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DETERMINING THE MODE FOR CONVOLUTION POWERS OF DISCRETE UNIFORM DISTRIBUTION

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We specify the smallest mode of the ordinary multinomials leading to the expression of the maximal probability of convolution powers of the discrete uniform distribution. The generating function for an extension of the maximal probability is given.

1. INTRODUCTION

Since the 18th century, the expression of the convolution power of the discrete uniform distribution has been very well known (e.g., de Moivre in 1711; see [6,7]). This probability distribution arises in many practical situations, including, in particular, games with equal chance, random affectation of tasks for many servers, and random walks. It is well known (see Dharmadhikari and Joak-Dev [5, pp. 108–109]) that the convolution of two discrete unimodal distributions might be nonunimodal. However, if these distributions are symmetric, we obtain a symmetric unimodal distribution. It is a discrete analog of Wintner's Theorem [12]. Knowing that the convolution power of the discrete uniform distribution is symmetric unimodal, the determination of the maximal probability (mode) of such a distribution and its argument remains a question for consideration. As a recent work on the problem, one can see the article by Mattner and Roos [8], in which they establish the upper bound for the maximal probability $c_{q,L} < \sqrt{6/\pi(q^2-1)L}$ ($c_{q,L}$ being the maximal probability of the Lth convolution power of the discrete uniform distribution on $\{0, 1, \ldots, q\}$). There were several works aimed at finding such a bound; we refer to Mattner and Roos [8] and Siegmund-Schultze and von Weizsäcker [10] for a historical survey.

Alternatively, our aim is to give an explicit expression of the mode of the Lth convolution powers of the discrete uniform distribution (Section 3). To do so, we

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use a simple combinatorial approach by means of the unimodality of the ordinary multinomials (Section 2). We end the article (Section 4) by giving the generating functions for two sequences of generalized ordinary multinomials.

2. ORDINARY MULTINOMIALS

The ordinary multinomials are a natural extension of binomial coefficients (see [1] for a recent overview on ordinary multinomials). Letting $q, L \in \mathbb{N}$, for an integer $k = 0, 1, \ldots, qL$, the ordinary multinomial $\binom{L}{k}_q$ is the coefficient of the kth term of the multinomial expansion

$$(1 + x + x^2 + \cdots + x^q)^L = \sum_{k\geq 0} \binom{L}{k}_q x^k \tag{1}$$

with $\binom{L}{k}_1 = \binom{L}{k}$ (being the usual binomial coefficient) and $\binom{L}{k}_q = 0$ for $k > qL$. Using the classical binomial coefficient, one has

$$\binom{L}{k}_q = \sum_{j_1+j_2+\cdots+j_q=k} \binom{L}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{q-1}}{j_q}. \tag{2}$$

The combinatorial interpretation is as follows. The left-hand side can be interpreted as the number of ways to distribute k balls into n cells with at most q balls by cell. For the right-hand side, this can be computed also as the number of ways to first distribute j_1 balls into n cells and then take the j_1 cells containing one ball each and choose j_2 from those and add a second ball, and so on. In the last step, we choose from the j_{q-1} cells containing $(q-1)$ balls and choose from those j_q balls and add another ball, which gives the results for all possibilities satisfying $j_1 + j_2 + \cdots + j_q = k$ with $j_1 \geq j_2 \geq \cdots \geq j_q \geq 0$.

Readily established properties are the symmetry relation

$$\binom{L}{k}_q = \binom{L}{qL-k}_q \tag{3}$$

and the recurrence relation

$$\binom{L}{k}_q = \sum_{m=0}^{q} \binom{L-1}{k-m}_q. \tag{4}$$

As an illustration of the latter recurrence relation, we give the triangles of pentanomial (Table 1) and hexanomial (Table 2) coefficients, which are just an extension well known in the combinatorial literature, of the standard Pascal triangle. (For example, we have $18 + 15 + 10 + 6 + 3 = 52$).

Let us investigate the unimodality of the sequence $\{\binom{L}{k}_q\}_{k=0}^m$. A finite sequence of real numbers $\{a_k\}_{k=0}^m$ $(m \geq 1)$ is called *unimodal* if there exists an integer

Table 1. Triangle of Pentanomial coefficients: $\binom{L}{k}_4$

L\ k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	
0	**1**														
1	1	1	**1**	1	1										
2	1	2	3	4	**5**	4	3	2	1						
3	1	3	6	10	15	18	**19**	18+	15+	10+	6+	3+	1		
4	1	4	10	20	35	52	68	80	**85**	80	68	=52	35	20	···
5	1	5	15	35	70	121	185	255	320	365	**381**	365	320	255	···

Table 2. Triangle of hexanomial coefficients: $\binom{L}{k}_5$

L\ k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	
0	**1**															
1	1	1	**1**	**1**	1	1										
2	1	2	3	4	5	**6**	5	4	3	2	1					
3	1	3	6	10	15	21	25	**27**	**27**	25	21	15	10	6	3	1
4	1	4	10	20	35	56	80	104	125	140	**146**	140	125	104	80	···
5	1	5	15	35	70	126	205	305	420	540	651	735	**780**	**780**	735	···

