

THE INFINITESIMAL INDEX

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(Received 1 November 2011; accepted 29 February 2012;
first published online 18 May 2012)

Abstract In this note, we study an invariant associated with the zeros of the moment map generated by an action form, the *infinitesimal index*. This construction will be used to study the compactly supported equivariant cohomology of the zeros of the moment map and to give formulas for the multiplicity index map of a transversally elliptic operator.

Keywords: index theory; equivariant cohomology

AMS 2000 *Mathematics subject classification:* Primary 19L10; 19L47; 19K56; 55N91

Introduction

Let G be a compact Lie group acting on a manifold N . Then G acts on the cotangent bundle $M = T^*N$ in a Hamiltonian way. Set $\mathfrak{g} := T_1G$ as its Lie algebra. The set M^0 of zeros of the moment map $\mu : M \rightarrow \mathfrak{g}^*$ is the union of the conormals to the G -orbits in N . An element S of the equivariant K -theory $K_G(M^0)$ of M^0 is called a transversally elliptic symbol, and Atiyah and Singer (see [1]) associated with S a trace class representation $\text{index}(S)$ of G . If \hat{G} is the dual of G , the representation $\text{index}(S)$ gives rise to a function $m(\tau)$ on \hat{G} such that $\text{index}(S) = \sum_{\tau \in \hat{G}} m(\tau)\tau$, called the multiplicity index map.

The analogue of the equivariant K -theory of M^0 is the equivariant cohomology with compact supports $H_{G,c}^*(M^0)$. Here we construct a map infdex_G^μ , called the infinitesimal index, associating with an element $[\alpha] \in H_{G,c}^*(M^0)$ an invariant distribution on \mathfrak{g}^* . We prove a certain number of functorial properties of this map, mimicking the properties of the index map formalized by Atiyah, Segal and Singer. However, although our proofs are similar to those of [4, 5, 20, 22], our point of view is dual. Indeed in previous works, the equivariant index, or integrals of equivariant cohomology classes, are (generalized) functions on G , or \mathfrak{g} , while we work directly on the dual space \mathfrak{g}^* .

More generally, we consider the case where M is a G -manifold provided with a G -invariant 1-form σ (and we do not assume that $d\sigma$ is non-degenerate). This allows us to obtain a map $\text{infdex}_G^\mu : H_{G,c}^*(M^0) \rightarrow \mathcal{D}'(\mathfrak{g}^*)^G$, where M^0 is the set of zeros of the

associated moment map $\mu : M \rightarrow \mathfrak{g}^*$ and $\mathcal{D}'(\mathfrak{g}^*)^G$ the space of G -invariant distributions on \mathfrak{g}^* . Our construction is strongly related to Paradan’s localization on M^0 of the equivariant cohomology of M (see [24]).

An outline of the article

Let us summarize the content of this article.

In the first section, we give a ‘de Rham’ definition of the equivariant cohomology with compact supports $H_{G,c}^*(Z)$ of a topological space Z which is a closed invariant subspace of a G -manifold M : a representative of a class $[\alpha]$ is an equivariant differential form $\alpha(x)$ on M with compact support and such that the equivariant differential $D\alpha$ of α vanishes in a neighborhood of Z . In the appendix, we show that under mild assumptions on M and Z our space $H_{G,c}^*(Z)$ is naturally isomorphic to the (topological) equivariant cohomology of Z with compact supports.

In the second section, we define the infinitesimal index. Let M be a G -manifold provided with a G -invariant 1-form σ (we will say that σ is an action form). Let v_x be the vector field on M associated with $x \in \mathfrak{g}$ and i_x the derivation on forms induced by contraction with v_x . The moment map $\mu : \mathfrak{g} \rightarrow C^\infty(M)$ or $\mu : M \rightarrow \mathfrak{g}^*$ is defined by $\mu(x) = -\langle \sigma, v_x \rangle = -i_x(\sigma)$. Then

$$\Omega(x) = \mu(x) + d\sigma = D\sigma(x)$$

is a closed (in fact exact) equivariant form on M . The symbol D denotes in this paper the equivariant differential as defined in the Cartan model (see formula (2)).

Our main remark is the following.

Proposition 0.1. *If f is a smooth function on \mathfrak{g}^* with compact support, and α is a compactly supported equivariant form such that the differential $D\alpha$ vanishes in a neighborhood of $M^0 := \mu^{-1}(0)$, then the double integral*

$$\int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

is independent of s , for s sufficiently large.

Here $\hat{f}(x)$ is the Fourier transform of f . Some comment is in order: if $\alpha(x)$ is closed (and compactly supported) on M , it is clear that the integral $\int_M e^{is\Omega(x)} \alpha(x)$ is independent of s as $\Omega(x) = D\sigma(x)$ is an exact equivariant form. In our context, $\alpha(x)$ is compactly supported, but $\alpha(x)$ is **not closed** on M : only its restriction to a neighborhood of M^0 is closed. This is however sufficient for proving that

$$\langle \text{infdex}_G^\mu([\alpha]), f \rangle = \lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx \tag{1}$$

is a well defined map from $H_{G,c}^*(M^0)$ to invariant distributions on \mathfrak{g}^* . This we call the *infinitesimal index*. The infinitesimal index does not depend on some deformations of the form σ ; see Theorem 3.5.

In the third and fourth sections, we prove a certain number of functorial properties of the infinitesimal index: the locality (excision) property in § 4.1, the functoriality with respect to subgroups in § 4.5, and the stability with respect to immersions in § 4.8.

One of the most important properties is the free action property that we prove in § 4.11. Consider the situation where the compact Lie group L acts freely on M and 0 is a regular value of μ . Then the infinitesimal index of a class $[\alpha]$ is a polynomial density on \mathfrak{l}^* . Its value at 0 is the integral of the cohomology class corresponding to $[\alpha]$ under the Kirwan map over the reduced space $\mu^{-1}(0)/L$. This is essentially the Witten nonabelian localization theorem [25]. We give also the double equivariant version, where a compact Lie group G acts on M , commuting with the free action of L .

We then deduce from these properties the stability with respect to induction in § 5.13.1, and a comparison formula with the infinitesimal index for the maximal torus of G in § 5.15.

Let us comment on previous work around this theme.

The use of the form $e^{isD\sigma}$ in order to ‘localize’ integrals is the main principle in the Witten nonabelian localization theorem [14, 25] and our definition of the infinitesimal index is strongly inspired by this principle.

Paradan has studied systematically the situation of a manifold M provided with a G -invariant action form σ . Indeed, he constructed in [20] a closed equivariant form P_σ on M , congruent to 1 in cohomology and supported near M^0 . Paradan’s form P_σ is constructed using equivariant cohomology with $C^{-\infty}$ coefficients. Multiplying $\alpha(x)$ by Paradan’s form $P_\sigma(x)$ leads to a closed compactly supported equivariant form on M and $I(x) := \int_M P_\sigma(x)\alpha(x)$ is a generalized function on \mathfrak{g} . As we explain in Remark 3.8, our infinitesimal index is the Fourier transform of $I(x)$. Properties of the infinitesimal index could thus be deduced by means of the Fourier transform from the functorial properties of P_σ proven in [20, 22]. For example, the independence of the infinitesimal index with respect to some deformations of the form σ is an important tool, similar to the independence for P_σ proven in a more general setting in [20] (Proposition 2.6). The formula for infdex_G^μ in term of a maximal torus of G is similar to a remarkable formula in [20] (Theorem 4.5). However, we have chosen here to prove directly properties of the infinitesimal index by using our limit definition. There are two advantages in doing so. First, we believe that the proofs are easier. Secondly, this is the framework that we will use in [11] to produce piecewise polynomial densities (also called spline distributions) directly on \mathfrak{g}^* from some compactly supported equivariant classes.

In the case where $M = N \oplus N^*$, where N is a representation space for a linear action of a torus G , we determined $K_G(M^0)$ as a space of functions on \hat{G} using the multiplicity index map in the article [9]. In a companion article [10], we have used the infinitesimal index to identify $H_{G,c}^*(M^0)$ with a space of spline distributions on \mathfrak{g}^* of which the functions describing the index are a discrete analogue.

Finally using the analogies between splines and discrete functions, we have compared in [11] the equivariant cohomology with compact supports and the equivariant K -theory of M^0 , by relating the infinitesimal index and the multiplicity index map.

1. Equivariant de Rham cohomology

Let M be a C^∞ manifold with a C^∞ action of a compact Lie group G . We are going to define its equivariant cohomology with compact supports following the Cartan definition (see [13]).

We define the space of compactly supported equivariant forms as

$$\mathcal{A}_{G,c}(M) = (S(\mathfrak{g}^*) \otimes \mathcal{A}_c(M))^G$$

with the grading given setting \mathfrak{g}^* in degree 2. Here $\mathcal{A}_c(M)$ is the algebra of differential forms on M with compact supports. Thus an element of $\mathcal{A}_{G,c}(M)$ can be written as $\alpha(x) = \sum_{a=1}^R P_a(x)\alpha^a$ where $P_a(x)$ are polynomial functions on \mathfrak{g} , and α^a are differential forms with compact support on M .

Each element $x \in \mathfrak{g}$ of the Lie algebra of G induces a vector field v_x on M , the *infinitesimal generator* of the action: here the sign convention is that $v_x = \frac{d}{d\epsilon} \exp(-\epsilon x) \cdot m$ in order that the map $x \rightarrow v_x$ be a Lie algebra homomorphism. A vector field V on M induces a derivation ι_V on forms, such that $\iota_V(df) = V(f)$, and for simplicity we denote ι_{v_x} by ι_x .

One defines the differential as follows. Given $\alpha \in \mathcal{A}_{G,c}(M)$, we think of α as an equivariant polynomial map on \mathfrak{g} with values in $\mathcal{A}_c(M)$; thus, for any $x \in \mathfrak{g}$, we set

$$D\alpha(x) := d(\alpha(x)) - \iota_x(\alpha(x)) \tag{2}$$

where d is the usual de Rham differential.

It is easy to see that D increases the degree by 1 and that $D^2 = 0$. Thus we can take the cohomology and we get the G -equivariant cohomology of M with compact supports.

Now take a G -stable closed set Z in a manifold M . Consider the open set $U = M \setminus Z$. Then U is a manifold and we have an inclusion of complexes $\mathcal{A}_{G,c}(U) \subset \mathcal{A}_{G,c}(M)$ given by extension by zero. We set

$$\mathcal{A}_{G,c}(Z, M) := \mathcal{A}_{G,c}(M) / \mathcal{A}_{G,c}(U).$$

Definition 1.1. The equivariant de Rham cohomology with compact supports $H_{G,c}^*(Z)$ is the cohomology of the complex $\mathcal{A}_{G,c}^*(Z, M)$.

Notice that $\mathcal{A}_{G,c}(U)$ is an ideal in $\mathcal{A}_{G,c}(M)$, so $\mathcal{A}_{G,c}(Z, M)$ is a differential graded algebra and $H_{G,c}^*(Z)$ is a graded algebra (without 1 if Z is not compact).

In this model, a representative of a class in $H_{G,c}^*(Z)$ is an equivariant form $\alpha(x)$ with **compact support** on M . The form α is not necessary equivariantly closed on M , but there exists a neighborhood of Z such that the restriction of $\alpha(x)$ to this neighborhood is equivariantly closed.

If Z is compact, the class 1 belongs to $H_{G,c}^*(Z)$: a representative of 1 is a G -invariant function χ on M with compact support and identically equal to 1 on a neighborhood of Z in M .

Remark 1.2. Our model for $H_{G,c}^*(Z)$ seems to depend on the ambient manifold M . However, in the appendix we are going to see that, under mild assumptions on M and Z , $H_{G,c}^*(Z)$ is naturally isomorphic to the equivariant singular cohomology of Z with compact supports.

By the very definition of $H_{G,c}^*(Z)$, we also deduce the following proposition.

Proposition 1.3. *Let M be a G -space, and $j : Z \rightarrow M$ the inclusion of a closed G -stable subset. Denote by $i : U \rightarrow M$ the inclusion of $U := M \setminus Z$. We have a long exact sequence*

$$\dots \rightarrow H_{G,c}^h(U) \xrightarrow{i^*} H_{G,c}^h(M) \xrightarrow{j^*} H_{G,c}^h(Z) \longrightarrow H_{G,c}^{h+1}(U) \rightarrow \dots \tag{3}$$

If $i : Z \rightarrow M$ is a closed G -submanifold of a manifold M , the restriction of forms gives rise to a well defined map $i^* : \mathcal{A}_{G,c}(Z, M) \rightarrow \mathcal{A}_{G,c}(Z)$.

Proposition 1.4. *If Z is a closed G -invariant submanifold of a manifold M admitting an equivariant tubular neighborhood, the map i^* induces an isomorphism in cohomology.*

Proof. We reduce to the case in which M is a vector bundle on Z by restriction to a tubular neighborhood. Put a G -invariant metric on this bundle and let $p : M \rightarrow Z$ be the projection. Choose a C^∞ function f on \mathbb{R} with compact support and equal to 1 in a neighborhood of 0. We map an equivariant form $\omega \in \mathcal{A}_{G,c}(Z)$ to $f(\|m\|^2)p^*\omega(m)$ and then to its class modulo $\mathcal{A}_{G,c}(U)$. It is easily seen that this map is an inverse in cohomology of the map i^* . \square

Assume that M is a $L \times G$ -manifold and that L acts freely on M . Then M/L is a manifold with a G -action. Let Z be a $G \times L$ closed subset of M . Denote by $p : M \rightarrow M/L$ the projection. The pull-back of forms on M/L induces a map from $p^* : H_{G,c}^*(Z/L) \rightarrow H_{L \times G,c}^*(Z)$.

Proposition 1.5. *The pull-back*

$$p^* : H_{G,c}^*(Z/L) \rightarrow H_{L \times G,c}^*(Z)$$

is an isomorphism.

Proof. The fact that the pull-back of forms induces an isomorphism between $H_{G,c}^*(M/L)$ and $H_{L \times G,c}^*(M)$, and between $H_{G,c}^*((M \setminus Z)/L)$ and $H_{L \times G,c}^*(M \setminus Z)$, is proven as in the work of Cartan (see [13] or [12]). Our statement then follows from Proposition 1.3. \square

2. Basic definitions

2.1. The action form and the moment map

Let G be a Lie group and M a G -manifold.

Definition 2.2. An *action form* is a G -invariant real 1-form σ on M .

The prime examples of this setting are when M is even dimensional and $d\sigma$ is non-degenerate. In this case $d\sigma$ defines a symplectic structure on M .

Example 2.3. For every manifold N , we may take its cotangent bundle $M := T^*N$ with projection $\pi : T^*N \rightarrow N$. The canonical action form σ on a tangent vector v at a point (n, ϕ) , $n \in N$, $\phi \in T_n^*N$ is given by

$$\langle \sigma | v \rangle := \langle \phi | d\pi(v) \rangle.$$

In this setting, $d\sigma$ is a canonical symplectic structure on T^*N and, if $r = \dim(N)$, the form $\frac{d\sigma^r}{r!}$ determines an orientation and a measure, the *Liouville measure*, on T^*N . If a group G acts on N , then it acts also on T^*N , preserving the canonical action form and hence the symplectic structure and the Liouville measure.

Remark 2.4. If M is a manifold with a G -invariant Riemannian structure, we can consider an invariant vector field instead of a 1-form.

Definition 2.5. Given an action form σ we define the *moment map* $\mu_\sigma : M \rightarrow \mathfrak{g}^*$ associated with σ by

$$\mu_\sigma(m)(x) := -\langle \sigma | v_x \rangle(m) = -\iota_x(\sigma)(m) \tag{4}$$

for $m \in M, x \in \mathfrak{g}$.

Remark 2.6. Due to our sign convention for v_x , we have

$$\mu(m)(x) := \left\langle \sigma \left| \frac{d}{d\epsilon} \exp(\epsilon x) \cdot m \right. \right\rangle.$$

The moment map is a G -equivariant map, where on \mathfrak{g}^* we have the *coadjoint action*.

