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ON DENSITY OF FOURIER COEFFICIENTS

BY
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1. Let f be an L integrable real valued function of period 2π and let

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx) \tag{1}$$

be its Fourier series. It is known that if f is of bounded variation then all na_n and nb_n ($n=1, 2, 3, \ldots$) lie in the interval $[-V(f)/\pi, V(f)/\pi]$ where $V(f)$ is the total variation of f. M. Izumi and S. Izumi [3] have recently asserted the following theorem A about the density of the positive and negative Fourier sine coefficients of a function of bounded variation.

THEOREM A. *Suppose that f is of bounded variation and there is at least one point of discontinuity. Let $x_0=0$ and d_0 be the jump of f, if f is discontinuous at $x_0=0$, otherwise $d_0=0$; further let $x_1, x_2, \ldots$ be its points of discontinuity in the interval $(0, 2\pi)$ and let $d_k=d(x_k)=f(x_k+0)-f(x_k-0)$ be jump at x_k. We write*

$$D = \sum_{j\geq 0} d(x_j) = \sum_{j\geq 0} d_j.$$

(I) *If $d_0+D>0$, then $\mu^+(N)$, the number of positive sine coefficients b_n with $n\leq N$ satisfy the relation*

$$\varliminf_{N\to\infty}\frac{\mu^+(N)}{N}\geq\frac{d_0+D}{V(f)}.$$

(II) *If $d_0+D<0$, then $\mu^-(N)$, the number of negative sine coefficients b_n with $n\leq N$ satisfy the relation*

$$\varliminf_{N\to\infty}\frac{\mu^-(N)}{N}\geq\frac{|d_0+D|}{V(f)}.$$

(III) *If $d_0+D=0$ and*

$$\varliminf_{N\to\infty}\sum_{n=1}^{N}\left|\sum_{j>0} d(x_j)\cos nx_j\right|>0 \tag{2}$$

then

$$\varliminf_{N\to\infty}\frac{\mu^+(N)}{N}>0,$$

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and

$$\varliminf_{N\to\infty} \frac{\mu^-(N)}{N} > 0.$$

M. Izumi and S. Izumi have based the proof of above theorem on the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} nb_n = \frac{d_0+D}{\pi} \tag{3}$$

which, unfortunately, holds only if $D=0$. This follows from the following theorem of Féjer [2].

THEOREM B. *If f is of bounded variation, then the sequence*

$$\{n(b_n \cos nx - a_n \sin nx)\}$$

is summable $(C, 1)$ *to $d(x)/\pi$ at every point x where*

$$d(x) = f(x+0)-f(x-0).$$

In theorem B, we put $x=0$, we shall obtain,

$$\lim_{N\to\infty} \sum_{n=1}^{N} nb_n = \frac{d_0}{\pi}. \tag{4}$$

From (3) and (4) we conclude that in theorem A, d_0+D should be replaced everywhere by d_0. In other words, if we define,

$$q_n = \begin{cases} 1 & (b_n > 0); \\ 0 & (b_n \leq 0); \end{cases}$$

and

$$r_n = \begin{cases} 1 & (b_n < 0); \\ 0 & (b_n \geq 0); \end{cases}$$

then theorem A can be expressed in the following corrected form;

THEOREM A′. *Let f be of bounded variation in* $[-\pi, \pi]$ *with its Fourier series* (1) *and let d_0 be the jump of f at zero.*

(I) *If $d_0>0$ then* $\lim_{N\to\infty} \inf 1/N \sum_{n=1}^{N} q_n \geq d_0/V(f)$*;*

(II) *If $d_0<0$ then* $\lim_{N\to\infty} \inf 1/N \sum_{n=1}^{N} r_n \geq |d_0|/V(f)$*;*

(III) *of Theorem* A *can be expressed in an analogous way.*

The main aim of this note is to generalize and make precise the theorem of M. and S. Izumi [3] by replacing (C, 1) summability by any summability method satisfying suitable restrictions. Among other things we also show that the hypothesis (2) in case (III) of Theorem A is superfluous and all the results are true uniformly in $p=0, 1, 2, \ldots$. In Theorem 3, we have studied about the density of positive and negative Fourier cosine coefficients. Theorem 3 is true without any additional hypothesis of M. and S. Izumi.

