A domain-theoretic Banach–Alaoglu theorem

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Dedicated to Klaus Keimel on the occasion of his 65th birthday

We give a domain-theoretic analogue of the classical Banach–Alaoglu theorem, showing that the patch topology on the weak* topology is compact. Various theorems follow concerning the stable compactness of spaces of valuations on a topological space. We conclude with reformulations of the patch topology in terms of polar sets or Minkowski functionals, showing, in particular, that the 'sandwich set' of linear functionals is compact.

1. Introduction

One of Klaus Keimel's many mathematical interests is the interaction between order theory and functional analysis. In recent years this has led to the beginnings of a 'domain-theoretic functional analysis', which may be considered to be a topic within 'positive analysis' in the sense of Jimmie Lawson (Lawson 2004). In this, 'notions of positivity and order play a key role', as do lower semicontinuity and (thus) T_0 spaces. The present paper contributes a domain-theoretic analogue of the classical Banach–Alaoglu theorem for continuous d-cones, that is, domains endowed with a compatible cone structure (Tix *et al.* 2005).

We begin with some historical remarks to set the present work in context. There have been quite extensive developments within functional analysis concerning positivity and order. The topics investigated include lattice-ordered vector spaces, also called Riesz Spaces (Luxemburg and Zaanen 1971), Banach lattices (Schaefer 1975), and, more generally, ordered vector spaces and positive operators; there have also been developments in which vector spaces were replaced by ordered cones (Fuchssteiner and Lusky 1981). However, in these contexts the topologies considered were always Hausdorff.

In the early 80s Keimel became interested in the work of Boboc, Bucur and Cornea on axiomatic potential theory (Boboc *et al.* 1981). A student of his, Matthias Rauch, considered their work from the viewpoint of domain theory (Rauch 1982), showing, among other things, that a special class of their ordered cones, the standard H-cones, can be viewed as continuous lattice-ordered d-cones, with addition and scalar multiplication being Lawson continuous. Next, starting in the late 80s, Keimel worked on ordered cones with Walter Roth, with a monograph appearing in 1992 (Keimel and Roth 1992). 'Convex'

quasi-uniform structures on cones arose there, replacing the standard uniform structure on locally convex topological vector spaces; these quasi-uniform structures subsume order and topology. This may have been the first time that non-Hausdorff topologies had been considered in functional analysis.

Roth wrote several papers in this area, including his 2000 paper (Roth 2000) on Hahn–Banach-type theorems for locally convex cones. Later, in her 1999 Ph.D. thesis (Tix 1999), Keimel's student Regina Tix gave a domain-theoretic version of these theorems in the framework of d-cones, where the order is now that of a dcpo (directed complete partial order); see also Tix (2001). These Hahn–Banach theorems include sandwich-type theorems, separation theorems and extension theorems. Plotkin subsequently gave another separation theorem, which was incorporated, together with other improvements, into a revised version of Tix's thesis (Tix *et al.* 2005).

The present paper can be seen as providing another contribution of that kind. The classical Banach–Alaoglu Theorem states that in a topological vector space the polar of a neighbourhood of zero is weak*-compact (Rudin 1975). We give an analogue for continuous d-cones. We have a certain advantage in that the range of our functionals, the non-negative reals extended with a point at infinity, is Lawson compact. It turns out that, under an appropriate assumption, an entire topology is compact: the patch topology on the weak*-upper topology of the dual space of the cone. It follows that various kinds of polar sets are weak*-compact.

The work on Hahn–Banach-type theorems has found application in theoretical computer science, *viz.* the study of powerdomains. In her thesis, Tix considered powerdomains for combinations of ordinary and probabilistic nondeterminism; more precisely, she combined each of the three classical powerdomains for nondeterminism (lower, upper and convex) with the powerdomain of all valuations. It was a pleasant surprise that the separation theorems found application in this development, and we anticipate that so too will the domain-theoretic Banach–Alaoglu theorem given here.

We take Gierz *et al.* (2003) as a standard reference on domain theory and related topology; we refer the reader particularly to the material on stably compact spaces, and also to Jung (2004) and Alvarez-Manilla *et al.* (2004) for more recent material on that topic, where it is argued that stably compact spaces are the correct T_0 analogue of compact Hausdorff spaces. The required background on d-cones can be found in Tix *et al.* (2005, Chapter 2). We cover it much more briefly here in Section 2, which concerns technical preliminaries. We derive our domain-theoretic Banach–Alaoglu theorem in Section 3, and then discuss some reformulations of the weak*-upper topology and its dual in terms of polar sets and functional bounds in Section 4.