The form $d\sigma$ is a closed 2-form on M . Then $D\sigma(x) = \mu(x) + d\sigma$ is a closed (and in fact exact) equivariant form on M .

Let us present a few examples.

Example 2.7. In Example 2.3, take $N = S^1 = \{e^{i\theta}\}$. The form $d\theta$ gives a trivialization $T^*S^1 = S^1 \times \mathbb{R}$. The vector field $\frac{\partial}{\partial \theta}$ gives a canonical generator of the Lie algebra of S^1 , and $d\theta$, a generator for the dual. The circle group S^1 acts freely by rotations on itself. If $[e^{i\theta}, t]$ is a point of T^*S^1 with $t \in \mathbb{R}$, the action form σ is $\sigma = td\theta$. The function t is the moment map and $dt \wedge d\theta$ the symplectic form.

More generally, take $N = G$, a Lie group. Denote by

$$L(g) : h \mapsto gh, \quad R(g) : h \mapsto hg^{-1}$$

the left and right actions of G on G and by extension also on T^*G . Let us now trivialize $T^*G = G \times \mathfrak{g}^*$ using left invariant forms. Then, in this trivialization, for $h \in G$ and $\xi \in \mathfrak{g}^*$,

$$L(g)(h, \xi) = (gh, \xi), \quad R(g)(h, \xi) = (hg^{-1}, g\xi).$$

Call $\pi : T^*G \rightarrow G$ the canonical projection. Fix a basis ψ_1, \dots, ψ_r of left invariant 1-forms on G such that a point of $T^*G = G \times \mathfrak{g}^*$ is a pair $(g, \zeta) = (g, \sum_i \zeta_i \psi_i)$. Clearly the action form is $\sigma = \sum_i \zeta_i \pi^*(\psi_i)$, and the symplectic form is $\sum_i d\zeta_i \wedge \pi^*(\psi_i) + \sum_i \zeta_i \pi^*(d\psi_i)$. In the noncommutative case, in general $d\psi_i \neq 0$; nevertheless when we compute the Liouville form, we immediately see that these terms disappear and

$$\frac{d\sigma^r}{r!} = d\zeta_1 \wedge \pi^*(\psi_1) \wedge \dots \wedge d\zeta_r \wedge \pi^*(\psi_r). \tag{5}$$

We can rewrite this as $\frac{d\sigma^r}{r!} = (-1)^{\frac{r(r-1)}{2}} d\zeta \wedge \pi^*(V_\psi)$ where we set $V_\psi := \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_r$ and $d\zeta := d\zeta_1 \wedge \dots \wedge d\zeta_r$. At this point it is clear that V_ψ gives a Haar measure dg

on G while $d\zeta$ gives a translation invariant measure on \mathfrak{g}^* . Thus we have (identifying top forms with measures according to the orientation of T^*G given by $\frac{d\sigma^r}{r!}$)

$$\frac{d\sigma^r}{r!} = d\zeta dg. \tag{6}$$

Remark 2.8. We can further normalize our choice of the basis of left invariant forms such that V_ω gives the normalized Haar measure giving volume 1 to G . This normalizes also the translation invariant measure on \mathfrak{g}^* . The only further choice consists in choosing an orientation for G such that we have an induced orientation for \mathfrak{g}^* giving the canonical orientation on T^*G .

Let us call μ_ℓ and μ_r the moment maps for the left and right actions of G respectively. By the very definition of σ , we have the following proposition.

Proposition 2.9.

$$\mu_\ell(g, \psi) = g\psi, \quad \mu_r(g, \psi) = -\psi, \quad \text{left trivialization.} \tag{7}$$

If we had used right invariant forms in order to trivialize the bundle, we would have

$$\mu_r(g, \psi) = -g\psi, \quad \mu_\ell(g, \psi) = \psi, \quad \text{right trivialization.} \tag{8}$$

Example 2.10. We get another example in the case of a symplectic vector space V with antisymmetric form B . Then $\sigma = \frac{1}{2}B(v, dv)$ is a 1-form on V invariant under the action of the symplectic group G , such that $d\sigma = \frac{1}{2}B(dv, dv)$ is a symplectic 2-form on V . The moment map $\mu : V \rightarrow \mathfrak{g}^*$ is given by $\mu(v)(x) = \frac{1}{2}B(v, xv)$.

For example, let $M := \mathbb{R}^2$ with coordinates $v := [v_1, v_2]$, $B = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$. The action form σ is $\frac{1}{2}(v_1dv_2 - v_2dv_1)$ and $d\sigma = dv_1 \wedge dv_2$. The compact part of the symplectic group is the circle group S^1 acting by rotations. The moment map is $\frac{v_1^2 + v_2^2}{2}$.

Remark 2.11. Given a vector space N , the space $N \oplus N^*$ has a canonical symplectic structure given by

$$\langle (u, \phi) | (v, \psi) \rangle := \langle \phi | v \rangle - \langle \psi | u \rangle. \tag{9}$$

The symplectic structure $d\sigma$ on the cotangent bundle to a vector space N gives a symplectic structure B to the vector space $T^*N = N \oplus N^*$.

The action form σ coming from the cotangent structure is not the same as the action form on $N \oplus N^*$ given by duality (9) (in the case $V = \mathbb{R}$, ydx versus $\frac{1}{2}(ydx - xdy)$), but the moment map relative to the subgroup $GL(N)$ acting by $(gn, {}^t g^{-1}\phi)$ is the same, as is $d\sigma$.

2.12. The cohomology groups $\mathcal{H}_{G,c}^\infty(M)$

We will need to extend the notion of equivariant cohomology groups. Consider the space $C^\infty(\mathfrak{g})$ of C^∞ functions on \mathfrak{g} . We may consider the $\mathbb{Z}/2\mathbb{Z}$ -graded spaces $\mathcal{A}_G^\infty(M)$ (or $\mathcal{A}_{G,c}^\infty(M)$) consisting of the G -equivariant C^∞ maps from \mathfrak{g} to $\mathcal{A}(M)$ (or to $\mathcal{A}_c(M)$). The equivariant differential D is well defined on $\mathcal{A}_G^\infty(M)$ (or on $\mathcal{A}_{G,c}^\infty(M)$) and

takes even forms to odd forms and vice versa. Thus we get the cohomology groups $\mathcal{H}_G^\infty(M)$, $\mathcal{H}_{G,c}^\infty(M)$. The group $\mathcal{H}_{G,c}^\infty(M)$ is a module over $\mathcal{H}_G^\infty(M)$, and in particular on $C^\infty(\mathfrak{g})^G = \mathcal{H}_G^\infty(pt)$.

Proceeding as in the previous case, we may define for any G -stable closed subspace Z of M the cohomology groups $\mathcal{H}_{G,c}^\infty(Z)$. An element in $\mathcal{H}_{G,c}^\infty(Z)$ is thus represented by an element in $\mathcal{A}_{G,c}^\infty(M)$ whose boundary has support in $M \setminus Z$. We have a natural map $H_{G,c}^*(Z) \rightarrow \mathcal{H}_{G,c}^\infty(Z)$.

In order to take Fourier transforms, we will need to use yet another space.

Consider the space $\mathcal{P}^\infty(\mathfrak{g})$ of C^∞ functions on \mathfrak{g} with at most polynomial growth. Equivalently, we say that $\mathcal{P}^\infty(\mathfrak{g})$ consists of functions with moderate growth. We may consider the spaces $\mathcal{A}_{G,c}^{\infty,m}(M)$ consisting of the G -equivariant C^∞ maps with at most polynomial growth from \mathfrak{g} to $\mathcal{A}_c(M)$. The index m indicates the moderate growth on \mathfrak{g} of the coefficients. We get the cohomology groups $\mathcal{H}_{G,c}^{\infty,m}(M)$. This new cohomology has $\mathcal{H}_G^{\infty,m}(pt) = \mathcal{P}^\infty(\mathfrak{g})^G$ and is a module over $\mathcal{P}^\infty(\mathfrak{g})^G$. We may define in the same way the groups $\mathcal{H}_{G,c}^{\infty,m}(Z)$ of the cohomology with compact supports, and with coefficients of at most polynomial growth, for any G -stable closed subspace Z of M .

2.13. Connection forms

We shall use a fundamental notion in Cartan’s theory of equivariant cohomology. Let us recall the following definition.

Definition 2.14. Given a free action of a compact Lie group L on a manifold P , a *connection form* is an L -invariant 1-form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ with coefficients in the Lie algebra of L such that $-\iota_x \omega = x$ for all $x \in \mathfrak{l}$.

If on P with free L -action we also have a commuting action of another compact group G , it is easy to see that there exists a $G \times L$ -invariant connection form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ on P for the free action of L .

Let $M = P/L$ and $y \in \mathfrak{g}$. Define the curvature R and the G -equivariant curvature R_y of the bundle $P \rightarrow M$ by

$$R := d\omega + \frac{1}{2}[\omega, \omega], \quad R_y := -i_y \omega + R. \tag{10}$$

Example 2.15. Consider $L = G$ and $P = G$ with left and right action. A connection form for the right action can be constructed as follows. Each element x of the Lie algebra of G defines the vector field v_x by right action. These are left invariant vector fields. Given a basis e_1, \dots, e_r of \mathfrak{g} , set $v_i := v_{e_i}$. This determines a dual basis and correspondingly left invariant forms ω_i with $i_{v_j}(\omega_i) = \langle \omega_i | v_j \rangle = \delta^i_j$ such that $-\sum_i \omega_i e_i$ is a connection form for the right action.

This form is also left invariant and $R = 0$, so by (10) the equivariant curvature is $-i_y \omega$ where now i_y is associated with the left action. We then have

$$R_y(g) = -\sum_i i_y(\omega_i)(g)e_i = -g^{-1}y. \tag{11}$$

The equivariant Chern–Weil homomorphism ([7, 8]; see [3]) associates with any L -invariant smooth function a on \mathfrak{l} a closed G -equivariant form, with C^∞ coefficients as in § 2.12, denoted by $y \rightarrow a(R_y)$, on $M = P/L$.

The formula for this form is obtained via the Taylor series of the function a as follows. Choose a basis e_j , $j = 1, \dots, r$, of \mathfrak{l} and write $R = \sum_j R_j \otimes e_j$. For a multi-index $I := (i_1, \dots, i_r)$, define $R^I := \prod_{j=1}^r R_j^{i_j}$. Then, given a point $p \in P$, we set the following definition.

Definition 2.16.

$$a(R_y)(p) := a(-i_y\omega + R) = a(-i_y\omega(p)) + \sum_I \partial_I a(-i_y\omega(p)) \frac{R^I}{I!} \tag{12}$$

which is a finite sum since R is a nilpotent element.

One easily verifies that this is independent of the chosen basis. Moreover one can prove (as in the construction of ordinary characteristic classes) the following proposition.

Proposition 2.17 ([7, 8], see [3]). *The differential form $a(R_y)$ is the pull-back of a G -equivariant closed form (still denoted by $a(R_y)$) on $M = P/L$. Its cohomology class in $\mathcal{H}_G^\infty(M)$ is independent of the choice of the connection.*

3. The definition of the infinitesimal index

3.1. The infinitesimal index

As before, consider a compact Lie group G and a G -manifold M equipped with an action form σ . We assume M oriented. Let $\mu := \mu_\sigma : M \rightarrow \mathfrak{g}^*$ be the corresponding moment map given by (4).

Set

$$M_G^0 := \mu^{-1}(0), \quad U := M \setminus M_G^0.$$

We simply denote M_G^0 by M^0 when the group G is fixed.

Consider the equivariant form

$$\Omega := d\sigma + \mu = D\sigma.$$

Let $\mathcal{D}'(\mathfrak{g}^*)$ be the space of distributions on \mathfrak{g}^* . It is a $S[\mathfrak{g}^*]$ module where \mathfrak{g}^* acts by differentiation. When G is noncommutative, we need to work with the space $\mathcal{D}'(\mathfrak{g}^*)^G$ of G -invariant distributions.

By Definition 1.1, a representative of a class $[\alpha] \in H_{G,c}^*(M^0)$ is a form $\alpha \in [S(\mathfrak{g}^*) \otimes \mathcal{A}_c(M)]^G$ such that $D\alpha$ is compactly supported in U .

Let us define a map called the infinitesimal index

$$\text{infdex}_G^\sigma : H_{G,c}^*(M^0) \rightarrow \mathcal{D}'(\mathfrak{g}^*)^G$$

as follows.

We fix a translation invariant Lebesgue measure $d\xi$ on \mathfrak{g}^* . We choose a square root i of -1 and define the Fourier transform

$$\hat{f}(x) := \int_{\mathfrak{g}^*} e^{-i(\xi|x)} f(\xi) d\xi.$$

We normalize dx on \mathfrak{g} such that the inverse Fourier transform is

$$f(\xi) = \int_{\mathfrak{g}} e^{i(\xi|x)} \hat{f}(x) dx. \tag{13}$$

The measure $dx d\xi$ is independent of the choice of $d\xi$.

Let $f(\xi)$ be a C^∞ function on \mathfrak{g}^* with compact support in a ball B_R of radius R in \mathfrak{g}^* (for a choice of Euclidean structure on \mathfrak{g}^*) and $\hat{f}(x)$ its Fourier transform, a rapidly decreasing function on \mathfrak{g} .

Consider the differential form on M depending on a parameter s :

$$\Psi(s, \alpha, f) = \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx,$$

and define

$$\begin{aligned} \langle \text{infdex}(s, \alpha, \sigma), f \rangle &:= \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx \\ &= \int_M \Psi(s, \alpha, f). \end{aligned} \tag{14}$$

This double integral on $M \times \mathfrak{g}$ is absolutely convergent, since α is compactly supported on M and depends polynomially on x , while $\hat{f}(x)$ is rapidly decreasing.

More precisely, write $\alpha(x) = \sum_{a=1}^R P_a(x) \alpha^a$ with α^a compactly supported forms on M and $P_a(x)$ polynomial functions of x . Then

$$\Psi(s, \alpha, f)(m) = \sum_a \left[\int_{\mathfrak{g}} \hat{f}(x) P_a(x) e^{is(\mu(m), x)} dx \right] e^{isd\sigma} \alpha^a.$$

By Fourier inversion (as in (13)),

$$\int_{\mathfrak{g}} \hat{f}(x) P_a(x) e^{is(\mu(m), x)} dx = (P_a(-i\partial)f)(s\mu(m)), \tag{15}$$

and thus

$$\Psi(s, \alpha, f) = \sum_a ((P_a(-i\partial)f) \circ (s\mu)) e^{isd\sigma} \alpha^a. \tag{16}$$

In particular, remark that $\Psi(s, \alpha, f)$ does not depend on the choice of $d\xi$. Another consequence of this analysis is the following proposition.

Proposition 3.2. *Let $K \subset M$ be the support of α and $C \subset \mathfrak{g}^*$ the support of f .*

The support of $\Psi(s, \alpha, f)$ is contained in $K \cap \mu^{-1}(C/s)$. In particular, if $s\mu(K) \cap C = \emptyset$, then $\Psi(s, \alpha, f) = 0$.

Given $s > 0$, set $V_s = \mu^{-1}(B_{R/s})$. We can then choose some $s_0 \gg 0$ so large that the restriction of α to the small neighborhood V_{s_0} of M^0 is equivariantly closed.

This is possible since $D\alpha$ has a compact support K in $U = M \setminus M^0$, and so, if $\rho := \min_{m \in K} \|\mu(m)\| > 0$, it suffices to take $s_0 > R/\rho$.

We have $(P_\alpha(-i\partial)f)(s\mu(m)) = 0$ if $\|s\mu(m)\| > R \iff \|\mu(m)\| > R/s$. Thus we see that, for $s \geq s_0$, if K is the support of α , $\Psi(s, \alpha, f)$ has compact support contained in $V_s \cap K$.

