J. Appl. Prob. **49,** 338–350 (2012)
Printed in England

A SYMMETRY PROPERTY FOR A CLASS OF RANDOM WALKS IN STATIONARY RANDOM ENVIRONMENTS ON $\mathbb{Z}$

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Abstract

A correspondence formula between the laws of *dual* Markov chains on $\mathbb{Z}$ with two transition jumps is established. This formula contributes to the study of random walks in stationary random environments. Counterexamples with more than two jumps are exhibited.

Keywords: Markov chain; duality; random walk; stationary random environment; conductance and resistance

2010 Mathematics Subject Classification: Primary 60J10; 60K37

1. Introduction

If $(S_n)_{n\geq 0}$ refers to a random walk of law ν on the group $\mathbb{Z}$ of integers and if $(S_n^*)_{n\geq 0}$ is the random walk whose law $\overset{\vee}{\nu}$ is the inverse distribution of ν, we obviously have, for any nonnegative integer n and any x in $\mathbb{Z}$,

$$\mathrm{P}[S_n^* = -x \mid S_0^* = 0] = \mathrm{P}[S_n = x \mid S_0 = 0]. \tag{1}$$

In the case of a Markov chain $(S_n)_{n\geq 0}$ on $\mathbb{Z}$, with transition probabilities $(p_{y,y+z})_{y,z\in\mathbb{Z}}$, we can consider a *dual* Markov chain $(S_n^*)_{n\geq 0}$ by taking as new transition probabilities

$$p^*_{y,y+z} := p_{y,y-z}, \qquad y, z \in \mathbb{Z}.$$

In [5] it was shown that the law of the return time at 0 for a birth-and-death Markov chain is easily expressed using the law of such a dual Markov chain. This kind of duality also appears in [3] and [4] for the study of random walks in stationary random environments on $\mathbb{Z}$.

For a Markov chain $(S_n)_{n\geq 0}$, equality (1) is obviously wrong in the general case as soon as $n \geq 2$.

The situation is slightly different in the case of random walks in stationary random environments. In what follows, we establish a *quenched* formula, the *annealed* version of which corresponds to (1) for random walks in stationary random environments on $\mathbb{Z}$ with the number of possible jumps at each site equal to exactly two. In Section 6 we give counterexamples when three jumps are allowed.

Received 12 July 2010; revision received 28 November 2011.
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2. Dual Markov chains, random walks in stationary random environments, and main results

Let a and b be two elements of $\mathbb{Z}$ with $a > b$.

In the following, the Markov chains we consider are (time-homogeneous) Markov chains on $\mathbb{Z}$ with exactly two different jumps, a and b, or possibly $-a$ and $-b$.

Let us denote by $(S_n)_{n\geq 0}$ such a Markov chain with jumps a and b. The transition probabilities of $(S_n)_{n\geq 0}$ are simply given by the function

$$\mathcal{P}\colon \mathbb{Z} \to (0, 1), \qquad y \mapsto p_y,$$

where $p_y := p_{y,y+a}$. The probability $1 - p_{y,y+a}$, which equals $p_{y,y+b}$, is also denoted by q_y.

In this paper, the dual Markov chain of $(S_n)_{n\geq 0}$ is the Markov chain $(S_n^*)_{n\geq 0}$ on $\mathbb{Z}$ with jumps $-a$ and $-b$ whose transition probabilities are given by

$$p^*_{y,y-a} = p_{y,y+a} \quad \text{and} \quad p^*_{y,y-b} = p_{y,y+b}.$$

Theorem 1 below shows a simple correspondence formula between the laws of $(S_n)_{n\geq 0}$ and $(S_n^*)_{n\geq 0}$. Our main motivation in this paper is to apply this theorem to random walks in stationary random environments on $\mathbb{Z}$.

Theorem 1. *With the above notation, for any nonnegative integers n, n_a, and n_b with $n_a + n_b = n$, we have, for $x = n_a a + n_b b$,*

$$\begin{aligned}\mathrm{P}[S_n^* = -x \mid S_0^* = 0] = {}& \mathrm{P}[S_n = a \mid S_0 = -x + a] \\ &+ (\mathrm{P}[S_n > a \mid S_0 = -x + a] - \mathrm{P}[S_n > b \mid S_0 = -x + b])\end{aligned}$$

for $n_a \geq 1$, and

$$\mathrm{P}[S_n^* = -x \mid S_0^* = 0] = \mathrm{P}[S_n = b \mid S_0 = -x + b]$$

for $n_a = 0$ ($x = nb$).

Remark. The last equality of Theorem 1 is easy to prove:

$$\begin{aligned}\mathrm{P}[S_n^* = -nb \mid S_0^* = 0] &= p^*_{0,-b} p^*_{-b,-2b} \cdots p^*_{-(n-1)b,-nb} \\ &= p_{0,b} p_{-b,0} \cdots p_{-(n-1)b,-(n-2)b} \\ &= \mathrm{P}[S_n = b \mid S_0 = -nb + b].\end{aligned}$$

The equality corresponding to the case $n_a = n$ could be established in the same way.

In the next three sections, we consider $n_a \geq 1$, first with an example (Section 3) and then in the general case (Sections 4 and 5).

