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Large-time behaviour of solutions to the dissipative nonlinear Schrödinger equation

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We study the Cauchy problem for the nonlinear Schrödinger equation with dissipation

$$\left.\begin{aligned} u_t + \mathcal{L}u + \mathrm{i}|u|^2 u = 0, \qquad & x \in \boldsymbol{R}, \quad t > 0, \\ u(0,x) = u_0(x), \quad & x \in \boldsymbol{R}, \end{aligned}\right\} \tag{A}$$

where $\mathcal{L}$ is a linear pseudodifferential operator with dissipative symbol $\operatorname{Re} L(\xi) \geqslant C_1|\xi|^2/(1+\xi^2)$ and $|L'(\xi)| \leqslant C_2(|\xi| + |\xi|^n)$ for all $\xi \in \boldsymbol{R}$. Here, $C_1, C_2 > 0$, $n \geqslant 1$. Moreover, we assume that $L(\xi) = \alpha\xi^2 + O(|\xi|^{2+\gamma})$ for all $|\xi| < 1$, where $\gamma > 0$, $\operatorname{Re}\alpha > 0$, $\operatorname{Im}\alpha \geqslant 0$. When $L(\xi) = \alpha\xi^2$, equation (A) is the nonlinear Schrödinger equation with dissipation $u_t - \alpha u_{xx} + \mathrm{i}|u|^2 u = 0$. Our purpose is to prove that solutions of (A) satisfy the time decay estimate

$$\|u(t)\|_\infty \leqslant C(1+t)^{-1/2}(1+\log(1+t))^{-1/2\sigma}$$

under the conditions that $u_0 \in \boldsymbol{H}^{n,0} \cap \boldsymbol{H}^{0,1}$ have the mean value

$$\hat{u}_0(0) = \frac{1}{\sqrt{2}\pi}\int u_0(x)\,\mathrm{d}x \neq 0$$

and the norm $\|u_0\|_{\boldsymbol{H}^{n,0}} + \|u_0\|_{\boldsymbol{H}^{0,1}} = \varepsilon$ is sufficiently small, where $\sigma = 1$ if $\operatorname{Im}\alpha > 0$ and $\sigma = 2$ if $\operatorname{Im}\alpha = 0$, and

$$\boldsymbol{H}^{m,s} = \{\phi \in \boldsymbol{S}';\ \|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\| < \infty\}, \quad m, s \in \boldsymbol{R}.$$

Therefore, equation (A) is considered as a critical case for the large-time asymptotic behaviour because the solutions of the Cauchy problem for the equation $u_t - \alpha u_{xx} + \mathrm{i}|u|^{p-1}u = 0$, with $p > 3$ have the same time decay estimate $\|u\|_{\boldsymbol{L}^\infty} = O(t^{-1/2})$ as that of solutions to the linear equation. On the other hand, note that solutions of the Cauchy problem (A) have an additional logarithmic time decay. Our strategy of the proof of the large-time asymptotics of solutions is to translate (A) to another nonlinear equation in which the mean value of the nonlinearity is zero for all time.

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1. Introduction

We consider the Cauchy problem for the following nonlinear Schrödinger equation with dissipation

$$\left.\begin{aligned} u_t + \mathcal{L}u + \mathrm{i}|u|^2 u = 0, &\qquad x \in \boldsymbol{R}, \quad t > 0, \\ u(0,x) = u_0(x), &\qquad x \in \boldsymbol{R}, \end{aligned}\right\} \tag{1.1}$$

where the linear pseudodifferential operator $\mathcal{L}$ is defined as $\mathcal{L}\phi = \mathcal{F}^{-1}L(\xi)\hat{\phi}(\xi)$, here and below $\mathcal{F}\phi$ (or $\hat{\phi}$) is the Fourier transform of ϕ defined by the formula

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\mathrm{i}x\xi} \phi(x)\,\mathrm{d}x$$

and $\mathcal{F}^{-1}\phi(x)$ (or $\check{\phi}(x)$) is the inverse Fourier transform of ϕ, i.e.

$$\check{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{\mathrm{i}x\xi} \phi(\xi)\,\mathrm{d}\xi.$$

We assume that the linear operator $\mathcal{L}$ is dissipative, that is, its symbol satisfies

$$\operatorname{Re} L(\xi) \geqslant \frac{C_1|\xi|^2}{1+|\xi|^2} \quad \text{and} \quad |L'(\xi)| \leqslant C_2(|\xi| + |\xi|^n) \tag{1.2}$$

for all $\xi \in \boldsymbol{R}$, where $C_1, C_2 > 0$, $n \geqslant 1$. Moreover, we assume that the symbol has the following asymptotic representation for small values of ξ

$$L(\xi) = \alpha\xi^2 + O(|\xi|^{2+\gamma}) \quad \text{for all } |\xi| < 1, \tag{1.3}$$

where $\gamma > 0$, $\operatorname{Re}\alpha > 0$, $\operatorname{Im}\alpha \geqslant 0$. In particular, if we choose $L(\xi) = \alpha\xi^2$, then (1.1) is the nonlinear Schrödinger equation with dissipation (it is also known as the generalized Ginzburg–Landau equation) $u_t - \alpha u_{xx} + \mathrm{i}|u|^2 u = 0$. Our main purpose is to study the large-time asymptotics of solutions to the Cauchy problem (1.1). Large-time asymptotics of solutions to the Cauchy problems for dissipative equations has been extensively studied previously (see [1–4, 6–8, 12, 17, 19, 21–25] and the literature cited there in). Critical cases for the Cauchy problems with nonlinearities involving derivatives of unknown functions were considered in [5, 9, 16, 18, 20]. The fact that the nonlinearity has the form of the full derivative (and so the mean value of the nonlinearity is zero for all time) enables us to find the large-time asymptotics of solutions easily. In [11, 14], the Cauchy problem to the nonlinear heat equation $u_t - u_{xx} + |u|^{p-1}u = 0$ was considered and it was shown that the solution u satisfies the time decay estimate $\|u(t)\|_{\boldsymbol{L}^\infty} = O(t^{-1/2}(\log(1+t))^{-1/2})$ when $p = 3$ and $\|u(t)\|_{\boldsymbol{L}^\infty} = O(t^{-1/2})$ when $p > 3$. This fact means that the case $p = 3$ is the critical one. These results on the large-time asymptotics of solutions were extended to the case of the porous media equation with critical exponents (see [10]). The aim of the present paper is to obtain the large-time asymptotics of solutions to the critical Cauchy problem (1.1). The methods of papers [11, 14] do not work for the complex Ginzburg–Landau equation. On the other hand, our method stated below in §§ 2 and 3 is also applicable to the Cauchy problem for the nonlinear heat equation $u_t - u_{xx} + |u|^2 u = 0$.