2. Technical preliminaries

We are concerned with semimodules for two (unitary) semirings: $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$, where by a semimodule we mean a module for a semiring, see Golan (1999). The first semiring is that of the non-negative reals with the usual addition and multiplication; the second extends the first with an infinite element and the extensions of the semiring operations with $\infty + x = \infty$, $\infty \cdot 0 = 0$ and $\infty \cdot x = \infty$, if $x \neq 0$. Then a *d-cone* is an $\overline{\mathbb{R}}_+$-semimodule

in the category of dcpos, where $\overline{\mathbb{R}}_+$ is endowed with the usual ordering with least element 0, making it a continuous lattice; an *ordered cone* is an $\mathbb{R}_+$-semimodule in the category of posets, endowing $\mathbb{R}_+$ with the usual ordering; and a *topological cone* is an $\mathbb{R}_+$-semimodule in the category of topological spaces, endowing $\mathbb{R}_+$ with the upper topology. Our definition of a d-cone differs inessentially from that in Tix *et al.* (2005), where the infinite element is avoided.

In any cone the action of the semiring induces an action of the multiplicative group $(0, \infty)$, and therefore all such actions are d-cone automorphisms, and so, in particular, automorphisms of the way below relation on d-cones. Further, 0 is always the least element, taking the specialisation ordering in the topological case.

A function between semimodules is *homogeneous* if it preserves the semiring action, *additive* if it preserves semimodule addition, and *linear* if it preserves both. If the semimodule is preordered, such a function f is *subadditive* if $f(x + y) \leqslant f(x) + f(y)$ always holds, *superadditive* if $f(x + y) \geqslant f(x) + f(y)$ always holds, *sublinear* if it is homogeneous and subadditive, and *superlinear* if it is homogeneous and superadditive. We may sometimes specify the semiring under consideration if it is not clear which we mean, writing, for example, '$\mathbb{R}_+$-homogeneous.'

A *functional* on a set X is simply a function on X with range $\overline{\mathbb{R}}_+$. Given a collection $\mathscr{F}$ of such functionals on a set X and a topology on $\overline{\mathbb{R}}_+$, the corresponding *weak** topology on $\mathscr{F}$ is the weakest topology making all point evaluation functions $\mathrm{ev}_x : f \mapsto f(x)$ continuous: so we speak of the *weak*-upper*, or *weak*-Scott*, the *weak*-lower* and the *weak*-Lawson* topologies on $\mathscr{F}$. The weak*-upper topology has as a subbasis the sets

$$W_{x,r} =_{\mathrm{def}} \{f \in \mathscr{F} \mid f(x) > r\}$$

where $r \in (0, \infty)$; the weak*-lower topology has as a subbasis the sets

$$L_{x,r} =_{\mathrm{def}} \{f \in \mathscr{F} \mid f(x) < r\}$$

where $r \in (0, \infty)$; and the weak*-Lawson topology is the join of the other two weak* topologies. The weak*-lower topology is always a separating dual topology for the weak*-upper topology (see Tix *et al.* (2005, Definition VI-6.17)), and its specialisation ordering is the pointwise one.

We will be particularly interested in C^*, the collection of all linear functionals on a cone C, taking these to be continuous or monotone as appropriate to the kind of cone considered. One can endow C^* with a cone structure, when it is called the *dual* cone: the operations are defined pointwise, and then, taking the pointwise order, we have notions of dual cone for the dcpo and poset case, and this, taking the weak*-upper topology, gives one for topological cones.

Two examples of d-cones are: $\mathscr{L}(X)$, the collection of continuous functionals on a topological space X, taking the upper topology on $\overline{\mathbb{R}}_+$, with the pointwise ordering; and $\mathscr{V}(X)$ the collection of continuous valuations on X, again with the pointwise ordering. Their properties are treated in detail in Tix *et al.* (2005, Chapter 2); we note here a 'Riesz Representation Theorem', which states that $\Lambda : \mathscr{V}(X) \cong \mathscr{L}(X)^*$ is a d-cone isomorphism, where $\Lambda_v = f \mapsto \int f \, dv$.

Both d-cones and topological cones whose underlying topology is T_0 yield ordered cones, taking the the underlying order and the specialisation order, respectively. A continuous d-cone, that is, one whose underlying dcpo is continuous, yields a topological cone, taking the Scott topology: the point is that addition is then continuous in the product topology.

3. The Banach–Alaoglu theorem

We begin with a Banach–Alaoglu theorem for ordered cones. The proof follows the general lines of the usual proof of the standard Banach–Alaoglu Theorem, for example, see Rudin (1975), embedding the dual space in a compact one of functionals and then showing that the set we wish to prove to be compact is closed in the induced topology.

Lemma 1. Let τ and τ_d be separating dual topologies and subtopologies of a compact Hausdorff topology $\bar{\tau}$. Then τ is stably compact, τ_d is its co-compact topology and $\bar{\tau}$ is its patch topology.

Proof. The join of the two separating topologies is Hausdorff and thus equal to $\bar{\tau}$; we can then apply Gierz *et al.* (2003, Theorem VI-6.18) to obtain the desired conclusions. $\square$

Theorem 1. Let C be an ordered cone. Then the weak*-upper topology on C^* is stably compact, and its dual is the weak*-lower topology.