We have then the formula

$$\langle \text{infdex}(s, \alpha, \sigma), f \rangle = \int_M \Psi(s, \alpha, f) = \int_{V_s} \Psi(s, \alpha, f). \tag{17}$$

Note that from the formula (17), the lemma follows.

Lemma 3.3. *If α has support in U then, for s large, $\Psi(s, \alpha, f) = 0$.*

We will often make use of the following lemma.

Lemma 3.4. *We have*

$$-i \frac{d}{ds} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx = \int_M \int_{\mathfrak{g}} \sigma e^{is\Omega(x)} D(\alpha)(x) \hat{f}(x) dx.$$

Proof. Indeed, since $\Omega(x) = D\sigma(x)$,

$$\begin{aligned} -i \frac{d}{ds} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx &= \int_M \int_{\mathfrak{g}} D\sigma(x) e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx \\ &= v + r \end{aligned}$$

with

$$v = \int_{\mathfrak{g}} \int_M D(\sigma e^{is\Omega(x)} \alpha(x)) \hat{f}(x) dx$$

and

$$r = \int_M \int_{\mathfrak{g}} \sigma e^{is\Omega(x)} D(\alpha)(x) \hat{f}(x) dx$$

since $D(\Omega) = 0$ and D is a derivation, we have $D(e^{is\Omega(x)}) = 0$.

As $\alpha(x)$ is compactly supported, $v = 0$, and we obtain the lemma. □

Let us see that

$$\langle \text{infdex}(s, \alpha, \sigma), f \rangle = \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

does not depend on the choice of $s \geq s_0$.

We use Lemma 3.4 above to compute $\frac{d}{ds} \langle \text{infdex}(s, \alpha, \sigma), f \rangle$. By the hypotheses made, the form $\sigma D\alpha$ has compact support in U ; thus by Lemma 3.3, the differential form $\Psi(s, \sigma D\alpha, f) = \int_{\mathfrak{g}} \sigma e^{is\Omega(x)} D(\alpha)(x) \hat{f}(x) dx$ is identically equal to 0 for $s \geq s_0$. This implies that for $s \geq s_0$,

$$\frac{d}{ds} \langle \text{infdex}(s, \alpha, \sigma), f \rangle = 0,$$

and hence the independence of the choice of $s \geq s_0$.

We now see the independence of the choice of the representative α . In fact, take a different representative $\alpha + \beta$ with β compactly supported on U . Then

$$\lim_{s \rightarrow \infty} \langle \text{infdex}(s, \beta, \sigma), f \rangle = 0$$

by Lemma 3.3.

Next let us show that $\lim_{s \rightarrow \infty} \langle \text{infdex}(s, \alpha, \sigma), f \rangle$ depends only on the cohomology class of α . Take $\alpha = D\beta$, with β compactly supported on M ; we see that

$$\begin{aligned} \langle \text{infdex}(s, \alpha, \sigma), f \rangle &= \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} D\beta(x) \hat{f}(x) dx \\ &= \int_{\mathfrak{g}} \int_M D(e^{is\Omega(x)} \beta(x)) \hat{f}(x) dx = 0. \end{aligned}$$

Finally, let us consider two action forms σ_1, σ_0 , with $\sigma_0 = \sigma$. Then the moment map for $\sigma_t = t\sigma_1 + (1 - t)\sigma_0$ is $\mu_t = t\mu_1 + (1 - t)\mu_0$, with $\mu_0 = \mu$. We assume that the closed set $\mu_t^{-1}(0)$ remains equal to M^0 , for $t \in [0, 1]$. Let us see that $\text{infdex}(s, \alpha, \sigma_1) = \text{infdex}(s, \alpha, \sigma_0)$, for s large.

Indeed, consider $\Omega(t) = D\sigma_t$. Let

$$I(t, s) = \int_M \int_{\mathfrak{g}} e^{is\Omega(t,x)} \alpha(x) \hat{f}(x) dx.$$

We obtain

$$\begin{aligned} -i \frac{d}{dt} I(t, s) &= s \int_M \int_{\mathfrak{g}} D(\sigma_1 - \sigma_0)(x) e^{is\Omega(t,x)} \alpha(x) \hat{f}(x) dx \\ &= v + r \end{aligned}$$

with

$$v = s \int_{\mathfrak{g}} \int_M D((\sigma_1 - \sigma_0)e^{is\Omega(t,x)} \alpha(x)) \hat{f}(x) dx$$

and

$$r = s \int_M \int_{\mathfrak{g}} (\sigma_1 - \sigma_0) e^{is\Omega(t,x)} D(\alpha)(x) \hat{f}(x) dx.$$

As $\alpha(x)$ is compactly supported, $v = 0$.

As for r , we remark that $\Omega(t, x) = \langle \mu_t, x \rangle + q(t)$ where $q(t)$ is a 2-form. The integral r involves the value of f , and its derivatives, at the points $s\mu_t(m)$. As the compact support K of $D\alpha$ is disjoint from M^0 , our assumption implies that $\mu_t(m)$ is never equal to 0 for $m \in K$ and $t \in [0, 1]$. Thus, if $\rho := \min_{m \in K, t \in [0, 1]} \|\mu_t(m)\| > 0$, arguing as for the formulas (15) and (16), we deduce that $r = 0$ if we take $s_0 > R/\rho$.

One has still to verify that this linear map satisfies the continuity properties that make it a distribution. We leave this to the reader.

In conclusion we have shown the following theorem.

Theorem 3.5. *Let σ be an action form with moment map μ . Let $M^0 = \mu^{-1}(0)$. Then we can define a map*

$$\text{infdex}_G^\sigma : H_{G,c}^*(M^0) \rightarrow \mathcal{D}'(\mathfrak{g}^*)^G$$

setting, for any $[\alpha] \in H_{G,c}^*(M^0)$ and for any smooth function with compact support f on \mathfrak{g}^* ,

$$\langle \text{infdex}_G^\sigma([\alpha]), f \rangle := \lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx.$$

The map infdex_G^σ is a well defined homomorphism of $S[\mathfrak{g}^*]^G$ modules.

If the 1-form σ moves along a smooth curve σ_t with moment map μ_t such that $\mu_t^{-1}(0)$ remains equal to M^0 , then

$$\text{infdex}_G^{\sigma_t} = \text{infdex}_G^\sigma.$$

In particular, if two action forms σ_1, σ_2 have the same moment map μ , the two infinitesimal indices $\text{infdex}_G^{\sigma_1}$ and $\text{infdex}_G^{\sigma_2}$ coincide. Indeed, the moment map μ_t associated with $(1-t)\sigma_1 + t\sigma_2$ is constant. In view of this property, we denote simply by infdex_G^μ the map infdex_G^σ . We call it the infinitesimal index map associated with μ , or, for short, the index map.

Remark 3.6. In general, the maps infdex_G^μ and $\text{infdex}_G^{-\mu}$ are different (cf. Example 3.14), although the zeros of the moment maps associated with σ and $-\sigma$ are the same. Thus the stability condition that the set $\mu_t^{-1}(0)$ remains constant when moving along σ_t is essential in order to insure the independence of the infinitesimal index.

Let us give another formula for infdex_G^μ . From this formula, it will be clear that infdex_G^μ belongs to the space $\mathcal{S}'(\mathfrak{g}^*)^G$ of invariant tempered distributions on \mathfrak{g}^* .

Let f be a Schwartz function on \mathfrak{g}^* . If α is a representative of $[\alpha] \in H_{G,c}^*(M^0)$, we see that $\int_{\mathfrak{g}} e^{is\Omega(x)} (D\alpha)(x) \hat{f}(x) dx$ is a rapidly decreasing function of s : $D\alpha$ being identically equal to 0 on a neighborhood of M^0 , this is expressed in terms of the value of the function f , and its derivatives, at points $s\mu(m)$, where $\mu(m)$ is nonzero. Thus we can define the compactly supported differential form $\Phi(\alpha, f)$ on M by

$$\Phi(\alpha, f) := \int_{\mathfrak{g}} \alpha(x) \hat{f}(x) dx + i\sigma \int_{s=0}^\infty \left(\int_{\mathfrak{g}} e^{is\Omega(x)} D\alpha(x) \hat{f}(x) dx \right) ds. \tag{18}$$

Proposition 3.7. We have

$$\langle \text{infdex}_G^\mu(\alpha), f \rangle = \int_M \Phi(\alpha, f).$$

Proof. Let f be a function with compact support on \mathfrak{g}^* . Then

$$\lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

is equal to

$$\int_M \int_{\mathfrak{g}} \alpha(x) \hat{f}(x) dx + \int_0^\infty \frac{d}{ds} \left(\int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx \right) ds.$$

By Lemma 3.4, we obtain the proposition. □

Remark 3.8. It is possible to define equivariant forms on M with $C^{-\infty}$ coefficients [16]. Such a form is an equivariant map from test densities on \mathfrak{g} to differential forms on M . The equivariant differential D extends and we obtain the group $\mathcal{H}_G^{-\infty}(M)$, and similarly the group $\mathcal{H}_{G,c}^{-\infty}(M)$. If $\alpha \in H_{G,c}^*(M^0)$, and g is a test function on \mathfrak{g} , we may define the differential form

$$(p(\alpha), gdx) = \int_{\mathfrak{g}} \alpha(x)g(x)dx + i\sigma \int_{s=0}^{\infty} \left(\int_{\mathfrak{g}} e^{is\Omega(x)} D\alpha(x)g(x)dx \right) ds.$$

It is easy to see that $p(\alpha)$ is a compactly supported equivariant form on M with $C^{-\infty}$ coefficients such that $D(p(\alpha)) = 0$. Indeed, we have $p(\alpha) = \alpha - \sigma \frac{D\alpha}{D\sigma}$, where $\frac{D\alpha}{D\sigma}$ is well defined in the distribution sense by $-i \int_{s=0}^{\infty} e^{isD\sigma} D\alpha ds$.

We see that $\alpha \mapsto p(\alpha)$ defines a map from $H_{G,c}^*(M^0)$ to $\mathcal{H}_{G,c}^{-\infty}(M)$. In this framework, our distribution $\text{infdex}_G^\mu(\alpha)$ on \mathfrak{g}^* is the Fourier transform of the generalized function $\int_M p(\alpha)$ on \mathfrak{g} .

Paradan defined, associated with an action form σ , a particular element $P_\sigma \in \mathcal{H}_G^{-\infty}(M)$ representing 1 and supported in a neighborhood of M^0 [19]. This element is the form $p(1)$ defined above (when M^0 is compact). Most of our subsequent theorems could be obtained by Fourier transforms of theorems proven in [20, 22] where basic functorial properties of P_σ are proved. However, we will work on \mathfrak{g}^* instead that on \mathfrak{g} and we will give direct proofs.

3.9. Extension of the definition of the infinitesimal index

Let us see that the definition of the infinitesimal index extends to $\mathcal{H}_{G,c}^{\infty,m}(M^0)$.

If $\alpha \in \mathcal{A}_{G,c}^{\infty,m}(M)$ is such that $D\alpha = 0$ in a neighborhood of M^0 , we see that Lemma 3.4 still holds, f being a Schwartz function on \mathfrak{g}^* :

$$-i \frac{d}{ds} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx = \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \sigma D\alpha(x) \hat{f}(x) dx.$$

Since α is of at most polynomial growth, the function of x given by $D\alpha(x) \hat{f}(x)$ is still a Schwartz function of x . Thus by Fourier inversion, we again see that $-i \frac{d}{ds} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$ is a rapidly decreasing function of s and we may define

$$\langle \text{infdex}_G^\mu(\alpha), f \rangle = \lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx.$$

We have again the formula

$$\langle \text{infdex}_G^\mu(\alpha), f \rangle = \int_M \Phi(\alpha, f)$$

where $\Phi(\alpha, f)$ is given by equation (18).

This formula shows that $\text{infdex}_G^\mu(\alpha)$ is a G -invariant tempered distribution on \mathfrak{g}^* . With similar arguments, we obtain the following theorem.

Theorem 3.10. *We can define a map*

$$\text{infdex}_G^\mu : \mathcal{H}_{G,c}^{\infty,m}(M^0) \rightarrow \mathcal{S}'(\mathfrak{g}^*)^G$$

setting for any $[\alpha] \in \mathcal{H}_{G,c}^{\infty,m}(M^0)$ and for any Schwartz function f on \mathfrak{g}^*

$$\langle \text{infdex}_G^\mu([\alpha]), f \rangle := \lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx.$$

If σ moves smoothly along a curve σ_t such that $\mu_t^{-1}(0)$ remains equal to M^0 , the map infdex_G^μ remains constant.

Furthermore, using the Fourier transform \mathcal{F} of tempered distributions,

$$\mathcal{F}(\text{infdex}_G^\mu([\alpha])) = \lim_{s \rightarrow \infty} \int_M e^{is\Omega(x)} \alpha(x). \tag{19}$$

Remark 3.11. If f has compact support and the Fourier transform of $\alpha(x)$ is a distribution with compact support on \mathfrak{g}^* , the value $\int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$ is independent of s when s is sufficiently large.

Let us state some immediate properties of the infinitesimal index. We recall that our construction of the infinitesimal index map is strongly inspired by the Witten nonabelian localization theorem [25]. In particular, we have the following ‘nonabelian localization’ result.

Theorem 3.12. Let $[\alpha] \in \mathcal{H}_{G,c}^{\infty,m}(M)$ and $I(x) = \int_M \alpha(x)$, a function on \mathfrak{g} with moderate growth. Let σ be an action form, and let M^0 be the zeros of the moment map. Then $[\alpha]$ defines an element $[\alpha_0]$ in $\mathcal{H}_{G,c}^{\infty,m}(M^0)$ and

$$\mathcal{F}(\text{infdex}_G^\mu([\alpha_0]))(x) = I(x). \tag{20}$$

Proof. This is clear from formula (19) as $\int_M e^{is\Omega(x)} \alpha(x)$ does not depend on s , since $\Omega(x)$ is exact and α is closed with compact support. □

The left-hand side of (20) depends only on the restriction of α on a small neighborhood of M^0 . Thus Theorem 3.12 says that we can compute the equivariant integral of α on M , knowing α on a small neighborhood of M^0 .

Remark 3.13. Let M be a G -manifold equipped with a G -invariant Riemannian metric. Take a G -invariant vector field V on M such that V_m at each point $m \in M$ is tangent to the orbit Gm and let σ be the 1-form associated with V using the metric. Then M^0 is the set of zeros of the vector field V .

- If G is abelian, we may choose $V = v_x$ with x generic in \mathfrak{g} , and then $M^0 = M^G$, the set of fixed points of G on M . Theorem 3.12 leads to the ‘abelian localization theorem’ of Atiyah, Bott, Berline and Vergne [2, 6].

- When G is not necessarily abelian and M is provided with a Hamiltonian structure with symplectic moment map $\nu : M \rightarrow \mathfrak{g}^*$, then the Kirwan vector field $V_m = \exp(\epsilon\nu(m))m$ is such that M^0 coincides with the critical points of the function $\|\nu\|^2$ (we used an identification $\mathfrak{g}^* = \mathfrak{g}$). Then one of the connected components of M^0 is the zero of the symplectic moment map ν , and μ and ν coincide near this component. This is the situation considered by Witten (and extensively studied by Paradan [19]) with applications to intersection numbers of reduced spaces $\nu^{-1}(0)/G$ (as in [15]).

Example 3.14. • If $G := \{1\}$ is trivial, $H_{G,c}^*(M^0)$ is equal to $H_c^*(M)$ and the infinitesimal index maps to constants, by just integration of compactly supported cohomology classes.

• If $M = \{pt\}$ is a point, the moment map and $\Omega(x)$ are both 0 while $M^0 = M = \{pt\}$. Its equivariant cohomology is $S[\mathfrak{g}^*]^G$.

