^2 > A/4$ whenever $0 \leqslant s \leqslant \alpha$ and $\beta \leqslant t \leqslant 1$, so the above-mentioned monotonicity of $\mathcal{F}$ implies that

$$\mathcal{F}\left(\frac{i}{N}, \frac{j}{N}\right) \leqslant \mathcal{F}(\alpha, \beta) \quad \text{for all } i \in [1, \alpha N] \text{ and } j \in [\beta N, N]. \qquad (8.6)$$

We now make the observation that

$$\{\sigma \in \mathcal{S}_N(\tau) : \max\{\sigma_1, \ldots, \sigma_{\lfloor \alpha N \rfloor}\} > \beta N\} \subset \bigcup_{i=1}^{\lfloor \alpha N \rfloor} \bigcup_{j=\lceil \beta N \rceil}^{N} \mathcal{S}_{N,i,j}(\tau). \qquad (8.7)$$

Therefore, equations (8.7), (8.5), and (8.6) imply that

$$P_N^{\tau}\{\max\{\pi_1, \ldots, \pi_{\lfloor \alpha N \rfloor}\} > \beta N\} \leqslant N^2 \mathcal{F}(\alpha, \beta)^N \frac{L(\tau)^N}{|\mathcal{S}_N(\tau)|}.$$

We know $\mathcal{F}(\alpha, \beta) < 1$ by condition (8.2), and hence (8.3) holds. □

9. Leftmost elements of $\sigma \in \mathcal{S}_N(k\tau_2\cdots\tau_k)$

The purpose of this section is to prove Proposition 9.2, which roughly says the following: for certain permutations τ and for some $\delta > 0$ (depending on τ), there is an exponentially small (in N) probability that the first δN entries of a randomly chosen τ-avoiding permutation are not small.

We first define an operation that inserts a pattern tightly into a permutation at a prescribed height.

Definition 9. Let $\sigma \in \mathcal{S}_N$ and $\psi \in \mathcal{S}_k$.

(a) Let $h \in \{1,\ldots,N\}$. Let $J = \min\{j : \sigma_j \geqslant h\}$. Define the permutation θ in $\mathcal{S}_{N+k}$ as follows:

$$\theta_i = \begin{cases} \sigma_i & \text{if } i < J, \\ h + \psi_{j+1} - 1 & \text{if } i = J + j \text{ for } j \in \{0,\ldots,k-1\}, \\ \sigma_{i-k} & \text{if } i \geqslant J + k \text{ and } \sigma_{i-k} < h, \\ \sigma_{i-k} + k & \text{if } i \geqslant J + k \text{ and } \sigma_{i-k} \geqslant h. \end{cases}$$

Then we denote θ by $\mathcal{I}_\psi(\sigma; h)$.

(b) Assume $N \geqslant h_1 > h_2 > \cdots > h_r \geqslant 1$. Let $\sigma^{(0)} = \sigma$ and let $\sigma^{(w)} = \mathcal{I}_\psi(\sigma^{(w-1)}; h_w)$ for $w = 1,\ldots,r$. Then we denote $\sigma^{(r)}$ by $\mathcal{I}_\psi(\sigma; h_1,\ldots,h_r)$.

Note that $\mathcal{I}_{\alpha_t} = \mathcal{I}_t$, where $\mathcal{I}_t$ is as introduced in Definition 8.

Recall Definition 4. Observe that for $\tau = (k_0+1)\bar{\tau}$, where $\bar{\tau} \in \mathcal{S}_{k_0}$, if $\sigma \in \mathcal{S}_N(\tau)$ and $\bar{\tau}$ occurs tightly in σ at position i and if $h \leqslant \sigma_{i+l} \leqslant h + k_0 - 1$ (for $l = 0, 1, \ldots, k_0 - 1$), then $\sigma_j < h$ for all $j \in \{1,\ldots,i-1\}$.

Definition 10. A pattern $\bar{\tau} \in \mathcal{S}_{k_0}$ is said to satisfy the Tight Pattern Insertion Property (TPIP) if $\mathcal{I}_{\bar{\tau}}(\sigma; h) \in \mathcal{S}_{k_0+N}(\tau)$ for every $\sigma \in \mathcal{S}_N(\tau)$ and $h \in \{1,\ldots,N\}$, where $\tau = (k_0+1)\bar{\tau}$.

