Proceedings of the Edinburgh Mathematical Society (2006) **49**, 667–681 ©
DOI:10.1017/S0013091505000015 Printed in the United Kingdom

QUANTUM DEFORMATIONS OF CATALAN'S CONSTANT, MAHLER'S MEASURE AND THE HÖLDER–SHINTANI DOUBLE SINE FUNCTION

NOBUSHIGE KUROKAWA

Department of Mathematics, Tokyo Institute of Technology, Oo-okayama, Meguro-ku, Tokyo 152-8551, Japan (kurokawa@math.titech.ac.jp)

(Received 21 January 2005)

Abstract We study quantum deformations of Catalan's constant, Mahler's measure and the double sine function. We establish quantum deformations of basic relations between these three objects.

Keywords: quantum deformation; Mahler measure; Catalan constant

2000 *Mathematics subject classification:* Primary 11M36

1. Introduction

The Catalan constant [**3**]

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\ &= 0.915\,965\,594\,177\,219\,0\cdots \end{aligned} \tag{1.1}$$

is a famous mysterious constant appearing in many places in mathematics and physics. Using Euler's dilogarithm

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

G can be written as

$$G = \frac{1}{2\mathrm{i}}(\mathrm{Li}_2(\mathrm{i}) - \mathrm{Li}_2(-\mathrm{i})). \tag{1.2}$$

The integral representation

$$G = \int_0^1 \frac{\tan^{-1}(x)}{x}\,\mathrm{d}x \tag{1.3}$$

follows from the expression

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

The basic interpretation of G considers it as the special value of the zeta function

$$G = L(2, \chi_{-4}),$$

where

$$L(s, \chi_{-4}) = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s} \tag{1.4}$$

is the Dirichlet L-function for the non-principal Dirichlet character χ_{-4} modulo 4:

$$\chi_{-4}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the algebraic nature of G would be clarified via the $K_3(\mathbb{Z}[\sqrt{-1}])$-regulator according to the Lichtenbaum–Beilinson conjecture, since

$$G = \frac{\zeta_{\mathbb{Q}(\sqrt{-1})}(2)}{\zeta(2)} = \frac{6}{\pi^2}\zeta_{\mathbb{Q}(\sqrt{-1})}(2).$$

Recently, Rivoal and Zudilin [**19**] made progress with the irrationality problem for G: they proved that one of $L(2, \chi_{-4})\,(= G)$, $L(4, \chi_{-4})$, ..., $L(14, \chi_{-4})$ is irrational.

In previous papers [**9**, **11**, **13**] we showed that

$$G = 2\pi \log(F(\tfrac{1}{4})2^{-1/8}), \tag{1.5}$$

where

$$F(x) = \mathrm{e}^x \prod_{n=1}^{\infty} \biggl(\biggl(\frac{1 - x/n}{1 + x/n}\biggr)^n \mathrm{e}^{2x}\biggr) \tag{1.6}$$

is the double sine function defined by Hölder [**7**] in 1886. This is a quasi-periodic function satisfying

$$F(x + 1) = F(x)(-2\sin \pi x).$$

We refer to Manin [**17**] for an excellent survey of multiple sine functions. The formula (1.5) comes from

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^2} &= \frac{1}{2\mathrm{i}}(\mathrm{Li}_2(\mathrm{e}^{2\pi \mathrm{i}x}) - \mathrm{Li}_2(\mathrm{e}^{-2\pi \mathrm{i}x})) \\ &= 2\pi \log(F(x)(2\sin \pi x)^{-x}), \end{aligned} \tag{1.7}$$

which is valid for $0 < x < 1$ and where the left-hand side is the Clausen function (see [**14**] and [**18**]).

On the other hand, there is a formula due to Smyth [**20**] for G using the Mahler measure originating from the theory of transcendental numbers (Mahler [**16**]):

$$G = \tfrac{1}{2}\pi m(x + y - xy + 1). \tag{1.8}$$

See [**2**, **4**, **5**, **15**, **16**, **20**, **20**] for background on the Mahler measure. We recall that the Mahler measure of a rational function $f(x_1,\dots,x_n) \in \mathbb{C}(x_1,\dots,x_n)$ is given by

$$\begin{aligned} m(f) &= \int_0^1 \cdots \int_0^1 \log|f(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\dots,\mathrm{e}^{2\pi\mathrm{i}\theta_n})|\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \\ &= \mathrm{Re}\int_0^1 \cdots \int_0^1 \log(f(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\dots,\mathrm{e}^{2\pi\mathrm{i}\theta_n}))\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n. \end{aligned} \tag{1.9}$$