We are now interested in the context of random walks in stationary random environments. The transition probabilities of $(S_n)_{n\geq 0}$ (and, thus, also of $(S_n^*)_{n\geq 0}$) are then given by realizations of a stationary sequence of random variables.

More precisely, considering an invertible measure-preserving transformation $\theta\colon \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{F}, \mu)$ (see [7] for instance), we introduce a measurable function $p\colon \Omega \to [0, 1]$ and, for a fixed ω in Ω, we define the transition probabilities of $(S_n)_{n\geq 0}$ by

$$p_y(\omega) := p(\theta^y \omega), \qquad y \in \mathbb{Z}.$$

Thus, for each ω, we now have a probability $\mathrm{P} = \mathrm{P}^\omega$ that depends on ω; it is the *quenched law* of the random walk in the environment given by ω. The averaging probability with respect to the environments $\overline{\mathrm{P}} = \mathrm{P}^\omega(\cdot)\mu(\mathrm{d}\omega)$ is called the *annealed law* of the random walk in stationary random environments.

In this context, Theorem 1 can be rewritten as follows.

Theorem 2. (Theorem 1 reformulated.) *With the above notation, for almost all ω and any nonnegative integers n, n_a, and n_b with $n_a + n_b = n$, we have*

$$\begin{aligned}\mathrm{P}^\omega[S_n^* = -x \mid S_0^* = 0] = \mathrm{P}^{\theta^{-x+a}\omega}[S_n = x \mid S_0 = 0]& \\ + (\mathrm{P}^{\theta^{-x+a}\omega}[S_n > x \mid S_0 = 0] - \mathrm{P}^{\theta^{-x+b}\omega}&[S_n > x \mid S_0 = 0])\end{aligned}$$

for $n_a \geq 1$, and

$$\mathrm{P}^\omega[S_n^* = -x \mid S_0^* = 0] = \mathrm{P}^{\theta^{-x+b}\omega}[S_n = -x \mid S_0 = 0]$$

for $n_a = 0$, where $x = n_a a + n_b b$.

As a corollary of Theorem 2, we obtain our second theorem which is concerned with the *annealed* case.

Theorem 3. *In mean with respect to the environments, for any nonnegative integer n, the law of S_n and the law of $-S_n^*$ are the same. In other words, for any x in $\mathbb{Z}$,*

$$\overline{\mathrm{P}}[S_n^* = -x \mid S_0^* = 0] = \overline{\mathrm{P}}[S_n = x \mid S_0 = 0].$$

We deduce in particular the following result.

Corollary 1. *For any nonnegative integer n, we have the identity*

$$\overline{\mathrm{E}}((S_n^*)^2 \mid S_0^* = 0) = \overline{\mathrm{E}}(S_n^2 \mid S_0 = 0).$$

2.1. Remark on the reversible case

A *conductance* between two successive integers y and $y+1$ is a positive number $c(y, y+1)$ and its inverse, $r(y, y+1)$, is the *resistance* between y and $y+1$.

To a given family $(c(y, y+1))_{y\in\mathbb{Z}}$ of conductances, we can associate a nearest-neighbours Markov chain $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ whose transition probabilities are proportional to the conductances. Thus, we have

$$\mathrm{P}[S_{n+1} = y+1 \mid S_n = y] = \frac{c(y, y+1)}{\overline{c}(y)}$$

and

$$\mathrm{P}[S_{n+1} = y-1 \mid S_n = y] = \frac{c(y-1, y)}{\overline{c}(y)},$$

where $\overline{c}(y) := c(y-1, y) + c(y, y+1)$.

When the conductances are obtained as realizations of a stationary sequence of positive random variables, we obtain a random walk in stationary random environments.

In the ergodic case, we can prove that, for almost all environments ω of conductances, the sequence $(\mathrm{E}^\omega(S_n^2 \mid S_0 = 0)/n)_{n\geq 1}$ converges to the asymptotic variance

$$\sigma^2 = \frac{1}{\int c \,\mathrm{d}\mu \int (1/c)\,\mathrm{d}\mu},$$

with the convention that $1/+\infty = 0$ (see [1], [2], and [6]). Therefore, we observe a symmetry between c and $1/c$ asymptotically in time.

For a fixed environment of conductances, if we replace conductances by resistances, we change the Markov chain governed by transition probabilities proportional to the conductances into a Markov chain whose transition probabilities are inversely proportional to the conductances. Thus, the Markov chain $(S_n)_{n\geq 0}$ becomes the dual chain $(S_n^*)_{n\geq 0}$ and we can apply Corollary 1. This shows that the previous symmetry between c and $1/c$ appears in fact at any fixed time n when averaging with respect to the environments.

3. Initial observations: an example

For this example, we consider a Markov chain $(S_n)_{n\geq 0}$ with jumps $a = 1$ and $b = -1$ and we are interested in the probability

$$\mathrm{P}^{\omega}[S_n = x \mid S_0 = 0]$$

for $n = 10$ and $x = 2$.

We recall the following notation:

$$p_y = p_{y,y+1} \quad \text{and} \quad q_y = p_{y,y-1}, \qquad y \in \mathbb{Z}.$$

First, let us consider the probability that $(S_0, S_1, S_2, \ldots, S_{10})$ follows the path

$$\Gamma := (0, -1, 0, 1, 2, 3, 4, 3, 2, 3, 2),$$

starting at 0 and ending at 2; see Figure 1.