The following lemma provides some examples of permutations satisfying the TPIP. The proof of the lemma appears at the end of this section. Proposition 9.2 below is the motivation for introducing the TPIP.

Lemma 9.1. *Let $\tau = k\tau_2\cdots\tau_k = k\bar{\tau}$ such that $\tau_2 \in \{1,2\}$ and $\bar{\tau} \neq 21$. Then $\mathcal{I}_{\bar{\tau}}(\sigma; h)$ avoids τ for any $\sigma \in \mathcal{S}_N(\tau)$ and $h \in \{1,\ldots,N\}$, that is, the TPIP is satisfied for $\bar{\tau}$.*

Proposition 9.2. *Let $\tau = (k_0+1)\tau_2\cdots\tau_{k_0+1} \in \mathcal{S}_{k_0+1}$ and suppose that the TPIP is satisfied for $\bar{\tau} = \tau_2\cdots\tau_{k_0+1} \in \mathcal{S}_{k_0}$. Fix $\epsilon \in (0,1)$. Then there exists a $\delta \in (0,1)$ (depending on $\bar{\tau}$) such that*

$$\limsup_{N\to\infty} \left[P_N^\tau\{\max\{\pi_1,\ldots,\pi_{\lfloor \delta N\rfloor}\} > \epsilon N\}\right]^{1/N} < 1.$$

https://doi.org/10.1017/S0963548313000576 Published online by Cambridge University Press

Combining Proposition 9.2 with Lemma 9.1 yields the following. In particular, it says that Proposition 9.2 holds for $\tau = 4231$.

Corollary 9.3. *Let $\tau = k\tau_2 \cdots \tau_k = k\bar{\tau}$ such that $\tau_2 \in \{1,2\}$ and $\bar{\tau} \neq 21$. Then Proposition 9.2 holds.*

Before proving Proposition 9.2, we establish some preliminary results.

Definition 11. Let $\psi \in \mathcal{S}_k$. For $\sigma \in \mathcal{S}_N$, define

$$\mathrm{Heights}_\psi(\sigma) = \{h : \psi \text{ occurs tightly in } \sigma \text{ at height } h\}.$$

For $1 \leqslant i < j \leqslant N-k+1$, let $\mathbb{H}_N^\psi[i,j]$ be the set of σ in $\mathcal{S}_N$ such that

$$\mathrm{Heights}_\psi(\sigma) \cap [i,j] = \emptyset.$$

Lemma 9.4. *Assume that the TPIP is satisfied for $\psi \in \mathcal{S}_k$. Assume $1 \leqslant i < j \leqslant N-k+1$ and $\sigma, \tilde{\sigma} \in \mathbb{H}_N^\psi[i,j]$,*

(a) *Assume $h, \tilde{h} \in [i,j]$. If $\mathcal{I}_\psi(\sigma;h) = \mathcal{I}_\psi(\tilde{\sigma};\tilde{h})$, then $\sigma = \tilde{\sigma}$ and $|h-\tilde{h}| \leqslant 2(k-1)$.*

(b) *Assume $j \geqslant h_1 > \cdots > h_r \geqslant i$ and $j \geqslant \tilde{h}_1 > \cdots > \tilde{h}_r \geqslant i$. If*

$$\mathcal{I}_\psi(\sigma;h_1,\ldots,h_r) = \mathcal{I}_\psi(\tilde{\sigma};\tilde{h}_1,\ldots,\tilde{h}_r),$$

then $\sigma = \tilde{\sigma}$ and $|h_\ell - \tilde{h}_\ell| \leqslant 2(k-1)$ for every $\ell = 1,\ldots,r$.

Proof. **(a)** Suppose that $\mathcal{I}_\psi(\sigma;h) = \mathcal{I}_\psi(\tilde{\sigma};\tilde{h})$. Since σ and $\tilde{\sigma}$ do not contain any tight occurrence of ψ at any height in $[i,j]$, using Definition 9, tight insertion of ψ at height $h \in [i,j]$ can lead to the creation of at most $2k-1$ new tight occurrences of ψ patterns in $\mathcal{I}_\psi(\sigma;h)$. More precisely, the pattern ψ inserted tightly at height h must contribute to the creation of any new tight occurrence of ψ at say height $\bar{h} \in [i,j]$, so

$$\bar{h} \in \{h-k+1, h-k+2, \ldots, h-1, h, h+1, \ldots, h+k-1\}$$

and hence $|h-\tilde{h}| \leqslant 2(k-1)$. Then σ and $\tilde{\sigma}$ are obtained by removing the pattern ψ at heights h and $\tilde{h}$ respectively and by Definition 9 it is clear that the two permutations obtained from this procedure are identical, that is, $\sigma = \tilde{\sigma}$.