By Theorem 3.5 it is then enough to compute the infinitesimal index of the class 1. This is given by

$$f \mapsto \int_{\mathfrak{g}} \hat{f}(x) dx = f(0)$$

by the Fourier inversion formula. So, in this case the infinitesimal index of 1 is the δ -function δ_0 .

More generally, we have extended the definition of infindex_G^μ to the space $\mathcal{P}^\infty(\mathfrak{g})^G$ of invariant functions on \mathfrak{g} with at most polynomial growth. If $\alpha(x)$ is any G -invariant function on \mathfrak{g} with polynomial growth and $\hat{\alpha}$ is its Fourier transform (a distribution on \mathfrak{g}^*), we obtain

$$\text{infindex}_G^\mu(\alpha) = \hat{\alpha}. \tag{21}$$

• Consider $M = T^*S^1$ with the canonical action form as in Example 2.7. Then $M^0 = S^1$. We compute the infinitesimal index of the class $1 \in H_{G,c}^*(M^0) = \mathbb{R}$. Let $\chi(t)$ be a function identically equal to 1 in a neighborhood of $t = 0$. Then $D\sigma(x) = xt + dt \wedge d\theta$, and by definition

$$\begin{aligned} \langle \text{infindex}_G^\mu(1), f \rangle &= \lim_{s \rightarrow \infty} \int_{T^*S^1} \left(\int_{-\infty}^\infty \chi(t) e^{isxt + isdt d\theta} \hat{f}(x) dx \right) \\ &= \lim_{s \rightarrow \infty} \int_{T^*S^1} \chi(t) f(st) e^{isdt d\theta} = \lim_{s \rightarrow \infty} 2\pi i s \int_{\mathbb{R}} \chi(t) f(st) dt \\ &= \lim_{s \rightarrow \infty} 2\pi i \int_{\mathbb{R}} \chi(t/s) f(t) dt. \end{aligned}$$

Taking the limit, we see that

$$\langle \text{infindex}_G^\mu(1), f \rangle = 2\pi i \int_{\mathbb{R}} f(t) dt,$$

that is the distribution $\text{infindex}_G^\mu(1)$ is just $2\pi i$ times the integration with respect to the Lebesgue measure dt .

• More generally, consider $M = T^*G$ with the canonical action form σ as in Example 2.7 and the canonical $G \times G$ -action by left and right multiplications. Set $r := \dim G$ and orient M via the Liouville form $d\sigma^r$. We take (g, ζ) with $g \in G$ and $\zeta \in \mathfrak{g}^*$ as coordinates on $M = G \times \mathfrak{g}^*$. We write an element of $\mathfrak{g} \oplus \mathfrak{g}$ as (y, x) . We have $M^0 = G$ and want to compute the infinitesimal index of the class $1 \in H_{G \times G, c}^0(M^0) = \mathbb{R}$. Let χ be a function with compact support, G -invariant and identically equal to 1 in a neighborhood of 0 in \mathfrak{g}^* . This function gives also a function on $T^*G = G \times \mathfrak{g}^*$, still denoted by χ . Then $\chi(g, \zeta) = \chi(\zeta)$ is a representative of 1. Let f be a function on $\mathfrak{g}^* \oplus \mathfrak{g}^*$. Then (using

formulas (7) and (6) and Fourier inversion), we have

$$\begin{aligned} \langle \text{infindex}_{G \times G}^\mu(1), f \rangle &= \lim_{s \rightarrow \infty} \int_{T^*G} \int_{\mathfrak{g} \oplus \mathfrak{g}} \chi(\zeta) e^{is d\sigma} e^{is \langle \zeta, g^{-1}y-x \rangle} \hat{f}(x, y) dx dy \\ &= \lim_{s \rightarrow \infty} \int_{T^*G} \int_{\mathfrak{g} \oplus \mathfrak{g}} \chi(\zeta) (is)^r \frac{(d\sigma)^r}{r!} e^{is \langle \zeta, g^{-1}y-x \rangle} \hat{f}(x, y) dx dy \\ &= i^r \lim_{s \rightarrow \infty} \int_{G \times \mathfrak{g}^*} \int_{\mathfrak{g} \oplus \mathfrak{g}} \chi(\zeta) s^r e^{is \langle \zeta, g^{-1}y-x \rangle} \hat{f}(x, y) dx dy d\zeta dg \\ &= i^r \lim_{s \rightarrow \infty} \int_{G \times \mathfrak{g}^*} \chi(\zeta) f(sg\zeta, -s\zeta) d\zeta dg \\ &= i^r \lim_{s \rightarrow \infty} \int_{G \times \mathfrak{g}^*} \chi(\zeta/s) f(g\zeta, -\zeta) d\zeta dg. \end{aligned}$$

Taking the limit, we obtain

$$\langle \text{infindex}_{G \times G}^\mu(1), f \rangle = i^r \int_{G \times \mathfrak{g}^*} f(g\zeta, -\zeta) d\zeta dg. \tag{22}$$

• Consider now $M = \mathbb{R}^2$ as in Example 2.10. As we have seen, the action form σ is $\frac{1}{2}(v_1 dv_2 - v_2 dv_1)$, so $D\sigma(x) = dv_1 \wedge dv_2 + x\|v\|^2/2$. Then $M^0 = 0$.

We compute the infinitesimal index of the class $1 \in H_{G,c}^*(M^0)$. Let $\chi(t)$ be a function on \mathbb{R} with compact support and identically equal to 1 in a neighborhood of $t = 0$. We then get

$$\langle \text{infindex}_G^\mu(1), f \rangle = \lim_{s \rightarrow \infty} \int_{\mathbb{R}^2} \int_{-\infty}^\infty \chi(\|v\|^2) e^{isx \frac{\|v\|^2}{2} + isdv_1 dv_2} \hat{f}(x) dx.$$

Using polar coordinates on \mathbb{R}^2 , and inversion of the Fourier transform, we see that

$$\langle \text{infindex}_G^\mu(1), f \rangle = 2\pi i \lim_{s \rightarrow \infty} \int_0^\infty \chi(t/s) \int_0^\infty f(t) dt.$$

Taking the limit, we obtain

$$\langle \text{infindex}_G^\mu(1), f \rangle = 2\pi i \int_0^\infty f(t) dt,$$

that is the distribution $\text{infindex}_G^\mu(1)$ is just $2\pi i$ times the Heaviside distribution supported on \mathbb{R}^+ .

4. Properties of the infinitesimal index

There are several functorial properties of the infinitesimal index that we need to investigate: *locality, product, restriction, the map i , and free action.*

4.1. Locality

The easiest property is *locality*.

Let M be a G -action manifold with moment map μ and $i : U \rightarrow M$ an invariant open set; then we have a mapping $i_* : \mathcal{A}_{G,c}(U) \rightarrow \mathcal{A}_{G,c}(M)$ which induces also a mapping

$$i_* : H_{G,c}^*(U^0) \rightarrow H_{G,c}^*(M^0).$$

Proposition 4.2. *The mapping i_* is compatible with the infinitesimal index.*

Proof. This is immediate from the definitions. □

4.3. The product of manifolds

If we have a product $M_1 \times M_2$ of two manifolds relative to two different groups $G_1 \times G_2$, we have

$$(M_1 \times M_2)^0 = M_1^0 \times M_2^0$$

and the cohomology is the product.

Proposition 4.4. *The infinitesimal index of the external product of two cohomology classes is the external product of the two distributions.*

Proof. This is immediate from the definitions. □

4.5. Restriction to subgroups

Let $L \subset G$ be a compact subgroup of G such that \mathfrak{l} , the Lie algebra of L , is a subalgebra of \mathfrak{g} . The moment map μ_L for L is just the composition of μ_G with the restriction $p : \mathfrak{g}^* \rightarrow \mathfrak{l}^*$. Thus $\mu_L^{-1}(0) \supset \mu_G^{-1}(0)$.

If f is a test function on \mathfrak{l}^* , then p^*f is a smooth function on \mathfrak{g}^* constant along the fibers of the projection.

Definition 4.6. We will say that a distribution Θ on \mathfrak{g}^* is a distribution with compact support along the fibers if for any test function f on \mathfrak{l}^* , the distribution $(p^*f)\Theta$ has compact support on \mathfrak{g}^* .

If Θ is a distribution on \mathfrak{g}^* with compact support along the fibers, we may define $p_*\Theta$ as a distribution on \mathfrak{l}^* by

$$\langle p_*\Theta, f \rangle := \int_{\mathfrak{g}^*} (p^*f)\Theta. \tag{23}$$

The right-hand side is computed as the limit when T tends to ∞ of $\langle \Theta, (p^*f)\chi_T \rangle$ when χ_T is a smooth function with compact support and equal to 1 on the ball B_T of \mathfrak{g}^* .

Let Z_G be a closed G -invariant subset of M containing $\mu_L^{-1}(0)$ (if L is normal in G and in particular if G is abelian, we can take $Z_G = \mu_L^{-1}(0)$). Then we have two maps

$$j : H_{G,c}^*(Z_G) \rightarrow H_{G,c}^*(\mu_G^{-1}(0))$$

and

$$r : H_{G,c}^*(Z_G) \rightarrow H_{L,c}^*(\mu_L^{-1}(0)).$$

Theorem 4.7. *If $[\alpha] \in H_{G,c}^*(Z_G)$ then $\text{infdex}_G^{\mu_G}(j[\alpha])$ is compactly supported along the fibers of the map $p : \mathfrak{g}^* \rightarrow \mathfrak{l}^*$, and*

$$p_*(\text{infdex}_G^{\mu_G}(j[\alpha])) = \text{infdex}_L^{\mu_L}(r[\alpha]). \tag{24}$$

Proof. Write $\mathcal{F}^{\mathfrak{g}^*}(h)$ for the Fourier transform \hat{h} of a function h on \mathfrak{g}^* .

Let f be a test function on \mathfrak{l}^* with support on a ball B_R . We have, for χ a test function on \mathfrak{g}^* ,

$$\langle (p^*f)\text{infdex}_G^{\mu_G}(j[\alpha]), \chi \rangle = \lim_{s \rightarrow \infty} \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \mathcal{F}^{\mathfrak{g}^*}((p^*f)\chi)(x) dx.$$

By our assumption on α , there exists $\epsilon > 0$ such that $D\alpha$ is equal to 0 on the subset $\|\mu_L(m)\| = \|p\mu_G(m)\| < \epsilon$ of M . The support C of $(p^*f)\chi$ is contained in the set of $y \in \mathfrak{g}^*$ such that $\|p(y)\| < R$. The support K of $D\alpha$ is contained in the set of points m such that $\|p\mu_G(m)\| > \epsilon$. Thus by Proposition 3.2 and the argument of Lemma 3.4, the distribution

$$\chi \rightarrow \int_M \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \mathcal{F}^{\mathfrak{g}^*}(\chi p^*f)(x) dx$$

stabilizes as soon as $s > R/\epsilon$.

Write for $s_0 > R/\epsilon$

$$\begin{aligned} \langle (p^*f)\text{infdex}_G^{\mu_G}(j[\alpha]), \chi \rangle &= \int_M \int_{\mathfrak{g}} e^{is_0\Omega(x)} \alpha(x) \mathcal{F}^{\mathfrak{g}^*}(\chi p^*f)(x) dx \\ &= \int_M \Psi(s_0, \alpha, \chi p^*f) \end{aligned}$$

where

$$\begin{aligned} \Psi(s_0, \alpha, \chi p^*f)(m) &= \sum_a \left[\int_{\mathfrak{g}} P_a(x) e^{is_0\langle \mu(m), x \rangle} \mathcal{F}^{\mathfrak{g}^*}(\chi p^*f)(x) dx \right] e^{is_0 d\sigma} \alpha^a \\ &= \sum_a ((P_a(-i\partial)(\chi p^*f) \circ (s_0\mu)) e^{is_0 d\sigma} \alpha^a. \end{aligned} \tag{25}$$

Applying Proposition 3.2, we have that, if K is the compact support of α , as s_0 is greater than R/ϵ , the form $\Psi(s_0, \alpha, \chi p^*f)$ is supported on the compact subset $s_0\mu_G(K)$ in \mathfrak{g}^* . This shows the first statement that $\text{infdex}_G^{\mu_G}(j[\alpha])$ is compactly supported along the fibers of p .

We pass next to formula (24). We then have

$$\langle (p^*f)\text{infdex}_G^{\mu_G}(j[\alpha]), \chi_T \rangle = \int_M \Psi(s_0, \alpha, \chi_T p^*f)$$

for any T large.

Using formula (25), when T is sufficiently large, as χ_T is equal to 1 on the compact subset $s_0\mu_G(K)$, we thus have that $\Psi(s_0, \alpha, \chi_T p^*f)$ is simply

$$\sum_a ((P_a(-i\partial)p^*f) \circ (s_0\mu)) e^{is_0 d\sigma} \alpha^a.$$

As p^*f is constant along the fibers, if we denote by α_0 the restriction of $\alpha(x)$ to \mathfrak{l} , we see that $\Psi(s_0, \alpha, \chi_T p^*f)$ is equal to the differential form $\Psi(s_0, \alpha_0, f)$ as all derivatives in the $\ker p$ direction annihilate p^*f . We thus obtain our theorem. \square

4.8. The Thom class and the map $i_!$

Let Z be an oriented G -manifold of dimension d and $i : M \hookrightarrow Z$ a G -stable oriented submanifold of dimension $n = d - k$. Assume that M is an action manifold with moment map μ and that Z is equipped with an action form σ_Z such that the associated moment map μ_Z extends μ . Thus $Z^0 \cap M = M^0$. Under these assumptions, we will define a map

$$i_! : H_{G,c}^*(M^0) \rightarrow H_{G,c}^*(Z^0)$$

preserving the infdex.

Let us recall the existence of an equivariant Thom class ([18]; see [13], p. 158, and [21]). We assume first that M has a G -stable tubular neighborhood N in Z , with projection $p : N \rightarrow M$. Then there exists a unique class τ_M of equivariantly closed forms on N with compact support along the fibers such that the integral $p_*\tau_M$ is identically equal to 1 along each fiber. Thus for any equivariant form $\alpha(x)$ on M with compact support (but not necessarily closed), we have that

$$\int_M \alpha = \int_N p^*\alpha \wedge \tau_M.$$

In general, let us take a class $[\alpha] \in H_{G,c}^*(M^0)$ where $\alpha \in \mathcal{A}_{G,c}(M)$ and $D\alpha$ has support K in $M \setminus M^0$.

Consider a G -stable open set $U \subset M$ with the following properties.

- (i) The support of α is contained in U .
- (ii) The closure of U is compact and has an open neighborhood A in Z , and $M \cap A$ has a G -stable tubular neighborhood in A .

By locality, we can then substitute U with M and thus assume that the pair (Z, M) has all the properties which insure the existence of a Thom class τ_M .

Consider a G -invariant Riemannian metric on the normal bundle \mathcal{N} to M in Z . Define S^ϵ as the (open) disk bundle of radius ϵ in \mathcal{N} . Then we can take our tubular neighborhood in such a way that it is diffeomorphic to S^ϵ for some ϵ .

We claim that we can take S^ϵ so close to M that $p^{-1}K \cap S^\epsilon \cap Z^0 = \emptyset$. Indeed, $p^{-1}K \cap \overline{S^\epsilon}$ is a compact set and, since K is disjoint from M^0 and hence from Z^0 , for a sufficiently small ϵ , $p^{-1}K \cap \overline{S^\epsilon}$ is disjoint from Z^0 .

Let us now fix the Thom form τ_M in $\mathcal{A}_{G,c}(\mathcal{N})$ with support in S^ϵ .

Consider then the form $p^*\alpha \wedge \tau_M$. We have that $D(p^*\alpha \wedge \tau_M) = p^*D\alpha \wedge \tau_M$ has support in $p^{-1}K \cap S^\epsilon \subset Z \setminus Z^0$. It follows that $p^*\alpha \wedge \tau_M$ defines an element in $H_{G,c}^*(Z^0)$.