The main idea of the present paper is to make a change of the dependent variable $u = v\mathrm{e}^{-\varphi(t)+\mathrm{i}\psi(t)}$ in order to rewrite (1.1) in the form

$$v_t + \mathcal{L}v + \mathrm{i}\mathrm{e}^{-2\varphi}|v|^2 v - (\varphi' - \mathrm{i}\psi')v = 0,$$

in which the new nonlinearity $\mathrm{i}\mathrm{e}^{-2\varphi}|v|^2 v - (\varphi' - \mathrm{i}\psi')v$ has the zero mean value for all time. Then we apply the method similar to that of [15, 18, 20].

To state our result precisely we introduce the following notation and function spaces.

Let

$$\partial = \partial_x = \frac{\partial}{\partial x} \quad \text{and} \quad \partial_t = \frac{\partial}{\partial t}.$$

As usual, we denote the Lebesgue space by $\boldsymbol{L}^p = \{\phi \in \boldsymbol{S}'; \|\phi\|_p < \infty\}$, where

$$\|\phi\|_p = \left(\int_{\boldsymbol{R}} |\phi(x)|^p \,\mathrm{d}x\right)^{1/p}$$

if $1 \leqslant p < \infty$ and $\|\phi\|_\infty = \operatorname{ess\,sup}\{|\phi(x)|; x \in \boldsymbol{R}\}$ if $p = \infty$. For simplicity, we let $\|\phi\| = \|\phi\|_2$. The weighted Sobolev space

$$\boldsymbol{H}^{m,s} = \{\phi \in \boldsymbol{S}'; \|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\| < \infty\}, \quad m, s \in \boldsymbol{R}.$$

By $\boldsymbol{C}(\boldsymbol{I}; \boldsymbol{B})$, we denote the space of continuous functions from a time-interval $\boldsymbol{I}$ to the Banach space $\boldsymbol{B}$. Denote $\boldsymbol{W} = \boldsymbol{H}^{n,0} \cap \boldsymbol{H}^{0,1}$ and $\|\phi\|_{\boldsymbol{W}} = \|\phi\|_{n,0} + \|\phi\|_{0,1}$, where n is the same as in (1.2). Different positive constants might be denoted by the same letter C. Denote also

$$\mu_1 = -\operatorname{Im} \frac{\theta^2}{4\pi\sqrt{2|\alpha|^2 + \alpha^2}} \geqslant 0,$$

$$\mu_2 = \frac{\theta^4}{3(4\pi\alpha)^2} \log \frac{9(3-\sqrt{5})}{3+\sqrt{5}},$$

$$\omega = \operatorname{Re} \frac{\theta^2 \mu_\sigma^{-1/\sigma}}{4\pi\sqrt{2|\alpha|^2 + \alpha^2}},$$

$$\theta = \left|\int u_0(x)\,\mathrm{d}x\right| = \sqrt{2\pi}|\hat{u}_0(0)| > 0.$$

In the present paper we prove the following result.

THEOREM 1.1. *Let the symbol $L(\xi)$ satisfy conditions (1.2), (1.3) with $\operatorname{Re}\alpha > 0$ and $\operatorname{Im}\alpha \geqslant 0$. We assume that the initial data $u_0 \in \boldsymbol{W}$ have the mean value*

$$\hat{u}_0(0) = \frac{1}{\sqrt{2\pi}} \int u_0(x)\,\mathrm{d}x \neq 0$$

and the norm $\|u_0\|_{\boldsymbol{W}} = \varepsilon$ is sufficiently small. Then there exists a unique solution of the Cauchy problem (1.1) such that $u(t,x) \in \boldsymbol{C}^\infty((0,\infty) \times \boldsymbol{R}) \cap \boldsymbol{C}((0,\infty); \boldsymbol{W})$ with the following time decay estimate

$$\|u(t)\|_\infty \leqslant \frac{3\varepsilon}{\sqrt{1+t}(1+\mu_\sigma \log(1+t))^{1/2\sigma}},$$

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where $\sigma = 1$ *if* $\operatorname{Im}\alpha > 0$ *and* $\sigma = 2$ *if* $\operatorname{Im}\alpha = 0$*. Moreover, the solution has the following asymptotics for large time* $t \to \infty$ *uniformly with respect to* $x \in \boldsymbol{R}$

$$u(t,x) = \frac{\theta}{2\pi\sqrt{2\alpha}\sqrt{\mu_1}\sqrt{t\log t}} \mathrm{e}^{-x^2/4\alpha t + \mathrm{i}\omega \log\log t} + O\biggl(\frac{\varepsilon}{\sqrt{t\log t}\log\log t}\biggr) \tag{1.4}$$

in the case $\sigma = 1$ *and*

$$u(t,x) = \frac{\theta}{2\pi\sqrt{2\alpha}t^{1/4}(\log t)^{1/4}} \mathrm{e}^{-x^2/4\alpha t + \mathrm{i}\omega\sqrt{\log t}} + O\biggl(\frac{\varepsilon}{\sqrt{t\log t}}\biggr) \tag{1.5}$$

in the case $\sigma = 2$*.*

Remark 1.2. Note that the asymptotic behaviour of solutions described in theorem 1.1 in the case $\operatorname{Im}\alpha > 0$ have an additional logarithmic decay in comparison with the linear evolution (as it happens in the case of the semilinear heat equation with the critical exponent $u_t - u_{xx} + u^3 = 0$ (see [11, 14])). This logarithmic decay is defined by the first approximation of the perturbation theory (see the proof of theorem 1.1 below). However, in the case $\operatorname{Im}\alpha = 0$, the coefficient μ_1 vanishes and the logarithmic decay of the solutions become slower (it is defined by the second approximation of the perturbation theory).

We organize our paper as follows. In § 2 we prepare some preliminary estimates in lemmas 2.1 and 2.2. Section 3 is devoted to the proof of theorem 1.1.

2. Preliminaries

The solution of the linear Cauchy problem

$$\left.\begin{aligned} u_t + \mathcal{L}u &= f(t,x), \quad x \in \boldsymbol{R}, \quad t > 0, \\ u(0,x) &= u_0(x), \quad x \in \boldsymbol{R}, \end{aligned}\right\}$$

can be written in the following form

$$u(t) = \mathcal{K}(t)u_0 + \int_0^t \mathcal{K}(t-\tau)f(\tau)\,\mathrm{d}\tau,$$

where the Green operator $\mathcal{K}$ is given by $\mathcal{K}(t)\phi = \mathcal{F}^{-1}\mathrm{e}^{-L(\xi)t}\hat{\phi}(\xi)$. We denote also

$$\mathcal{G}(t)\phi = \mathcal{F}^{-1}\mathrm{e}^{-\alpha\xi^2 t}\hat{\phi}(\xi) = \int G(t,x-y)\phi(y)\,\mathrm{d}y,$$

where the heat kernel

$$G(t,x) = \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-x^2/4\alpha t}.$$