$l \in \{0, \ldots, m\}$ such that the subsequence $\{a_k\}_{k=0}^{l}$ increases and $\{a_k\}_{k=l}^{m}$ decreases. If $a_0 \le a_1 \le \cdots \le a_{l_0-1} < a_{l_0} = \cdots = a_{l_1} > a_{l_1+1} \ge \cdots \ge a_m$, then the integers $l_0, \ldots, l_1$ are the *modes* of $\{a_k\}_{k=0}^{m}$. In the case where $l_0 = l_1$, we talk about a *peak*; otherwise the set of the values of the mode is called a *plateau*. For positive nonincreasing and nondecreasing sequences, unimodality is implied by log-concavity. A sequence $\{a_k\}_{k=0}^{m}$ is said to be *logarithmically concave* (log-concave for short) or *strongly unimodal* if $a_l^2 \ge a_{l-1}a_{l+1}$, $1 \le l \le m-1$. Additionally, if the sequence is *strictly log-concave* (SLC for short) (i.e. if the previous inequalities are strict), then the sequences have at most two consecutive modes (a peak or a plateau). For these notions, one can see Bertin and Theodorescu [2], Brenti [3], Comtet [4], Dharmadhikari and Joak-Dev [5], and Stanley [11]. In the following, $\lfloor a \rfloor$ denotes the greatest integer in a.

The first main result of this article is the following.

Theorem 1: *Let $q \ge 1$ and $L \ge 0$ be integers. Then the sequence $\{\binom{L}{k}_q\}_{k=0}^{qL}$ is unimodal and its smallest mode is given by*

$$k_L := \arg\max_k \binom{L}{k}_q = \lfloor (qL+1)/2 \rfloor .$$

Furthermore, we have the following recurrence relation:

$$\binom{L}{k_L}_q = \sum_{i \in I_q} \binom{L-1}{k_{L-1}+i}_q ,$$

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where

$$I_q = \begin{cases} \{-q/2, \ldots, q/2\} & \textit{if } q \textit{ is even} \\ \{-(q+1)/2, \ldots, (q-1)/2\} & \textit{if } q \textit{ and } L \textit{ are odd} \\ \{-(q-1)/2, \ldots, (q+1)/2\} & \textit{otherwise.} \end{cases}$$

PROOF: It suffices for each of the two cases—q odd and q even—to proceed by induction over L using the recurrence relation (4). ■

Remark 2: For even qL, we have a plateau of two modes: $qL/2$ *and* $qL/2+1$. Otherwise we have a peak: $(qL+1)/2$.

3. DETERMINING THE MAXIMAL PROBABILITY FOR CONVOLUTION POWERS OF DISCRETE UNIFORM DISTRIBUTION

We are now able to achieve our purpose: The expression of the maximal probability of the Lth convolution powers of the discrete uniform distribution. Let U_q be the random variable of the discrete uniform distribution over $\{0, 1, \ldots, q\}$ and let $U_q^{\star L}$ be its Lth convolution powers:

$$U_q := \frac{1}{q+1}(\delta_0 + \delta_1 + \cdots + \delta_q) \qquad (\delta_a \text{ is the Dirac measure}).$$

In [1], Belbachir, Bouroubi, and Khelladi established a link between the ordinary multinomials and the density probability of convolution powers of discrete uniform distribution. With respect to the counting measure, such a density is given by

$$P(U_q^{\star L} = k) = \frac{\binom{L}{k}_q}{(q+1)^L}, \qquad k = 0, 1, \ldots, qL.$$

Remark 3: From Odlyzko and Richmond [9] we know that for L sufficiently large, the sequence of probabilities $\{P(U_q^{\star L} = k)\}_k$ is strongly unimodal.

From Theorem 1, as a second main result, we give the values of $c_{q,L} := \max_k \binom{L}{k}_q / (q+1)^L$.

THEOREM 4: *The maximal probability of the Lth convolution power of the discrete uniform distribution over* $\{0, 1, \ldots, q\}$ *is*

$$c_{q,L} = \frac{1}{(q+1)^L} \binom{L}{\lfloor (qL+1)/2 \rfloor}_q.$$

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4. SOME GENERATING FUNCTIONS

As a third main result, we give the generating functions for the sequence of generalized ordinary multinomials, the sequences $\{\binom{z}{n}_q\}_n$ and $\{\binom{nz}{n}_q\}_n$, $z \in \mathbb{C}$, and the *extended* sequence of maximal probabilities for the convolution power of discrete uniform distribution: $\{c_{q,2n/q}\}_n$ (here $2n/q$ is not necessarily an integer). The following definition is motivated by (2).