Proof. By Lemma 1, it is enough to prove that the weak*-Lawson topology on C is compact. The weak*-Lawson topology on the collection $\overline{\mathbb{R}}_+^{|C|}$ of all functionals is the $|C|$-fold power of the Lawson topology on $\overline{\mathbb{R}}_+$, and thus compact by the Tychonoff theorem. The weak*-Lawson topology on C^* is evidently the subspace topology induced by the weak*-Lawson topology on $\overline{\mathbb{R}}_+^{|C|}$, and thus compact if we can show that it is a closed subset of $\overline{\mathbb{R}}_+^{|C|}$ in that topology.

To that end, we show, successively, that the subsets of monotone, homogeneous and additive functionals are closed. The subset of monotone functionals can be written in the form

$$\bigcap_{x \leqslant y} \langle \mathrm{ev}_x, \mathrm{ev}_y \rangle^{-1}(\leqslant),$$

and is therefore closed, since the order relation on $\overline{\mathbb{R}}_+$ is closed in the product Lawson topology on $\overline{\mathbb{R}}_+^2$ and the point evaluation functionals are continuous with respect to the weak*-Lawson topology.

The subset of homogeneous functionals can be written in the form

$$\bigcap_{\lambda,x} \langle \mathrm{ev}_{\lambda \cdot x}, (\lambda \cdot -) \circ \mathrm{ev}_x \rangle^{-1}(=),$$

and is therefore closed since the equality relation is closed and multiplication is continuous in the Lawson topology.

Finally, the subset of additive functionals can be written in the form

$$\bigcap_{x,y} \langle \mathrm{ev}_x, \mathrm{ev}_y, \mathrm{ev}_{x+y} \rangle^{-1}(+),$$

and is therefore closed, since addition is continuous in the Lawson topology and so its graph is a closed subset of $\overline{\mathbb{R}}_+^3$. □

This theorem does not extend to d-cones. Consider the d-cone $C = \mathscr{V}(\Omega)$ where Ω consists of the natural numbers, with the usual ordering, extended with a point at infinity: then the weak*-Lawson topology on C^* is not compact (so we also have C^* is not a closed subset of $\overline{\mathbb{R}}_+^{|C|}$). For the set $\{F \in C^* \mid F(\eta_\infty) \geqslant 1\}$ is weak*-Lawson closed and covered by the increasing sequence $W_{\eta_n,0}$ of weak*-upper open sets, but by no member of it, as $\int f_n \, d-$ is in $\{F \in C^* \mid F(\eta_\infty) \geqslant 1\}$, but not in $W_{\eta_n,0}$, where $f_n(m) = 0$ if $m \leqslant n$ and $= 1$, otherwise (η_x is the point valuation at x).

To proceed further, we consider the relation between the continuous functionals on a dcpo P and the monotone functionals on it, which we write as $\mathscr{M}(P)$. There is an evident inclusion:

$$\phi : \mathscr{L}(P) \hookrightarrow \mathscr{M}(P).$$

As both $\mathscr{L}(P)$ and $\mathscr{M}(P)$ are complete lattices and the inclusion preserves all sups, ϕ has a right adjoint $\psi : \mathscr{M}(P) \to \mathscr{L}(P)$, which assigns to any monotone functional its *(Scott continuous) lower envelope*, being the greatest continuous function below it; note here that ψ is a retraction with ϕ the corresponding section, so $\langle \phi, \psi \rangle$ is an embedding–projection pair. If P is continuous, the lower envelope is given by a standard formula

$$\psi(f)(x) = \bigvee_{a \ll x} f(a).$$

The idea of using arguments involving both closed subsets and lower envelopes to prove stable compactness appears first in a paper of Jung (Jung 2004): the application there was to show the stable compactness of spaces of valuations. We show below that, as may be expected, results of that kind follow from the domain-theoretic Banach–Alaoglu theorem.

Proposition 1. *Let C be a continuous d-cone. Then, for any $f \in \mathscr{M}(P)$, $\psi(f)$ is $\mathbb{R}_+$-homogeneous if f is, subadditive if f is, and, assuming $\ll$ additive on C, superadditive if f is.*

Proof. For the preservation of $\mathbb{R}_+$-homogeneity, $\psi(f)$ is clearly strict if f is, and taking $r \in (0, \infty)$, we have

$$
\begin{aligned}
\psi(f)(r \cdot x) &= \textstyle\bigvee_{a \ll r \cdot x} f(a) \\
&= \textstyle\bigvee_{b \ll x} f(r \cdot b) \quad (r \cdot - \text{ acts automorphically}) \\
&= r \cdot (\psi(f)(x)) \quad \text{(by the homogeneity of } f \\
&\qquad\qquad\qquad\quad \text{and the continuity of the action).}
\end{aligned}
$$