We have

$$\mathrm{P}[(S_0, S_1, S_2, \ldots, S_{10}) = \Gamma] = q_0\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, q_4\, q_3\, p_2\, q_3.$$

Expanding this product with respect to each factor $q_x = 1 - p_x$, we obtain a sum with sixteen terms

$$\begin{aligned}
\Sigma := {} & 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, 1\, p_2\, 1 + 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, 1\, p_2\, (-p_3) \\
& + 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, (-p_3)\, p_2\, 1 + p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, (-p_3)\, p_2\, (-p_3) \\
& + 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, 1\, p_2\, 1 + 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, 1\, p_2\, (-p_3) \\
& + 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, (-p_3)\, p_2\, 1 \\
& + 1\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, (-p_3)\, p_2\, (-p_3) \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, 1\, p_2\, 1 + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, 1\, p_2\, (-p_3) \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, (-p_3)\, p_2\, 1 \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, 1\, (-p_3)\, p_2\, (-p_3) \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, 1\, p_2\, 1 \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, 1\, p_2\, (-p_3) \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, (-p_3)\, p_2\, 1 \\
& + (-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, (-p_3)\, p_2\, (-p_3).
\end{aligned}$$

The main idea consists in transforming each term of Σ in such a way to make appear the probabilities that the dual chain $(S_n^*)_{0\leq n\leq 10}$ follows some paths.

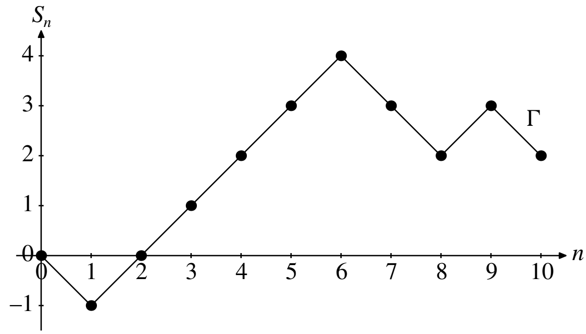

FIGURE 1: Representation of the path Γ.

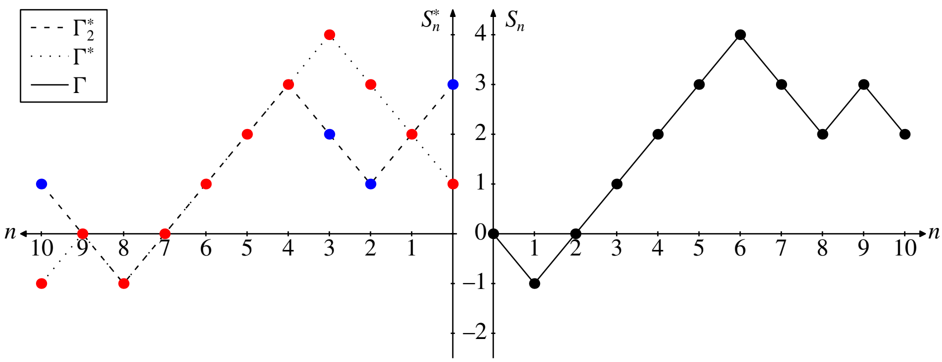

FIGURE 2: Representations of the paths Γ, Γ^*, and Γ_2^*.

Let us illustrate these transformations with the thirteenth term of Σ:

$$(-p_0)\, p_{-1}\, p_0\, p_1\, p_2\, p_3\, (-p_4)\, 1\, p_2 1.$$

To take into account the order in which the factors of this product appear, we consider the 10-tuple

$$\xi := (-p_0, p_{-1}, p_0, p_1, p_2, p_3, -p_4, 1, p_2, 1).$$

Then, we apply a mirror permutation, i.e.

$$(1, p_2, 1, -p_4, p_3, p_2, p_1, p_0, p_{-1}, -p_0),$$

followed by a shift of the negative signs to the left (and cyclically), ignoring the terms equal to 1:

$$(1, -p_2, 1, p_4, p_3, p_2, p_1, p_0, -p_{-1}, p_0) =: \eta.$$

Considering now the transition probabilities of the dual chain (S_n^*), this 10-tuple η reveals the path

$$\Gamma^* := (1, 2, 3, 4, 3, 2, 1, 0, -1, 0, -1);$$

see Figure 2. Indeed, the product

$$q_1\, q_2\, q_3\, p_4\, p_3\, p_2\, p_1\, p_0\, q_{-1}\, p_0 = \mathrm{P}[(S_0^*, S_1^*, S_2^*, S_3^*, S_4^*, S_5^*, S_6^*, S_7^*, S_8^*, S_9^*, S_{10}^*) = \Gamma^*],$$

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expanded with respect to the quantities $q_x = 1 - p_x$, contains the term

$$1\,(-p_2)\,1\,p_4\,p_3\,p_2\,p_1\,p_0\,(-p_{-1})\,p_0,$$

the product of the coordinates of η.

