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General uniqueness results and blow-up rates for large solutions of elliptic equations

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Making use of the Karamata regular variation theory and the López-Gómez localization method, we establish the uniqueness and asymptotic behaviour near the boundary $\partial\Omega$ for the large solutions of the singular boundary-value problem

$$\begin{aligned} \Delta u &= b(x)f(u), \quad x \in \Omega, \\ u(x) &= +\infty, \qquad x \in \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. The weight function $b(x)$ is a non-negative continuous function in the domain, which can vanish on the boundary $\partial\Omega$ at different rates according to the point $x_0 \in \partial\Omega$. $f(u)$ is locally Lipschitz continuous such that $f(u)/u$ is increasing on $(0,\infty)$ and $f(u)/u^p = H(u)$ for sufficiently large u and $p > 1$, here $H(u)$ is slowly varying at infinity. Our main result provides a sharp extension of a recent result of Xie with f satisfying $\lim_{u\to\infty} f(u)/u^p = H$ for some positive constants $H > 0$ and $p > 1$.

1. Introduction and main results

We are concerned with the uniqueness and exact asymptotic behaviour of large solutions for the following elliptic problem:

$$\left.\begin{aligned} \Delta u &= b(x)f(u), \quad x \in \Omega, \\ u(x) &= +\infty, \qquad x \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. By a solution to (1.1) we take to mean a function $u \in C^1_{\mathrm{loc}}(\Omega)$ which verifies the equation in the weak sense and

$$\lim_{d(x)\to 0} u(x) = \infty,$$

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where $d(x) = \operatorname{dist}(x, \partial\Omega)$ for each $x \in \Omega$. These solutions are often referred to as positive large solutions, boundary blow-up solutions or explosive solutions.

The basic structural assumptions of f are the following:

(f_1) $f \geqslant 0$ is locally Lipschitz continuous on $[0, \infty)$ and $f(u)/u$ is increasing on $(0, \infty)$;

(f_2) there exist a slowly varying function H (see definition 2.1) and $p > 1$ such that

$$\lim_{u\to\infty} \frac{f(u)}{H(u)u^p} = 1.$$

We assume that the following conditions on $b(x)$ are satisfied. For each $x_0 \in \partial\Omega$, define the boundary normal sections $b_{x_0}(r)$ as $b_{x_0}(r) = b(x_0 - r\boldsymbol{n}_{x_0})$, $r > 0$, $r \sim 0$, where $\boldsymbol{n}_{x_0}$ stands for the outward unit normal at $x_0 \in \partial\Omega$. For any $x_0 \in \partial\Omega$, assume that there exists $\tau > 0$, such that $b(x) \in C^1(\bar{\Omega}_\tau(x_0) \cap \bar{\Omega})$ and

$$b_{x_0}(r) \in C^1(0, \tau), \quad b'_{x_0}(r) > 0 \quad \text{for each } r \in (0, \tau), \tag{1.2}$$

and

$$\lim_{x\in\partial\Omega, x\to x_0, r\to 0} \frac{b_x(r)}{b_{x_0}(r)} = 1, \tag{1.3}$$

where $\Omega_\tau(x_0)$ is a ball in $\mathbb{R}^N$ of radius τ centred at x_0.

The main purpose of this paper is, using the Karamata regular variation theory approach introduced by Cîrstea and Rădulescu [8–12], to study the uniqueness and asymptotic behaviour of large solutions of (1.1) in a general framework. Our main results are summarized in the following theorems.

THEOREM 1.1. *Suppose that* (f_1) *and* (f_2) *hold and that* $b(x) > 0$ *in* Ω *satisfies (1.2)–(1.3). Then problem (1.1) possesses a unique positive solution* $u(x)$ *in* Ω*. Let*

$$\mathcal{B}_{x_0}(r) = \int_0^r \int_0^s (H \circ \mathcal{B}_{x_0}^{-\beta}(t)) b_{x_0}(t)\,\mathrm{d}t\,\mathrm{d}s, \tag{1.4}$$

where H *appears in* (f_2) *and* $(H \circ \mathcal{B}_{x_0}^{-\beta})(t) = H(\mathcal{B}_{x_0}^{-\beta}(t))$*. Then, for each* $x_0 \in \partial\Omega$*, any positive solution* $u(x)$ *of (1.1) satisfies*

$$\lim_{r\to 0} \frac{u(x_0 - r\boldsymbol{n}_{x_0})}{K(x_0)\mathcal{B}_{x_0}^{-\beta}(r)} = 1, \tag{1.5}$$

where

$$K(x_0) = [\beta(\beta+1)C(x_0) - \beta]^\beta, \quad \beta = \frac{1}{p-1}, \tag{1.6}$$

$$C(x_0) = \lim_{r\to 0} \frac{[\mathcal{B}'_{x_0}(r)]^2}{\mathcal{B}_{x_0}(r) b_{x_0}(r) H \circ \mathcal{B}_{x_0}^{-\beta}(r)}. \tag{1.7}$$