For the preservation of subadditivity, we calculate

$$
\begin{aligned}
\psi(f)(x + y) &= \textstyle\bigvee_{c \ll x+y} f(c) \\
&\leqslant \textstyle\bigvee_{a \ll x, b \ll y} f(a + b) \quad \text{(by the continuity of } + \\
&\qquad\qquad\qquad\qquad\quad \text{and the monotonicity of } f) \\
&\leqslant \textstyle\bigvee_{a \ll x, b \ll y} f(a) + f(b) \quad \text{(by the subadditivity of } f) \\
&= \psi(f)(x) + \psi(f)(y).
\end{aligned}
$$

And for the preservation of superadditivity, we calculate

$$
\begin{aligned}
\psi(f)(x+y) &= \bigvee_{c\ll x+y} f(x+y) \\
&\geqslant \bigvee_{a\ll x, b\ll y} f(a+b) \quad \text{(by the additivity of } \ll) \\
&\geqslant \bigvee_{a\ll x, b\ll y} f(a)+f(b) \quad \text{(by the superadditivity of } f) \\
&= \psi(f)(x)+\psi(f)(y).
\end{aligned}
$$

$\square$

Note that the preservation of homogeneity and subadditivity has already been shown by Tix: see, for example, Tix (2001).

We also need a different topology from the weak*-Lawson topology. Given a collection $\mathscr{F}$ of continuous functionals on a topological space X, define the *open-lower* topology on $\mathscr{F}$ to have as subbasis all sets of the form

$$
L_{U,r} = \{f \in \mathscr{F} \mid \exists x \in U. f(x) < r\}
$$

for U open and $r \in (0, \infty)$.

Lemma 2. Let $\mathscr{F}$ be a collection of continuous functionals on a domain P. Then the open-lower topology is a separating dual topology for the weak*-upper topology.

Proof. First suppose that $f \leqslant g$ in the weak*-Scott specialisation ordering, which is the same as the pointwise one. Then, if $g \in L_{U,r}$, we also clearly have that $f \in L_{U,r}$. Conversely, suppose we have $f \not\leqslant g$ in the pointwise ordering. Then there is an x and $r \in (0, \infty)$ such that $g(x) < r < f(x)$. So, as f is continuous, there is an $a \ll x$ such that $f(a) > r$, and it follows that $f \in W_{a,r}$ and $g \in L_{a\!\uparrow,r}$; note that the sets $W_{a,r}$ and $L_{a\!\uparrow,r}$ are disjoint. It follows that $g \not\leqslant f$ in the open-lower specialisation ordering, as otherwise we would have that $f \in L_{a\!\uparrow,r} \cap W_{a,r}$. So the topologies are dual and separating, as required. $\square$

We now have everything needed for the domain-theoretic analogue of the Banach–Alaoglu theorem.

Theorem 2. Let C be a continuous d-cone with an additive way-below relation. Then the weak*-upper topology on C^* is stably compact, and its co-compact topology is the open-lower topology.

Proof. We know from Proposition 1 that ψ cuts down to a function from $(C_m)^*$ to C^*, and ϕ evidently cuts down to a function in the opposite direction. Both these functions are continuous with respect to the weak*-upper topology. This is obvious for ϕ, and for ψ we calculate

$$
\begin{aligned}
\psi^{-1}(\{f \in C^* \mid f(x) > r\}) &= \{f \in (C_m)^* \mid \bigvee_{b\ll x} f(b) > r\} \\
&= \bigcup_{b\ll x} \{f \in (C_m)^* \mid f(b) > r\}.
\end{aligned}
$$

So, as C^* is a weak*-upper retract of $(C_m)^*$ and as, by Theorem 1, that topology is stably compact, the weak*-upper topology on C^* is also stably compact since retracts of stably compact spaces are stably compact (Lawson 1988; Jung 2004).

If the $L_{U,r}$ are co-compact, it follows, using Lemma 1 (and Lemma 2) that the open-lower topology is the co-compact topology for the stably compact weak*-upper topology. So we show that all sets of the form $\{f \in C^* \mid \forall x \in U. f(x) \geqslant r\}$ with $r \in (0, \infty)$ are

weak*-upper compact, and that follows from the equation

$$\psi(\{g \in (C_m)^* \mid \forall x \in U.\, g(x) \geqslant r\}) = \{f \in C^* \mid \forall x \in U.\, f(x) \geqslant r\}$$

as ψ is weak*-upper continuous and $\{g \in (C_m)^* \mid \forall x \in U.\, g(x) \geqslant r\}$ is weak*-upper compact by Theorem 1, being a weak*-lower closed subset of $(C_m)^*$. To see that this equation holds, first note that, as ψ acts as the identity on continuous functionals, the right-hand side is included in the left-hand side. Conversely, suppose that g is in the left-hand side, and $x \in U$. Take any $s < r$. Then, as U is open, $x \gg$ some $a \in U$, we see that $g(a) \geqslant r > s$, and thus that $\psi(g)(x) > s$. It follows that $\psi(g)(x) \geqslant r$, as required. $\qquad\square$