We claim that this element depends only on the class $[\alpha]$. So first take another Thom form τ'_M with the same properties. Then there is a form $r_M \in \mathcal{A}_{G,c}(S^\epsilon)$ such that $\tau_M - \tau'_M = Dr_M$ and

$$p^*\alpha \wedge \tau_M - p^*\alpha \wedge \tau'_M = p^*\alpha \wedge Dr_M = D(p^*\alpha \wedge r_M) - p^*D\alpha \wedge r_M$$

where $p^*\alpha \wedge r_M$ has compact support and $p^*D\alpha \wedge r_M$ has support in $Z \setminus Z^0$.

Next assume that α is supported outside M^0 ; then again we may take τ_M such that $p^*\alpha \wedge \tau_M$ is supported outside Z^0 .

Finally, if $\alpha = D\beta$, we have $p^*\alpha \wedge \tau_M = D(p^*\beta \wedge \tau_M)$. Hence we can set

$$i_![\alpha] := [p^*\alpha \wedge \tau_M]. \tag{26}$$

Theorem 4.9. *Assume that M is an action manifold with action form σ and moment map μ and that Z is equipped with an action form σ_Z such that the associated moment map μ_Z extends μ . Then the morphism*

$$i_! : H_{G,c}^*(M^0) \rightarrow H_{G,c}^*(Z^0)$$

preserves the infinitesimal index.

Remark 4.10. We do not need to assume that the restriction of σ_Z to M is the action form σ on M , only that the moment map μ_Z restricts to μ .

Proof. First let us see that $\text{infdex}_G^{\mu_Z}(i_![\alpha])$ does not depend on the choice of the form σ_Z on Z , if the moment map μ_Z restricts to μ . We can assume that $Z = N$. Let $\beta = p^*\alpha \wedge \tau_M$. The form β is compactly supported.

Let σ_1, σ_0 be two 1-forms on Z and consider $\sigma_t = t\sigma_1 + (1 - t)\sigma_0$ and μ_t the corresponding moment map. Set $\Omega(t) = D\sigma_t$. We assume that the map μ_t coincides with μ on M for all t . Thus, provided we choose τ_M with support sufficiently close to M , there exists an $h > 0$ such that on the support of $D\beta$, we have $\|\mu_t\| > h > 0$.

Define

$$I(t, s) := \int_N \int_{\mathfrak{g}} \beta(x) e^{is\Omega(t,x)} \beta(x) \hat{f}(x) dx.$$

We can prove that $\frac{d}{dt} I(t, s) = 0$ in the same way as we proved the invariance of the infinitesimal index $\text{infdex}_G^{\mu_t}$ along a smooth curve μ_t (proof of Theorem 3.5); thus we skip the proof.

Having established the independence from σ , we choose for the final computation $\sigma_Z := p^*\sigma$. In this case, since $\beta = p^*\alpha \wedge \tau_M$,

$$\langle \text{infdex}_G^{\mu_Z}([\beta]), f \rangle = \lim_{s \rightarrow \infty} \int_{\mathfrak{g}} \int_N p^*(e^{is\Omega(x)} \alpha(x)) \wedge \tau_M(x) \hat{f}(x) dx. \tag{27}$$

As τ_M has integral 1 over each fiber of the projection $p : N \rightarrow M$, we obtain that (27) is equal to

$$\lim_{s \rightarrow \infty} \int_{\mathfrak{g}} \int_M e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

which is our statement. □

4.11. Free action

Let G and L be two compact groups. Consider now an oriented manifold N under $G \times L$ -action, with action 1-form σ_N and moment map $\mu_{G \times L} = (\mu_G, \mu_L) : N \rightarrow \mathfrak{g}^* \oplus \mathfrak{l}^*$. We set $N^0 = \mu_{G \times L}^{-1}(0)$.

Assume that:

- the group L acts freely on N ;
- 0 is a regular value of μ_L .

Define $P = \mu_L^{-1}(0)$. By assumption, P is a manifold with a free L -action so

$$M := \mu_L^{-1}(0)/L$$

is a G -manifold. We will see in a short while that the orientation on N determines a natural orientation on M .

We denote by π the projection $\pi : P \rightarrow M$. The invariance of σ_N under L action implies the following proposition.

Proposition 4.12. *The restriction $\bar{\sigma}$ of σ_N to P verifies $\iota_x \bar{\sigma} = 0$ for any $x \in \mathfrak{l}$ and descends to a G -invariant action form σ_M on M ; thus M is an action manifold and $\bar{\sigma} = \pi^*(\sigma_M)$.*

We denote by μ the moment map on M associated with σ_M . The map μ is obtained by factoring the restriction of μ_G to P which is L -invariant, that is, $\mu_G = \mu \circ \pi$ on P . Since N^0 is the subset of P where μ_G equals 0 , we see that $M^0 = \mu^{-1}(0)$, the fiber at zero of μ , is $M^0 = N^0/L$.

Recall (Proposition 1.5) that since the action of L is free, we have an isomorphism $\pi^* : H_{G,c}^*(M^0) \rightarrow H_{G \times L,c}^*(N^0)$.

Our goal in this section is, given a class $[\gamma] \in H_{G,c}^*(M^0)$, to compare $\text{infindex}_G^\mu([\gamma])$ and $\text{infindex}_{G \times L}^{\mu_G \times L}(\pi^*([\gamma]))$.

As 0 is a regular value of μ_L , any L -stable compact subset K in P has an L -stable neighborhood in N isomorphic to $K \times \mathfrak{l}^*$ with the moment map μ_L being the projection on the second factor. Since the computations of the infinitesimal index of a given class with compact support are local around N^0 (by Proposition 4.2), we may assume that $N = P \times \mathfrak{l}^*$ and that the moment map μ_L is the projection on the second factor. We write an element of N as (p, ζ) with $p \in P, \zeta \in \mathfrak{l}^*$.

The composition of the projection $\eta : N = P \times \mathfrak{l}^* \rightarrow P$ and of $\pi : P \rightarrow M$ is a fibration with fiber $L \times \mathfrak{l}^* = T^*L$. We orient M using the orientation of N and that given by the symplectic structure on T^*L (see formula (5)).

4.12.1. An auxiliary form. Let us choose now a connection form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ for the free action of L on P . We want to apply Definition 2.16 to the following functions. For ζ a point in \mathfrak{l}^* , define $\theta_\zeta \in C^\infty(\mathfrak{l})^L$ by

$$\theta_\zeta(x) := \int_L e^{i\langle x, l\zeta \rangle} dl = \int_{\mathfrak{l}^*} e^{i\langle f, x \rangle} d\beta_\zeta(f)$$

where dl is a Haar measure on L or in an equivalent way where $d\beta_\zeta(f)$ is a L -invariant measure on the orbit $L\zeta \subset \mathfrak{l}^*$.

Thus for any $\zeta \in \mathfrak{l}^*$, we may consider, using the curvature R (cf. formula (10)), the G -equivariant closed form $\theta_\zeta(R_y)$ on M given by

$$\theta_\zeta(R_y) = \int_L e^{i\langle R_y, l\zeta \rangle} dl = \int_L e^{i\langle -\iota_y \omega, l\zeta \rangle} e^{i\langle R, l\zeta \rangle} dl. \tag{28}$$

We need some growth properties of the function $y \rightarrow \theta_\zeta(R_y)$. If we fix $p \in P$ and $\zeta \in \mathfrak{l}^*$, let us see the following lemma.

Lemma 4.13. *The function $y \rightarrow \theta_\zeta(R_y)(p)$ is the Fourier transform of a compactly supported measure $d\mu_{p,\zeta}$ on \mathfrak{g}^* (with values in $\bigwedge T_p^*P$).*

Proof. Indeed, let $f \in \mathfrak{l}^*$. The function $\langle -\iota_y\omega(p), f \rangle$ is linear in $y \in \mathfrak{g}$, so we write $\langle -\iota_y\omega(p), f \rangle = \langle y, h(p, f) \rangle$ with $h(p, f) \in \mathfrak{g}^*$ depending smoothly on p, f . We see that

$$\theta_\zeta(R_y)(p) = \int_{\mathfrak{l}^*} e^{i\langle y, h(p, f) \rangle} e^{i\langle f, R \rangle} d\beta_\zeta(f),$$

where $d\beta_\zeta(f)$ is an L -invariant measure supported on the orbit $L\zeta \subset \mathfrak{l}^*$.

Let us integrate over the fiber of the map $h_p : \mathfrak{l}^* \rightarrow \mathfrak{g}^*$ given by $f \rightarrow h(p, f) = \xi$. We obtain that

$$\theta_\zeta(R_y)(p) = \int_{\mathfrak{g}^*} e^{i\langle y, \xi \rangle} (h_p)_*(e^{i\langle f, R \rangle} d\beta_\zeta(f)). \tag{29}$$

In this formula, $(h_p)_*(e^{i\langle f, R \rangle} d\beta_\zeta(f))$ is a measure supported on the compact set $h_p(L\zeta)$ as $d\beta_\zeta(f)$ is supported in the compact set $L\zeta$. In particular, we see that, over a compact subset of P , $y \rightarrow \theta_\zeta(R_y)(p)$ is a bounded function of y as well as all of its derivatives in y , and estimates are uniform in ζ if ζ varies in a compact set of \mathfrak{l}^* . \square

If $[\gamma] \in H_{G,c}^*(M^0)$, we choose a representative $\gamma(y)$ which is a form with compact support on M , depending of y in a polynomial way. Set

$$\tilde{\gamma}_\zeta(y) := \gamma(y)\theta_\zeta(R_y). \tag{30}$$

Proposition 4.14. *The equivariant form $\tilde{\gamma}_\zeta(y)$ is of at most polynomial growth in y . It represents a class in $\mathcal{H}_{G,c}^{\infty,m}(M^0)$ which does not depend on the choice of the connection ω but only on the choice of the Haar measure dl .*

Proof. The fact that $\tilde{\gamma}_\zeta(y)$ is of at most polynomial growth follows from the preceding discussion. The second statement is proved as in ([7, 8]; see [3]). \square

Remark that $\theta_0(R_y) = \text{vol}(L, dl)$ where $\text{vol}(L, dl)$ is the volume of the compact Lie group L for the Haar measure dl such that $dld\zeta$ is the canonical measure on $T^*L = L \times \mathfrak{l}^*$ (by right or left trivialization).

4.14.1. The main formula. With the notation of the previous paragraph, given $[\gamma] \in H_{G,c}^*(M^0)$, we may apply the infinitesimal index construction (Theorem 3.10) to the cohomology class $[\tilde{\gamma}_\zeta] \in \mathcal{H}_{G,c}^{\infty,m}(M^0)$ of the equivariant form $\tilde{\gamma}_\zeta(y) = \gamma(y)\theta_\zeta(R_y)$. We have the following theorem.

Theorem 4.15. *Let f_1 be a test function on \mathfrak{l}^* and f_2 be a test function on \mathfrak{g}^* . Then $\langle \text{infdex}_G^\mu([\tilde{\gamma}_\zeta]), f_2 \rangle$ is a smooth function of ζ and*

$$\langle \text{infdex}_{G \times L}^{\mu_{G \times L}}(\pi^*([\gamma])), f_1 f_2 \rangle = i^{\dim L} \int_{\mathfrak{l}^*} \langle \text{infdex}_G^\mu([\tilde{\gamma}_\zeta]), f_2 \rangle f_1(\zeta) d\zeta. \tag{31}$$

Remark 4.16. Formula (22) is a particular case of the above theorem. Indeed consider $M = T^*L$ with the double action of $L \times L$. We take $G = L$ as the first copy, and L as the second copy acting freely on the right. Then $P = L$ and $M = L/L = \{pt\}$ is a point. The equivariant curvature R_y , a form on P with value in \mathfrak{l} , is $R_y(l) = -ly$ (formula (11)), so $\tilde{\gamma}_\zeta(y)$ is the invariant function $\int_L e^{-i\langle \zeta, ly \rangle} dl$ and $\langle \text{index}_G^\mu([\tilde{\gamma}_\zeta]), f_2 \rangle$ (the distribution Fourier transform of the function $\tilde{\gamma}_\zeta(y)$) is $\int_L f_2(-l\zeta) dl$. The theorem above gives

$$\langle \text{index}_{G \times L}^{\mu_{G \times L}}(1), f_1 f_2 \rangle = i^{\dim L} \int_{L \times \mathfrak{l}^*} f_1(\zeta) f_2(-l\zeta) dl d\zeta$$

which is formula (22).

Let us first write a corollary of this theorem.

Corollary 4.17. *Let f_2 be a test function on \mathfrak{g}^* . Then the distribution $f_1 \rightarrow \langle \text{index}_{G \times L}^{\mu_{G \times L}}(\pi^*([\gamma])), f_1 f_2 \rangle$ on \mathfrak{l}^* is a smooth density $D(\zeta) d\zeta$. The value of D at 0 is equal to $i^{\dim L} \text{vol}(L, dl) \langle \text{index}_G^\mu([\gamma]), f_2 \rangle$.*

We now prove Theorem 4.15.

Proof. Denote by $\eta : P \times \mathfrak{l}^* \rightarrow P$ the projection $\eta : (p, \zeta) \mapsto p$, and set $\xi = \pi \circ \eta : N \rightarrow M$, $\xi(p, \zeta) := \pi(p)$, $p \in P$, $\zeta \in \mathfrak{l}^*$.

Let $\gamma(y)$ be a compactly supported G -equivariant form on M representative of $[\gamma]$. Any $G \times L$ -equivariant form ψ with compact support on $N = P \times \mathfrak{l}^*$ which restricted to P coincides with $\pi^* \gamma$ can be taken as a representative for the cohomology class $\pi^*[\gamma] \in H_{G \times L, c}^*(N^0)$.

In order to construct ψ , take an L -invariant function $\rho : \mathfrak{l}^* \rightarrow \mathbb{R}$ supported near zero and such that ρ equals 1 on a neighborhood of 0 and define the form ψ , which is still L -invariant and G -equivariant, by

$$\psi(y)(p, \zeta) := \rho(\zeta) \xi^* \gamma(y). \tag{32}$$

Recall that $\bar{\sigma}$ is the restriction of σ_N on P and consider the 1-form $\eta^*(\bar{\sigma})$ on $N = P \times \mathfrak{l}^*$, the pull-back of $\bar{\sigma}$ under the projection $\eta : P \times \mathfrak{l}^* \rightarrow P$. Let $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{l}$ be our connection form. Then we have the following lemma.

Lemma 4.18. *$\langle \omega, \zeta \rangle$ is an action form on N , with the moment map for L the second projection. Its moment map for G vanishes on P .*

Consider $\sigma_0 = \sigma_N$ and $\sigma_1 = \eta^*(\bar{\sigma}) + \langle \omega, \zeta \rangle$ with moment maps μ_0, μ_1 .

Lemma 4.19. *The moment map $\mu_t = t\mu_1 + (1 - t)\mu_0$ associated with $t\sigma_1 + (1 - t)\sigma_0$ is such that $\mu_t^{-1}(0) = N^0$ for all $t \in [0, 1]$.*

Proof. This follows from the fact that the component under L of these maps is the second projection, so $\mu_t^{-1}(0) \subset P$ for all t and moreover μ_1, μ_0 coincide on P . Thus $\mu_t^{-1}(0) = P^0 = N^0$. □

According to Theorem 3.5, we may thus assume that $\sigma_N = \eta^*(\bar{\sigma}) + \langle \omega, \zeta \rangle$ and compute with this ‘normal form’ the values of $\text{infdex}_{G \times L}^{\mu_{G \times L}}$.

Recall that $\mu : M \rightarrow \mathfrak{g}^*$ is the moment map relative to G associated with σ_M . By abuse of notation, we still denote by μ its pull-back by $\pi \eta$ to N . This is the moment map associated with $\eta^*(\bar{\sigma})$.