Thus we have a triangle:

$$\begin{array}{ccc} G & = & \tfrac{1}{2}\pi m(x+y-xy+1) \\ \| & & \| \\ & 2\pi\log(F(\tfrac{1}{4})2^{-1/8}). & \end{array}$$

The purpose of this paper is to make a quantum deformation (or a q-deformation) of this triangle to

$$\begin{array}{ccc} G_q & = & \tfrac{1}{2}\pi m_q\left(\dfrac{x-1}{x+1}y+1\right) \\ \| & & \| \\ & \tfrac{1}{2}\mathrm{i}\log F_q(\tfrac{1}{4}) \pmod{\pi\mathbb{Z}} & \end{array}$$

for $0<q<1$. (Actually, we treat all $q \geqslant 0$.) When letting $q \uparrow 1$ (the 'classical limit') we recover the original triangle: note that

$$\begin{aligned} m\left(\frac{x-1}{x+1}y+1\right) &= m((x-1)y+x+1) \qquad (\text{since } m(x+1)=0) \\ &= m(xy-y+x+1) \qquad (\text{taking } y \mapsto -y) \\ &= m(-xy+y+x+1). \end{aligned}$$

On the other hand, taking the 'crystal limit' $q \downarrow 0$ we have

$$\begin{array}{ccc} G_0 & = & \tfrac{1}{2}\pi m_0\left(\dfrac{x-1}{x+1}y+1\right) \\ \| & & \| \\ & \tfrac{1}{2}\mathrm{i}\log F_0(\tfrac{1}{4}) & \end{array}$$

with

$$G_0 = \tfrac{1}{4}\pi, \qquad m_0\left(\frac{x-1}{x+1}y+1\right) = \tfrac{1}{2}, \qquad F_0(\tfrac{1}{4}) = -\mathrm{i}.$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

The q-deformations are defined as follows. We define

$$G_q = \int_0^1 \frac{\tan^{-1}(x)}{x}\,\mathrm{d}_q x \tag{1.10}$$

by the Jackson integral. For simplicity, we restrict ourselves here to the case $0 < q < 1$ (see § 2 for the general case), then the Jackson integral is given as

$$\int_0^1 f(x)\,\mathrm{d}_q x = \sum_{n=0}^{\infty} f(q^n)(q^n - q^{n+1}). \tag{1.11}$$

It is shown that

$$\begin{aligned} G_q &= \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n[n]_q} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[2n+1]_q}, \end{aligned} \tag{1.12}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Using the quantum dilogarithm

$$\mathrm{Li}_{2,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n[n]_q} \tag{1.13}$$

(see [**8**] and [**6**]) we see that

$$G_q = \frac{1}{2\mathrm{i}}(\mathrm{Li}_{2,q}(\mathrm{i}) - \mathrm{Li}_{2,q}(-\mathrm{i})). \tag{1.14}$$

The q-Mahler measure was introduced in a previous paper [**10**] as

$$m_q(f) = \mathrm{Re} \int_0^1 \cdots \int_0^1 l_q(f(\mathrm{e}^{2\pi\mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi\mathrm{i}\theta_n}))\,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \tag{1.15}$$

for a rational function $f(x_1, \dots, x_n) \in \mathbb{C}(x_1, \dots, x_n)$ with the q-logarithm

$$l_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{[n]_q} \tag{1.16}$$

originally defined for $|x-1| < 1$ and analytically continued to $x \in \mathbb{C}$ via

$$l_q(x) = (1-q) \sum_{m=0}^{\infty} \frac{x-1}{x-1+q^{-m}}. \tag{1.17}$$

Lastly, we put

$$F_q(x) = \prod_{n=0}^{\infty} \left(\frac{1-q^n \mathrm{e}^{2\pi\mathrm{i}x}}{1-q^n \mathrm{e}^{-2\pi\mathrm{i}x}} \right)^{1-q}, \tag{1.18}$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

which can also be expressed as

$$F_q(x) = \exp\left(-2\mathrm{i}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n[n]_q}\right) \tag{1.19}$$

and

$$F_q(x) = \exp(\mathrm{Li}_{2,q}(\mathrm{e}^{-2\pi\mathrm{i}x}) - \mathrm{Li}_{2,q}(\mathrm{e}^{2\pi\mathrm{i}x})). \tag{1.20}$$

In § 4, we also make another deformation of $F(x)$ using the regularized double sine function $F(x,(1,\tau))$ introduced by Shintani [**21**] to investigate Kronecker's Jugendtraum for real quadratic fields.