Note that C^* is always compact in the weak*-upper topology, for any d-cone C; this is just because 0 is its least element. So the force of the conclusion is more the local stable compactness of C^*. The d-cone $\mathscr{V}(\Omega)$ used in the counterexample above satisfies the conditions of the theorem (see Tix *et al.* (2005, Chapter 2.2)) and thus also provides an example in which the open-lower and the weak*-lower topologies disagree: the set $\{F \in \mathscr{V}(\Omega)^* \mid F(\eta_\infty) \geqslant 1\}$ is closed in the latter, but not the former.

Comparing our domain-theoretic Banach–Alaoglu theorem with the standard one, one may notice the assumption that $\ll$ is additive, and also the difference in the proofs, where we consider projections as well as subsets. It is shown in Tix *et al.* (2005) that the condition on the way-below relation is equivalent to the requirement that addition is *quasi-open*, meaning that $(U + V)\!\uparrow$ is open whenever U and V are. In the case of topological vector spaces, not only is addition an open map, but a stronger condition holds, that each map $x + -$ is open. This entails that any linear functional (to $\mathbb{R}$ or $\mathbb{C}$) bounded on a neighbourhood of 0 is continuous, and so difficulties with continuity do not arise in that setting.

Some theorems of Jung and Tix (1998), Jung (2004), Tix *et al.* (2005) and Alvarez-Manilla *et al.* (2004) concerning the stable compactness of spaces of valuations on a topological space X follow from Theorem 2. We sometimes slightly weaken the hypothesis on X from stable compactness to local stable compactness or strengthen the conclusion by identifying the co-compact topology. We write $\mathscr{V}_{\leqslant 1}(X)$ for the collection of subprobability valuations, and $\mathscr{V}_1(X)$ for the collection of probability valuations.

Corollary 1. Let X be a stably locally compact topological space. Then $\mathscr{V}(X)$ is stably compact in the weak*-upper topology with co-compact topology the open-lower topology. The same is true of $\mathscr{V}_{\leqslant 1}(X)$, and also of $\mathscr{V}_1(X)$ if X is stably compact.

Proof. Since X is stably locally compact, we have, by Tix *et al.* (2005, Propositions 2.25 and 2.28), that $\mathscr{L}(X)$ is a continuous d-cone with additive $\ll$, so, by Theorem 2, $\mathscr{L}(X)^*$ is weak*-upper stably compact, with co-compact topology the open-lower topology.

The isomorphism $\Lambda : \mathscr{V}(X) \cong \mathscr{L}(X)^*$ induces a corresponding pair of topologies on $\mathscr{V}(X)$. We will show that these include the weak*-upper and co-compact topologies, respectively. Then, since, by Lemma 2, these are a separating dual pair of topologies, it follows by Lemma 1 that the weak*-upper topology is indeed stably compact with co-compact topology the open-lower one.

For the inclusion of the weak*-upper topology we need only observe that

$$\Lambda(\{v \in \mathcal{V}(X) \mid v(U) > r\}) = \{F \in \mathcal{L}(X)^* \mid F(\chi_U) > r\}$$

for any open set U and $r \in (0, \infty)$. For the inclusion of the open-lower topology it suffices to prove that

$$\Lambda(\{v \mid \forall U \in O. v(U) \geqslant r\}) = \bigcap_{U \in O, s \in (0,1)} \{F \mid \forall f \in (s\chi_U)\dagger. F(f) \geqslant sr\})$$

for O an open set of $\mathcal{O}(X)$ and $r \in (0, \infty)$, since the set on the right is closed in the open-lower topology on $\mathcal{L}(X)^*$. The inclusion from left to right is clear. In the other direction, take a Λ_v in the set on the right, and a $U \in O$ to prove $v(U) \geqslant r$. Then there is a $U' \ll U$ with $U' \in O$, since O is open. Choose $s \in (0,1)$. Then, by Tix *et al.* (2005, Lemma 2.26), $\chi_U \gg s\chi_{U'}$, and thus $v(U) \geqslant sr$. But s is an arbitrary element of $(0,1)$, so we see that $v(U) \geqslant r$, as required.