DEFINITION 5: *For $z \in \mathbb{C}$, we define the generalized ordinary multinomials, as follows:*

$$\binom{z}{k}_q := \sum_{\substack{k_1+k_2+\cdots+k_q=k \\ k_1 \geq k_2 \geq \cdots \geq k_q}} \frac{z(z-1)\cdots(z-k_1+1)}{(k_1-k_2)!(k_2-k_3)!\cdots(k_{q-1}-k_q)!k_q!}. \quad \textbf{(5)}$$

THEOREM 6: *Let $z \in \mathbb{C}$; the generating function for generalized ordinary multinomials is given by*

$$\sum_{n \geq 0} \binom{z}{n}_q t^n = (1+t+t^2+\cdots+t^q)^z.$$

PROOF:

$$\begin{aligned}\sum_{n\geq 0}\binom{z}{n}_q t^n &= \sum_{n\geq 0}\sum_{h_1+2h_2+\cdots+qh_q=n} \frac{z(z-1)\cdots(z-(h_1+h_2+\cdots+h_q)+1)}{h_1!h_2!\cdots h_{q-1}!h_q!}t^n \\ &= \sum_{n\geq 0}\sum_{m\geq 0}\sum_{\substack{h_1+2h_2+\cdots+qh_q=n \\ h_1+h_2+\cdots+h_q=m}} \binom{z}{m}\frac{m!}{h_1!h_2!\cdots h_{q-1}!h_q!}t^n \\ &= \sum_{m\geq 0}\binom{z}{m}\sum_{h_1+h_2+\cdots+h_q=m}\frac{m!}{h_1!h_2!\cdots h_{q-1}!h_q!}t^{h_1+2h_2+\cdots+qh_q} \\ &= \sum_{m\geq 0}\binom{z}{m}(t+t^2+\cdots+t^q)^m \\ &= \left(1+t+t^2+\cdots+t^q\right)^z. \end{aligned}$$

■

Remark 7: Problem 19 of Comtet [4, Vol. 1, p. 172] states that

$$\sum_{n\geq 0} x^n \mathsf{C}_{t^n}(1+t+t^2)^n = (1-2x-3x^2)^{-1/2},$$

using the fact that the coefficient of t^n in the development of $(1+t+t^2)^n$: $\mathsf{C}_{t^n}(1+t+t^2)^n = \binom{n}{n}_2$ is $\max_k \binom{n}{k}_2$, we obtain the following combinatorial identity:

$$G_2(t) := \sum_{n\geq 0} c_{2,n}t^n = \left(1+\frac{t}{3}\right)^{-1/2}(1-t)^{-1/2}.$$

This last identity can be shown as the generating function of the sequence $\{c_{2,n}\}_n$.

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THEOREM 8: *Let $z \in \mathbb{C}$; the generating function of the sequence $\{\binom{nz}{n}_q\}_n$ is given by*

$$\sum_{n\geq 0}\binom{nz}{n}_q t^n = u\left(1 - z\frac{u + 2u^2 + \cdots + qu^q}{1 + u + u^2 + \cdots + u^q}\right)^{-1},$$

where u is a solution of the equation $t = u(1 + u + u^2 + \cdots + u^q)^{-z}$.

PROOF: Use Hermite's Theorem [4] for the function $t \mapsto t(1 + t + \cdots + t^q)^{-z}$. ■

THEOREM 9: *For q even, the generating function of the sequence $\{c_{q,2n/q}\}_n$ is given by*

$$G_q(t) := \sum_{n\geq 0} t^n c_{q,2n/q} = \left(1 - \frac{2}{q}\frac{u + 2u^2 + \cdots + qu^q}{1 + u + u^2 + \cdots + u^q}\right)^{-1}$$
$$= \frac{q}{2}\frac{1 + \sum_{k=1}^{q/2}(u^{-k} + u^k)}{\sum_{k=1}^{q/2} k(u^{-k} - u^k)},$$

where u is a solution of the equation

$$t = u\left(\frac{q+1}{1 + u + u^2 + \cdots + u^q}\right)^{2/q} = \left(\frac{q+1}{1 + \sum_{k=1}^{q/2}(u^{-k} + u^k)}\right)^{2/q}.$$

PROOF: Use Theorem 9 for $z = 2/q$ and the change of variable $t \to (q + 1)t$. ■

Remark 10: The sequence $\{c_{q,2n/q}\}_n$ contains strictly the subsequence $\{c_{q,L}\}_L$.

COROLLARY 11: *For $q = 4$, the generating function of $\{c_{4,n/2}\}_n$ for $t \in]-\sqrt{5}, 1[$ is given by*

$$G_4(t) := \sum_{n\geq 0} t^n c_{4,n/2} = \left(1 - \frac{1}{4}t^2 - \frac{1}{8}t^4 - \frac{1}{200}t(5t^2 + 20)^{3/2}\right)^{-1/2}.$$

COROLLARY 12: *We have the following identities*

$$\sum_{n\geq 0}(-5)^{-n}\binom{n/2}{n}_4 = 2 \quad \textit{and} \quad \sum_{n\geq 0}(-1)^n c_{4,n/2} = 2/\sqrt{5}.$$

Remark 13: The generating function of the sequence $\{c_{4,n}\}_n$ for $t \in]-1, 1[$ is given by

$$\sum_{n\geq 0} t^n c_{4,n} = (G_4(\sqrt{|t|}) + G_4(-\sqrt{|t|}))/2.$$

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