Concerning the parameter q, we investigate certain dualities such as $G_q \longleftrightarrow G_{q^{-1}}$, $m_q(f) \longleftrightarrow m_{q^{-1}}(f)$ and $F_q \leftrightarrow F_{q^{-1}}$.

Lastly we notice that our result can be generalized to some extent to other polynomials and rational functions. We refer to the paper [**12**] for related matters.

2. Quantum Catalan constant

We already defined G_q for $0 < q < 1$. In the case $q > 1$ we also define

$$G_q = \int_0^1 \frac{\tan^{-1}(x)}{x}\,\mathrm{d}_q x$$

via the Jackson integral

$$\int_0^1 f(x)\,\mathrm{d}_q x = \sum_{n=1}^{\infty} f(q^{-n})(q^{1-n} - q^{-n}).$$

We easily deduce the following properties of G_q.

Theorem 2.1.

(1) $G_q = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)[2n+1]_q}$ *for* $0 < q < 1$ *and* $q > 1$.

(2) $\displaystyle\lim_{q\uparrow 1} G_q = G$.

(3) $\displaystyle\lim_{q\downarrow 0} G_q = \tfrac{1}{4}\pi$.

(4) $\displaystyle\lim_{q\uparrow +\infty} G_q = 1$.

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

Proof. (1) Let $0 < q < 1$.

At this point we must be careful because of the non-absolute convergence of $\tan^{-1}(1)$. We have

$$\begin{aligned}
G_q &= (1-q)\sum_{n=0}^{\infty} \tan^{-1}(q^n) \\
&= (1-q)\tan^{-1}(1) + (1-q)\sum_{n=1}^{\infty} \tan^{-1}(q^n) \\
&= \tfrac{1}{4}\pi(1-q) + (1-q)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m q^{(2m+1)n}}{2m+1} \\
&= (1-q)\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} + (1-q)\sum_{m=0}^{\infty} \frac{(-1)^m q^{2m+1}}{(2m+1)(1-q^{2m+1})} \\
&= (1-q)\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}\left(1 + \frac{q^{2m+1}}{1-q^{2m+1}}\right) \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)[2m+1]_q}.
\end{aligned}$$

The case $q > 1$ is exactly similar.

Parts (2), (3) and (4) follow from (1) by noting the uniform convergence of the series for G_q on taking the following limits:

$$\lim_{q\uparrow 1}[n]_q = n,$$
$$\lim_{q\downarrow 0}[n]_q = 1$$

and

$$\lim_{q\uparrow\infty}[n]_q = \begin{cases} 1 & \text{if } n = 1, \\ +\infty & \text{if } n > 1. \end{cases}$$

□

3. Quantum Mahler measure

We prove the following result.

Theorem 3.1. *For all $q > 0$,*

$$G_q = \tfrac{1}{2}\pi m_q\left(\frac{x-1}{x+1}y + 1\right).$$

We fix the notation by putting

$$l_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{[n]_q},$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

which is absolutely convergent in

$$|x-1| < \max\{1, q\},$$

where

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1, \\ n & \text{if } q = 1. \end{cases}$$

The function $l_1(x) = \log(x)$ is the well-known logarithm. For $q \neq 1$, the analytic continuation of $l_q(x)$ to all $x \in \mathbb{C}$ is given by

$$l_q(x) = \begin{cases} (1-q)\displaystyle\sum_{m=0}^{\infty} \frac{x-1}{x-1+q^{-m}} & \text{if } 0 < q < 1, \\ (q-1)\displaystyle\sum_{m=1}^{\infty} \frac{x-1}{x-1+q^{m}} & \text{if } q > 1. \end{cases}$$

Both calculations are similar and easy. For example, when $0 < q < 1$,

$$\begin{aligned} l_q(x) &= (1-q)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{1-q^n} \\ &= (1-q)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} (-1)^{n-1}(x-1)^n q^{nm} \\ &= (1-q)\sum_{m=0}^{\infty} \frac{(x-1)q^m}{1+(x-1)q^m} \\ &= (1-q)\sum_{m=0}^{\infty} \frac{x-1}{x-1+q^{-m}}. \end{aligned}$$

Thus $l_q(x)$ is meromorphic on $\mathbb{C}$ for $0 < q < 1$ or $q > 1$. We notice that

$$l_0(x) = 1 - 1/x$$

and

$$l_\infty(x) = x - 1$$

are obtained as $\lim_{q\downarrow 0} l_q(x)$ and $\lim_{q\uparrow +\infty} l_q(x)$.