The set of subprobability valuations $\{v \in \mathcal{V}(X) \mid v(X) \geqslant 1\}$ is weak*-upper closed in $\mathcal{V}(X)$ and, when X is compact, the set $\{v \in \mathcal{V}(X) \mid v(X) \geqslant 1\}$ is closed in the open-lower topology on $\mathcal{V}(X)$, so the set of probability valuations is closed in the patch topology. The rest of the theorem follows from these two observations, using Jung (2004, Proposition 2.16). $\square$

When X is locally compact, the open-lower topology on $\mathcal{V}(X)$ has a subbasis of closed sets of the form

$$\{v \in \mathcal{V}(X) \mid \forall U \supset K. v(U) \geqslant r\}$$

for K compact and $r \in (0, \infty)$. This form of the co-compact topology was noted, without proof, for $\mathcal{V}_{\leqslant 1}(X)$ and $\mathcal{V}_1(X)$ in Jung (2004); it is then evident that one can restrict to $r \in (0,1)$. The restriction to stably compact X in the last part of the corollary is necessary as a topological space Y is compact if $\mathcal{V}_1(Y)$ is compact in the weak*-upper topology. (To see this, suppose $\mathcal{V}_1(Y)$ is compact and let U_i be a directed covering of Y by open sets. Then $W_i =_{\text{def}} \{v \mid v(U_i) > 0\}$ is a directed covering of $\mathcal{V}_1(Y)$ by weak*-upper open sets, so some W_i includes it. But then U_i includes Y, as $x \in U_i$ holds iff $\eta_x \in W_i$ does.)

One can specialise these results to domains following, for example, Tix *et al.* (2005). A domain, *qua* topological space, is stably locally compact iff it is coherent, and stably compact iff its Lawson topology is compact. If P is a domain, then both $\mathcal{V}(P)$ and $\mathcal{V}_{\leqslant 1}(P)$ are also – but, in general, $\mathcal{V}_1(P)$ is not. Finally, on both $\mathcal{V}(P)$ and $\mathcal{V}_{\leqslant 1}(P)$ the weak*-upper topology coincides with the Scott topology (Kirch 1993; Tix 1995). So we see from the corollary that if P is a coherent domain, then both $\mathcal{V}(P)$ and $\mathcal{V}_{\leqslant 1}(P)$ are Lawson compact.

Lawson has proved a certain converse: for a domain P, if $\mathcal{V}(P)$ is Lawson compact, then P is coherent, see Tix *et al.* (2005, Theorem 2.10 (d)). The necessity of the additivity condition of Theorem 2 follows. Take any domain P and assume that the weak*-upper topology on $\mathcal{L}(P)^*$ is stably compact. Then, following the proof of the corollary, the weak*-upper topology on $\mathcal{V}(P)$ is also stably compact, and thus, by Lawson's result, the Scott topology on P is stably locally compact, and it follows, by Tix *et al.* (2005, Proposition 2.28), that $\mathcal{L}(P)$ has an additive way below relation. So if we take any

non-coherent domain P, we see that $\mathcal{L}(P)$ is continuous, but that $\ll$ is not additive, by Tix *et al.* (2005, Proposition 2.29), and then that $\mathcal{L}(P)^*$ is not stably compact.

Finally, we mention a natural question: having Banach–Alaoglu theorems for ordered cones and d-cones, is there also one for topological cones? In this respect note that the conclusion of Theorem 2 relates to the dual of C considered as a topological cone.

4. Polar sets and Minkowski functionals

The weak*-upper topology and its dual can be defined in two other ways: using polar sets and using Minkowski functionals, more precisely, their domain-theoretic analogues.

Definition 1. Let X be a subset of a d-cone C. Then we define its *lower polar* to be

$$X_\circ = \{f \in C^* \mid \forall x \in X.\, f(x) \leqslant 1\},$$

and its *upper polar* to be

$$X^\circ = \{f \in C^* \mid \forall x \in X.\, f(x) \geqslant 1\}.$$

Proposition 2. Let C be a d-cone.

1. The weak*-upper topology has as a subbasis of closed sets all lower polars, and, also, all lower polars of non-empty Scott-closed convex sets.
2. The open-lower topology has as subbasis of closed sets all upper polars of open sets (not containing 0), and, also, if C is continuous, all upper polars of convex open sets (not containing 0).

   Proof.

1. Regarding the first assertion, every lower polar set is evidently closed in the weak*-upper topology on C^*, and, conversely, $W_{a,r}$ is the complement of $\{r^{-1} \cdot a\}_\circ$. For the second, we can clearly disregard lower polars of empty sets, and the lower polar of a set is easily seen to be the same as the least Scott-closed convex set containing it.
2. Regarding the first assertion, every upper polar set is clearly closed in the topology generated by the $L_{U,r}$, and, conversely, the complement of $L_{U,r}$ is $(r^{-1} \cdot U)^\circ$. We can clearly disregard upper polars of sets containing 0. The second assertion follows from the fact that when C is continuous it is locally convex in the sense that every neighbourhood contains a convex open one, see Tix *et al.* (2005, Proposition 2.5). □

Our main alternative description of the polar topology is in terms of functional bounds; the connection between the two is given using *Minkowski functionals*. For any subset X of a d-cone C, define its *upper* and *lower* Minkowski functionals by

$$\mu_X(x) = \bigvee \{r \in (0, \infty) \mid x \in r \cdot X\}$$

and

$$v_X(x) = \bigwedge \{r \in (0, \infty) \mid x \in r \cdot X\},$$

yielding two monotone functions $\mu : \mathscr{P}(C) \to \overline{\mathbb{R}}_+$ and $v : \mathscr{P}(C) \to (\overline{\mathbb{R}}_+)^{op}$. Equivalently, we could let r range over $\overline{\mathbb{R}}_+$, but we find the above form of the definition more convenient.