Lemma 2.1. *Let dissipation condition (1.2) and asymptotic representation (1.3) for the symbol* $L(\xi)$ *of the operator* $\mathcal{L}$ *be fulfilled. Assume that a function* $f(t,x)$ *has the zero mean value* $\hat{f}(t,0) = 0$ *and satisfies the estimates*

$$\|f(t)\|_{\boldsymbol{W}} \leqslant \varepsilon(1+t)^{-3/4-\beta}g^{-k/\sigma}(t) \quad \textit{for all } t > 0,$$

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where $\beta \in [0, \frac{1}{4})$, $k = 0, 1, 2$. *We also assume that the function* $g(t)$ *is such that* $\frac{1}{2}(1 + \mu_\sigma \log(1+t)) \leqslant g(t) \leqslant 2(1 + \mu_\sigma \log(1+t))$ *for all* $t > 0$. *Then the following inequalities are valid*

$$\sqrt[4]{1+t}\left\| \int_0^t \mathrm{d}\tau\, g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)f(\tau)\right\|_{n,0} + \frac{1}{\sqrt[4]{t}}\left\| \int_0^t \mathrm{d}\tau\, g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)f(\tau)\right\|_{0,1} \leqslant C\varepsilon,$$

$$\left\| \int_0^t \mathrm{d}\tau\, g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)f(\tau)\right\|_{\infty} \leqslant C\varepsilon g^{-(k+1)/\sigma}(t)(1+t)^{-1/2-\beta}$$

and

$$\left\| \int_0^t \mathrm{d}\tau\, g^{-1/\sigma}(\tau)(\mathcal{K}(t-\tau) - \mathcal{G}(t-\tau))f(\tau)\right\|_{\infty} \leqslant C\varepsilon g^{-(k+1)/\sigma}(t)(1+t)^{-1/2-\gamma/2-\beta}$$

for all $t > 0$.

Proof. Since the function $f(t, x)$ has the zero mean value $\hat{f}(t, 0) = 0$, applying dissipation condition (1.2), the Plancherel theorem and the inequality

$$|\hat{f}(\tau, \xi)| = |\hat{f}(\tau, \xi) - \hat{f}(\tau, 0)| \leqslant C\sqrt{|\xi|}\|f(\tau)\|_{0,1},$$

we get

$$\begin{aligned}\|\partial_x^l \mathcal{K}(t-\tau)f(\tau)\|^2 &\leqslant C\|f(\tau)\|_{0,1}^2 \int_{|\xi|\leqslant 1} \mathrm{e}^{-C\xi^2(t-\tau)}|\xi|^{1+2l}\,\mathrm{d}\xi \\ &\quad + \int_{|\xi|\geqslant 1} \mathrm{e}^{-C(t-\tau)}|\xi|^{2l}|\hat{f}(\tau,\xi)|^2\,\mathrm{d}\xi \\ &\leqslant \frac{C(\|f(\tau)\|_{l,0} + \|f(\tau)\|_{0,1})^2}{(1+t-\tau)^{1+l}} \end{aligned} \tag{2.1}$$

for all $t > \tau > 0$, where $0 \leqslant l \leqslant n$. In the same manner, we obtain

$$\begin{aligned}\|\mathcal{K}(t-\tau)f(\tau)\|_{0,1}^2 &\leqslant C\|\mathcal{K}(t-\tau)f(\tau)\|^2 + C\int |\partial_\xi(\mathrm{e}^{-L(\xi)(t-\tau)}\hat{f}(\tau,\xi))|^2\,\mathrm{d}\xi \\ &\leqslant C\sqrt{t-\tau}\|f(\tau)\|_{0,1}^2 \int_{|\xi|\leqslant 1} \mathrm{e}^{-C\xi^2(t-\tau)}\,\mathrm{d}\xi \\ &\quad + C\int_{|\xi|\geqslant 1} \mathrm{e}^{-C(t-\tau)}|\xi|^{2n}|\hat{f}(\tau,\xi)|^2\,\mathrm{d}\xi \\ &\leqslant C\|f(\tau)\|_{\boldsymbol{W}}^2 \\ &\leqslant C\varepsilon(1+\tau)^{-3/2-2\beta}.\end{aligned}$$

Whence, since $g^{-1/\sigma}(t) \leqslant 2$, we get

$$\begin{aligned}\sqrt[4]{1+t}\bigg\|\int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)f(\tau)\,\mathrm{d}\tau\bigg\|_{n,0} \\ + \frac{1}{\sqrt[4]{t}}\bigg\|\int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)f(\tau)\,\mathrm{d}\tau\bigg\|_{0,1} \\ \leqslant C\varepsilon\int_0^t\bigg(\frac{\sqrt[4]{1+t}}{\tau^{3/4}\sqrt{t-\tau}}+\frac{1}{\tau^{3/4}\sqrt[4]{t}}\bigg)\,\mathrm{d}\tau \leqslant C\varepsilon.\end{aligned}$$

We now prove the rest two estimates of the lemma. Note that, for $0 \leqslant t \leqslant 4$, these estimates follow from the first one. So consider now $t > 4$. Inequality (2.1) yields

$$\begin{aligned}\|\mathcal{K}(t-\tau)f(\tau)\|_\infty^2 &\leqslant C\|\mathcal{K}(t-\tau)f(\tau)\|\|\partial_x\mathcal{K}(t-\tau)f(\tau)\| \\ &\leqslant C(1+t-\tau)^{-3/2}\|f(\tau)\|_{\boldsymbol{W}}^2 \\ &\leqslant C(1+t-\tau)^{-3/2}(1+\tau)^{-3/2-2\beta}g(\tau)^{-2k/\sigma}\end{aligned}$$

for all $t > \tau > 0$. From the condition of the lemma for the function $g(t)$, we have the estimates $(1+t)^{-1/8+\beta/2} \leqslant Cg^{-(k+1)/\sigma}(t)$ for $0 \leqslant \beta < \frac{1}{4}$ and

$$\begin{aligned}\sup_{\tau\in[\sqrt{t},t]} g^{-(k+1)/\sigma}(\tau) &\leqslant C(1+\mu_\sigma\log(1+\sqrt{t}))^{-(k+1)/\sigma} \\ &\leqslant C(1+\tfrac{1}{2}\mu_\sigma\log(1+t))^{-(k+1)/\sigma} \\ &\leqslant Cg^{-(k+1)/\sigma}(t),\end{aligned}$$

whence

$$\begin{aligned}&\bigg\|\int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)f(\tau)\,\mathrm{d}\tau\bigg\|_\infty \\ &\quad\leqslant C\varepsilon\int_0^t\frac{g^{-(k+1)/\sigma}(\tau)\,\mathrm{d}\tau}{(t-\tau)^{3/4}\tau^{3/4+\beta}} \\ &\quad\leqslant Ct^{-3/4}\bigg(\int_0^{\sqrt{t}}\frac{\mathrm{d}\tau}{\tau^{3/4+\beta}}+g^{-(k+1)/\sigma}(t)\int_{\sqrt{t}}^{t/2}\frac{\mathrm{d}\tau}{\tau^{3/4+\beta}} \\ &\qquad\qquad+g^{-(k+1)/\sigma}(t)\int_{t/2}^t\frac{\mathrm{d}\tau}{(t-\tau)^{3/4+\beta}}\bigg) \\ &\quad\leqslant Ct^{-1/2-\beta}(t^{-1/8+\beta/2}+g^{-(k+1)/\sigma}(t)) \\ &\quad\leqslant Cg^{-(k+1)/\sigma}(t)(1+t)^{-1/2-\beta}\end{aligned}$$