Proof of Theorem 3.1. We calculate

$$m_q = m_q\left(a\frac{x-1}{x+1}y + 1\right)$$

for $0 < a \leqslant 1$, and show that

$$\begin{aligned} m_q &= \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_q} \\ &= \frac{1}{\pi \mathrm{i}}(\mathrm{Li}_{2,q}(\mathrm{i}a) - \mathrm{Li}_{2,q}(-\mathrm{i}a)). \end{aligned} \tag{3.1}$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

Then we see easily that

$$m_q\left(\frac{x-1}{x+1}y+1\right)=\frac{2}{\pi}G_q$$

from Theorem 2.1 (1). We prove (3.1) for $0<q<1$. The calculation is similar for $q>1$. By definition,

$$\begin{aligned} m_q &= \operatorname{Re}\int_0^1\int_0^1 l_q\left(a\frac{e^{2\pi i\theta_1}-1}{e^{2\pi i\theta_1}+1}e^{2\pi i\theta_2}+1\right)d\theta_1\,d\theta_2 \\ &= (1-q)\operatorname{Re}\sum_{m=0}^{\infty}\int_0^1\int_0^1\frac{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}+q^{-m}}\,d\theta_1\,d\theta_2. \end{aligned}$$

We show that

$$\int_0^1\int_0^1\frac{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}+q^{-m}}\,d\theta_1\,d\theta_2=\frac{2}{\pi}\tan^{-1}(aq^{-m}). \tag{3.2}$$

Then, using the absolutely convergent series

$$\tan^{-1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

for $|x|<1$, we have

$$\begin{aligned} m_q &= \frac{2}{\pi}(1-q)\sum_{m=0}^{\infty}\tan^{-1}(aq^{-m}) \\ &= \frac{2}{\pi}(1-q)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(aq^{-m})^{2n+1}}{2n+1} \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_q}, \end{aligned}$$

which is valid for $0<a<1$. In the case $a=1$ we again take care, because of the non-absolutely converging series $\tan^{-1}(1)$:

$$\begin{aligned} m_q &= \frac{2}{\pi}(1-q)\sum_{m=0}^{\infty}\tan^{-1}(q^m) \\ &= \frac{2}{\pi}(1-q)\tan^{-1}(1)+\frac{2}{\pi}(1-q)\sum_{m=1}^{\infty}\tan^{-1}(q^m) \\ &= \tfrac{1}{2}(1-q)+\frac{2}{\pi}(1-q)\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(2n+1)m}}{2n+1} \\ &= \frac{2}{\pi}(1-q)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}+\frac{2}{\pi}(1-q)\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}}{(2n+1)(1-q^{2n+1})} \end{aligned}$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

$$= \frac{2}{\pi}(1-q)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(1+\frac{q^{2n+1}}{1-q^{2n+1}}\right)$$
$$= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)[2n+1]_q}$$
$$= \frac{2}{\pi}G_q.$$

We now return to the calculation of the integral (3.2):

$$I_m = \int_0^1\int_0^1 \frac{a\mathrm{i}\tan(\pi\theta_1)\mathrm{e}^{2\pi\mathrm{i}\theta_2}}{a\mathrm{i}\tan(\pi\theta_1)\mathrm{e}^{2\pi\mathrm{i}\theta_2}+q^{-m}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$
$$= 1 - q^{-m}\int_0^1\int_0^1 \frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{a\mathrm{i}\tan(\pi\theta_1)\mathrm{e}^{2\pi\mathrm{i}\theta_2}+q^{-m}}.$$

Notice that

$$\int_0^1 \frac{\mathrm{d}\theta_2}{a\mathrm{i}\tan(\pi\theta_1)\mathrm{e}^{2\pi\mathrm{i}\theta_2}+q^{-m}} = \begin{cases} q^m & \text{if } |a\mathrm{i}\tan(\pi\theta_1)| < q^{-m}, \\ 0 & \text{if } |a\mathrm{i}\tan(\pi\theta_1)| > q^{-m}. \end{cases}$$

Take the number α_m in $0 < \alpha_m < \frac{1}{2}$ that satisfies

$$\cot(\pi\alpha_m) = aq^m.$$

Then,

$$|a\mathrm{i}\tan(\pi\theta_1)| < q^{-m} \iff |\cot(\pi\theta_1)| > aq^m$$
$$\iff 0 < \theta_1 < \alpha_m \ \text{ or } \ 1-\alpha_m < \theta_1 < 1.$$