It is easy to see that $\mathcal{B}_{x_0}(r)$ is an increasing C^2-function on some interval $(0, \delta)$ with sufficiently small $\delta > 0$, and that $\lim_{r\to 0} \mathcal{B}_{x_0}(r) = 0$. Thus, according to [7, lemma 3.2], we know that $C(x_0) \geqslant 1$. Moreover, $C(x_0) > 1$ if and only if $\mathcal{B}'_{x_0}(r)$ is normalized regularly varying at zero with index $1/(C(x_0)-1)$, and $C(x_0)=1$ if and only if

$$\lim_{r\to 0}\frac{r\mathcal{B}''_{x_0}(r)}{\mathcal{B}'_{x_0}(r)}=0,$$

with $\mathcal{B}_{x_0}(r)$ being of the form

$$\mathcal{B}_{x_0}(r)=c\exp\left\{-\int_r^\eta \frac{\mathrm{d}s}{\zeta(s)}\right\},$$

for some positive constants c, η and a positive function $\zeta\in C^1(0,\eta)$ such that $\lim_{r\to 0}\zeta'(r)=0$.

A point worth emphasizing is that the behaviour at infinity for a slowly varying function cannot be predicted. For example, $L(u)=\exp\{(\log u)^{1/3}\cos((\log u)^{1/3})\}$ is slowly varying with $\lim_{u\to\infty}\inf L(u)=0$ and $\lim_{u\to\infty}\sup L(u)=\infty$. So if f satisfies (f_2), the behaviour at infinity for $f(u)/u^p$ cannot be completely described. Thus, theorem 1.1 extends the results of [4,5,14,22–26,29], since the main results of these works were obtained for f satisfying (f_1) and that

(f_3) there exist $p>1$ and some positive constants H such that

$$H=\lim_{u\to\infty} f(u)/u^p>0,$$

which implies that f behaves like a pure power u^p near infinity.

It is also interesting to note that (f_2) implies that f is regularly varying at infinity with index p, written $f(u)\in \mathrm{RV}_p$ (see definition 2.1). For more details, see [19, remark 1.1].

Continuing the studies of [4,5,22,23], Huang *et al*. [19] also considered the boundary behaviour of large solutions to problem (1.1) under the same conditions, but the formula of the asymptotic behaviour is different, because the function $\mathcal{B}_{x_0}(r)$ used for setting the exact boundary blow-up rate of the solution to (1.1) is apparently different.

Some slight generalizations of problem (1.1) are possible, and still we can get the results in theorem 1.1 for a porous media logistic equation

$$\begin{aligned}\Delta u^m &= b(x)f(u), && x\in\Omega,\\ u(x)&=+\infty, && x\in\partial\Omega,\end{aligned}$$

where $m>1$ (for more results for the porous media equation, see [13,14] and the references therein).

Another possible generalization is to consider q-Laplacian equations, that is,

$$\begin{aligned}\Delta_q u &= b(x)f(u), && x\in\Omega,\\ u(x)&=+\infty, && x\in\partial\Omega,\end{aligned}$$

where $\Delta_q u=\mathrm{div}(|\nabla u|^{q-2}\nabla u)$ denotes the q-Laplacian operator with $q>1$.

With a little more effort, the results in this paper can be extended easily to equations with nonlinear gradient terms

$$\begin{aligned}\Delta u\pm|\nabla u|^q &= b(x)f(u), && x\in\Omega,\\ u(x)&=+\infty, && x\in\partial\Omega.\end{aligned}$$

For more results concerning elliptic boundary blow-up problems with nonlinear gradient terms, see [2, 16, 17, 30, 31].

For more results concerning elliptic boundary blow-up problems (apart from the above-mentioned references), the reader is referred to [1, 6, 15, 18, 20, 21, 32] and the references therein.

The rest of the paper is organized as follows. In § 2 we collect some preliminary results that are needed throughout this paper. In § 3 we give the asymptotic behaviour of the solutions of an auxiliary problems. Theorem 1.1 will be proved in § 4 by the localization method introduced in [22].

2. Preliminaries

The main purpose of this section is to provide some concepts from the theory of regular variation. For detailed accounts of the theory of regular variation, its extensions and many of its applications, we refer the interested reader to [3, 27, 28]. Unless otherwise stated, and where there is no possibility of confusion, regular variation is assumed to occur at infinity.