(b) The proof will be given by induction on r. By the proof of part (a), the statement in part (b) holds for $r=1$. Suppose now that the statement is true for $r-1$. We show that it is also true for r. Assume that

$$N \geqslant h_1 > h_2 > \cdots > h_r \geqslant 1, \quad N \geqslant \tilde{h}_1 > \tilde{h}_2 > \cdots > \tilde{h}_r \geqslant 1$$
$$\text{and} \quad \mathcal{I}_\psi(\sigma;h_1,\ldots,h_r) = \mathcal{I}_\psi(\tilde{\sigma};\tilde{h}_1,\ldots,\tilde{h}_r).$$

Then, by Definition 9(b),

$$\mathcal{I}_\psi(\mathcal{I}_\psi(\sigma;h_1,\ldots,h_{r-1});h_r) = \mathcal{I}_\psi(\mathcal{I}_\psi(\tilde{\sigma};\tilde{h}_1,\ldots,\tilde{h}_{r-1});\tilde{h}_r).$$

Thus it follows from part (a) that $|h_r - \tilde{h}_r| \leqslant 2(k-1)$ and

$$\mathcal{I}_\psi(\sigma;h_1,\ldots,h_{r-1}) = \mathcal{I}_\psi(\tilde{\sigma};\tilde{h}_1,\ldots,\tilde{h}_{r-1}).$$

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Hence, using the assumption of the induction, $|h_\ell - \tilde{h}_\ell| \leqslant 2(k-1)$ for every $\ell = 1,\ldots,r$ and $\sigma = \tilde{\sigma}$. □

Lemma 9.5. *Let $\bar{\tau} \in \mathcal{S}_{k_0}$ and $\tau = (k_0+1)\bar{\tau} \in \mathcal{S}_{k_0+1}$. Assume that the TPIP is satisfied for $\bar{\tau}$. For $1 \leqslant i < j \leqslant N-k_0+1$ and $1 \leqslant r \leqslant j-i+1$,*

$$|\mathbb{H}_N^{\bar{\tau}}[i,j] \cap \mathcal{S}_N(\tau)| \binom{j-i+1}{r} \leqslant |\mathcal{S}_{N+k_0 r}(\tau)|(2k_0)^r.$$

Proof. This follows from Lemma 9.4(b). □

The proof of Proposition 9.2 relies on the following calculation.

Corollary 9.6. *Assume that the TPIP is satisfied for $\bar{\tau} \in \mathcal{S}_{k_0}$, and let $\tau = (k_0+1)\bar{\tau}$. Assume $1 \leqslant j \leqslant N$. Then*

$$|\mathbb{H}_N^{\bar{\tau}}[1,j] \cap \mathcal{S}_N(\tau)| \leqslant \frac{j^{5/2}}{c_3} L(\tau)^N (2k_0 L(\tau)^{k_0} t)^{tj} \quad \text{for every } t \in (0,1).$$

Proof. Let $r = \lfloor tj \rfloor$. By Lemma 9.5 with $i=1$, as well as equation (2.1) and Lemma 2.1(c),

$$\begin{aligned} |\mathbb{H}_N^{\bar{\tau}}[1,j] \cap \mathcal{S}_N(\tau)| &\leqslant \frac{|\mathcal{S}_{N+k_0 r}(\tau)|(2k_0)^r}{\binom{j}{r}} \\ &\leqslant \frac{j^{5/2}}{c_3} L(\tau)^{N+k_0 r}(t^t(1-t)^{1-t})^j (2k_0)^{tj} \\ &\leqslant \frac{j^{5/2}}{c_3} L(\tau)^N (2k_0 L(\tau)^{k_0} t)^{tj}. \end{aligned}$$ □

Proof of Proposition 9.2. For $j,k \in \{1,\ldots,N\}$ with $j \geqslant k$, let

$$A_N^\tau(j;k) = \{\sigma \in \mathcal{S}_N(\tau) : \max\{\sigma_1,\ldots,\sigma_k\} = j\}.$$

For $N \geqslant k$, consider the map $\mathcal{F}_N^k : \mathcal{S}_N \to \mathcal{S}_k \times \mathcal{S}_{N-k}$ defined by

$$\mathcal{F}_N^k(\sigma_1\cdots\sigma_N) = (\mathrm{Patt}(\sigma_1\cdots\sigma_k), \mathrm{Patt}(\sigma_{k+1}\cdots\sigma_N)).$$

For example, $\mathcal{F}_9^4(614892537) = (3124, 51324)$.