2. Let $\Lambda=(\lambda_{n,k})$ $(n, k=0, 1, 2, \ldots)$ be an infinite matrix of real numbers. A sequence $\{S_k\}$ is said to be summable Λ if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} S_k$$

exists; It is said to be summable F_Λ if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} S_{k+p}$$

exists uniformly in $p=0, 1, 2, \ldots$. F_Λ summability reduces to almost convergence if Λ is chosen to be the matrix of arithmetic mean. We shall call a matrix $\Lambda=(\lambda_{n,k})$ admissible if

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| = M < \infty;$$

It will be called positive admissible if

(A) $$\lambda_{n,k} \geq 0 \quad \text{for all } n \text{ and } k,$$

(B) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} = 1.$$

Obviously, every positive admissible matrix is admissible but the converse is not necessarily true. We shall denote

$$\mu_n^+(p) = \sum_{k=0}^{\infty} \lambda_{n,k} q_{k+p};$$

and

$$\mu_n^-(p) = \sum_{k=0}^{\infty} \lambda_{n,k} r_{k+p};$$

then we shall prove the following;

THEOREM 1. *Suppose that f is of bounded variation and there is at least one point of discontinuity. Let d_0 be the jump of f at zero, if f is discontinuous at zero, otherwise $d_0=0$. Suppose $\Lambda=(\lambda_{n,k})$ is positive admissible matrix such that $\{e^{ikt}\}$ is summable Λ to zero for all* $t \not\equiv 0 \pmod{2\pi}$.

(I) *If $d_0>0$ then $\lim_{n\to\infty} \inf \mu_n^+(p) \geq d_0/V(f)$ uniformly in $p=0, 1, 2, \ldots$.*

(II) *If $d_0<0$ then $\lim_{n\to\infty} \inf \mu_n^-(p) \geq |d_0|/V(f)$ uniformly in $p=0, 1, 2, \ldots$.*

(III) *If $d_0=0$ and there is at least one value of x for which the sum of jumps of f at $\pm x$ is not zero then*

$$\liminf_{n\to\infty} \mu_n^+(p) > 0$$

uniformly in $p=0, 1, 2, \ldots$, and

$$\liminf_{n\to\infty} \mu_n^-(p) > 0$$

uniformly in $p=0, 1, 2, \ldots$.

Now it is necessary to state few other results which we shall use in the proof of our theorems.

Let $V[0, 2\pi]$ denote the class of all real or complex valued functions f of bounded variation in $[0, 2\pi]$ such that for all $x, f(x)=[f(x+0)+f(x-0)]/2$ and $f(x+2\pi)-f(x)=f(2\pi)-f(0)$ and let $\sum_{-\infty}^{\infty} c_k e^{ikx}$ be its Fourier-Stieljes series, then Siddiqi [4] has proved:

THEOREM C. *Let Λ be an admissible matrix. Then*

(I) $$\{A_k(x)\} = \left\{c_k e^{ikx}+c_{-k}e^{-ikx}-\pi^{-1}\sum_{j=0}^{\infty} d_j \cos k(x-x_j)\right\}$$

is summable F_Λ to zero for every $f \in V[0, 2\pi]$ and every $x \in [0, 2\pi]$ if and only if $\{e^{ikt}\}$ is summable Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$;

(II) *$\{c_k e^{ikx}-c_{-k}e^{-ikx}\}$ is summable F_Λ to zero for every $f \in V[0, 2\pi]$ and every $x \in [0, 2\pi]$ if and only if $\{\sin kt\}$ is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$.*

We note that, if we suppose that $f(x+2\pi)=f(x)$ and if the Fourier series of f, in its real form, is given by (1), then, for $k>0$

$$c_k = \tfrac{1}{2}ik(a_k-ib_k); \qquad c_{-k} = -\tfrac{1}{2}ik(a_k+ib_k).$$

Thus

$$A_k(x) = k(-a_k \sin kx+b_k \cos kx)-\pi^{-1}\sum_{j=0}^{\infty} d_j \cos k(x-x_j);$$

and

$$(c_k e^{ikx}-c_{-k}e^{-ikx}) = ik(a_k \cos kx+b_k \sin kx).$$

In [5] he has further proved a result which, when expressed in terms of the Fourier series of f in its real form, gives us

THEOREM D. *If Λ is a positive admissible matrix such that $\{\cos kt\}$ is summable F_Λ to zero for all $t \not\equiv 0[\pmod{2\pi}]$ then for every $f \in V[0, 2\pi]$ which is of period 2π and for every $x \in [0, 2\pi]$ the sequence $\{|A_k(x)|\}$ and the sequence*

$$\{|B_k(x)|\} = \left\{\left| k(a_k \cos kx+b_k \sin kx)-\pi^{-1}\sum_{j=0}^{\infty} d_j \sin k(x-x_j) \right|\right\}$$

are summable F_Λ to zero.