Our Minkowski functionals are defined by an obvious analogy with the standard ones; they, and some of their properties, are also implicit in the proof of Tix's Separation Theorem, see, for example, Tix *et al.* (2005, Theorem 3.4).

In the other direction, given any functional g on C we define

$$S(g) = \{x \in C \mid g(x) > 1\}$$

and

$$L(g) = \{x \in C \mid g(x) \leqslant 1\}.$$

Both μ and v are complete lattice homomorphisms, and thus left adjoints. The next lemma provides relevant information on these two adjunctions; we do not distinguish notationally between functions and their various restrictions and corestrictions.

Lemma 3. Let C be a d-cone. Then:

1 μ cuts down to an isomorphism of the complete lattice of the Scott open subsets of C not containing 0 and that of the homogeneous continuous functionals on C, with the pointwise ordering; it has inverse S. This cuts down, in its turn, to an isomorphism of the complete lattice of convex open subsets of C not containing 0 and that of the superlinear continuous functionals on C. Further, for any homogeneous continuous functional g and open set U not containing 0, we have that $g \geqslant \mu_U$ iff $g \in U^\circ$.

2 v cuts down to an adjunction between the complete lattice of the subsets of C containing 0 and that of the (opposite of) the $\mathbb{R}_+$-homogeneous functionals on C; it has right adjoint L. They, in turn, cut down to an adjunction between the complete lattice of the non-empty convex down-closed subsets of C and that of the (opposite of) the $\mathbb{R}_+$-sublinear monotone functionals on C. Further, for any $\mathbb{R}_+$-homogeneous functional g and non-empty set X, we have that $g \leqslant v_X$ iff $g \in X_\circ$.

Proof.

1 That μ sends (convex) open sets not containing 0 to (superadditive) homogeneous continuous functionals is a straightforward verification; the corresponding properties of S are immediate. Next, μ is monotone and S clearly is too, and we prove that they are inverses. To see that $\mu_{S(g)} = g$ for any homogeneous continuous function g, note that $x \in r \cdot S(g)$ iff $g(x) > r$, for any $x \in C$ and $r \in (0,1)$. To see that $S(\mu_U) = U$, for any open set U not containing 0, note that, for any $x \in C$,

$$
\begin{array}{lll}
x \in S(\mu_U) & \text{iff} & \mu_U(x) > 1 \\
& \text{iff} & \exists r \in (1, \infty).\, x \in r \cdot U \\
& \text{iff} & x \in U \qquad \text{(as } U \text{ is open)}.
\end{array}
$$

All these equivalences are obvious except the last. The 'only if' holds as U is open and therefore upper closed; the 'if' holds as for any $x \in C$ we have that $x = \bigvee r_n \cdot x$ where r_n is any increasing sequence of positive reals tending to 1. So, if $x \in U$, we have $r_n \cdot x \in U$ for some r_n, and thus $x \in r_n^{-1} \cdot U$.

By the isomorphism, for any positively homogeneous continuous functional g and open set U, we have $g \geqslant \mu_U$ iff $S(g) \supset U$, and we now show that the latter is equivalent to $g \in U^\circ$. Only the implication from right to left is in question, so suppose that $g(z) \geqslant 1$

for all $z \in U$, and choose $x \in U$. As U is open, we have $r_n \cdot x \in U$ for some r_n (the r_n are as before), so $g(r_n \cdot x) \geq 1$, which implies that $g(x) > 1$, as required.

2 That v sends (convex down-closed) subsets of C not containing 0 to $\mathbb{R}_+$-homogeneous (subadditive monotone) functionals is a straightforward verification; the corresponding properties of L are immediate. To see that the right adjoint is S, we calculate

$$
\begin{aligned}
g \leq v_X \quad &\text{iff} \quad \forall x.\, g(x) \leq \bigwedge\{r \in (0, \infty) \mid x \in r \cdot X\} \\
&\text{iff} \quad \forall x.\, \forall r \in (0, \infty).\, x \in r \cdot X \supset g(x) \leq r \\
&\text{iff} \quad \forall x.\, x \in X \supset g(x) \leq 1 \qquad \text{(as g is $\mathbb{R}_+$-homogeneous)} \\
&\text{iff} \quad X \subset \mathrm{L}(g).
\end{aligned}
$$

The final assertion follows from the adjunction and the fact that $X \subset \mathrm{L}(g)$ iff $g \in X_\circ$. $\qquad\square$

We remark that in part 2, L is actually a closure operation: one easily verifies that $v_{\mathrm{L}(g)} = g$ for any $\mathbb{R}_+$-homogeneous functional g.