for all $t > 4$. If $\operatorname{Re}L(\xi) \geqslant \operatorname{Re}\alpha\xi^2$, using condition (1.3), we write the estimate

$$\begin{aligned}|\mathrm{e}^{-L(\xi)t}-\mathrm{e}^{-\alpha\xi^2t}| &\leqslant \mathrm{e}^{-\operatorname{Re}\alpha\xi^2t}|\mathrm{e}^{-L(\xi)t+\alpha\xi^2t}-1| \\ &\leqslant C|\xi|^{2+\gamma}t\mathrm{e}^{-\operatorname{Re}\alpha\xi^2t} \\ &\leqslant C|\xi|^\gamma\mathrm{e}^{-C\xi^2t}.\end{aligned}$$

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The case $\operatorname{Re} L(\xi) \leqslant \operatorname{Re} \alpha\xi^2$ is considered in the same manner. Hence

$$
\begin{aligned}
&\|\partial^l(\mathcal{K}(t-\tau)-\mathcal{G}(t-\tau))f(\tau)\|^2 \\
&\qquad \leqslant C\|f\|_{0,1}^2 \int_{|\xi|\leqslant 1} |\xi|^{2\gamma+2l+1} \mathrm{e}^{-C\xi^2(t-\tau)}\,\mathrm{d}\xi + C\mathrm{e}^{-C(t-\tau)} \int_{|\xi|\geqslant 1} |\xi^l \hat{f}(\tau,\xi)|^2\,\mathrm{d}\xi \\
&\qquad \leqslant C\|f(\tau)\|_{\boldsymbol{W}}^2 (1+t-\tau)^{-1-l-\gamma} \\
&\qquad \leqslant C\varepsilon^2(1+t-\tau)^{-1-l-\gamma}(1+\tau)^{-3/2-2\beta} g(\tau)^{-2k/\sigma}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\left\| \int_0^t g^{-1/\sigma}(\tau)(\mathcal{K}(t-\tau)-\mathcal{G}(t-\tau))f(\tau)\,\mathrm{d}\tau \right\|_\infty \\
&\qquad \leqslant C \int_0^t g^{-1/\sigma}(\tau)\|(\mathcal{K}(t-\tau)-\mathcal{G}(t-\tau))f(\tau)\|^{1/2} \\
&\qquad\qquad \times \|\partial_x(\mathcal{K}(t-\tau)-\mathcal{G}(t-\tau))f(\tau)\|^{1/2}\,\mathrm{d}\tau \\
&\qquad \leqslant C\varepsilon \int_0^t \frac{g^{-(k+1)/\sigma}(\tau)\,\mathrm{d}\tau}{(t-\tau)^{3/4+\gamma/2}\tau^{3/4+\beta}} \\
&\qquad \leqslant C\varepsilon t^{-3/4-\gamma/2}\biggl(\int_0^{\sqrt{t}} \frac{\mathrm{d}\tau}{\tau^{3/4+\beta}} + g^{-(k+1)/\sigma}(t) \int_{\sqrt{t}}^{t/2} \frac{\mathrm{d}\tau}{\tau^{3/4+\beta}} \\
&\qquad\qquad + g^{-(k+1)/\sigma}(t) t^{-\beta+\gamma/2} \int_{t/2}^t \frac{\mathrm{d}\tau}{(t-\tau)^{3/4+\gamma/2}} \biggr) \\
&\qquad \leqslant C\varepsilon t^{-1/2-\beta-\gamma/2}(t^{-1/8+\beta/2} + g^{-(k+1)/\sigma}(t)) \\
&\qquad \leqslant C\varepsilon g^{-(k+1)/\sigma}(t)(1+t)^{-1/2-\beta-\gamma/2}
\end{aligned}
$$

for all $t > 4$. Lemma 2.1 is proved. □

Denote $\Phi(t) = \mathcal{K}(t)v_0$ and

$$
\Psi(t) = -\mathrm{i} \int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)\biggl(|\Phi|^2\Phi(\tau) - \frac{\Phi(\tau)}{\theta} \int |\Phi|^2\Phi(\tau,x)\,\mathrm{d}x \biggr)\,\mathrm{d}\tau.
$$

Lemma 2.2. *Let dissipation condition (1.2) and asymptotic representation (1.3) be fulfilled. Also we assume that* $v_0 \in \boldsymbol{W}$ *and the norm* $\|v_0\|_{\boldsymbol{W}} = \varepsilon$ *is sufficiently small. Suppose that a function* $v(t,x)$ *satisfies the estimates* $\|v(t)\|_\infty \leqslant \varepsilon(1+t)^{-1/2}$, $\|v(t)\| \leqslant \varepsilon(1+t)^{-1/4}$ *for all* $t > 0$, *and have the asymptotic representation*

$$
\|v(t) - \Phi(t) - (\sigma-1)\Psi(t)\|_\infty \leqslant \frac{C\varepsilon^{1+2\sigma}}{g(t)\sqrt{t+1}}
$$

for all $t > 0$. *We also assume that the function* $g(t)$ *is such that*

$$
\tfrac{1}{2}(1+\mu_\sigma \log(1+t)) \leqslant g(t) \leqslant 2(1+\mu_\sigma \log(1+t))
$$

for all $t > 0$ *and* $|g'(t)| \leqslant C\theta^{2\sigma}/(t+1)$, *with* $\theta = \hat{v}_0(0)$ *when* $\sigma = 2$. *Then the following inequalities are valid*

$$
\left| \frac{1}{\theta} \operatorname{Im} \int g^{-(\sigma-1)/\sigma}(t)|v|^2 v(t,x)\,\mathrm{d}x \right| \leqslant \frac{C\varepsilon^{2\sigma}}{t+1} \tag{2.2}
$$

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and

$$\tfrac{1}{2}(1+2\sigma\mu_\sigma\log(t+1)) \leqslant 1-\frac{2\sigma}{\theta}\operatorname{Im}\int_0^t g^{(\sigma-1)/\sigma}(\tau)\int |v|^2 v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 2(1+2\sigma\mu_\sigma\log(1+t)) \tag{2.3}$$

for all $t>0$.