3. **Proof of Theorem 1.** (*Case I*). If we use Theorem C part I at $x=0$, we shall get

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}(k+p)b_{k+p} = \frac{d_0}{\pi} + \lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k}\pi^{-1} \sum_{j=1}^{\infty} d_j \cos(k+p)x_j$$

uniformly in $p=0, 1, 2, \ldots$; but the last term of right-hand side of above expression tends to zero as $n\to\infty$ (see [4]), and it therefore follows that,

(5) $$\sum_{k=0}^{\infty} \lambda_{n,k}(k+p)b_{k+p} = \frac{d_0}{\pi}+o(1),$$

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uniformly in $p=0, 1, 2, \ldots$. We can write

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+p)b_{k+p} = \Sigma^{+}+\Sigma^{-}$$

where Σ^{+} and Σ^{-} denote positive sum and negative sum respectively. Since b_{k+p} are Fourier coefficients of a function of bounded variation hence,

$$-\frac{V(f)}{\pi} \le (k+p)b_{k+p} \le \frac{V(f)}{\pi} \tag{6}$$

where $V(f)$ is the total variation of f. Now using the definition of $\mu_n^{+}(p)$ and (6) we get,

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+p)b_{k+p} \le \Sigma^{+}\lambda_{n,k}(k+p)b_{k+p} \le \mu_n^{+}(p)\frac{V(f)}{\pi}.$$

Taking the limit as $n\to\infty$ and using (5) we get,

$$\frac{d_0}{\pi} \le \liminf_{n\to\infty} \mu_n^{+}(p)\frac{V(f)}{\pi}$$

uniformly in p. We can write this as

$$\liminf_{n\to\infty} \mu_n^{+}(p) \ge \frac{d_0}{V(f)}$$

uniformly in $p=0, 1, 2, \ldots$ which is case I of Theorem 1.

(*Case II*). If we apply the same arguments of Case I on $-f$ instead of f, we can get,

$$\liminf_{n\to\infty} \bar{\mu}_n(p) \ge \frac{|d_0|}{V(f)}$$

uniformly in $p=0, 1, 2, \ldots$ which is Case II.

(*Case III*). The sequence $\{e^{ikt}\}$ is summable Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$ if and only if $\{\cos kt\}$ is summable F_Λ to zero for all $t\not\equiv 0 \pmod{2\pi}$. It follows from Theorem D that $\{|A_k(x)|\}$ is summable F_Λ to zero. Applying this result at $x=0$, and by the fact that the difference of moduli is not greater than the modulus of difference, we shall obtain,

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+p)\,|b_{k+p}| = \frac{d_0}{\pi}+\pi^{-1}\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty} d_j\cos(k+p)x_j\right|+o(1) \tag{7}$$

uniformly in $p=0, 1, 2, \ldots$. Adding (5) and (7) and using the definition of absolute value we get,

$$2\Sigma^{+}\lambda_{n,k}(k+p)b_{k+p} = \frac{2d_0}{\pi}+\pi^{-1}\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty} d_j\cos(k+p)x_j\right|+o(1), \tag{8}$$

uniformly in p, where Σ^{+} denotes the positive sum. Now from (6) and (8) and by the definition of $\mu_n^{+}(p)$ we obtain,

$$2\mu_n^{+}(p)\frac{V(f)}{\pi} \ge \frac{2d_0}{\pi}+\pi^{-1}\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty} d_j\cos(k+p)x_j\right|+o(1) \tag{9}$$

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uniformly in p. Now it is sufficient to show that

$$\liminf_{n\to\infty}\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j\cos(k+p)x_j\right|>0 \tag{10}$$

uniformly in $p=0, 1, 2, \ldots.$

We first note that, as is well known [4];

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\lambda_{n,k}\sum_{j=1}^{\infty}d_j\cos(k+p)x_j=0 \tag{11}$$

uniformly in $p=0, 1, 2, \ldots.$ We next observe, since $(\lambda_{n,k})$ is a positive admissible matrix, and the inner sum in (11) is bounded, there is a real number $M'>0$ such that

$$\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j\cos(k+p)x_j\right|\geq M'\sum_{k=0}^{\infty}\lambda_{n,k}\left[\left|\sum_{j=1}^{\infty}d_j\cos(k+p)x_j\right|\right]^2.$$