With the same transformations as above, the first term of Σ, i.e.

$$1\,p_{-1}\,p_0\,p_1\,p_2\,p_3\,1\,1\,p_2\,1,$$

gives the 10-tuple $(1, p_2, 1, 1, p_3, p_2, p_1, p_0, p_{-1}, 1)$, which is associated to the path $\Gamma_1^* := (1, 2, 1, 2, 3, 2, 1, 0, -1, -2, -1)$.

For the second term of Σ,

$$1\,p_{-1}\,p_0\,p_1\,p_2\,p_3\,1\,1\,p_2\,(-p_3),$$

we obtain the 10-tuple $(p_3, p_2, 1, 1, p_3, p_2, p_1, p_0, -p_{-1}, 1)$, which is associated to the path $\Gamma_2^* := (3, 2, 1, 2, 3, 2, 1, 0, -1, 0, 1)$; see Figure 2.

Continuing the same way reveals that

(i) each term of Σ reveals a path Γ_i^* $(1 \le i \le 16)$;

(ii) each of the paths Γ_i^* starts at 1 and ends at -1, or starts at 3 and ends at 1;

(iii) none of the paths Γ_i^* and Γ_j^* such that $\Gamma_i^* \neq \Gamma_j^*$ are translations of each other;

(iv) considering all the terms of Σ associated to the same path Γ_i^* and summing the products of their transformed 10-tuple coordinates, we obtain only a part of the probability

$$\mathrm{P}[(S_0^*, S_1^*, S_2^*, S_3^*, S_4^*, S_5^*, S_6^*, S_7^*, S_8^*, S_9^*, S_{10}^*) = \Gamma_i^*].$$

From (ii), it follows that a simple shift by '-1' or '-3' of the indexes of the p_y makes the resultant paths Γ_i^* all start at 0 and end at -2.

Now, we claim that, when considering all the possible paths of length 10 starting at 0 and ending at 2, we can reconstitute exactly the probability

$$\mathrm{P}[S_{10}^* = -2 \mid S_0^* = 0]$$

by using the same procedure described above (see Corollary 2 below).

The following section makes precise and generalizes these remarks.

4. Results needed for the proof of Theorem 1

In this section we consider the general case of a Markov chain on $\mathbb{Z}$ with jumps a and b $(a > b)$ whose transition probabilities are given by the function

$$\mathscr{P} : \mathbb{Z} \to (0, 1), \qquad y \mapsto p_y,$$

where $p_y = p_{y,y+a}$.

The sequence $(p_y)_{y\in\mathbb{Z}}$ will sometimes be regarded as a sequence of indeterminates, also denoted by $\mathscr{P}$. Distinguishing between the two uses is left to the reader.

Consider a positive integer n, and let $x = n_a a + n_b b$ be an S_n-reachable state, where n_a and n_b are nonnegative integers such that $n_a + n_b = n$.

Our aim is to establish a relationship between the law of S_n^* and the probability

$$\mathrm{P}[S_n = x \mid S_0 = 0]$$

that the walk $(S_n)_{n\ge 0}$ reaches x at time n when it starts at 0.

We shall proceed in a combinatorial way and consider, as in the previous section, the probability for S_n and S_n^* to follow peculiar paths. To do this, we need to introduce further notation.

We denote by $\mathcal{C}_0^{(n)}$ the set of paths of length n starting at 0 whose jumps are equal to a or b:

$$\mathcal{C}_0^{(n)} := \{\Gamma = (\gamma_0, \gamma_1, \dots, \gamma_n) \in \mathbb{Z}^{n+1} \mid \gamma_0 = 0,\ \gamma_{i+1} - \gamma_i \in \{a, b\},\ i = 0, 1, \dots, n-1\}.$$

We denote by $\mathcal{C}_{0,s}^{(n)}$ for s in $\mathbb{Z}$ the set of the paths in $\mathcal{C}_0^{(n)}$ which end at s:

$$\mathcal{C}_{0,s}^{(n)} := \{\Gamma \in \mathcal{C}_0^{(n)} \mid \gamma_n = s\}.$$

We successively introduce

$$\mathcal{E}(\mathcal{P}) := \bigcup_{y \in \mathbb{Z}} \{-p_y, 1, p_y\},$$

and, for $\Gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ in $\mathcal{C}_0^{(n)}$,

$$\mathcal{D}_\Gamma(\mathcal{P}) := \{\xi = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathcal{E}(\mathcal{P})^n \mid \xi_i = p_{\gamma_i} \text{ if } \gamma_{i+1} = \gamma_i + a \\ \text{or } \xi_i \in \{1, -p_{\gamma_i}\} \text{ if } \gamma_{i+1} = \gamma_i + b\}.$$

Thus, we have

$$\mathrm{P}[(S_0, S_1, \dots, S_n) = \Gamma] = p_{\gamma_0,\gamma_1} p_{\gamma_1,\gamma_2} \cdots p_{\gamma_{n-1},\gamma_n} = \sum_{\xi \in \mathcal{D}_\Gamma(\mathcal{P})} \prod_{i=0}^{n-1} \xi_i$$

(this last identity is obtained by expanding the product $p_{\gamma_0,\gamma_1} p_{\gamma_1,\gamma_2} \cdots\ p_{\gamma_{n-1},\gamma_n}$ with respect to each factor $q_y = 1 - p_y$).