Proof. For $t\in[0,1]$, the estimates (2.2), (2.3) are trivial. Consider the case $t>1$. We have $\Phi(t)=\mathcal{K}(t)v_0=\theta G(t)+R$, where

$$G(t,x)=\frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-x^2/4\alpha t}$$

is the heat kernel and the remainder term $R=(\mathcal{G}(t)v_0-\theta G(t))+(\mathcal{K}(t)-\mathcal{G}(t))v_0$ has the zero mean value $\hat{R}(t,0)=0$. By virtue of estimate (1.3) and Plancherel's theorem (as in lemma 2.1), we get

$$\|\partial^l R\|^2 \leqslant C\varepsilon^2\int |\xi|^{2l}(|\xi|+|\xi|^\gamma)\mathrm{e}^{-C\xi^2 t}\,\mathrm{d}\xi \leqslant C\varepsilon^2(1+t)^{-1/2-l-2\beta},$$

where $\beta=\min(\frac{1}{4},\frac{1}{2}\gamma)$. In the same manner, we have $\|R\|_{0,1}\leqslant C\varepsilon(1+t)^{1/4-\beta}$ and $\|R\|_\infty\leqslant C\varepsilon(1+t)^{-1/2-\beta}$. First consider the case $\operatorname{Im}\alpha>0$, that is, $\sigma=1$. We have

$$\begin{aligned}\||v|^2v-\theta^3|G|^2G\|_1 &= \|\theta R\bar{G}(G+\Phi)+\bar{R}\Phi^2+r\bar{\Phi}(\Phi+v)+\bar{r}v^2\|_1\\ &\leqslant C(\|R\|_\infty+\|r\|_\infty)(\theta^2\|G\|^2+\|\Phi\|^2+\|v\|^2)\\ &\leqslant \frac{C\varepsilon^5}{g(t)(1+t)}+\frac{C\varepsilon^3}{(1+t)^{1+\beta}},\end{aligned}$$

where $r=v-\Phi$. By an easy computation, we obtain

$$\begin{aligned}-\operatorname{Im}\theta^3\int |G|^2G(t,x)\,\mathrm{d}x &= -\operatorname{Im}\frac{\theta^3}{(4\pi t)^{3/2}|\alpha|\sqrt{\alpha}}\int \mathrm{e}^{-x^2/4t(2/\alpha+1/\bar{\alpha})}\,\mathrm{d}x\\ &= -\operatorname{Im}\frac{\theta^3}{4\pi t\sqrt{2|\alpha|^2+\alpha^2}}\\ &= \frac{\mu_1\theta}{t}>0.\end{aligned}$$

Then, since $g^{-1}(t)\leqslant C(1+\mu_\sigma\log(1+t))^{-1}$, we get

$$\left|\operatorname{Im}\frac{1}{\theta}\int |v|^2v(t,x)\,\mathrm{d}x+\frac{\mu_1}{t+1}\right| \leqslant \frac{C\varepsilon^5}{\theta(t+1)(1+\mu_1\log(t+1))}+\frac{C\varepsilon^3}{\theta(t+1)^{1+\beta}}$$

for all $t>0$, whence estimate (2.2) follows and

$$\begin{aligned}&\left|\frac{1}{\theta}\operatorname{Im}\int_0^t \mathrm{d}\tau\int |v|^2v(\tau,x)\,\mathrm{d}x+\mu_1\log(1+t)\right|\\ &\qquad\leqslant \frac{C\varepsilon^5}{\theta}\int_0^t\frac{\mathrm{d}\tau}{(1+\tau)(1+\mu_1\log(1+\tau))}+\frac{C\varepsilon^3}{\theta}\int_0^t\frac{\mathrm{d}\tau}{(1+\tau)^{1+\beta}}\\ &\qquad\leqslant \varepsilon+\varepsilon\log(1+\mu_1\log(1+t))\\ &\qquad\leqslant \tfrac{1}{4}(1+\mu_1\log(1+t)).\end{aligned}$$

Thus we obtain (2.3) in the case $\operatorname{Im}\alpha > 0$. Now we turn to the case $\operatorname{Im}\alpha = 0$, i.e. $\sigma = 2$. Now,

$$\operatorname{Im}\theta^3 \int |G|^2 G(t,x)\,\mathrm{d}x = 0,$$

so we have to analyse in more details the second approximation Ψ of the perturbation theory. We have $|\Phi|^2\Phi(t) = \theta^3 G^3(t) + P(t)$, where the remainder term is estimated as

$$\|P(t)\|_{\boldsymbol{W}} = \|\theta RG(\theta G + \Phi) + \bar{R}\Phi^2\|_{\boldsymbol{W}} \leqslant \frac{C\varepsilon^3}{(1+t)^{3/4+\beta}}$$

for all $t > 0$. Hence

$$|\Phi|^2\Phi(t) - \frac{\Phi(t)}{\theta}\int |\Phi|^2\Phi(t,x)\,\mathrm{d}x = \theta^3\biggl(G^3(t) - G(t)\int G^3(t,x)\,\mathrm{d}x\biggr) + Q(t),$$

where the remainder

$$Q(t) = \biggl(P(t) - G(t)\int P(t,x)\,\mathrm{d}x\biggr) + \frac{R(t)}{\theta}\int |\Phi|^2\Phi(t,x)\,\mathrm{d}x$$

is estimated as $\|Q(t)\|_{\boldsymbol{W}} \leqslant C\varepsilon^3(1+t)^{-3/4-\beta}$ and satisfies the condition

$$\int Q(t,x)\,\mathrm{d}x = 0.$$

Whence, by lemma 2.1, we obtain

$$\begin{aligned}
&\biggl\| \int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)\biggl(|\Phi|^2\Phi(\tau) - \frac{\Phi(\tau)}{\theta}\int |\Phi|^2\Phi(\tau,x)\biggr)\,\mathrm{d}x \\
&\quad - g^{-1/\sigma}(\tau)\theta^3\mathcal{G}(t-\tau)\biggl(G^3(\tau) - G(\tau)\int G^3(\tau,x)\biggr)\,\mathrm{d}x\,\mathrm{d}\tau\biggr\|_\infty \\
&\qquad \leqslant \biggl\|\int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)Q(\tau)\,\mathrm{d}\tau\biggr\|_\infty \\
&\qquad\quad + \theta^3\biggl\|\int_0^t g^{-1/\sigma}(\tau)(\mathcal{K}(t-\tau) \\
&\qquad\qquad - \mathcal{G}(t-\tau))\biggl(G^3(\tau) - G(\tau)\int G^3(\tau,x)\,\mathrm{d}x\biggr)\,\mathrm{d}\tau\biggr\|_\infty \\
&\qquad \leqslant C\varepsilon^3(1+t)^{-1/2-\beta}g(\tau)^{-1/\sigma}(t),
\end{aligned}$$

where $\beta = \min(\frac{1}{4}, \frac{1}{2}\gamma)$. By an easy computation, we get

$$\frac{1}{\sqrt{4\pi\alpha\tau(t-\tau)}}\int \exp\biggl(-\frac{(x-y)^2}{4\alpha(t-\tau)} - \frac{by^2}{4\alpha\tau}\biggr)\,\mathrm{d}y = \frac{\exp(-x^2/4\alpha(t-((b-1)/b)\tau))}{\sqrt{bt-(b-1)\tau}}$$

for any $b \neq 0$ and

$$\int G^3(t,y)\,\mathrm{d}y = \frac{1}{(4\pi\alpha t)^{3/2}}\int \mathrm{e}^{-3y^2/4\alpha t}\,\mathrm{d}y = (4\pi\alpha t\sqrt{3})^{-1}.$$