Suppose now δ_j is the sum of jumps at $\pm x_j$. Grouping together the terms corresponding to x_j and $-x_j$ we get,

$$\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j\cos(k+p)x_j\right|\geq M'\sum_{k=0}^{\infty}\lambda_{n,k}\left[\left|\sum_{j=1}^{\infty}\delta_j\cos(k+p)x_j\right|\right]^2 \tag{12}$$

where the sum on right in square brackets is taken over the terms which are such that, for all i, j

$$x_i\pm x_j\not\equiv 0 \pmod{2\pi}. \tag{13}$$

Since there is at least one value of x for which the sum of the jumps of f at $\pm x$ is not zero, hence we can find at least one $\delta_j\neq 0$. If we expand the square in brackets of (12), its right hand side becomes equal to

$$\begin{aligned}
&= M'\left[\sum_{k=0}^{\infty}\lambda_{n,k}\left(\sum_{j=1}^{\infty}\delta_j^2\cos^2(k+p)x_j+2\sum_{i,j}\delta_i\delta_j\cos(k+p)x_j\cos(k+p)x_i\right)\right]\\
&= \frac{M'}{2}\left[\sum_{k=0}^{\infty}\lambda_{n,k}\sum_{j=1}^{\infty}\delta_j^2+\sum_{k=0}^{\infty}\lambda_{n,k}\sum_{j=1}^{\infty}\delta_j^2\cos 2(k+p)x_j\right.\\
&\qquad\left.+\sum_{k=0}^{\infty}2\lambda_{n,k}\sum_{i,j}\delta_i\delta_j\cos(k+p)(x_i+x_j)+\sum_{k=0}^{\infty}2\lambda_{n,k}\sum_{i,j}\delta_i\delta_j\cos(k+p)(x_i-x_j)\right]\\
&= \frac{M'}{2}\left(\sum_{k=0}^{\infty}\lambda_{n,k}\sum_{j=1}^{\infty}\delta_j^2+I_1+I_2+I_3\right).
\end{aligned}$$

Now consider I_1. We have, the inversion in the order of summation being justified by absolute convergence hence

$$I_1=\frac{M'}{2}\left(\sum_{j=1}^{\infty}\delta_j^2\sum_{k=0}^{\infty}\lambda_{n,k}\cos 2(k+p)x_j\right)$$

which tends to zero provided that there is no $x_j \equiv \pi (\text{mod } 2\pi)$. If, however, we have $x_j \equiv \pi (\text{mod } 2\pi)$ for $j=s$, (say), then

$$I_1 \to \tfrac{1}{2} M' \delta_s^2.$$

Now again due to inversion in the order of summation, we can write

$$I_2 = M' \sum_{i,j} \delta_i \delta_j \sum_{k=0}^{\infty} \lambda_{n,k} \cos(k+p)(x_i+x_j).$$

But, for every i, j we have $x_i + x_j \not\equiv 0$ (mod 2π) hence

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} \cos(k+p)(x_i+x_j) = 0.$$

Now from an argument similar to the argument of (11), we can prove $I_2 \to 0$ uniformly in p. Similarly we can prove $I_3 \to 0$ uniformly in p. Now collecting the terms of I_1, I_2, and I_3, we can write (12) into

$$\sum_{k=0}^{\infty} \lambda_{n,k} \left| \sum_{j=1}^{\infty} d_j \cos(k+p)x_j \right| \geq M'/2 \sum_{k=0}^{\infty} \lambda_{n,k} \sum_{j=1}^{\infty} \delta_j^2 + o(1). \tag{14}$$

Since there is some $\delta_j \neq 0$, the right-hand side of (14) does not tend to zero. Hence (10) holds. From (9) and (10) we get

$$\liminf_{n\to\infty} \mu_n^+(p) > 0$$

uniformly in $p=0, 1, 2, \ldots$ which is the first relation of Case III. Similarly we can prove the second relation. Hence Theorem 1 is completely proved.