Definition 2.1. A positive measurable function f defined on $[D, \infty)$ for some $D > 0$, is called regularly varying (at infinity) with index $p \in \mathbb{R}$ (written $f \in \mathrm{RV}_p$) if, for all $\xi > 0$,

$$\lim_{u\to\infty} \frac{f(\xi u)}{f(u)} = \xi^p.$$

When the index of regular variation p is zero, we say that the function is slowly varying. The transformation $f(u) = u^p L(u)$ reduces regular variation to slow variation.

Proposition 2.2. *Assume that L is slowly varying. Then the convergence*

$$L(\xi u)/L(u) \to 1 \quad \text{as } u \to \infty$$

holds uniformly on each compact ε-set in $(0, \infty)$.

Proposition 2.3. *If L is slowly varying, then*

(i) $\ln L(u)/\ln u \to 0$ *as* $u \to \infty$,

(ii) *for any* $\alpha > 0$, $u^\alpha L(u) \to \infty$ *and* $u^{-\alpha} L(u) \to 0$ *as* $u \to \infty$,

(iii) $(L(u))^\alpha$ *varies slowly for every* $\alpha \in \mathbb{R}$,

(iv) *if* L_1 *varies slowly, so do* $L(u)L_1(u)$ *and* $L(u) + L_1(u)$.

Now we collect some important results which will be used in the proof of theorem 1.1.

Definition 2.4. A function $\underline{u} \in C^2(\Omega)$ is a (classical) subsolution to problem (1.1), if $\underline{u} = +\infty$ on $\partial\Omega$ and

$$\Delta \underline{u} \geqslant b(x) f(\underline{u}), \quad x \in \Omega.$$

Similarly, $\bar{u}$ is a (classical) supersolution to problem (1.1), if $\bar{u} = +\infty$ on $\partial\Omega$ and

$$\Delta \bar{u} \leqslant b(x) f(\bar{u}), \quad x \in \Omega.$$

The following comparison principle plays an important role in the proof of theorem 1.1, and will be used in later sections.

Proposition 2.5. *Let f be continuous on $(0,\infty)$ such that $f(u)/u$ is increasing for $u>0$, and let $b(x)\in C(\Omega)$ be a non-negative function. Assume that $u_1,u_2\in C^2(\Omega)$ are positive such that*

$$\begin{gathered}\Delta u_1 - b(x)f(u_1)\leqslant 0\leqslant \Delta u_2 - b(x)f(u_2),\quad x\in\Omega,\\ \limsup_{\mathrm{d}(x,\partial\Omega)\to 0}(u_2-u_1)(x)\leqslant 0.\end{gathered}$$

Then we have $u_1\geqslant u_2$ in Ω.

3. Some auxiliary problems

To prove theorem 1.1, first consider the boundary blow-up rate of problem (1.1) if Ω is a ball in $\mathbb{R}^N$ and the weight function $b(x)$ is a radially symmetric function on the ball.

Theorem 3.1. *Suppose that $\Omega=\Omega_R(x_0)$ is a ball in $\mathbb{R}^N$ of radius R centred at x_0 and $f(u)$ satisfies (f_1) and (f_2), and*

$$b(x)=b(R-\|x-x_0\|)=b(d(x))=b(\mathrm{dist}(x,\partial\Omega_R(x_0)))$$

is a radially symmetric function on the ball. $b\in C([0,R]:[0,\infty))$. Define

$$B(r)=\int_0^r\int_0^s(H\circ B^{-\beta}(t))b(t)\,\mathrm{d}t\,\mathrm{d}s, \tag{3.1}$$

$$C_0=\lim_{r\to 0}\frac{[B'(r)]^2}{B(r)b(r)H\circ B^{-\beta}(r)}. \tag{3.2}$$

Then the problem (1.1) has a unique solution u satisfying

$$\lim_{d(x)\to 0}\frac{u(x)}{KB^{-\beta}(d(x))}=1, \tag{3.3}$$

where

$$K=[\beta(\beta+1)C_0-\beta]^{\beta},\quad \beta=\frac{1}{p-1}.$$

Proof. For clarity, we divide the lengthy proof into several steps.

Step 1 (existence). (f_2) implies that $f(u)\in \mathrm{RV}_p$, taking into account [9, theorem 1.1], we derive that problem (1.1) possesses a large solutions.

Step 2 (local supersolution and subsolution to (1.1)). Let u denote an arbitrary large solution of (1.1).

Define $u_{\pm}(x)=\xi^{\pm}B^{-\beta}(d(x))$, $0<d(x)<\delta$, where

$$\xi^{\pm}=\left[\frac{\beta(\beta+1)C_0-\beta}{1\mp\varepsilon}\right]^{\beta},\quad \beta=\frac{1}{p-1},$$

where $\varepsilon>0$ is small, and where δ is to be determined later.