Denoting by $\mathcal{D}(0, x, n, \mathcal{P})$ the disjoint union of the sets $\mathcal{D}_\Gamma(\mathcal{P})$ over Γ in $\mathcal{C}_{0,x}^{(n)}$, we obtain the following proposition.

Proposition 1. *With the above notation,*

$$\mathrm{P}[S_n = x \mid S_0 = 0] = \sum_{\xi \in \mathcal{D}(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} \xi_i .$$

Similarly, we have

$$\mathrm{P}[S_n^* = -x \mid S_0^* = 0] = \sum_{\eta \in \mathcal{D}^*(0,-x,n,\mathcal{P})} \prod_{i=0}^{n-1} \eta_i ,$$

where $\mathcal{D}^*(0, -x, n, \mathcal{P})$ is the set

$$\bigcup_{\Gamma \in -\mathcal{C}_{0,x}^{(n)}} \{\eta = (\eta_0, \eta_1, \dots, \eta_{n-1}) \in \mathcal{E}(\mathcal{P})^n \mid \eta_i \in \{1, -p_{\gamma_i}\} \textit{ if } \gamma_{i+1} = \gamma_i - b \\ \textit{or } \eta_i = p_{\gamma_i} \textit{ if } \gamma_{i+1} = \gamma_i - a\}.$$

Let us now suppose that n_a is greater than or equal to 1 and introduce the notation

$$i(\xi) := \max\{i \in \{0, 1, \dots, n-1\} \mid \xi_i \neq 1\} \quad \text{for } \xi \in \mathcal{D}(0, x, n, \mathcal{P})$$

and

$$\mathcal{D}_+(0, x, n, \mathcal{P}) := \{\xi \in \mathcal{D}(0, x, n, \mathcal{P}) \mid \xi_{i(\xi)} \in \{p_y \mid y \in \mathbb{Z}\}\}.$$

Remark. Given an element ξ of $\mathcal{D}_+(0, x, n, \mathcal{P})$, then, by construction, the unique path Γ in $\mathcal{C}_{0,x}^{(n)}$ associated to ξ appears to make a jump equal to a between the times $i(\xi)$ and $i(\xi) + 1$, and continues to make jumps equal to b thereafter until it reaches x. Hence, we have

$$\gamma_{i(\xi)} = x - a - (n - i(\xi) - 1)b.$$

Furthermore, since Γ has exactly n_b jumps equal to b, we necessarily have

$$i(\xi) \geq n - n_b - 1 = n_a - 1.$$

The following proposition gives a probabilistic interpretation of the set $\mathcal{D}_+(0, x, n, \mathcal{P})$.

Proposition 2. *For any positive integer* n *and any* $x = n_a a + n_b b$ *with* $n = n_a + n_b$ *and* $n_a \geq 1$,

$$\mathrm{P}[S_n \geq x \mid S_0 = 0] = \sum_{\xi \in \mathcal{D}_+(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} \xi_i.$$

Proof. Let $\Gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be a path in $\mathcal{C}_0^{(n)}$ such that $\gamma_n \geq x$.

We begin by noting that, since the jumps of γ are equal to a or b, the function $i \mapsto -bi + \gamma_i$ is increasing with jumps equal to 0 or $a - b$. Its initial value $-b \cdot 0 + \gamma_0 = 0$ is less than or equal to $(n_a - 1)\ (a - b) = -bn + x - (a - b)$ ($a > b$ and $n_a \geq 1$) and its final value $-bn + \gamma_n$ is greater than or equal to $-bn + x$. Thus, there exists a unique instant i_0 in $\{0, 1, \ldots, n - 1\}$ such that

$$-bi_0 + \gamma_{i_0} = -bn + x - (a - b) \quad \text{and} \quad -b(i_0 + 1) + \gamma_{i_0+1} = -bn + x.$$

Furthermore, i_0 must be greater than $n_a - 1$.

It follows that

$$\begin{aligned}\mathrm{P}&[S_n \geq x \mid S_0 = 0] \\ &= \sum_{i_0=n_a-1}^{n-1} \mathrm{P}[i_0 + S_{i_0} = -bn + x - (a - b),\ (i_0 + 1) + S_{i_0+1} = -bn + x \mid S_0 = 0].\end{aligned}$$

Using the homogeneity in time of the transition probabilities of $(S_n)_{n\geq 0}$ and Proposition 1, we obtain, thanks to the remark before Proposition 2,

$$\begin{aligned}\mathrm{P}[S_n \geq x \mid S_0 = 0] &= \sum_{i_0=n_a-1}^{n-1} \mathrm{P}[S_{i_0} = -i_0 - bn + x - (a - b) \mid S_0 = 0] p_{-i_0-bn+x-(a-b)} \\ &= \sum_{i_0=n_a-1}^{n-1} \sum_{\xi \in \mathcal{D}(0,-i_0-bn+x-(a-b),i_0,\mathcal{P})} \left(\prod_{j=0}^{i_0-1} \xi_j\right) p_{-i_0-bn+x-(a-b)} \\ &= \sum_{\xi \in \mathcal{D}_+(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} \xi_i.\end{aligned}$$

This completes the proof.