We note that, taking

$$\begin{aligned} \lambda_{n,k} &= \frac{1}{n+1} \qquad k \leq n \\ &= 0 \qquad\quad k > n \end{aligned}$$

and $p=0$, we obtain theorem A′ as a special case of our Theorem 1. Now we define,

$$q_n(x) = \begin{cases} 1 & \left(\dfrac{x}{\pi} < nb_n \leq \dfrac{V(f)}{\pi}\right); \\ 0 & \text{otherwise,} \end{cases}$$

and

$$r_n(x) = \begin{cases} 1 & \left(-\dfrac{V(f)}{\pi} \leq nb_n < \dfrac{x}{\pi}\right); \\ 0 & \text{otherwise.} \end{cases}$$

We also define

$$\mu_n^+(p)(x) = \sum_{k=0}^{\infty} \lambda_{n,k} q_{k+p}(x)$$

$$\mu_n^-(p)(x) = \sum_{k=0}^{\infty} \lambda_{n,k} r_{k+p}(x).$$

Then we prove the following;

THEOREM 2. *Let f be a function of bounded variation and have more than two points of discontinuity, including origin. Let d_0 be the jump of f at zero, if f is discontinuous at zero, otherwise $d_0=0$. Suppose $\Lambda=(\lambda_{n,k})$ is a positive admissible matrix of real numbers such that $\{e^{ikt}\}$ is summable Λ to zero for all $t\not\equiv 0$* (mod 2π).

(I) *If $d_0>x$ then*

$$\liminf_{n\to\infty} \mu_n^+(p)(x) \geq \frac{|(d_0-x)|}{V(f)-|d_0|+|d_0-x|+|x|}$$

uniformly in $p=0, 1, 2, \ldots$.

(II) *If $d_0<x$ then*

$$\liminf_{n\to\infty} \mu_n^-(p)(x) \geq \frac{|(d_0-x)|}{V(f)-|d_0|+|d_0-x|+|x|}$$

uniformly in $p=0, 1, 2, 3, \ldots$.

Proof of Theorem 2. Here we shall give the proof of Case II of this theorem. Case I can be proved in a similar way. Consider the function

$$g(t) = f(t)-\frac{x}{\pi}\phi(t) \tag{15}$$

where

$$\phi(t) = \frac{\pi-t}{2} \qquad (0 < t < 2\pi)$$

$$\phi(0) = \phi(2\pi) = 0$$

and outside $[0, 2\pi]$ by periodicity. Then we observe ϕ is an odd and is continuous except at the origin where it has jump $\phi(+0)-\phi(-0)=\pi$. Hence g has jump (d_0-x) at the origin and has jump d_j at x_j (j=1, 2, 3, ...) and further

$$b_n(g) = b_n(f)-\frac{x}{\pi n} \qquad (n = 1, 2, 3, \ldots) \tag{16}$$

where $b_n(g)$ denotes the Fourier sine coefficients of g. Now applying Theorem 1, Case II to the function g, we get,

$$\liminf_{n\to\infty} \mu_n^-(p)(x) \geq \frac{|(d_0-x)|}{V(g)} \tag{17}$$

uniformly in $p=0, 1, 2, \ldots$.

Since the variation of $(x/\pi)\phi(t)$ in $(0+, 2\pi-)$ is $|x|$, the variation of g in $(0+, 2\pi-)$ can not exceed that of f by more than $|x|$. Hence, adding the jumps at the origin, we can get

$$V(g) \leq V(f)-|d_0|+|d_0-x|+|x|.$$

This and (17) gives

$$\liminf_{n\to\infty} \mu_n^-(p)(x) \geq \frac{|(d_0-x)|}{V(f)-|d_0|+|d_0-x|+|x|}$$

uniformly in $p=0, 1, 2, \ldots$ which is Case II of Theorem 2.

From Theorem 2 we can also deduce the following:

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COROLLARY 2.1. *Under the hypothesis of Theorem 2 we can prove,*

(I) *If* $d_0 > x \geq 0$ *then*

$$\liminf_{n\to\infty} \mu_n^+(p)(x) \geq \frac{(d_0 - x)}{V(f)}$$

uniformly in $p=0, 1, 2, \ldots$.

(II) *If* $d_0 < x \leq 0$ *then*

$$\liminf_{n\to\infty} \mu_n^-(p)(x) \geq \frac{|(d_0 - x)|}{V(f)}$$

uniformly in $p=0, 1, 2, \ldots$.

As a special case for $x=0$, Theorem 2 reduces to Theorem 1.