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Therefore,

$$\begin{aligned}\mathcal{G}(t-\tau)\Big(G^3(\tau)-G(\tau)\int G^3(\tau,x)\,\mathrm{d}x\Big)&=\int G(t-\tau,x-y)\Big(G^3(\tau,y)-\frac{G(\tau,y)}{4\pi\alpha\tau\sqrt{3}}\Big)\,\mathrm{d}y\\&=\frac{1}{(4\pi\alpha\tau)^{3/2}\sqrt{4\pi\alpha(t-\tau)}}\int \mathrm{e}^{-(x-y)^2/4\alpha(t-\tau)}\Big(\mathrm{e}^{-3y^2/4\alpha t}-\frac{\mathrm{e}^{-y^2/4\alpha t}}{\sqrt{3}}\Big)\,\mathrm{d}y\\&=\frac{1}{\sqrt{3}(4\pi\alpha)^{3/2}\tau}\Big(\frac{\exp(-x^2/4\alpha(t-\frac{2}{3}\tau))}{\sqrt{t-2\tau/3}}-\frac{\mathrm{e}^{-x^2/4\alpha t}}{\sqrt{t}}\Big)\end{aligned}$$

uniformly in $x\in\boldsymbol{R}$. In particular, we see that

$$\|\Psi(t)\|_\infty\leqslant\frac{C\varepsilon^3}{\sqrt{t+1}}g^{-1/\sigma}(t)$$

and

$$\Psi(t,x)=-\frac{\mathrm{i}\theta^3}{\sqrt{3}(4\pi\alpha)^{3/2}}\int_0^t g^{-1/\sigma}(\tau)\Big(\frac{\exp(-x^2/4\alpha(t-\frac{2}{3}\tau))}{\sqrt{t-2\tau/3}}-\frac{\mathrm{e}^{-x^2/4\alpha t}}{\sqrt{t}}\Big)\frac{\mathrm{d}\tau}{\tau}+O(\varepsilon^3(1+t)^{-1/2-\beta}).$$

Then, by the fact that $|\Phi|^2=\theta^2|G|^2+\text{remainder terms}$, we obtain

$$\begin{aligned}-\operatorname{Im}\int|\Phi(t,x)|^2\Psi(t,x)\,\mathrm{d}x&=\frac{\theta^5}{\sqrt{3}(4\pi\alpha)^{5/2}t}\int\mathrm{d}x\,\mathrm{e}^{-x^2/2\alpha t}\\&\quad\times\int_0^t g^{-1/\sigma}(\tau)\Big(\frac{\exp(-x^2/4\alpha(t-\frac{2}{3}\tau))}{\sqrt{t-2\tau/3}}-\frac{\mathrm{e}^{-x^2/4\alpha t}}{\sqrt{t}}\Big)\frac{\mathrm{d}\tau}{\tau}\\&\qquad+O(\varepsilon^5(1+t)^{-1-\beta})\\&=\frac{\theta^5}{\sqrt{3}(4\pi\alpha)^{5/2}t}\int_0^t\frac{\mathrm{d}\tau}{\tau}g^{-1/\sigma}(\tau)\\&\quad\times\int\Big(\frac{\mathrm{e}^{-x^2/2\alpha t-x^2/4\alpha(t-2\tau/3)}}{\sqrt{t-2\tau/3}}-\frac{\mathrm{e}^{-3x^2/4\alpha t}}{\sqrt{t}}\Big)\,\mathrm{d}x+O(\varepsilon^5(1+t)^{-1-\beta})\\&=\frac{\theta^5}{3(4\pi\alpha)^2t}\int_0^t\frac{\mathrm{d}\tau}{\tau}g^{-1/\sigma}(\tau)\Big(\frac{1}{\sqrt{1-4\tau/9t}}-1\Big)+O\Big(\frac{\varepsilon^5}{(1+t)^{1+\beta}}\Big).\end{aligned}\tag{2.4}$$

Denote

$$\nu(\tau)=\int_0^\tau\Big(\frac{1}{\sqrt{1-z}}-1\Big)\frac{\mathrm{d}z}{z}=\log\Big(\frac{4(1-\sqrt{1-\tau})}{\tau(1+\sqrt{1-\tau})}\Big).$$

Since $|\nu(\tau)|\leqslant C\sqrt{\tau}$, and via the condition $|g'(t)|\leqslant C\theta^{2\sigma}/(t+1)$, we get

$$\Big|\int_0^t\nu\Big(\frac{4\tau}{9t}\Big)(g^{-1/2}(\tau))'\,\mathrm{d}\tau\Big|\leqslant\frac{C\theta^4}{\sqrt{t}}\int_0^t g^{-3/2}(\tau)\frac{\mathrm{d}\tau}{\sqrt{\tau}}\leqslant C\theta^4g^{-3/2}(t),$$

hence integration by parts with respect to τ yields

$$\int_0^t \frac{\mathrm{d}\tau}{\tau} g^{-1/2}(\tau)\left(\frac{1}{\sqrt{1-4\tau/9t}}-1\right) = \nu(\tfrac{4}{9})g^{-1/2}(t) + O(\varepsilon^4 g^{-3/2}(t)). \tag{2.5}$$

Substitution of (2.5) into (2.4) gives

$$-\operatorname{Im}\int |\Phi(t,x)|^2\Psi(t,x)\,\mathrm{d}x = \frac{\mu_2\theta}{t}g^{-1/2}(t) + O\left(\frac{\varepsilon^4}{t}g^{-3/2}(t)\right) \tag{2.6}$$

for all $t > 1$. By the condition of the lemma, we have

$$\||v|^2v - |\Phi|^2\Phi - 2|\Phi|^2\Psi - \Phi^2\bar{\Psi}\|_1 \leqslant \frac{C\varepsilon^7}{(t+1)g(t)}$$

and via (2.6) we obtain

$$\begin{aligned} -\operatorname{Im}\int |v(t,x)|^2 v(t,x)\,\mathrm{d}x &= -\operatorname{Im}\int |\Phi(t,x)|^2\Psi(t,x)\,\mathrm{d}x + O\left(\frac{\varepsilon^7}{tg(t)}\right) \\ &= \frac{\mu_{2\theta}}{t\sqrt{g(t)}} + O\left(\frac{\varepsilon^7}{tg(t)}\right), \end{aligned}$$

whence the representation (2.2) follows. Moreover, we have

$$\begin{aligned} &\left|\frac{1}{\theta}\operatorname{Im}\int_0^t \sqrt{g(\tau)}\,\mathrm{d}\tau \int |v|^2 v(\tau,x)\,\mathrm{d}x + \mu_2\log(1+t)\right| \\ &\qquad \leqslant \varepsilon + \frac{C\varepsilon^7}{\theta}\int_0^t \frac{\mathrm{d}\tau}{(1+\tau)\sqrt{1+\mu_1\log(1+\tau)}} + \frac{C\varepsilon^5}{\theta}\int_0^t \frac{\mathrm{d}\tau}{(1+\tau)^{1+\beta}} \\ &\qquad \leqslant \varepsilon + \varepsilon\sqrt{1+\mu_2\log(1+t)} \\ &\qquad \leqslant \tfrac{1}{8}(1+\mu_2\log(1+t)). \end{aligned}$$

Whence we get (2.3) for the case $\sigma = 2$. Lemma 2.2 is proved. □

3. Proof of theorem 1.1

First we state the local existence of classical solutions for the Cauchy problem (1.1) (for the proof, see [13, 19]).