A simple calculation yields ($|\nabla d(x)| = 1$),

$$\nabla u_\pm(x) = -\xi^\pm \beta B^{-\beta-1}(d(x))B'(d(x))\nabla d(x),$$

$$\begin{aligned}\Delta u_\pm(x) &= \xi^\pm \beta(\beta+1)B^{-\beta-2}(d(x))[B'(d(x))]^2 - \xi^\pm \beta B^{-\beta-1}(d(x))B''(d(x))\\ &\quad - \xi^\pm \beta B^{-\beta-1}(d(x))B'(d(x))\Delta d(x).\end{aligned}$$

Consequently,

$$\Delta u_\pm(x) - b(x)f(u_\pm(x)) = b(d(x))f(u_\pm(x))[B_1(d(x)) - B_2(d(x)) - B_3(d(x)) - 1],$$

where

$$B_1(t) = \frac{\xi^\pm \beta(\beta+1)B^{-\beta-2}(t)[B'(t)]^2}{b(t)f(\xi^\pm B^{-\beta}(t))},$$

$$B_2(t) = \frac{\xi^\pm \beta B^{-\beta-1}(t)B''(t)}{b(t)f(\xi^\pm B^{-\beta}(t))},$$

$$B_3(t) = \frac{\xi^\pm \beta B^{-\beta-1}(t)B'(t)\Delta d(x)}{b(t)f(\xi^\pm B^{-\beta}(t))}.$$

From (f_2), definition 2.1 and (3.2), we find that

$$\begin{aligned}\lim_{t\to 0} B_1(t) &= \xi^\pm \beta(\beta+1)\lim_{t\to 0}\frac{B^{-\beta-2}(t)[B'(t)]^2}{b(t)f(\xi^\pm B^{-\beta}(t))}\\ &= \xi^\pm \beta(\beta+1)\lim_{t\to 0}\frac{f(B^{-\beta}(t))}{f(\xi^\pm B^{-\beta}(t))}\frac{H(B^{-\beta}(t))B^{-p\beta}(t)}{f(B^{-\beta}(t))}\frac{[B'(t)]^2}{B(t)b(t)H(B^{-\beta}(t))}\\ &= \frac{\beta(\beta+1)C_0}{(\xi^\pm)^{p-1}}.\end{aligned} \tag{3.4}$$

By virtue of (3.1), we know that

$$B'(t) = \int_0^t H\circ B^{-\beta}(s)b(s)\,\mathrm{d}s, \qquad B''(t) = H(B^{-\beta}(t))b(t).$$

Then

$$\begin{aligned}\lim_{t\to 0} B_2(t) &= \xi^\pm \beta \lim_{t\to 0}\frac{B^{-\beta-1}(t)B''(t)}{b(t)f(\xi^\pm B^{-\beta}(t))}\\ &= \xi^\pm \beta \lim_{t\to 0}\frac{f(B^{-\beta}(t))}{f(\xi^\pm B^{-\beta}(t))}\frac{H(B^{-\beta}(t))B^{-p\beta}(t)}{f(B^{-\beta}(t))}\\ &= \frac{\beta}{(\xi^\pm)^{p-1}}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\lim_{t\to 0} B_3(t) &= \xi^\pm \beta \lim_{t\to 0}\frac{B^{-\beta-1}(t)B'(t)\Delta d(x)}{b(t)f(\xi^\pm B^{-\beta}(t))}\\ &= \xi^\pm \beta \lim_{t\to 0}\frac{f(B^{-\beta}(t))}{f(\xi^\pm B^{-\beta}(t))}\frac{H(B^{-\beta}(t))B^{-p\beta}(t)}{f(B^{-\beta}(t))}\frac{B'(t)}{b(t)H(B^{-\beta}(t))}\Delta d(x)\\ &= 0.\end{aligned} \tag{3.6}$$

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Thus, taking into account (3.4)–(3.6), we obtain

$$\lim_{d(x)\to 0}[B_1(d(x)) - B_2(d(x)) - B_3(d(x)) - 1] = \mp\varepsilon,$$

which implies that there exists $\delta > 0$ sufficiently small that, for $x \in \Omega$ with $0 < d(x) < \delta$,

$$\left.\begin{aligned}\Delta u_+(x) - b(x)f(u_+(x)) \leqslant 0,\\ \Delta u_-(x) - b(x)f(u_-(x)) \geqslant 0,\end{aligned}\right\} \tag{3.7}$$

which implies that $u_\pm(x)$ are a local supersolution and subsolution to (1.1).