4. Now we shall consider the Fourier cosine coefficients a_n of the function of bounded variation. We denote,

$$q_n^* = \begin{cases} 1 & (a_n > 0); \\ 0 & (a_n \leq 0); \end{cases}$$

$$r_n^* = \begin{cases} 1 & (a_n < 0); \\ 0 & (a_n \geq 0); \end{cases}$$

and also denote

$$\nu_n^+(p) = \sum_{k=0}^{\infty} \lambda_{n,k} q_{k+p}^*$$

and

$$\nu_n^-(p) = \sum_{k=0}^{\infty} \lambda_{n,k} r_{k+p}^*$$

then we shall prove;

THEOREM 3. *Let* f *be a function of bounded variation and has points of discontinuity different from origin. Let* $x_j (j=1, 2, 3, \ldots)$ *be its points of discontinuity and* d_j *be its jump at* x_j. *Suppose that there is at least one value of* x *for which the difference of jumps of* f *at* $\pm x$ *is not zero. Suppose that* $\Lambda = (\lambda_{n,k})$ *is a positive admissible matrix of real numbers such that* $\{e^{ikt}\}$ *is summable* Λ *to zero for all* $t \not\equiv 0$ (mod 2π) *then*

$$\liminf_{n\to\infty} \nu_n^+(p) > 0$$

uniformly in $p=0, 1, 2, \ldots$ *and also*

$$\liminf_{n\to\infty} \nu_n^-(p) > 0$$

uniformly in $p=0, 1, 2, \ldots$.

Proof of Theorem 3. The hypothesis, $\{e^{ikt}\}$ is summable Λ to zero for all $t \not\equiv 0$ (mod 2π), implies that $\{\sin kt\}$ is summable F_Λ to zero for all $t \not\equiv 0$ (mod 2π). Hence the Theorem C part (II) at $x=0$ gives

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+p)a_{k+p} = o(1) \tag{18}$$

uniformly in $p=0, 1, 2, \ldots$ and Theorem D implies $\{|B_k(x)|\}$ is summable F_Λ to zero, which, at $x=0$, means

$$\sum_{k=0}^{\infty}\lambda_{n,k}(k+p)\,|a_{k+p}| = \sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j \sin(k+p)x_j\right| + o(1) \tag{19}$$

uniformly in p. Adding (18) and (19) we get,

$$2\Sigma^{+}\lambda_{n,k}(k+p)a_{k+p} = \sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j \sin(k+p)x_j\right| + o(1) \tag{20}$$

uniformly in p, where Σ^{+} denotes positive sum.

Since f is a function of bounded variation,

$$-\frac{V(f)}{\pi} \leq na_n \leq \frac{V(f)}{\pi} \tag{21}$$

for all $n \geq 1$. Now using (21) and the definition of $\nu_n^{+}(p)$ in (20) we get,

$$2\nu_n^{+}(p)\frac{V(f)}{\pi} \geq \sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j \sin(k+p)x_j\right| + o(1) \tag{22}$$

uniformly in $p=0, 1, 2, \ldots$. It is sufficient to show now that

$$\liminf_{n\to\infty}\sum_{k=0}^{\infty}\lambda_{n,k}\left|\sum_{j=1}^{\infty}d_j \sin(k+p)x_j\right| > 0 \tag{23}$$

uniformly in $p=0, 1, 2, \ldots$. But, from hypothesis, we have that there is at least one point of discontinuity different from origin, hence (23) can be shown valid by an argument similar to (10). There is one point which is relevant in connection of the proof of (23). If we have any $x_j \equiv \pi (\mathrm{mod}\ 2\pi)$, then the expression inside the modulus of (23) will be unchanged if this term is omitted from the sum. Hence there is no loss of generality in supposing that this case does not arise. Now we shall not repeat the proof of (23). From (22) and (23) we conclude

$$\liminf_{n\to\infty}\nu_n^{+}(p) > 0$$

uniformly in $p=0, 1, 2, \ldots$ which is the first relation of Theorem 3. Similarly we can prove the second relation of this theorem.

If we choose Λ to be the matrix of arithmetic mean and $p=0$ then our Theorem 3 gives a sharpened version of the theorem of M. and S. Izumi [3]. We also note that the expressions on the right of Cases I and II, of Theorem 1 cannot be replaced by larger numbers. For consider the function

$$\begin{aligned} f(t) &= 1 \qquad t \in (0, \pi) \\ &= 0 \qquad t \in (\pi, 2\pi) \end{aligned}$$

for which $\mu_n^{+}(p) \simeq \frac{1}{2} = |d_0|/V(f)$.

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