THEOREM 3.1. *Let the initial data $u_0 \in \boldsymbol{W}$. Then, for some $T > 0$, there exists a unique solution $u(t,x) \in \boldsymbol{C}^\infty((0,T)\times\boldsymbol{R}) \cap \boldsymbol{C}((0,T);\boldsymbol{W})$ of the Cauchy problem (1.1). Moreover, if the initial data are sufficiently small $\|u_0\|_{\boldsymbol{W}} \leqslant \varepsilon$, then there exists a time $T > 1$ such that the solution $u(t,x)$ of the Cauchy problem (1.1) satisfies the estimates $\|u\|_{\boldsymbol{W}} \leqslant 2\varepsilon$ for all $t \in [0,T]$.*

We make a change of the dependent variable $u(t,x) = \mathrm{e}^{-\varphi(t)+\mathrm{i}\psi(t)}v(t,x)$. Then we get, from (1.1),

$$v_t + \mathcal{L}v + \mathrm{i}\mathrm{e}^{-2\varphi}|v|^2 v - (\varphi' - \mathrm{i}\psi')v = 0. \tag{3.1}$$

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Now let us demand that the real-valued functions $\varphi(t)$ and $\psi(t)$ satisfy the following condition

$$\int (\mathrm{i}\mathrm{e}^{-2\varphi}|v|^2 v - (\varphi' - \mathrm{i}\psi')v)\,\mathrm{d}x = 0,$$

whence, via (3.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int v(t,x)\,\mathrm{d}x = 0 \quad \text{for all } t > 0.$$

Therefore, if we choose $\varphi(0) = 0$ and $\psi(0) = \arg \hat{u}_0(0)$, we can write

$$\hat{v}(t,0) = \hat{v}_0(0) = \mathrm{e}^{\varphi(0)-\mathrm{i}\psi(0)}\hat{u}_0(0) \equiv \frac{\theta}{\sqrt{2\pi}} > 0$$

and equations

$$\varphi' = -\frac{1}{\theta}\mathrm{e}^{-2\varphi}\,\mathrm{Im}\int |v|^2 v\,\mathrm{d}x \quad \text{and} \quad \psi' = \frac{1}{\theta}\mathrm{e}^{-2\varphi}\,\mathrm{Re}\int |v|^2 v\,\mathrm{d}x. \tag{3.2}$$

Thus we obtain a system

$$\left.\begin{aligned} v_t + \mathcal{L}v + \mathrm{i}\mathrm{e}^{-2\varphi}\biggl(|v|^2 - \frac{1}{\theta}\int |v|^2 v\,\mathrm{d}x\biggr)v = 0, \quad x \in \boldsymbol{R}, \quad t > 0,\\ \varphi' = -\frac{1}{\theta}\mathrm{e}^{-2\varphi}\,\mathrm{Im}\int |v|^2 v\,\mathrm{d}x, \quad x \in \boldsymbol{R}, \quad t > 0,\\ v(0,x) = v_0(x), \quad \varphi(0) = 0, \quad x \in \boldsymbol{R}, \end{aligned}\right\} \tag{3.3}$$

where

$$v_0(x) = u(0,x)\exp\biggl(-\mathrm{i}\arg\int u(0,x)\,\mathrm{d}x\biggr).$$

After system (3.3) has been solved, the function ψ can be computed from (3.2). Multiplying the second equation of system (3.3) by the factor $\mathrm{e}^{2\sigma\varphi(t)}$ and then integrating with respect to time $t > 0$ and making a change of the dependent variables $v = \Phi + (\sigma - 1)\Psi + r$ and $\mathrm{e}^{2\sigma\varphi(t)} = g(t)$, we get the system of integral equations $(r, g) = (\mathcal{A}(r, g), \mathcal{B}(r, g))$, where $\Phi(t) = \mathcal{K}(t)v_0$ is the first approximation of the perturbation theory,

$$\Psi(t) = -\mathrm{i}\int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)\biggl(|\Phi|^2\Phi(\tau) - \frac{\Phi(\tau)}{\theta}\int |\Phi|^2\Phi(\tau,x)\,\mathrm{d}x\biggr)\,\mathrm{d}\tau$$

is the second approximation of the perturbation theory and the operators

$$\left.\begin{aligned} \mathcal{A}(r,g)(t) &= -\mathrm{i}\int_0^t g^{-1/\sigma}(\tau)\mathcal{K}(t-\tau)\\ &\quad\times\biggl(|v|^2 v(\tau) - (\sigma-1)|\Phi|^2\Phi(\tau) - \frac{v(\tau)}{\theta}\int |v|^2 v(\tau,x)\,\mathrm{d}x\\ &\qquad + (\sigma-1)\frac{\Phi(\tau)}{\theta}\int |\Phi|^2\Phi(\tau,x)\,\mathrm{d}x\biggr)\,\mathrm{d}\tau,\\ \mathcal{B}(r,g)(t) &= 1 - \frac{2\sigma}{\theta}\,\mathrm{Im}\int_0^t g^{(\sigma-1)/\sigma}(\tau)\int |v|^2 v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}\right\} \tag{3.4}$$

We prove that the transformation $(\mathcal{A},\mathcal{B})$ gives us the contraction mapping in the set

$$\boldsymbol{X} = \Big\{ r \in \boldsymbol{C}((0,\infty);\boldsymbol{W});\ g \in \boldsymbol{C}(0,\infty) : \\ \sup_{t>0}((1+t)^{1/2} g(t)\|r(t)\|_\infty + \sqrt[4]{1+t}\|r(t)\|_{n,0} \\ + (1+t)^{-1/4}\|r(t)\|_{0,1}) \leqslant C\varepsilon^{3+2\sigma}, \\ \tfrac{1}{2}(1+2\sigma\mu_\sigma \log(t+1)) \leqslant g(t) \leqslant 2(1+2\sigma\mu_\sigma\log(1+t)), \\ |g'(t)| \leqslant \frac{C\varepsilon^{2\sigma}}{1+t} \text{ for all } t>0 \Big\}.$$