STEP 3 (supersolution to (1.1)). Define $\Omega_{\alpha,\beta} = \{x \in \Omega\colon \alpha < d(x) < \beta\}$. Fix $\sigma \in (0, \frac{1}{4}\delta)$, and set

$$u_{+,\sigma}(x) = u_+(d - \sigma, s) + M_+,$$

where (d, s) are the local coordinates of $x \in \Omega_{\sigma,\delta/2}$, and $M_+ > 0$ is sufficiently large that, for $\sigma \in (0, \frac{1}{4}\delta)$ and $s \in \partial\Omega$,

$$u_{+,\sigma}(\tfrac{1}{2}\delta, s) = u_+(\tfrac{1}{2}\delta - \sigma, s) + M_+ \geqslant u(\tfrac{1}{2}\delta, s). \tag{3.8}$$

Note that

$$\lim_{d\to\sigma} u_{+,\sigma}(x) = \infty. \tag{3.9}$$

On the other hand, in view of (3.7) and (f_1), we obtain, for $x \in \Omega_{\sigma,\delta/2}$,

$$\begin{aligned}\Delta u_{+,\sigma}(x) &= \Delta u_+(d - \sigma, s)\\ &\leqslant b(x)f(u_+(d - \sigma, s))\\ &\leqslant b(x)f(u_+(d - \sigma, s) + M_+)\\ &= b(x)f(u_{+,\sigma}(x)).\end{aligned}$$

This fact, combined with (3.8), (3.9) and proposition 2.5, shows that, for every $\sigma \in (0, \frac{1}{4}\delta)$,

$$u(x) \leqslant u_{+,\sigma}(x) = u_+(d - \sigma, s) + M_+, \quad x \in \Omega_{\sigma,\delta/2}.$$

Letting $\sigma \to 0$, we conclude that $u(x) \leqslant u_+(x) + M_+$ for all $x \in \Omega$ with $0 < d(x) < \frac{1}{2}\delta$.

STEP 4 (subsolution to (1.1)). Define

$$u_\sigma(x) = \theta u_-(d + \sigma, s), \quad x \in \Omega_{\delta/2} := \{x \in \Omega\colon 0 < d(x) < \tfrac{1}{2}\delta\},$$

where $\theta \in (0, 1)$ is chosen sufficiently small that

$$u_\sigma(\tfrac{1}{4}\delta, s) = \theta u_-(\tfrac{1}{4}\delta + \sigma, s) \leqslant u(\tfrac{1}{4}\delta, s) \quad \text{for all } \sigma \in (0, \tfrac{1}{4}\delta), \text{ for all } s \in \partial\Omega. \tag{3.10}$$

Note that

$$\limsup_{d(x)\to 0}(u_\sigma - u)(x) = -\infty. \tag{3.11}$$

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Using (3.7), we obtain, for $x \in \Omega_{\delta/4}$,

$$\begin{aligned} \Delta u_\sigma &= \theta \Delta u_-(d+\sigma, s) \\ &\geqslant \theta b(x) f(u_-(d+\sigma, s)) \\ &\geqslant b(x) f(\theta u_-(d+\sigma, s)) \\ &= b(x) f(u_\sigma). \end{aligned} \tag{3.12}$$

Thus, taking into account (3.10)–(3.12) and proposition 2.5 again, we obtain

$$u_\sigma(x) = \theta u_-(d+\sigma, s) \leqslant u(x), \quad x \in \Omega_{\delta/4}.$$

Letting $\sigma \to 0$, we find $\theta u_-(x) \leqslant u(x)$, $x \in \Omega_{\delta/4}$.

Set

$$u_{\sigma,\theta}(x) = u_-(d+\sigma, s) - (1-\theta) u_-(\delta^*, s), \quad x \in \Omega_{\delta^*},\ \sigma \in (0, \delta/4 - \delta^*).$$

Then, for $x \in \Omega_{\delta^*}$, we have

$$\Delta u_{\sigma,\theta}(x) = \Delta u_-(d+\sigma, s) \geqslant b(x) f(u_-(d+\sigma, s)) \geqslant b(x) f(u_{\sigma,\theta}).$$

By virtue of

$$\limsup_{d(x)\to 0} (u_{\sigma,\theta} - u)(x) = -\infty$$

and

$$u_{\sigma,\theta}(\delta^*, s) = u_-(\delta^* + \sigma, s) - (1-\theta) u_-(\delta^*, s) \leqslant \theta u_-(\delta^*, s) \leqslant u(x),$$

we derive that

$$u_{\sigma,\theta}(x) = u_-(d+\sigma, s) - (1-\theta) u_-(\delta^*, s) \leqslant u(x), \quad x \in \Omega_{\delta^*}.$$

Letting $\sigma \to 0$, we get $u_-(d, s) - (1-\theta) u_-(\delta^*, s) \leqslant u(x)$, $x \in \Omega_{\delta^*}$.