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For all r in $\mathbb{Z}$, we now define a shift on $\mathcal{E}(\mathcal{P})$ by

$$\theta^r(p_y) := p_{y+r}, \qquad \theta^r(1) := 1, \qquad \theta^r(-p_y) := -p_{y+r}, \qquad y \in \mathbb{Z},$$

and we extend it on $\mathcal{E}(\mathcal{P})^n$ according to

$$\theta^r(\xi) := (\theta^r(\xi_0), \theta^r(\xi_1), \dots, \theta^r(\xi_{n-1})), \qquad \xi = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathcal{E}(\mathcal{P})^n.$$

We further define the three functions

$$\varphi_1, \varphi_2, \varphi_3 \colon \mathcal{E}(\mathcal{P})^n \to \mathcal{E}(\mathcal{P})^n$$

by

$$\varphi_1(\xi) := \begin{cases} \theta^{-x+a}(\xi) & \text{if } \xi \in \mathcal{D}_+(0, x, n, \mathcal{P}), \\ \theta^{-x+b}(\xi) & \text{otherwise,} \end{cases}$$

$$\varphi_2((\xi_0, \xi_1, \dots, \xi_{n-1})) := (\xi_{n-1}, \dots, \xi_1, \xi_0),$$

and, for all $i \in \{0, 1, \dots, n-1\}$,

$$(\varphi_3(\xi))_i := \begin{cases} 1 & \text{if } \xi_i = 1, \\ \operatorname{sgn}(\xi_{\operatorname{succ}_\xi(i)}) \mid \xi_i \mid & \text{if } \xi_i \neq 1, \end{cases}$$

where $\operatorname{succ}_\xi(i)$ denotes the index of the coordinate of ξ which is not equal to 1 and which 'cyclically follows' the coordinate ξ_i. More precisely, if the set $\{j \in \{i+1, i+2, \dots, n-1\} \mid \xi_j \neq 1\}$ is not empty, we set

$$\operatorname{succ}_\xi(i) := \min\{j \in \{i+1, i+2, \dots, n-1\} \mid \xi_j \neq 1\};$$

otherwise,

$$\operatorname{succ}_\xi(i) := \min\{j \in \{0, 1, \dots, i\} \mid \xi_j \neq 1\}.$$

Observe that φ_1 and φ_3 do not affect the coordinates of ξ that are equal to 1.

The composite mapping $\Phi := \varphi_3 \circ \varphi_2 \circ \varphi_1$ allows us to write a first relation between the laws of S_n and S_n^*.

Proposition 3. *For any positive integer n and any $x = n_a a + n_b b$ with $n = n_a + n_b$ and $n_a \geq 1$, the mapping Φ restricted to $\mathcal{D}(0, x, n, \mathcal{P})$ is one-to-one between the sets $\mathcal{D}(0, x, n, \mathcal{P})$ and $\mathcal{D}^*(0, -x, n, \mathcal{P})$ of Proposition 1.*

Proof. Since the sets $\mathcal{D}(0, x, n, \mathcal{P})$ and $\mathcal{D}^*(0, -x, n, \mathcal{P})$ have the same cardinality, it suffices to prove that $\Phi \colon \mathcal{D}(0, x, n, \mathcal{P}) \to \mathcal{E}(\mathcal{P})^n$ is an injective mapping with values in $\mathcal{D}^*(0, -x, n, \mathcal{P})$.

The function Φ is injective because the functions

$$\varphi_1 \colon \mathcal{D}(0, x, n, \mathcal{P}) \to \mathcal{E}(\mathcal{P})^n, \qquad \varphi_2 \colon \mathcal{E}(\mathcal{P})^n \to \mathcal{E}(\mathcal{P})^n, \quad \text{and} \quad \varphi_3 \colon \mathcal{E}(\mathcal{P})^n \to \mathcal{E}(\mathcal{P})^n$$

are also injective. The injectivity of $\varphi_1 \colon \mathcal{D}(0, x, n, \mathcal{P}) \to \mathcal{E}(\mathcal{P})^n$ is a consequence of the injectivity of the shifts θ^r, $r \in \mathbb{Z}$, and the fact that the images by φ_1 of the sets $\mathcal{D}_+(0, x, n, \mathcal{P})$ and its complementary in $\mathcal{D}(0, x, n, \mathcal{P})$ are disjoint. The injectivity of $\varphi_2 \colon \mathcal{E}(\mathcal{P})^n \to \mathcal{E}(\mathcal{P})^n$ is straightforward. The injectivity of $\varphi_3 \colon \mathcal{E}(\mathcal{P})^n \to \mathcal{E}(\mathcal{P})^n$ is a consequence of the fact that, if we do not take into account the coordinates of ξ in $\mathcal{E}(\mathcal{P})^n$ which are equal to 1 (and stay unchanged under the action of φ_3), the action of φ_3 on ξ is a cyclic permutation of the signs of the remaining coordinates.

Let ξ be an element of $\mathcal{D}(0, x, n, \mathcal{P})$. We prove the theorem by showing that $\Phi(\xi)$ belongs to the set $\mathcal{D}^*(0, -x, n, \mathcal{P})$. Set $\Phi(\xi) = (\eta_0, \eta_1, \dots, \eta_{n-1})$.