We next consider a 'homogenising' operation. For any functional f on a set X define

$$
H_u(f) = \bigvee_{r \in (0, \infty)} r^{-1} \cdot f(r \cdot x).
$$

Lemma 4. Let f be a strict continuous functional on a d-cone C. Then $H_u(f)$ is the least homogeneous continuous functional above it.

Proof. It is evident that $H_u(f)$ is continuous and below any homogeneous functional above f. To see that it is homogeneous, note that it is strict and that for any $s \in (0, \infty)$ and $x \in C$

$$
\begin{aligned}
H_u(f)(s \cdot x) &= \bigvee_{r \in (0, \infty)} r^{-1} \cdot f(r \cdot (s \cdot x)) \\
&= s \cdot \bigvee_{r \in (0, \infty)} (rs)^{-1} \cdot f(rs \cdot x) \\
&= s \cdot \bigvee_{t \in (0, \infty)} t^{-1} \cdot f(t \cdot x) \\
&= s \cdot H_u(f)(x) \qquad\qquad\qquad\qquad\qquad\quad \square
\end{aligned}
$$

It is interesting to note that the Minkowski functionals can be understood as homogenised characteristic functions, since $\mu(X) = H_u(\chi_X)$ and $v(X) = H_l(\chi_X)$, where H_l is defined analogously to H_u, but taking infs instead of sups.

We can now reformulate the weak*-upper and the open-lower topologies in terms of functional bounds.

Proposition 3. Let C be a d-cone.

1 The weak*-upper topology has subbases of closed sets of each the following forms: all sets of the form $\{g \in C^* \mid g \leq h\}$ with h any functional; all sets of that form with h an $\mathbb{R}_+$-sublinear monotone functional; and, if C is continuous, all sets of that form with h a sublinear continuous functional.

2 The open-lower topology has subbases of closed sets of each of the following forms: all sets of the form $\{g \in C^* \mid f \leq g\}$ with f a strict continuous functional; all sets of that form with f a homogeneous continuous functional; and, if C is continuous, all sets of that form with f a superlinear continuous functional.

Proof.

1 It is clear that the sets of the form $\{g \in C^* \mid g \leqslant h\}$ form a subbasis of closed sets for the weak*-upper topology. By Proposition 2.1, all lower polars of non-empty down-closed convex subsets X also form a subbasis, and by Lemma 3.2, these can be written in the form $\{g \in C^* \mid g \leqslant v_X\}$. So, as v_X is $\mathbb{R}_+$-sublinear and monotone, we can restrict the subbasis to the required form. Finally, if C is continuous, such lower polars can also be written as $\{g \in C^* \mid g \leqslant \psi(v_X)\}$ and, by Proposition 1, $\psi(v_X)$ is sublinear and continuous.

2 We know from Proposition 2.2 that the open-lower topology has a subbasis consisting of sets of the form U°, with U an open subset of C not containing 0. Lemma 3.1 tells us that $U^\circ = \{g \in C^* \mid \mu_U \leqslant g\}$ and that μ_U is homogeneous and continuous; further applying Lemma 3.1, we obtain, for any homogeneous continuous functional f, that $\{g \in C^* \mid f \leqslant g\} = \{g \in C^* \mid \mu_{S(f)} \leqslant g\} = S(f)^\circ$. We therefore conclude that the open-lower topology has a subbasis consisting of all sets of the form $\{g \in C^* \mid f \leqslant g\}$ with f a homogeneous continuous functional. Similar reasoning shows that we can restrict to superlinear continuous functionals f if C is continuous.

Finally, for any strict continuous functional f, by Lemma 4, we have that $H_u(f)$ is homogeneous and continuous, and also that $f \leqslant g$ iff $H_u(f) \leqslant g$, for any $g \in C^*$. So the sets of the form $\{g \in C^* \mid f \leqslant g\}$ with f homogeneous and continuous also form a subbasis. $\square$

This proposition clearly allows quite a number of equivalent formulations of the weak*-upper topology and its dual. We note the following immediate consequence of Theorem 2 and the proposition.

Corollary 2. Let C be a continuous d-cone with an additive way-below relation, and suppose that f is a continuous superlinear functional on C and h is an $\mathbb{R}_+$-sublinear functional on C. Then the 'sandwich set' of functionals

$$\{g \in C^* \mid f \leqslant g \leqslant h\}$$

is compact in the patch topology (on the weak*-upper topology).

This complements the Sandwich Theorem, see, for example, Tix *et al.* (2005, Theorem 3.2), which says – though without the assumption that way-below is additive – that the sandwich set is non-empty.

Acknowledgements

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