Step 5 (boundary behaviour). By steps 3 and 4, we arrive at

$$u_-(x) - (1-\theta) u_-(\delta^*, s) \leqslant u(x) \leqslant u_+(x) + M_+, \quad x \in \Omega_{\delta^*},$$

since $u_-(\delta^*, s), M_+$ are bounded and $\lim_{d(x)\to 0} B^{-\beta}(d(x)) = \infty$. Then

$$\xi^- \leqslant \liminf_{d(x)\to 0} \frac{u(x)}{B^{-\beta}(d(x))} \leqslant \limsup_{d(x)\to 0} \frac{u(x)}{B^{-\beta}(d(x))} \leqslant \xi^+.$$

Equation (3.3) follows by $\varepsilon \to 0$. □

Given a point $x_0 \in \mathbb{R}^N$, and positive real numbers $r_2 > r_1 > 0$, we denote

$$\Omega_{r_1,r_2}(x_0) = \{x \in \mathbb{R}^N : r_1 < |x - x_0| < r_2\}.$$

As an immediate consequence of theorem 3.1, we obtain the following.

Corollary 3.2. *Suppose that $\Omega = \Omega_{r_1,r_2}(x_0)$. Then any positive solution u of (1.1) satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)}{K B^{-\beta}(d(x))} = 1,$$

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where K, B *and* β *are defined in theorem 3.1,*

$$d(x) = d(x, \partial\Omega_{r_1,r_2}(x_0)) = \begin{cases} r_2 - |x - x_0| & \text{if } \frac{1}{2}(r_1 + r_2) \leqslant |x - x_0| < r_2, \\ |x - x_0| - r_1 & \text{if } r_1 < |x - x_0| < \frac{1}{2}(r_1 + r_2). \end{cases}$$

4. Proof of main theorem

The main goal of this section is to prove theorem 1.1 by the localization method introduced in [22], where the solutions of (1.1) are estimated by the large solutions in sufficiently small interior balls and sufficiently large exterior annuli.

Proof.

STEP 1. Fix $\varepsilon > 0$, according to (1.3), we obtain that there exist $\rho = \rho(\varepsilon) \in (0, \eta)$ and $\mu = \mu(\varepsilon)$ such that, for all $x \in \partial\Omega \cap \bar{\Omega}_\rho(x_0)$, $r \in (0, \mu)$,

$$1 - \varepsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{b(x - r\boldsymbol{n}_x)}{b(x_0 - r\boldsymbol{n}_{x_0})} < 1 + \varepsilon. \tag{4.1}$$

Set

$$\mathcal{B} = \{x - r\boldsymbol{n}_x : x \in \partial\Omega \cap \bar{\Omega}_\rho(x_0),\ r \in [0, \mu]\}.$$

Since $\partial\Omega$ is smooth, ρ, μ can be shortened, if necessary, so that for each $y \in \mathcal{B}$, there exists a unique $y_0 \in \partial\Omega \cap \bar{\Omega}_\rho(x_0)$, and $r(y) \in [0, \mu]$, such that

$$y = y_0 - r(y)\boldsymbol{n}_{y_0}, \qquad r(y) = |y - y_0| = \operatorname{dist}(y, \partial\Omega).$$

Furthermore, there exists $r_0 \in (0, \min\{\frac{1}{2}\rho, \frac{1}{2}\mu\})$, such that $\Omega_{r_0}(x_0 - r_0\boldsymbol{n}_{x_0}) \subset \Omega$, and $\bar{\Omega}_{r_0}(x_0 - r_0\boldsymbol{n}_{x_0}) \cap \partial\Omega = \{x_0\}$. Thus there exists $\sigma_0 > 0$ such that, for $\sigma \in (0, \sigma_0]$, $\bar{\Omega}_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0}) \subset \Omega \cap \operatorname{Int}\mathcal{B}$.

By (1.2) and (4.1), for $\sigma \in [0, \sigma_0]$ and $y \in \bar{\Omega}_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0})$, we infer that

$$\begin{aligned} b(y) &= b(y_0 - r(y)\boldsymbol{n}_{y_0}) \\ &\geqslant (1 - \varepsilon)b(x_0 - r(y)\boldsymbol{n}_{x_0}) \\ &= (1 - \varepsilon)b_{x_0}(r(y)) \\ &= (1 - \varepsilon)b_{x_0}(\operatorname{dist}(y, \partial\Omega)) \\ &\geqslant (1 - \varepsilon)b_{x_0}(\operatorname{dist}(y, \partial\Omega_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0}))). \end{aligned}$$

Subsequently, $b(y) \geqslant (1 - \varepsilon)b_{x_0}(r_\sigma)$, where $r_\sigma = \operatorname{dist}(y, \partial\Omega_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0}))$.

Consider

$$\left.\begin{aligned} \Delta u &= (1 - \varepsilon)b_{x_0}(r_\sigma)f(u), && x \in \Omega_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0}), \\ u(x) &= +\infty, && x \in \partial\Omega_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0}). \end{aligned}\right\} \tag{4.2}$$

Due to theorem 3.1, problem (4.2) has a unique solution $\mathcal{U}$ for each $\sigma \in [0, \sigma_0]$.