Lemma 4.20. *Let $\Omega := D\sigma_N$ for $(x, y) \in \mathfrak{l} \oplus \mathfrak{g}$. At a point $(p, \zeta) \in P \times \mathfrak{l}^*$, we have*

$$\Omega(x, y) = \langle x, \zeta \rangle - \langle \iota_y \omega, \zeta \rangle + \langle y, \mu \rangle + d\eta^*(\bar{\sigma}) + d\langle \omega, \zeta \rangle.$$

Proof. By the definition of a connection form (for the action of L), we have $\langle x, \zeta \rangle = -\langle \iota_x \omega, \zeta \rangle$, so $\langle x, \zeta \rangle - \langle \iota_y \omega, \zeta \rangle$ is the value of the moment map at (x, y) of $\langle \omega, \zeta \rangle$. As for $\eta^*(\bar{\sigma})$, by the definition of $P = \mu_L^{-1}(0)$, the part relative to L of its moment map is equal to 0. □

We write $\Omega(x, y) = \langle x, \zeta \rangle + \Omega'(y)$ with

$$\Omega'(y) = -\langle \iota_y \omega, \zeta \rangle + \langle y, \mu \rangle + \eta^* d(\bar{\sigma}) + d\langle \omega, \zeta \rangle$$

independent of x . We have

$$\Omega'(y) = \eta^*(D\bar{\sigma}) - \langle \iota_y \omega, \zeta \rangle + d\langle \omega, \zeta \rangle. \tag{33}$$

For s sufficiently large,

$$\langle \text{infdex}_{G \times L}^{\mu_{G \times L}}(\pi^*([\gamma])), f_1 f_2 \rangle = I(s)$$

with

$$I(s) = \int_N \int_{\mathfrak{g} \times \mathfrak{l}} e^{is\Omega(x,y)} \psi(y) \hat{f}_1(x) \hat{f}_2(y) dx dy.$$

Applying Fourier inversion:

$$\int_{\mathfrak{l}} e^{is\langle x, \zeta \rangle} \hat{f}_1(x) dx = f_1(s\zeta),$$

we obtain that

$$I(s) = \int_N \int_{\mathfrak{g}} e^{is\Omega'(y)} \psi(y) f_1(s\zeta) \hat{f}_2(y) dy$$

where $\psi(y)(p, \zeta) = \rho(\zeta) \xi^* \gamma(y)$ is defined by formula (32).

Write the connection form $\omega = \sum_{i=1}^r \omega_i e_i$ on an oriented basis $\{e_1, \dots, e_r\}$ of \mathfrak{l} , and set $\zeta_i = \langle e_i, \zeta \rangle$ for $i = 1, \dots, r$.

We have $\langle \omega, \zeta \rangle = \sum_{i=1}^r \zeta_i \omega_i$ and thus

$$d\langle \omega, \zeta \rangle = \sum_{i=1}^r \zeta_i d\omega_i + \sum_{i=1}^r d\zeta_i \wedge \omega_i. \tag{34}$$

Let us now integrate along the fiber \mathfrak{l}^* of the projection $\eta : N = P \times \mathfrak{l}^* \rightarrow P$. We thus need to identify the highest term of $e^{is\Omega'(y)}$ in the $d\zeta_i$. By (34) and (33), this highest term equals

$$(is)^r d\zeta_1 \wedge \omega_1 \wedge \dots \wedge d\zeta_r \wedge \omega_r = (-1)^{\frac{r(r+1)}{2}} (is)^r V_\omega \wedge d\zeta$$

where we set $V_\omega := \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_r$ and $d\zeta := d\zeta_1 \wedge \dots \wedge d\zeta_r$. We obtain

$$\begin{aligned}
 I(s) &= \int_N \int_{\mathfrak{g}} e^{is\Omega'(y)} \psi(y) f_1(s\zeta) \hat{f}_2(y) dy \\
 &= (-1)^{\frac{r(r+1)}{2}} i^r \int_{P \times \mathfrak{g}} e^{isD\bar{\sigma}} \gamma(y) \hat{f}_2(y) \left(\int_{\mathfrak{l}^*} s^r e^{-is\langle \iota_y \omega, \zeta \rangle} e^{i\langle d\omega, \zeta \rangle} \rho(\zeta) f_1(s\zeta) V_\omega \wedge d\zeta \right) dy.
 \end{aligned}$$

In the integral on \mathfrak{l}^* , we change ζ to $s\zeta$ and obtain

$$(-1)^{\frac{r(r+1)}{2}} i^r \int_{P \times \mathfrak{g}} e^{isD\bar{\sigma}} \gamma(y) \hat{f}_2(y) \left(\int_{\mathfrak{l}^*} e^{-i\langle \iota_y \omega, \zeta \rangle} e^{i\langle d\omega, \zeta \rangle} \rho(\zeta/s) f_1(\zeta) V_\omega \wedge d\zeta \right) dy.$$

On the compact support of $f_1(\zeta)$, if s is sufficiently large, $\rho(\zeta/s) = 1$. Also we may replace $d\omega$ by R as $R - d\omega = \frac{1}{2}[\omega, \omega]$ is annihilated by the wedge product with $\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_r$ and obtain (for s sufficiently large)

$$\langle \text{infdex}_{G \times L}^{\mu_{G \times L}} \psi, f_1 f_2 \rangle = (-1)^{\frac{r(r+1)}{2}} i^r \int_N \int_{\mathfrak{g}} e^{is\eta^* D\bar{\sigma}} \gamma(y) \hat{f}_2(y) e^{i\langle R_y, \zeta \rangle} f_1(\zeta) V_\omega \wedge d\zeta dy.$$

Now consider the fibration $N \rightarrow M \times \mathfrak{l}^*$ with fiber L . On each fiber, the form $V_\omega = \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_r$ induces an orientation and restricts to a Haar measure dl on L . Let us now integrate over the fiber. Recall that σ_M denotes the action form on M . Let $\Omega_M := D\sigma_M$; we have $\pi^* \sigma_M = \bar{\sigma}$, $\eta^* D\bar{\sigma} = \eta^* \pi^* \Omega_M$. By formula (6), recalling that $r = \dim L$ and using formula (28), we finally obtain that $I(s)$ is equal to

$$i^{\dim L} \int_{\mathfrak{l}^*} \left(\int_M \int_{\mathfrak{g}} e^{is\Omega_M(y)} \gamma(y) \theta_\zeta(R_y) \hat{f}_2(y) dy \right) f_1(\zeta) d\zeta.$$

Remark that when ζ varies in the compact support of f_1 , and over a compact subset K of M , the Fourier transform (in y) of $\theta_\zeta(R_y)$ stays supported on a fixed compact subset of \mathfrak{g}^* . Indeed, using formula (29), we see that the Fourier transform of $\theta_\zeta(R_y)$ is supported on the compact subset $h(\pi^{-1}K, L\zeta)$. By Remark 3.11, for $s \gg s_0$,

$$\int_{M \times \mathfrak{g}} e^{is\Omega_M(y)} \gamma(y) \theta_\zeta(R_y) \hat{f}_2(y) dy = \text{infdex}_G^\mu([\tilde{\gamma}_\zeta], f_2) \tag{35}$$

for any ζ in the support of f_1 .

Thus we obtain our claim. □

Another important particular case of the free action property is that where G is trivial. We then have $y = 0$ in all the steps of the proof of Theorem 4.15. We now summarize the result that we obtain in this particular case of Theorem 4.15. Let N be an oriented L -manifold with action form, and assume that the group L acts freely on N and that 0 is a regular value of μ_L . Let $M = N^0/L$ and let $[\gamma] \in H_{G,c}^*(N^0) = H_c^*(M)$.

Let R be the curvature of the fibration $N^0 \rightarrow M$. For any $\zeta \in \mathfrak{l}^*$, we consider the closed differential form on M given by

$$\theta_\zeta(R) = \int_L e^{i\langle R, l\zeta \rangle} dl. \tag{36}$$

Here, as R is an \mathfrak{l} -valued 2-form, $\theta_\zeta(R)$ is a polynomial function of ζ .

Then we obtain the following proposition.

Proposition 4.21. *The distribution $\text{infdex}_L^{\mu_L}([\gamma])$ is a polynomial density on \mathfrak{l}^* . More precisely,*

$$\langle \text{infdex}_L^{\mu_L}([\gamma]), f_1 \rangle = i^{\dim L} \int_{\mathfrak{l}^*} \left(\int_M \gamma \theta_\zeta(\mathbf{R}) \right) f_1(\zeta) d\zeta.$$

In particular the value of $\text{infdex}_L^{\mu_L}([\gamma])$ at 0 is well defined and computes the integral on the reduced space $\mu_L^{-1}(0)/L$ of the compactly supported cohomology class associated with $[\gamma]$. This is essentially the Witten localization formula [14, 25].

4.22. Extension of the properties of the infinitesimal index

We have extended the definition of the infinitesimal index to $\mathcal{H}_{G,c}^{\infty,m}(M^0)$. Analyzing the proofs of the properties *locality, product, and the map i* , we see that these properties hold for the infinitesimal index map on $\mathcal{H}_{G,c}^{\infty,m}(M^0)$. The proofs for the restriction property and the free action extend, provided we are in the situation of Remark 3.11: we consider the infinitesimal index on classes $[\alpha] \in H_{G,c}^{\infty,m}(M^0)$ such that the Fourier transform of $\alpha(x)$ is a distribution with compact support on \mathfrak{g}^* , and so the infinitesimal index stabilizes for s large. This will always be the situation in the applications to index formulas.

5. Some consequences of the functorial properties of the infinitesimal index

We list here some corollaries of the functorial properties: excision, product, restriction, push-forward, and free action, proved in the preceding section.

5.1. Diagonal action and convolution

Consider two G action manifolds M_1, M_2 with moment maps μ_1, μ_2 with zeros M_1^0, M_2^0 . Let Δ be the diagonal subgroup. The moment map for Δ is $\mu_1 + \mu_2$.

Let us assume that $(M_1 \times M_2)_\Delta^0 = M_1^0 \times M_2^0$. If $\alpha \in H_{G,c}^*(M_1^0)$ and $\beta \in H_{G,c}^*(M_2^0)$, we may apply the product property (Proposition 4.4) and the restriction property (Theorem 4.7). As the restriction map is such that $r^*f(\xi_1, \xi_2) = f(\xi_1 + \xi_2)$ ($\xi_1, \xi_2 \in \mathfrak{g}^*$), we obtain the following proposition.

Proposition 5.2. *Under the hypothesis $(M_1 \times M_2)_\Delta^0 = M_1^0 \times M_2^0$, the infinitesimal index $\text{infdex}_\Delta^{\mu_1 + \mu_2}(\alpha_1 \wedge \alpha_2)$ is the convolution product $\text{infdex}_G^{\mu_1}(\alpha_1) * \text{infdex}_G^{\mu_2}(\alpha_2)$ of the distributions $\text{infdex}_G^{\mu_1}(\alpha_1)$ and $\text{infdex}_G^{\mu_2}(\alpha_2)$.*

Let us give an important example of this situation.

Let M_X be a complex representation space for the action of a torus G , where $X = [a_1, a_2, \dots, a_m]$ is a list of nonzero weights $a_i \in \hat{G} \subset \mathfrak{g}^*$. We assume that X spans a pointed cone in \mathfrak{g}^* . Recall the definition of the *multivariate spline* T_X ; it is a tempered distribution defined by

$$\langle T_X | f \rangle = \int_0^\infty \dots \int_0^\infty f \left(\sum_{i=1}^m t_i a_i \right) dt_1 \dots dt_m. \tag{37}$$

Let us consider on $M_X = \mathbb{C}^m$ the action form such that $\mu(z_1, \dots, z_m) = \sum_i \frac{|z_i|^2}{2} a_i$. Then $M_X^0 = \{0\}$ and the class 1 is a class in $H_{G,c}^*(M_X^0)$. Using our computation in Example 3.14 of $\text{infindex}_G^\mu(1)$ in the case of $\mathbb{R}^2 = \mathbb{C}$, we obtain the following formula.

Proposition 5.3.

$$\text{infindex}_G^\mu(1) = (2\pi i)^m T_X.$$

We will use this calculation in [10] to identify $H_{G,c}^*((T^*M_X)^0)$ with a space of spline distributions on \mathfrak{g}^* .

Another example that we will use in § 5.15 is the case were one of the action forms, say σ_1 , is equal to 0, so $\mu_1 = 0$ and μ is the pull-back of μ_2 . Then

$$(M_1 \times M_2)_{\Delta}^0 = M_1 \times M_2^0.$$

In this case, the space $H_{G,c}^*(M_1^0)$ is simply $H_{G,c}^*(M_1)$ and $\int_{M_1} \alpha_1(x)$ is a polynomial function of $x \in \mathfrak{g}$. Thus $\text{infindex}_G^0(\alpha_1)$, the Fourier transform, is a distribution of support 0 on \mathfrak{g}^* .

Corollary 5.4.

$$\text{infindex}_{\Delta}^\mu[\alpha_1 \times \alpha_2] = \text{infindex}_G^0(\alpha_1) * \text{infindex}_G^{\mu_2}(\alpha_2).$$

5.5. Induction of distributions

Assume that $L \subset G$ is a subgroup; let $\mathfrak{l} \subset \mathfrak{g}$ be the corresponding Lie algebras. Choose Lebesgue measures on \mathfrak{g} , and \mathfrak{l} by fixing translation invariant top differential forms. This determines dual measures and forms on \mathfrak{g}^* , \mathfrak{l}^* and a Haar measure dg on G . If p is the restriction map $\mathfrak{g}^* \rightarrow \mathfrak{l}^*$, we let p_* be the integration over the fiber (with respect to the chosen forms and orientations). It sends a test function on \mathfrak{g}^* to a test function on \mathfrak{l}^* . Let

$$A(f)(\xi) = \int_G f(g\xi)dg. \tag{38}$$

The operator A transforms a test function on \mathfrak{g}^* to an invariant test function on \mathfrak{g}^* .

Definition 5.6. For a distribution V on \mathfrak{l}^* , we define the G -invariant distribution $\text{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} V$ on \mathfrak{g}^* by

$$\langle \text{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} V, f \rangle = \text{vol}(L, dl)^{-1} \langle V, p_*(A(f)) \rangle,$$

f being a test function on \mathfrak{g}^* .

It is easy to see that $\text{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} V$ is independent of the choices of measures.

5.7. Induction of action manifolds

Assume that $L \subset G$ is a subgroup. Take M an L -manifold with action form σ and moment map μ_L .

Consider T^*G as a $G \times L$ -action manifold where G acts on the left and L on the right, and the action form ω is the canonical 1-form on T^*G .

5.7.1. The induced action manifold. Set $N := T^*G \times M$ and p_1, p_2 be the first and second projections of this product manifold. We consider the action form $\psi = p_1^*\omega + p_2^*\sigma$ on N , and denote by $\tilde{\mu}_{G \times L} = \tilde{\mu}_G \oplus \tilde{\mu}_L$ the corresponding moment map.

Let us trivialize $T^*G = G \times \mathfrak{g}^*$ using left trivialization (7), so we make the identification $N = G \times \mathfrak{g}^* \times M$. According to the formula (7), if $(g, \xi, m) \in N$ we have

$$\tilde{\mu}_G(g, \xi, m) = -g\xi := -g\xi, \quad \tilde{\mu}_L(g, \xi, m) = -\xi|_{\mathfrak{l}} + \mu_L(m). \tag{39}$$

We denote by N^0 the zero fiber of the moment map $\tilde{\mu}_{G \times L}$ for $G \times L$, and by M^0 the zero fiber of the moment map μ_L on M for L .