We have to only establish the existence of a path $\Gamma^* = (\gamma_0, \gamma_1, \dots, \gamma_n)$ of $\mathcal{C}_0^{(n)}$ such that $\eta_i \in \{1, -p_{\gamma_i}\}$ when $\gamma_{i+1} = \gamma_i - b$ and $\eta_i = p_{\gamma_i}$ when $\gamma_{i+1} = \gamma_i - a$. (Note that such a path Γ^* necessarily ends at $-x$ since $\Phi(\xi)$ contains the same number of coordinates belonging to $\{p_y \mid y \in \mathbb{Z}\}$ than ξ.)

Let i_0 be the minimum i index such that $\eta_i \neq 1$ (i_0 exists because $n_a \geq 1$). Then existence of the path Γ^* is a consequence of the following assertions:

(i) $\eta_{i_0} \in \{-p_{-i_0 b},\ p_{-i_0 b}\}$;

(ii) for all i in $\{i_0 + 1, i_0 + 2, \dots, n - 1\}$, if $\eta_i \in \{-p_y, p_y\}$ then either $\eta_j = -p_{y+(i-j)b}$ or $\eta_j = p_{y+(i-j-1)b+a}$, where j is the greatest element of the set $\{i' \mid i_0 \leq i' \leq i - 1 \text{ and } \eta_{i'} \neq 1\}$.

Let us start by proving (i). Because of the definition of Φ and since ξ belongs to $\mathcal{D}(0, x, n, \mathcal{P})$, we have

$$\xi_{n-i_0} = \xi_{n-i_0+1} = \cdots = \xi_{n-1} = 1$$

(this assertion is empty when $i_0 = 0$) and

$$\xi_{n-1-i_0} \in \{-p_{x-(i_0+1)b}, p_{x-i_0 b-a}\}.$$

If $\xi_{n-1-i_0} = -p_{x-(i_0+1)b}$ then ξ belongs to $\mathcal{D}(0, x, n, \mathcal{P}) \backslash \mathcal{D}_+(0, x, n, \mathcal{P})$ and we successively obtain

$$(\varphi_1(\xi))_{n-1-i_0} = -p_{-i_0 b}, \qquad (\varphi_2 \circ \varphi_1(\xi))_{i_0} = -p_{-i_0 b},$$

and

$$\eta_{i_0} = (\varphi_3 \circ \varphi_2 \circ \varphi_1(\xi))_{i_0} \in \{-p_{-i_0 b},\ p_{-i_0 b}\}.$$

In the same way, if $\xi_{n-1-i_0} = p_{x-i_0 b-a}$ then ξ is an element of $\mathcal{D}_+(0, x, n, \mathcal{P})$ and we have

$$(\varphi_1(\xi))_{n-1-i_0} = p_{-i_0 b}, \qquad (\varphi_2 \circ \varphi_1(\xi))_{i_0} = p_{-i_0 b},$$

and

$$\eta_{i_0} = (\varphi_3 \circ \varphi_2 \circ \varphi_1(\xi))_{i_0} \in \{-p_{-i_0 b},\ p_{-i_0 b}\}.$$

This completes the proof of (i).

To prove (ii), suppose that $i \in \{i_0+1, i_0+2, \dots, n-1\}$ and that $\eta_i \in \{-p_y, p_y\}$. Denoting by j the greatest element of the set $\{i' \mid i_0 \leq i' \leq i-1 \text{ and } \eta_{i'} \neq 1\}$, we have, by the definition of Φ,

$$\xi_{n-1-i} \in \{-p_{y+r}, p_{y+r}\}, \qquad \text{where } r = x - a \text{ or } r = x - b$$

and

$$\xi_{n-i} = \xi_{n-i+1} = \cdots = \xi_{n-j-2} = 1.$$

This implies that if $\xi_{n-1-i} = -p_{y+r}$ then

$$\xi_{n-1-j} \in \{-p_{y+r+(i-j)b}, p_{y+r+(i-j)b}\},$$

from which we deduce the identity

$$\eta_j = -p_{y+(i-j)b}.$$

If $\xi_{n-1-i} = p_{y+r}$ then

$$\xi_{n-1-j} \in \{-p_{y+r+(i-j-1)b+a}, p_{y+r+(i-j-1)b+a}\}$$

and

$$\eta_j = p_{y+(i-j-1)b+a}.$$

This completes the proof of (ii) and, hence, the proposition.

Propositions 1 and 3 give at once the following result.

Corollary 2. *For any positive integer n and any $x = n_a a + n_b b$ with $n = n_a + n_b$ and $n_a \geq 1$,*

$$\mathrm{P}[S_n^* = -x \mid S_0^* = 0] = \sum_{\xi \in \mathcal{D}(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} (\Phi(\xi))_i.$$

5. Proof of Theorem 1

By the definition of Φ, for any ξ in $\mathcal{D}(0, x, n, \mathcal{P})$, we have

$$\prod_{i=0}^{n-1} (\Phi(\xi))_i = \prod_{i=0}^{n-1} \theta^r(\xi_i),$$

where $r = -x + a$ if ξ is in $\mathcal{D}_+(0, x, n, \mathcal{P})$ and $r = -x + b$ otherwise.