Furthermore,

$$\lim_{x \to \partial\Omega_{r_0}(x_0 - (r_0 + \sigma)\boldsymbol{n}_{x_0})} \frac{\mathcal{U}(x)}{K_1(x_0)\mathcal{B}_1(r_\sigma)} = (1 - \varepsilon)^\beta,$$

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where

$$\mathcal{B}_1(r) = \int_0^r \int_0^s (H \circ \mathcal{B}_1^{-\beta}(t)) b_{x_0}(t)\,\mathrm{d}t\,\mathrm{d}s,$$
$$K_1(x_0) = [\beta(\beta+1)C_{x_0} - \beta]^\beta, \quad \beta = \frac{1}{p-1},$$
$$C_{x_0} = \lim_{t\to 0} \frac{[\mathcal{B}_1'(t)]^2}{\mathcal{B}_1(t) b_{x_0}(t) H \circ \mathcal{B}_1^{-\beta}(t)}.$$

Define

$$\underline{u}_\sigma = u|_{\Omega_{r_0}(x_0-(r_0+\sigma)\boldsymbol{n}_{x_0})}.$$

Then $\underline{u}_\sigma$ is a bounded subsolution of (4.2). Hence, the maximum principle implies that, for each $\sigma \in [0, \sigma_0]$ and $x \in \Omega_{r_0}(x_0 - (r_0+\sigma)\boldsymbol{n}_{x_0})$,

$$\underline{u}_\sigma = u|_{\Omega_{r_0}(x_0-(r_0+\sigma)\boldsymbol{n}_{x_0})} \leqslant \mathcal{U},$$

which implies that

$$\limsup_{x\to\partial\Omega_{r_0}(x_0-(r_0+\sigma)\boldsymbol{n}_{x_0})} \frac{\underline{u}_\sigma}{K_1(x_0)\mathcal{B}_1(r_\sigma)} \leqslant (1-\varepsilon)^\beta.$$

Passing to the limit as $\sigma \to 0$ gives

$$\lim_{r\to 0} \frac{u(x_0 - r\boldsymbol{n}_{x_0})}{K_1(x_0)\mathcal{B}_1(r)} \leqslant (1-\varepsilon)^\beta.$$

This is valid for any sufficiently small $\varepsilon > 0$. Then

$$\lim_{r\to 0} \frac{u(x_0 - r\boldsymbol{n}_{x_0})}{K_1(x_0)\mathcal{B}_1(r)} \leqslant 1. \tag{4.3}$$

STEP 2. For any $x_0 \in \partial\Omega$, there exist $0 < r_1 < r_2$ and σ_0 such that

$$\Omega \subset \bigcap_{0\leqslant\sigma\leqslant\sigma_0} \Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\boldsymbol{n}_{x_0}), \qquad \partial\Omega \cap \bar{\Omega}_{r_1,r_2}(x_0 + r_1\boldsymbol{n}_{x_0}) = \{x_0\},$$

and r_1 is sufficiently small and r_2 is sufficiently large that $\Omega \subset \Omega_{r_1,r_2/3}(x_0+r_1\boldsymbol{n}_{x_0})$.

According to (4.1), we find that, for each $y \in \Omega_{2\eta}(x_0) \cap \bar{\Omega}$, where $\eta \in \min\{\rho, \mu\}$ is small,

$$\begin{aligned} b(y) = b(y_0 - r(y)\boldsymbol{n}_{y_0}) &\leqslant (1+\varepsilon) b_{x_0}(r(y)) \\ &= (1+\varepsilon) b_{x_0}(\operatorname{dist}(y, \partial\Omega)) \\ &\leqslant (1+\varepsilon) b_{x_0}(\operatorname{dist}(y, \partial\Omega_{r_1}(x_0 + r_1\boldsymbol{n}_{x_0}))) \\ &= (1+\varepsilon) b_{x_0}(\operatorname{dist}(y, \partial\Omega_{r_1,r_2}(x_0 + r_1\boldsymbol{n}_{x_0}))). \end{aligned}$$

Define a radially symmetric function $\tilde{b}\colon \Omega_{r_1,r_2}(x_0 + r_1\boldsymbol{n}_{x_0}) \mapsto [0,\infty)$ as

$$\tilde{b}(y) = (1+\varepsilon) b_{x_0}(r),$$

where $r = \operatorname{dist}(y, \partial\Omega_{r_1,r_2}(x_0 + r_1\boldsymbol{n}_{x_0}))$ and $y \in \Omega_{2\eta}(x_0) \cap \bar{\Omega}$. Moreover,

$$\tilde{b}(\operatorname{dist}(y, \partial\Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\boldsymbol{n}_{x_0}))) \geqslant b(y) \quad \text{for all } y \in \bar{\Omega} \text{ and all } \sigma \in [0, \sigma_0].$$