Lemma 5.8. *We have $N^0 = G \times M^0$.*

Proof. From the formula (39), the set of points of N where $\tilde{\mu}_G = 0$ is $G \times M$, and at these points we have $\tilde{\mu}_L(g, m) = \mu_L(m)$. □

Lemma 5.9. (i) *If we take the zero fiber of $\tilde{\mu}_L$, we obtain the manifold*

$$P := \tilde{\mu}_L^{-1}(0) = \{(g, \xi, m); g \in G, \xi \in \mathfrak{g}^*, m \in M; \xi|_{\mathfrak{l}} = \mu_L(m)\}. \tag{40}$$

(ii) *0 is a regular value for the moment map $\tilde{\mu}_L$.*

Proof. The first statement is immediate from formula (39). As for the second, fix g, m ; the map $\tau : \mathfrak{g}^* \rightarrow \mathfrak{l}^*$ given by $\tau : \xi \mapsto \tilde{\mu}_L(g, \xi, m) = -\xi|_{\mathfrak{l}} + \mu_L(m)$ clearly has a surjective differential for all ξ —hence the claim. □

We are thus in the situation of §4.11. The manifold N is a $G \times L$ -manifold, L acts freely on N and 0 is a regular value of the moment map $\tilde{\mu}_L$ for L . Consider the manifold $\mathcal{M} = P/L$. Applying Proposition 4.12, we see the following lemma.

Lemma 5.10. *The quotient $\mathcal{M} = P/L$ is a G -manifold. The action form on N restricted to P descends to \mathcal{M} .*

The induced moment map $\mu_G : P/L \rightarrow \mathfrak{g}^$ is obtained by taking the quotient from the moment map $\tilde{\mu}_G : (g, \xi, m) \rightarrow g\xi$ on P .*

Definition 5.11. We will say that \mathcal{M} is the *induced action manifold*.

By Lemma 5.8, the closed set N^0 , the zero fiber of the moment map $\tilde{\mu}_{G \times L}$, equals $G \times M^0$ and it is contained in P . Since, by definition, on $P = \tilde{\mu}_L^{-1}(0)$ the moment map $\tilde{\mu}_L$ equals 0, we have that on P the moment map $\tilde{\mu}_{G \times L}$ equals $\tilde{\mu}_G$. Therefore we obtain the following lemma.

Lemma 5.12. *Under the inclusions $N^0 \subset P$, $N^0/L \subset P/L$, the zero fiber $\mathcal{M}_G^0 \subset \mathcal{M}$ of the moment map μ_G is identified with $N^0/L = G \times_L M^0$.*

Denote by p_1, p_2 the two projections of $N^0 = G \times M^0$ on its factors. Denote by $\pi : G \times M^0 = N^0 \rightarrow N^0/L = G \times_L M^0$ the quotient map.

Thus we get the isomorphisms

$$H_{L,c}^*(M^0) \xrightarrow{p_2^*} H_{G \times L,c}^*(N^0) \xleftarrow{\pi^*} H_{G,c}^*(G \times_L M^0).$$

We set $j = \pi^{*-1} p_2^*$:

$$j : H_{L,c}^*(M^0) \xrightarrow{p_2^*} H_{G \times L,c}^*(N^0) \xrightarrow{(\pi^*)^{-1}} H_{G,c}^*(\mathcal{M}_G^0). \tag{41}$$

Remark 5.13. As in the usual case (see [12], page 33), the isomorphism j^{-1} can be described as follows. Let $\gamma(y)$, with $y \in \mathfrak{g}$, be an equivariant form on $P/L = \mathcal{M}$ representing $[\gamma] \in H_{G,c}^*(\mathcal{M}_G^0) = H_{G,c}^*(G \times_L M^0)$. We restrict γ to the L -invariant submanifold M embedded in \mathcal{M} by $m \mapsto (e, \mu_L(m), m)$ and obtain an L -equivariant form on M . We can represent $j^{-1}[\gamma]$ by $\gamma(x)|_M$ with $x \in \mathfrak{l}$.

5.13.1. The induction formula for the infinitesimal index. Given a class $[\alpha] \in H_{L,c}^*(M^0)$, our goal is to compare $\text{infdex}_L^{\mu_L}([\alpha])$ and $\text{infdex}_G^{\mu_G}(j([\alpha]))$, the first being a distribution on \mathfrak{l}^* and the second one a distribution on \mathfrak{g}^* . We shall show that $\text{infdex}_G^{\mu_G}(j([\alpha]))$ is induced by $\text{infdex}_L^{\mu_L}([\alpha])$, according to Definition 5.6.

Theorem 5.14. *Let $[\alpha] \in H_{L,c}^*(M^0)$; then*

$$\text{infdex}_G^{\mu_G}(j[\alpha]) = i^{\dim G - \dim L} \text{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*}(\text{infdex}_L^{\mu_L}([\alpha])). \tag{42}$$

Proof. Consider the form $\gamma := 1 \times \alpha$ on $G \times M$, where α is a representative of $[\alpha]$. By definition, $j = \pi^{*-1} p_2^*$, and we see that $[\gamma] = p_2^*[\alpha] = \pi^*j[\alpha]$.

Consider the $G \times L$ -manifold $N = T^*G \times M$. To this manifold we can apply Corollary 4.17. Let f_1 be a variable test function on \mathfrak{l}^* and f_2 be a given test function on \mathfrak{g}^* . The distribution $f_1 \rightarrow (\text{infdex}_{G \times L}^{\mu_{G \times L}}([\gamma]), f_2 f_1)$ is given by a smooth density $D(\zeta) d\zeta$ on \mathfrak{l}^* , and the value $D(0)$ equals $i^{\dim L} \text{vol}(L, dl) (\text{infdex}_G^{\mu_G}(j[\alpha]), f_2)$.

Let us compute $(\text{infdex}_{G \times L}^{\mu_{G \times L}}([\gamma]), f_2 f_1)$ using the fact that γ is the external product $1 \times \alpha$. We consider the product manifold $T^*G \times M$ provided with the action of $G_1 \times G_2$ where $G_1 = G \times G$ acts by left and right action on T^*G and $G_2 = L$ acts on M .

Consider next the embedding of $G \times L$ as a subgroup of $G \times G \times L$ through $\{(g, l, l), g \in G, l \in L\}$. Denote by

$$s : \mathfrak{g} \oplus \mathfrak{l} \rightarrow \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{l}, \quad (a, b) \mapsto (a, b, b)$$

the inclusion of Lie algebras. Denote by $p : \mathfrak{g}^* \rightarrow \mathfrak{l}^*$ the restriction map. Then for $\zeta \in \mathfrak{l}^*$ and $(\xi_1, \xi_2) \in \mathfrak{g}^* \oplus \mathfrak{g}^*$ the restriction map R associated with the inclusion s is

$$R := s^* : \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{l}^* \rightarrow \mathfrak{g}^* \oplus \mathfrak{l}^*, \quad (\xi_1, \xi_2, \zeta) \mapsto (\xi_1, \zeta + p(\xi_2)).$$

Remark that our given action form on N is $G \times G \times L$ -invariant and that $N^0 = G \times M^0$ is also the set of zeros of the moment map μ for the group $G \times G \times L$.

In order to compute $\text{infdex}_{G \times L}^{\mu_{G \times L}}([\gamma])$ we may thus apply first the external product property (Proposition 4.4) and then the restriction property, (Theorem 4.7)

obtaining

$$\text{infindex}_{G \times L}^{\tilde{\mu}_{G \times L}}(1 \times \alpha) = R_*(\text{infindex}_{G \times G}^{\mu}(1) \otimes \text{infindex}_L^{\mu L}([\alpha])).$$

We now make this formula more explicit. Let f_1 be a test function on \mathfrak{l}^* and f_2 a test function on \mathfrak{g}^* . Using the formula (23),

$$\langle R_*(\text{infindex}_{G \times G}^{\mu}(1) \otimes \text{infindex}_L^{\mu L}([\alpha])), f_1 f_2 \rangle = \lim_{T \rightarrow \infty} \langle \text{infindex}_{G \times G}^{\mu}(1) \otimes \text{infindex}_L^{\mu L}([\alpha]), R^*(f_1 f_2) \chi_T \rangle.$$

The function $R^*(f_1 f_2)(\xi_1, \xi_2, \zeta)$ is the function $f_1(\zeta + p(\xi_2))f_2(\xi_1)$. Using formula (22) for $\text{infindex}_{G \times G}^{\mu}(1)$ for T^*G , we obtain

$$\begin{aligned} & \lim_{T \rightarrow \infty} \langle \text{infindex}_{G \times G}^{\mu}(1) \otimes \text{infindex}_L^{\mu L}([\alpha]), f_1(\zeta + p(\xi_2))f_2(\xi_1) \chi_T \rangle \\ &= i^{\dim G} \langle \text{infindex}_L^{\mu L}([\alpha]), q(f_1, f_2) \rangle \end{aligned}$$

with (A is defined in (38))

$$q(f_1, f_2)(\zeta) = \int_{\mathfrak{g}^*} \int_G f_1(\zeta + p(\xi))f_2(-g\xi) dg d\xi = \int_{\mathfrak{g}^*} f_1(\zeta + p(\xi))Af_2(-\xi) d\xi.$$

Integrating first on the fiber $p : \mathfrak{g}^* \rightarrow \mathfrak{l}^*$, then on \mathfrak{l}^* , we see that

$$q(f_1, f_2)(\zeta) = f_1 * (p_*(Af_2))(\zeta)$$

where $u * v$ is the convolution product of test functions on \mathfrak{l}^* .

Then we obtain

$$\langle \text{infindex}_{G \times L}^{\tilde{\mu}_{G \times L}}([\gamma]), f_1 f_2 \rangle = i^{\dim G} \langle \text{infindex}_L^{\mu L}([\alpha]), f_1 * (p_*(Af_2))(\zeta) \rangle.$$

This is a smooth density with respect to $\zeta \in \mathfrak{l}^*$, and if f_1 tends to $\delta_0(\zeta)$, then $\langle \text{infindex}_{G \times L}^{\tilde{\mu}_{G \times L}}([\gamma]), f_1 f_2 \rangle$ tends to

$$i^{\dim G} \langle \text{infindex}_L^{\mu L}([\alpha]), p_*(Af_2)(\zeta) \rangle = i^{\dim G} \text{vol}(L, dl) \langle \text{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \text{infindex}_L([\alpha]), f_2(\zeta) \rangle.$$

We thus obtain the wanted formula, (42). □

5.15. Maximal tori

As usual, let M be a G -manifold with a G -invariant action form σ . Let $T \subset G$ be a maximal torus. We show next how to reduce the calculation of the infinitesimal index map for G to the calculation of the infinitesimal index map for T . Our construction is very similar to the construction of the map $K_G(T_G^*N) \rightarrow K_T(T_T^*N)$ at the level of K -theory given in [1].

Associated with σ , we have the moment maps $\nu_G : M \rightarrow \mathfrak{g}^*$ and $\nu_T = p \circ \nu_G : M \rightarrow \mathfrak{t}^*$, with $p : \mathfrak{g}^* \rightarrow \mathfrak{t}^*$ the restriction map.

Consider M as a T -manifold, and consider $N = T^*G \times M$, provided, as in § 5.7 (here the group L is T), with the action form $\psi = p_1^* \omega + p_2^* \sigma$ and the action of $G \times T$: the group G acts on T^*G by left action, and trivially on M , the group T acts on G by right action and acts on M . We denote by $\tilde{\mu}_{G \times T} = \tilde{\mu}_G \oplus \tilde{\mu}_T$ the corresponding moment map.

Recall, by formula (40), that

$$P = \tilde{\mu}_T^{-1}(0) = \{(g, \xi, m); g \in G, \xi \in \mathfrak{g}^*, m \in M; \xi|_{\mathfrak{t}} = \nu_T(m)\}$$

is a $G \times T$ -manifold on which G acts by $g_0 \cdot (g, \xi, m) = (g_0g, \xi, m)$ for $g_0 \in G, (g, \xi, m) \in P$, and T acts by $t \cdot (g, \xi, m) = (gt^{-1}, t\xi, tm)$.

We then consider $\mathcal{M} := P/T$, with moment map $\mu_G([g, \xi, m]) = g\xi$ (39). Recall that \mathcal{M}_G^0 is isomorphic to $G \times_T M_T^0$ embedded in P/T by $[g, 0, m]$.

For $[\alpha] \in H_{G,c}^*(M_G^0)$, we want to produce an element $r([\alpha]) \in H_{G,c}^*(\mathcal{M}_G^0) = H_{G,c}^*(G \times_T M_T^0)$ which has the same infinitesimal index as $[\alpha]$.

Proposition 5.16. *We can embed $G \times M$ in P through the map*

$$\gamma(g, m) = (g, \nu_G(g^{-1}m), g^{-1}m).$$

The map γ is $G \times T$ -equivariant, where G acts on $G \times M$ by diagonal action (left on G) while T acts by the right action on G and not on M .

Proof. First, $(g, \nu_G(g^{-1}m), g^{-1}m) \in P$ since $\nu_G(g^{-1}m)|_{\mathfrak{t}^*} = \nu_T(g^{-1}m)$. Next, $\gamma(hg, hm) = (hg, \nu_G(g^{-1}m), g^{-1}m)$ and

$$\gamma(gt^{-1}, m) = (gt^{-1}, \nu_G(tg^{-1}m), tg^{-1}m) = (gt^{-1}, t\nu_G(g^{-1}m), tg^{-1}m). \quad \square$$

Corollary 5.17. *The map γ induces, modulo the action of T , an embedding still denoted by $\gamma : G/T \times M \hookrightarrow \mathcal{M} = P/T$. Thus the manifold $G/T \times M$, with diagonal G -action, is identified with a G -invariant submanifold of \mathcal{M} .*

In fact more is true. Let $q : \mathcal{M} \rightarrow G/T \times M$ be the projection given by $q(g, \xi, m) = (gT, gm)$. Let $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{t}^\perp$ be the canonical T -invariant decomposition of \mathfrak{g}^* . Then we claim that the following proposition holds.

Proposition 5.18. *$q\gamma$ is the identity and $q : \mathcal{M} \rightarrow G/T \times M$ is a vector bundle with fiber \mathfrak{t}^\perp .*

Proof. The first claim comes from the definitions. As for the second, we may identify P with $G \times M \times \mathfrak{t}^\perp$ through the map

$$P \rightarrow G \times M \times \mathfrak{t}^\perp, \quad (g, \xi, m) \mapsto (g, m, \xi - \nu_T(m)). \quad \square$$

Lemma 5.19. *The restriction of the moment map μ_G on \mathcal{M} to $G/T \times M$ is just $(gT, m) \mapsto \nu_G(m)$ with zeros $G/T \times M_G^0$.*

Proof. We have $\mu_G(g, \xi, m) = g\xi$ by the previous discussion. An element (g, m) corresponds to the triple $(g, \nu_G(g^{-1}m), g^{-1}m)$, so the claim follows since ν_G is G -equivariant. \square

We now apply the construction γ_l of §4.8 to the manifold $G/T \times M$ embedded by γ in \mathcal{M} .

Recall that G/T is a even dimensional manifold. Take an equivariant form β on G/T with class

$$[\beta] = (-1)^{\frac{1}{2} \dim G/T} \frac{e(G/T)}{|W|} \tag{43}$$

where W is the Weyl group and $e(G/T)$ is the equivariant Euler class. Notice that since $|W|$ equals the Euler characteristic of G/T , $\int_{G/T} [\beta]$ is equal to $(-1)^{\frac{1}{2} \dim G/T}$.

Thus, by Theorem 3.12, the infinitesimal index of $[\beta]$ is just the δ -function on \mathfrak{g}^* . Let $[\alpha] \in H_{G,c}^*(M_G^0)$. We then construct the element $[\beta \wedge \alpha]$ in the compactly supported equivariant cohomology

$$[\beta \wedge \alpha] \in H_{G,c}^*((G/T \times M_G^0)) = H_{G,c}^*(G/T \times M_G^0).$$

Lemma 5.20. *The infinitesimal index of $[\beta \wedge \alpha]$ is equal to $(-1)^{\frac{1}{2} \dim G/T}$ times the infinitesimal index of $[\alpha]$.*

Proof. Apply Corollary 5.4. □

Under the embedding $\gamma : G/T \times M \hookrightarrow \mathcal{M}$ of action manifolds (cf. 5.17), by Theorem 4.9, we have now a homomorphism

$$\gamma_* : H_{G,c}^*(G/T \times M_G^0) \rightarrow H_{G,c}^*(\mathcal{M}_G^0)$$

preserving the infdex.