From this, using Corollary 1 and Propositions 1 and 2, we derive

$$\begin{aligned}
\mathrm{P}[S_n^* = -x \mid S_0^* = 0] &= \sum_{\xi \in \mathcal{D}(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} (\Phi(\xi))_i \\
&= \theta^{-x+a} \Bigg(\sum_{\xi \in \mathcal{D}_+(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} \xi_i \Bigg) \\
&\quad + \theta^{-x+b} \Bigg(\sum_{\xi \in \mathcal{D}(0,x,n,\mathcal{P}) \setminus \mathcal{D}_+(0,x,n,\mathcal{P})} \prod_{i=0}^{n-1} \xi_i \Bigg) \\
&= \theta^{-x+a}(\mathrm{P}[S_n \geq x \mid S_0 = 0]) \\
&\quad + \theta^{-x+b}(\mathrm{P}[S_n = x \mid S_0 = 0] - \mathrm{P}[S_n \geq x \mid S_0 = 0]) \\
&= \theta^{-x+a}(\mathrm{P}[S_n \geq x \mid S_0 = 0]) - \theta^{-x+b}(\mathrm{P}[S_n > x \mid S_0 = 0]) \\
&= \theta^{-x+a}(\mathrm{P}[S_n = x \mid S_0 = 0]) + \theta^{-x+a}(\mathrm{P}[S_n > x \mid S_0 = 0]) \\
&\quad - \theta^{-x+b}(\mathrm{P}[S_n > x \mid S_0 = 0]).
\end{aligned}$$

Here the probabilities have been regarded as polynomials in the indeterminates p_y, $y \in \mathbb{Z}$, and the shifts θ^{-x+a} and θ^{-x+b} have been extended to endomorphisms of the algebra $\mathbb{Z}[p_y; y \in \mathbb{Z}]$.

https://doi.org/10.1239/jap/1339878790 Published online by Cambridge University Press

Finally, for all r, s in $\mathbb{Z}$, we have

$$\theta^r(\mathrm{P}[S_n = s \mid S_0 = 0]) = \mathrm{P}[S_n = s + r \mid S_0 = r].$$

This allows us to complete the proof.

6. Counterexamples for random walks in stationary random environments with more than two jumps

In this section, to any invertible ergodic measure-preserving transformation $\theta\colon \Omega \to \Omega$ on a nonatomic probability space $(\Omega, \mathcal{F}, \mu)$, we construct a random walk in stationary random environments with jumps -1, 1, and 2 for which Theorem 2 does not hold.

By the hypotheses on $(\Omega, \mathcal{F}, \mu)$ and θ, there exists an element A of $\mathcal{F}$ with $\mu(A) > 0$ and $\theta^{-3}A, \theta^{-2}A, A, \theta A$ pairwise disjoint sets (see [7]). Thus, we can consider

$$\begin{aligned} r(\omega) &:= \tfrac{1}{6}(\mathbf{1}_A(\omega) + \mathbf{1}_\Omega(\omega)), \\ p(\omega) &:= \tfrac{1}{6}(\mathbf{1}_{\theta^{-1}A}(\omega) + \mathbf{1}_\Omega(\omega)), \\ q(\omega) &:= 1 - p(\omega) - r(\omega), \end{aligned}$$

and characterize the law of the random walk in stationary random environments by

$$\begin{aligned} \mathrm{P}^\omega[S_n = y + 2 \mid S_0 = y] &= r(\theta^y\omega), \\ \mathrm{P}^\omega[S_n = y + 1 \mid S_0 = y] &= p(\theta^y\omega), \\ \mathrm{P}^\omega[S_n = y - 1 \mid S_0 = y] &= q(\theta^y\omega). \end{aligned}$$

Then

$$\begin{aligned} &\overline{\mathrm{P}}[S_2 = 3 \mid S_0 = 0] - \overline{\mathrm{P}}[S_2^* = -3 \mid S_0^* = 0] \\ &\quad = \int (p(\omega)r(\theta\omega) + r(\omega)p(\theta^2\omega) - p(\omega)r(\theta^{-1}\omega) - r(\omega)p(\theta^{-2}\omega))\,\mathrm{d}\mu(\omega) \\ &\quad = \int r(\omega)(p(\theta^{-1}\omega) + p(\theta^2\omega) - p(\theta\omega) - p(\theta^{-2}\omega))\,\mathrm{d}\mu(\omega) \\ &\quad = \frac{1}{6}\int (p(\theta^{-1}\omega) + p(\theta^2\omega) - p(\theta\omega) - p(\theta^{-2}\omega))\,\mathrm{d}\mu(\omega) \\ &\qquad + \frac{1}{6^2}\int \mathbf{1}_A(\omega)(\mathbf{1}_A(\omega) + \mathbf{1}_{\theta^{-3}A}(\omega) - \mathbf{1}_{\theta^{-2}A}(\omega) - \mathbf{1}_{\theta A}(\omega))\,\mathrm{d}\mu(\omega) \\ &\quad = \frac{1}{6^2}\mu(A) \\ &\quad > 0, \end{aligned}$$

which yields the result.

Remark. We can similarly construct a random walk in stationary random environments with jumps -1, 0, and 1 for which Theorem 2 does not hold.

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