In view of corollary 3.2,

$$\begin{aligned}\Delta u &= \tilde{b}(r)f(u), \quad x \in \Omega_{r_1,r_2}(x_0+(r_1+\sigma)\boldsymbol{n}_{x_0}),\\ u(x) &= +\infty, \qquad x \in \partial\Omega_{r_1,r_2}(x_0+(r_1+\sigma)\boldsymbol{n}_{x_0}),\end{aligned}$$

has a unique large positive solution $\mathfrak{U}$, where $r = \operatorname{dist}(y, \partial\Omega_{r_1,r_2}(x_0+(r_1+\sigma)\boldsymbol{n}_{x_0}))$, and

$$\lim_{x\to\partial\Omega_{r_1,r_2}(x_0+(r_1+\sigma)\boldsymbol{n}_{x_0})} \frac{\mathfrak{U}(x)}{K_2(x_0)\mathcal{B}_2(r)} = (1+\varepsilon)^{\beta},$$

where

$$\mathcal{B}_2(r) = \int_0^r \int_0^s (H\circ\mathcal{B}_2^{-\beta}(t))b_{x_0}(t)\,\mathrm{d}t\,\mathrm{d}s,$$

$$K_2(x_0) = [\beta(\beta+1)C_{x_0} - \beta]^{\beta}, \quad \beta = \frac{1}{p-1},$$

$$C_{x_0} = \lim_{t\to 0} \frac{[\mathcal{B}_2'(t)]^2}{\mathcal{B}_2(t)b_{x_0}(r)H\circ\mathcal{B}_2^{-\beta}(t)}.$$

Moreover, since $\mathfrak{U}|_{\Omega}$ is a subsolution of (1.1), this implies that

$$\mathfrak{U}(x) \leqslant u(x) \quad \text{for all } \sigma\in[0,\sigma_0] \text{ and all } x\in\Omega_{r_1,r_2}(x_0+(r_1+\sigma)\boldsymbol{n}_{x_0})\cap\Omega.$$

This yields

$$\lim_{r\to 0}\frac{u(x_0 - r\boldsymbol{n}_{x_0})}{K_2(x_0)\mathcal{B}_2(r)} \geqslant (1+\varepsilon)^{\beta}.$$

Letting $\sigma \to$, we derive that

$$\liminf_{x\to x_0, x\in\Omega_{r_1,r_2}(x_0+r_1\boldsymbol{n}_{x_0})} \frac{u(x)}{K_2(x_0)\mathcal{B}_2(r)} \geqslant 1. \tag{4.4}$$

It can easily be seen that $\mathcal{B}_1(r) = \mathcal{B}_2(r)$ and $K_1(x_0) = K_2(x_0)$. Using (4.3) and (4.4), we obtain (1.5).

Step 3 (uniqueness of problem (1.1)). The uniqueness follows from theorem 1.1 by a standard argument. For completeness we include the short proof. Suppose that u_1 and u_2 are solutions of (1.1) on Ω. Then, by theorem 1.1, it follows that

$$\lim_{d(x)\to 0}\frac{u_1(x)}{u_2(x)} = 1.$$

Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(1-\varepsilon)u_2(x) \leqslant u_1(x) \leqslant (1+\varepsilon)u_2(x), \quad x\in\Omega_\delta,$$

where $\Omega_\delta = \{x\in\Omega\colon d(x,\partial\Omega) < \delta\}$. Then (f_1) implies that $u^{\pm}(x) = (1\pm\varepsilon)u_2(x)$ satisfy

$$\begin{aligned}\Delta u^{+} &\leqslant b(x)f(u^{+}), \quad x\in\Omega,\\ \Delta u^{-} &\geqslant b(x)f(u^{-}), \quad x\in\Omega.\end{aligned}$$

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Let ω be the unique solution of

$$\begin{aligned} \Delta\omega &= b(x)f(\omega), \quad x \in \Omega_0, \\ \omega &= u_1, \qquad\qquad x \in \partial\Omega_0, \end{aligned}$$

where $\Omega_0 = \{x \in \Omega\colon d(x, \partial\Omega) \geqslant \delta\}$. By the comparison principle, it follows that $u^-(x) \leqslant \omega(x) \leqslant u^+(x)$, $x \in \Omega_0$. The uniqueness of ω implies that $\omega = u_1$, $x \in \Omega_0$. Consequently, $(1-\varepsilon)u_2(x) \leqslant u_1(x) \leqslant (1+\varepsilon)u_2(x)$, $x \in \Omega = \Omega_\delta \cup \Omega_0$. Letting $\varepsilon \to 0$ we obtain that $u_1(x) = u_2(x)$. This completes the proof. □

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