For $\sigma \in A_N^\tau(j;k)$, it is obvious that $\mathrm{Patt}(\sigma_1\cdots\sigma_k)$ is in $\mathcal{S}_k(\tau)$ and that $\mathrm{Patt}(\sigma_{k+1}\cdots\sigma_N)$ is in $\mathcal{S}_{N-k}(\tau)$. We claim that also $\mathrm{Patt}(\sigma_{k+1}\cdots\sigma_N)$ is in $\mathbb{H}_{N-k}^{\bar{\tau}}[1, j-k-k_0]$. To prove the claim, write $\mathrm{Patt}(\sigma_{k+1}\cdots\sigma_N) = \theta_1\theta_2\cdots\theta_{N-k} = \theta$. Observe that $\theta_m \geqslant \sigma_{k+m} - k$ for $m = 1,\ldots,N-k$. If θ has a tight occurrence of $\bar{\tau}$ at some height $h \leqslant j-k-k_0$, then for some $\ell \in \{1,\ldots,N-k\}$ we have that $\theta_\ell\theta_{\ell+1}\cdots\theta_{\ell+k_0-1}$ forms a $\bar{\tau}$ pattern and $\theta_{\ell+i} \leqslant h+k_0-1 \leqslant j-k-1$ (for $i=0,\ldots,k_0-1$). Then $\sigma_{k+\ell}\sigma_{k+\ell+1}\cdots\sigma_{k+\ell+k_0-1}$ forms the pattern $\bar{\tau}$ and $\sigma_{k+\ell+i} \leqslant \theta_{\ell+i}+k \leqslant j-1$ (for $i=0,\ldots,k_0-1$). Combining this with the fact that $\sigma_i = j$ for some $i \leqslant k$ (by definition of $A_N^\tau(j;k)$), it follows that σ contains the pattern τ. This contradiction proves the claim. Thus we know that $\mathrm{Patt}(\sigma_{k+1}\cdots\sigma_N)$ is in $\mathcal{S}_{N-k}(\tau) \cap \mathbb{H}_{N-k}^{\bar{\tau}}[1, j-k-k_0]$ whenever $\sigma \in A_N^\tau(j;k)$.

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Let $\mathcal{F}_N^{(\tau,j;k)}$ be the restriction of $\mathcal{F}_N^k$ to $A_N^\tau(j;k)$. Observe that $\mathcal{F}_N^{(\tau,j;k)}$ is at most $\binom{j}{k}$-to-one. Let $t = 1/(4k_0 L(\tau)^{k_0})$. Then we have

$$\begin{aligned}|A_N^\tau(j;k)| &\leqslant |\mathcal{S}_k(\tau)||\mathcal{S}_{N-k}(\tau) \cap \mathbb{H}_{N-k}^{\bar{\tau}}[1, j-k-k_0]| \binom{j}{k} \\ &\leqslant L(\tau)^k \frac{j^{5/2}}{c_3} \frac{L(\tau)^{N-k}}{2^{t(j-k-k_0)}} \binom{j}{k} \qquad \text{(by (2.1) and Corollary 9.6).}\end{aligned}$$

Suppose $0 < \delta < \epsilon$, and let $k = \lfloor \delta N \rfloor$. Then we have

$$\begin{aligned}P_N^\tau\{\max\{\pi_1, \ldots, \pi_{\lfloor \delta N \rfloor}\} > \epsilon N\} &= \frac{\sum_{j>\epsilon N} |A_N^\tau(j;k)|}{|\mathcal{S}_N(\tau)|} \\ &\leqslant \frac{N^{7/2}}{c_3} \frac{L(\tau)^N}{|\mathcal{S}_N(\tau)|} \frac{2^{k_0 t}}{2^{t(\epsilon-\delta)N}} \binom{N}{k} \\ &\leqslant \frac{2c_4 N^{7/2}}{c_3} \frac{L(\tau)^N}{|\mathcal{S}_N(\tau)|} \frac{(\delta^\delta(1-\delta)^{1-\delta})^{-N}}{2^{t(\epsilon-\delta)N}},\end{aligned}$$

using Lemma 2.1(c) in the last step. Therefore

$$\limsup_{N\to\infty} \big[P_N^\tau\{\max\{\pi_1, \ldots, \pi_{\lfloor \delta N \rfloor}\} > \epsilon N\}\big]^{1/N} \leqslant \frac{1}{2^{t(\epsilon-\delta)}\delta^\delta(1-\delta)^{1-\delta}}. \tag{9.1}$$