First let us prove that the mapping $(\mathcal{A},\mathcal{B})$ transforms the set $\boldsymbol{X}$ into itself. When $(r,g) \in \boldsymbol{X}$, we get

$$\bigg\| |v|^2 v(t) - (\sigma-1)|\Phi|^2\Phi(t) - \frac{v(t)}{\theta}\int |v|^2 v(t,x)\,\mathrm{d}x \\ - (\sigma-1)\frac{\Phi(t)}{\theta}\int |\Phi|^2\Phi(t,x)\,\mathrm{d}x \bigg\|_{\boldsymbol{W}} \\ \leqslant C\varepsilon^3(1+t)^{-3/4} g^{-(\sigma-1)/\sigma}(t).$$

Therefore, applying lemma 2.1, we get the following estimates:

$$\|\mathcal{A}(r,g)(t)\|_\infty \leqslant C\varepsilon(1+t)^{-1/2}g^{-1}(t),$$
$$\|\mathcal{A}(r,g)(t)\|_{n,0} \leqslant \frac{C\varepsilon}{\sqrt[4]{1+t}},$$
$$\|\mathcal{A}(r,g)(t)\|_{0,1} \leqslant C\varepsilon\sqrt[4]{1+t}.$$

Furthermore, applying lemma 2.2, we get

$$\tfrac{1}{2}(1+\mu_\sigma\log(t+1)) \leqslant \mathcal{B}(r,g)(t) \leqslant 2(1+\mu_\sigma\log(1+t)), \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}(r,g)(t)\right| \leqslant \frac{C\varepsilon^{2\sigma}}{t+1}$$

for all $t>0$. Thus the set $\boldsymbol{X}$ is transformed into itself. We have, by lemma 2.1,

$$\sup_{t>0}(1+t)^{-\delta}\|\mathcal{A}(r_1,g_1)(t) - \mathcal{A}(r_2,g_2)(t)\|_\infty \\ \leqslant C\sup_{t>0}(1+t)^{-\delta}\bigg\|\int_0^t \mathcal{K}(t-\tau)\bigg(g_1^{-1/\sigma}(\tau)|v_1|^2 v_1 - g_2^{-1/\sigma}(\tau)|v_2|^2 v_2 \\ - g_1^{-1/\sigma}(\tau)\frac{v_1}{\theta}\int |v_1|^2 v_1\,\mathrm{d}x \\ + g_2^{-1/\sigma}(\tau)\frac{v_2}{\theta}\int |v_2|^2 v_2\,\mathrm{d}x\bigg)\,\mathrm{d}\tau\bigg\|_\infty \\ \leqslant C\sup_{t>0}(1+t)^{-\delta}\varepsilon^2\int_0^t \frac{\mathrm{d}\tau}{\tau^{1-\delta}}(\sup_{t>0}(1+t)^{-\delta}\|r_1(t)-r_2(t)\|_\infty \\ + \sup_{t>0}(1+t)^{-\delta}|g_1(t)-g_2(t)|) \\ \leqslant \tfrac{1}{2}(\sup_{t>0}(1+t)^{-\delta}\|r_1(t)-r_2(t)\|_\infty + \sup_{t>0}(1+t)^{-\delta}|g_1(t)-g_2(t)|)$$

with some small $\delta > 0$ and, via lemma 2.2, we get

$$\begin{aligned}
&\sup_{t>0}(1+t)^{-\delta}|\mathcal{B}(r_1,g_1)(t)-\mathcal{B}(r_2,g_2)(t)| \\
&\qquad \leqslant C\sup_{t>0}(1+t)^{-\delta}\left|\int_0^t\left(g_1^{-1/\sigma}(\tau)\int|v_1|^2v_1\,\mathrm{d}x - g_2^{-1/\sigma}(\tau)\int|v_2|^2v_2\,\mathrm{d}x\right)\mathrm{d}\tau\right| \\
&\qquad \leqslant \sup_{t>0} C\varepsilon^2(1+t)^{-\delta}\int_0^t\frac{\mathrm{d}\tau}{\tau^{1-\delta}}(\sup_{t>0}(1+t)^{-\delta}\|r_1(t)-r_2(t)\|_\infty \\
&\qquad\qquad + \sup_{t>0}(1+t)^{-\delta}|g_1(t)-g_2(t)|) \\
&\qquad \leqslant \tfrac{1}{2}(\sup_{t>0}(1+t)^{-\delta}\|r_1(t)-r_2(t)\|_\infty + \sup_{t>0}(1+t)^{-\delta}|g_1(t)-g_2(t)|).
\end{aligned}$$

Thus we see that the transformation $\mathcal{A}$, $\mathcal{B}$ is the contraction mapping. Therefore, there exists a unique solution r, g of the system of integral equations (3.4) from the space $\boldsymbol{X}$. From the proof of lemma 2.2, we see that

$$g(t) = \mu_1 \log t + O(\varepsilon \log \mu_1 \log t)$$

for $t \to \infty$ in the case $\sigma = 1$ and

$$g(t) = \mu_2 \log t + O(\varepsilon\sqrt{\mu_1 \log t})$$

for $t \to \infty$ in the case $\sigma = 2$. Also, we get

$$\begin{aligned}
\frac{1}{\theta}\operatorname{Re}\int|v|^2v(t,x)\,\mathrm{d}x &= \theta^2\operatorname{Re}\int|G|^2G(t,x)\,\mathrm{d}x + O\left(\frac{\varepsilon^2}{1+t}g^{-1/\sigma}(t)\right) \\
&= \frac{\omega\mu_\sigma^{1/\sigma}}{1+t} + O\left(\frac{\varepsilon^2}{1+t}g^{-1/\sigma}(t)\right),
\end{aligned}$$

whence we get the asymptotics

$$\begin{aligned}
\psi(t) &= \psi(0) + \frac{1}{\theta}\operatorname{Re}\int_0^t g^{-1/\sigma}(\tau)\int|v|^2v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \psi(0) + \int_0^t\frac{\omega\mu_\sigma^{1/\sigma}\,\mathrm{d}\tau}{(1+\tau)(1+\mu_\sigma\log(1+\tau))^{1/\sigma}} + O(\varepsilon) \\
&= \omega\log\log t + O(\varepsilon)
\end{aligned}$$

in the case $\sigma = 1$ and similarly $\psi(t) = \omega\sqrt{\log t} + O(\varepsilon\log\log t)$ in the case $\sigma = 2$. Therefore, via formulae

$$u(t,x) = \mathrm{e}^{-\varphi(t)+\mathrm{i}\psi(t)}v(t,x) = \mathrm{e}^{-\varphi(t)+\mathrm{i}\psi(t)}(\Phi + (\sigma-1)\Psi + r),$$

we obtain the asymptotics (1.4) and (1.5). Theorem 1.1 is proved.

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