We define

$$r([\alpha]) := \gamma_*([\beta \wedge \alpha]) \in H_{G,c}^*(\mathcal{M}_G^0). \tag{44}$$

We then have, combining Lemma 5.20 with Theorem 4.9,

$$\text{infdex}_G^{\nu G}([\alpha]) = \text{infdex}_G^{\mu G}(r([\alpha])). \tag{45}$$

On the other hand, we have the isomorphism

$$j : H_{T,c}^*(M_T^0) \rightarrow H_{G,c}^*(G \times_T M_T^0)$$

and we have shown in Theorem 5.14 that

$$\text{infdex}_G^{\mu G}(j[\theta]) = (-1)^{\frac{1}{2} \dim G/T} \text{Ind}_{\mathfrak{t}^*}^{\mathfrak{g}^*} \text{infdex}_T^{\nu T}([\theta])$$

for any $[\theta] \in H_{T,c}^*(M_T^0)$.

We deduce the following theorem.

Theorem 5.21. *Take the following commutative diagram:*

$$\begin{array}{ccccc}
 H_{G,c}^*(M_G^0) & \xrightarrow{r} & H_{G,c}^*(G \times_T M_T^0) & \xrightarrow{j^{-1}} & H_{T,c}^*(M_T^0) \\
 \text{infdex} \downarrow & & \text{infdex} \downarrow & & \text{infdex} \downarrow \\
 \mathcal{D}'(\mathfrak{g}^*)^G & \xrightarrow{id} & \mathcal{D}'(\mathfrak{g}^*)^G & \xleftarrow{\text{Ind}_{\mathfrak{t}^*}^{\mathfrak{g}^*}} & \mathcal{D}'(\mathfrak{t}^*)
 \end{array} \tag{46}$$

The element $[\lambda] := j^{-1}r([\alpha]) \in H_{T,c}^*(M_T^0)$ is such that

$$\text{infdex}_G^{\nu G}([\alpha]) = \text{Ind}_{\mathfrak{t}^*}^{\mathfrak{g}^*} \text{infdex}_T^{\nu T}(j^{-1}r([\alpha])). \tag{47}$$

Let us finally give an explicit formula for the element $[\lambda] = j^{-1}r([\alpha]) \in H_{T,c}^*(M_T^0)$ corresponding to $[\alpha] \in H_{G,c}^*(M_G^0)$.

Let $\text{Pf}(x) = \det_{\mathfrak{t}^\perp}^{1/2}(x)$ be the Pfaffian associated with the action of $x \in \mathfrak{t}$ in the oriented orthogonal space \mathfrak{t}^\perp .

We need the following proposition.

Proposition 5.22. *The restriction of the form $\beta(x)$ at the point $e \in G/T$ is the polynomial $|W|^{-1} (2\pi)^{-\frac{1}{2} \dim G/T} \text{Pf}(x)$.*

Proof. By construction, the equivariant Euler class is the restriction to G/T of the Thom class of the tangent bundle. The fiber of the tangent bundle at the T fixed point e is isomorphic to \mathfrak{t}^\perp . Thus this class restricts at the fixed point e as $(-2\pi)^{-\frac{1}{2} \dim G/T} \text{Pf}(x)$ ([18]; see [3], Theorem 7.41, and [21]). \square

Recall the decomposition $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{t}^\perp$. Let us consider the map $v_\perp : M \rightarrow \mathfrak{t}^\perp$ which is uniquely defined by the identity $v_G = v_T \oplus v_\perp$. Then $v_T^{-1}(0) \cap v_\perp^{-1}(0) = v_G^{-1}(0)$.

Denote by τ_0 the T -equivariant Thom class of the embedding $0 \rightarrow \mathfrak{t}^\perp$, a compactly supported equivariant class on \mathfrak{t}^\perp . Then $\tau_\perp := v_\perp^* \tau_0$ is a closed equivariant class on M supported on a small neighborhood A of $v_\perp^{-1}(0)$. It follows that we have the following lemma.

Lemma 5.23. *If $[\alpha] \in H_{G,c}^*(M_G^0)$, we can choose τ_0 such that the class $\tau_\perp \wedge \alpha$ defines a class in $H_{T,c}^*(M_T^0)$.*

Proof. Let $K \subset M \setminus M_G^0$ be the support of $D\alpha$; then $D(\tau_\perp \wedge \alpha) = \tau_\perp \wedge D\alpha$ is supported in $A \cap K$. Since $\emptyset = K \cap M_G^0 = K \cap M_T^0 \cap v_\perp^{-1}(0)$, we can choose τ_0 such that $A \cap K \cap M_T^0 = \emptyset$. \square

By Remark 5.13, an equivariant form representing $j^{-1}r([\alpha])$ is the restriction to $M = \{(e, 0, m), m \in M\}$ of $r(\alpha)$, when $x \in \mathfrak{t}$. We still denote it by $j^{-1}(r(\alpha))(x)$.

Theorem 5.24. *We can choose the Thom classes such that*

$$j^{-1}(r(\alpha))(x) = |W|^{-1} (2\pi)^{-\frac{1}{2} \dim G/T} \text{Pf}(x)\alpha(x) \wedge \tau_\perp(x).$$

Proof. Let $\tau_{G/T \times M}$ be a Thom class of the bundle $q : \mathcal{M} \rightarrow G/T \times M$ (Proposition 5.18). Then, by the η construction, the associated equivariant form on \mathcal{M} which we denoted by $r(\alpha)$ is $q^*(\beta \wedge \alpha) \wedge \tau_{G/T \times M}$.

Now the bundle $q : \mathcal{M} \rightarrow G/T \times M$ is trivial over $e \times M$ and isomorphic to $\mathfrak{t}^\perp \times M$ by $(\xi, m) \mapsto (e, \xi + v_G(m), m)$.

The restriction of the Thom class $\tau_{G/T \times M}$ gives a Thom class for this trivial bundle. We can then assume that the restriction of $\tau_{G/T \times M}$ is $\tau_0(\xi)$.

As (e, M) is embedded by $\xi = v_\perp(m)$, we obtain our theorem from Proposition 5.22. \square

Acknowledgements. The first two authors were partially supported by the Cofin 40%, MIUR. We wish to thank Paul-Émile Paradan for his comments.

Appendix A. Equivariant cohomology with compact supports

A.1. Compact supports

We are going to assume in this appendix that all our spaces are locally compact and paracompact and we are going to work with Alexander–Spanier cohomology groups, both ordinary and with compact supports, and with real coefficients. We shall denote them by H^* or, if we take compact supports, by H_c^* . H^* is a cohomology theory on spaces or pairs of spaces deduced from a functorial cochain complex $\mathcal{C}(X, Z)$, and H_c^* , the theory with compact supports, is associated with a natural subcomplex $\mathcal{C}_c(X, Z)$ (see [23] ch. 6).

Let us now recall a few properties. The first is (see [23] ch. 6, p. 321, Lemma 11).

Proposition A.2. *Let (X, Z) be a pair with X compact $Z \neq \emptyset$ closed. Set $U := X \setminus Z$. Then there are natural isomorphism $H_c^q(U) \simeq H^q(X, Z)$.*

In fact this is induced by the map of cochain complexes $\mathcal{C}_c(U) \rightarrow \mathcal{C}(X, Z)$, and composition of the inclusions $\mathcal{C}_c(U) \rightarrow \mathcal{C}_c(X) \rightarrow \mathcal{C}(X)$ and of the quotient $\mathcal{C}(X) \rightarrow \mathcal{C}(X, Z)$.

In particular, if we take an open set U in a compact space X (for example we could take the one-point compactification U^+ of a locally compact space U), we get that $H_c^*(U) = H^*(X, X \setminus U)$.

As an application of this, assume that $Z \subset U$ is closed and U is open in a compact space X . Set $Y = X \setminus U$ and take the triple (X, \tilde{Z}, Y) with $\tilde{Z} = Z \cup Y$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{C}_c^*(U \setminus Z) & \longrightarrow & \mathcal{C}_c^*(U) & \longrightarrow & \mathcal{C}_c^*(Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{C}^*(X, \tilde{Z}) & \longrightarrow & \mathcal{C}^*(X, Y) & \longrightarrow & \mathcal{C}^*(\tilde{Z}, Y) \longrightarrow 0
 \end{array}$$

Using the exactness of the bottom line we deduce the long exact sequence

$$\dots \rightarrow H_c^h(U \setminus Z) \xrightarrow{i_*} H_c^h(U) \xrightarrow{j^*} H_c^h(Z) \longrightarrow H_c^{h+1}(X \setminus Z) \rightarrow \dots$$

On the other hand, the top line induces a homomorphism of chain complexes

$$\mu : \mathcal{C}_c^*(U) / \mathcal{C}_c^*(U \setminus Z) \rightarrow \mathcal{C}_c^*(Z)$$

and since the vertical arrows induce isomorphism in cohomology, using the five lemma we easily deduce the following proposition.

Proposition A.3. *The homomorphism μ induces an isomorphism in cohomology.*

In order to compare the Alexander–Spanier and singular cohomologies, one needs to pass to the associated sheaves (see [23] ch. 6, p. 324). Thus, under suitable topological conditions, we obtain a natural isomorphism between Alexander–Spanier and singular cohomologies.

In particular, consider a C^∞ manifold M and a closed subset $Z \subset M$. Further, assume that Z is locally contractible (this is the case for T_G^*N in T^*N , as follows from the

description of the neighborhood of a G -orbit using the slice theorem). We then have (see [23] ch. 6, p. 341, Corollary 7) that, under these assumptions, we can use singular cochains and in fact, in the case of a manifold, singular C^∞ cochains to compute the cohomology since the Alexander–Spanier and singular cohomologies are naturally isomorphic in this case.

Integrating on singular C^∞ simplexes we get a commutative diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathcal{A}_c^*(M \setminus Z) & \longrightarrow & \mathcal{A}_c^*(M) \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \infty\mathcal{C}_c^*(M \setminus Z) & \longrightarrow & \infty\mathcal{C}_c^*(M)
 \end{array}$$

where \mathcal{A}_c^* is the complex of differential forms with compact supports. We deduce a homomorphism of cochain complexes

$$\nu : \mathcal{A}_c^*(M)/\mathcal{A}_c^*(M \setminus Z) \rightarrow \infty\mathcal{C}_c^*(M)/\infty\mathcal{C}_c^*(M \setminus Z).$$

Since the vertical arrows induce isomorphism in cohomology, we get a de Rham model for $H_c^*(Z)$.

Proposition A.4. *The homomorphism ν induces isomorphism in cohomology. In particular, $H_c^*(Z)$ is naturally isomorphic to the cohomology of the complex $\mathcal{A}_c^*(M)/\mathcal{A}_c^*(M \setminus Z)$.*

A.5. Classifying spaces

We now take a compact Lie group G and denote by B_G its classifying space (which is not locally compact). Recall that B_G is a polyhedron with finitely many cells in each dimension and it has a filtration $(B_G)_0 \subset \dots \subset (B_G)_n \subset (B_G)_{n+1} \subset \dots \subset B_G$ by compact manifolds with the property that the inclusion $(B_G)_n \subset B_G$ induces isomorphism in cohomology up to degree n . For example, if G is an s -dimensional torus, $B_G = \mathbb{C}P(\infty)^s$ and we may take $(B_G)_n = \mathbb{C}P(n)^s$ (indeed in this case, the inclusion induces an isomorphism up to degree $2n - 1$).

We denote by $\pi : E_G \rightarrow B_G$ the universal fibration and set $(E_G)_n = \pi^{-1}((B_G)_n)$. Thus $(E_G)_n$ is also a compact C^∞ manifold and a principal bundle over $(B_G)_n$.

Recall now that for any G -space Y , $H_G^*(Y) = H^*(Y \times_G E_G)$.

We can define the equivariant cohomology with compact supports of a G -space as follows. Take U locally compact. Embed U in its one-point compactification U^+ . The action of G extends to U^+ and we set the following definition.

Definition A.6. $H_{G,c}^*(U) = H_G^*(U^+, \infty)$.

Some remarks are in order.

- If U is compact, then U^+ is the disjoint union $U \cup \{\infty\}$ so $H_{G,c}^*(U) = H_G^*(U)$.
- If U is non-compact, then $H_{G,c}^*(U) = H^*(U^+ \times_G E_G, B_G)$ where $B_G = \{\infty\} \times_G E_G$.

- All the equivariant cohomologies are modules over $H_G^*(pt)$ and all the homomorphisms are module homomorphisms.

Recall that by the properties of $(B_G)_m$ for any $h \geq 0$, and for all m large enough, $H^r(B_G, R) = H^r((B_G)_m, R)$ for $0 \leq r \leq 2h$. So given a G -space X , the spectral sequences of the fibrations $X \times_G E_G \rightarrow B_G$ and $X \times_G (E_G)_m \rightarrow (B_G)_m$ have the same $E_r^{p,q}$ for all r and $p + q \leq h$. In particular, we get for any pair (X, Z) of G -spaces that for large m , $H_G^h(X, Z) = H^h(X \times_G (E_G)_m, Z \times_G (E_G)_m)$. From Proposition A.2, we then deduce the following proposition.

Proposition A.7. *Let X be a G -space with X compact Hausdorff and $Z \neq \emptyset$ a closed G -stable subspace. Set $U := X \setminus Z$. Then there is a natural isomorphism $H_{G,c}^q(U) \simeq H_G^q(X, Z)$.*

Furthermore for m large with respect to h , $H_{G,c}^h(U) \simeq H_c^h(U \times_G (E_G)_m)$.

Take now a C^∞ manifold M with a C^∞ action of G and a closed G -stable subset Z in M which we assume to be locally contractible—for instance if Z is locally triangular, as for instance when Z is semi-analytic [17]. The same is true for $Z \times_G (E_G)_m$ for any m , so we can apply Proposition A.4 and deduce that for m large with respect to h , $H_{G,c}^h(Z)$ is the h th cohomology group of the complex $\mathcal{A}_c^*(M \times_G (E_G)_m) / \mathcal{A}_c^*((M \setminus Z) \times_G (E_G)_m)$.

But one knows (see [13]) that for any m we have a natural morphism of complexes $\mathcal{A}_{G,c}(M) \rightarrow \mathcal{A}_c^*(M \times_G (E_G)_m)$ which induces isomorphism in cohomology in small degree. The same holds also for the open set $M \setminus Z$ so we get a commutative diagram:

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathcal{A}_{G,c}(M \setminus Z) & \longrightarrow & \mathcal{A}_{G,c}(M) \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{A}_c^*((M \setminus Z) \times_G (E_G)_m) & \longrightarrow & \mathcal{A}_c^*(M \times_G (E_G)_m)
 \end{array}$$

which induces a morphism of complexes

$$\rho : \mathcal{A}_{G,c}(M) / \mathcal{A}_{G,c}(M \setminus Z) \rightarrow \mathcal{A}_c^*(M \times_G (E_G)_m) / \mathcal{A}_c^*((M \setminus Z) \times_G (E_G)_m).$$

From this we immediately deduce the following proposition.

Proposition A.8. $H_{G,c}^*(Z)$ equals the cohomology of the complex $\mathcal{A}_{G,c}^*(Z, M) = \mathcal{A}_c^*(M) / \mathcal{A}_c^*(M \setminus Z)$.

Proof. From the above considerations we have, if m is large with respect to h , that ρ induces an isomorphism in cohomology in degree h . Since we have seen that in degree h the cohomology of the complex $\mathcal{A}_c^*(M \times_G (E_G)_m) / \mathcal{A}_c^*((M \setminus Z) \times_G (E_G)_m)$ is $H_{G,c}^h(Z)$, everything follows. \square

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