As δ decreases to 0, the right-hand side of equation (9.1) tends to $2^{-t\epsilon}$, which is less than 1. This proves the proposition. □

Proof of Lemma 9.1. Suppose to the contrary that $\mathcal{I}_{\bar{\tau}}(\sigma; h) =: \theta_1 \cdots \theta_{N+k-1}$ contains the pattern τ. So there exists a subsequence $\theta_{i[1]} \cdots \theta_{i[k]}$ which forms the pattern τ. Since σ avoids τ, the pattern $\bar{\tau}$ inserted tightly at height h of σ must contribute to the formation of τ in $\mathcal{I}_{\bar{\tau}}(\sigma; h)$. Hence there exists $1 \leqslant K \leqslant k$ such that $h \leqslant \theta_{i[K]} \leqslant h + k - 2$. Choose K to be the smallest integer with this property. We claim that $K = 1$. Suppose to the contrary that $K > 1$. Then $i[1] < J$, where J is chosen such that $\sigma_j < h$ for $1 \leqslant j < J$ and consequently $\theta_j < h$ for $1 \leqslant j < J$. Thus $\theta_{i[1]} < h$, but $\tau_1 = k = \max\{\tau_i\}_{1\leqslant i\leqslant k}$ implies that $\theta_{i[1]} > \theta_{i[K]} \geqslant h$, and so we have a contradiction. Therefore, $K = 1$ and $h \leqslant \theta_{i[1]} \leqslant h + k - 2$. Next we show that $\theta_{i[2]} < h$. To do this, we consider two cases.

(I) $\tau_2 = 1$. Since $\tau_1 = k$, we must have

$$\theta_{i[2]} \leqslant \theta_{i[1]} - (k-1) \leqslant h + k - 2 - (k-1) = h - 1.$$

(II) $\tau_2 = 2$ and $\bar{\tau} \neq 21$. In this case, $k \geqslant 4$. Assume $\theta_{i[2]} \geqslant h$. Since $\tau_1 = k$, we must have

$$h \leqslant \theta_{i[2]} \leqslant \theta_{i[1]} - (k-2) \leqslant h + k - 2 - (k-2) = h.$$

That is, $\theta_{i[2]} = h$ and $\theta_{i[1]} = h + k - 2$. Hence

$$\theta_{i[m]} \in [h, h+k-2] \quad \text{for every } m \in \{1, \ldots, k\} \setminus \{m_0\},$$

where m_0 is defined by $\tau_{m_0} = 1$. This forces

$$i[m] \in [J, J+k-2] \quad \text{for every } m \in \{1, \ldots, k\} \setminus \{m_0\}.$$

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Since $m_0 \neq 1$, we have $i[1] = J$. Therefore, by Definition 9(a),

$$\theta_{i[1]} = \theta_J = h + \bar{\tau}_1 - 1 = h + \tau_2 - 1 = h + 1.$$

Combining this with $\theta_{i[1]} = h + k - 2$, we conclude $k = 3$, which is a contradiction. Thus $\theta_{i[2]} < h$, and hence $i[2] > J + k - 1$. Therefore, $\theta_{J+k-1}\theta_{i[2]} \cdots \theta_{i[k]}$ is also a τ pattern in θ, which gives rise to the τ pattern $\sigma_J \sigma_{i[2]-k+1} \cdots \sigma_{i[k]-k+1}$ in σ. This is a contradiction. Hence $\mathcal{I}_{\bar{\tau}}(\sigma; h)$ avoids τ and the proof is complete. $\square$

Acknowledgements

The authors wish to thank Andrew Rechnitzer and Mike Zabrocki for helpful discussions, Mireille Bousquet-Mélou for pointers to the literature, Victor Tsetsulin for writing the computer program used here to generate random 4231-avoiding permutations (as an NSERC Undergraduate Student Research Award holder, supervised by NM), and a careful referee for helpful comments.

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