Hence

$$I_m = 1 - q^{-m}\left(\int_0^{\alpha_m} q^m\,\mathrm{d}\theta_1 + \int_{1-\alpha_m}^1 q^m\,\mathrm{d}\theta_1\right)$$
$$= 1 - 2\alpha_m$$
$$= 1 - \frac{2}{\pi}\cot^{-1}(aq^m)$$
$$= \frac{2}{\pi}\tan^{-1}(aq^m).$$

□

Here we introduce a reciprocity: $G_q \longleftrightarrow G_{q^{-1}}$.

Theorem 3.2. *$(G_q - \frac{1}{4}\pi)/\sqrt{q}$ is invariant under $q \mapsto q^{-1}$.*

Proof. This can be proved directly from the definition of G_q, but we prefer to obtain it from a more general reciprocity, $m_q(f) \longleftrightarrow m_{q^{-1}}(f)$, shown in Theorem 3.3 below, using Theorem 3.1. □

Theorem 3.3. *$(m_q(f) - m_0(f))/\sqrt{q}$ is invariant under $q \mapsto q^{-1}$.*

Proof. From the definition of $m_q(f)$ it is sufficient to show the invariance of

$$\frac{l_q(x) - l_0(x)}{\sqrt{q}}$$

under $q \mapsto q^{-1}$. This is obvious for $q = 1$. Let $0 < q < 1$. Then

$$l_q(x) = (1-q)\left(1 - \frac{1}{x}\right) + (1-q)\sum_{m=1}^{\infty} \frac{x-1}{x-1+q^{-m}}$$

and

$$l_{q^{-1}}(x) = (q^{-1} - 1)\sum_{m=1}^{\infty} \frac{x-1}{x-1+q^{-m}}.$$

Hence

$$l_q(x) = (1-q)l_0(x) + q l_{q^{-1}}(x).$$

Thus

$$\frac{l_q(x) - l_0(x)}{\sqrt{q}} = \frac{l_{q^{-1}}(x) - l_0(x)}{\sqrt{q^{-1}}}.$$

□

4. The quantum Hölder double sine function

Before introducing the quantum double sine function, we recall briefly the expression

$$F(x) = (2\sin \pi x)^x \exp\left(\frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^2}\right) \tag{4.1}$$

in $0 < x < 1$ for Hölder's double sine function $F(x)$. There are several ways to reach this, and we refer to [**11**, **13**] for a general treatment containing multiple sine functions; we note that $F(x) = \mathcal{S}_2(x)$ in [**11**, **13**] and that $\mathcal{S}_r(x)$ was treated for integers $r \geqslant 1$.

A simple way to show (4.1) is as follows. From the definition (1.6) of $F(x)$ we have

$$\log F(x) = x + \sum_{n=1}^{\infty}\left(n\left(\log\left(1 - \frac{x}{n}\right) - \log\left(1 + \frac{x}{n}\right)\right) + 2x\right)$$

and

$$\begin{aligned}\frac{F'(x)}{F(x)} &= 1 + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2} \\ &= \pi x \cot(\pi x).\end{aligned}$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

Hence, noting that $F(0)=1$, we see that

$$\begin{aligned} F(x) &= \exp\biggl(\int_0^x \pi t\cot(\pi t)\,\mathrm{d}t\biggr) \\ &= \exp\biggl([t\log(\sin\pi t)]_0^x - \int_0^x \log(\sin\pi t)\,\mathrm{d}t\biggr) \\ &= \exp\biggl(x\log(\sin\pi x) - \int_0^x \log(\sin\pi t)\,\mathrm{d}t\biggr). \end{aligned}$$

Here we use

$$\log(\sin\pi x) = -\log 2 - \sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n}$$

for $0<x<1$, then, by uniform convergence,

$$\begin{aligned} \int_0^x \log(\sin\pi t)\,\mathrm{d}t &= -x\log 2 - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^x \cos(2\pi nt)\,\mathrm{d}t \\ &= -x\log 2 - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2}. \end{aligned}$$

Thus, we obtain (4.1). Hence, letting $x=\frac{1}{4}$ we get

$$\begin{aligned} G &= 2\pi\log(F(\tfrac{1}{4})(2\sin\tfrac{1}{4}\pi)^{-1/4}) \\ &= 2\pi\log(F(\tfrac{1}{4})2^{-1/8}), \end{aligned}$$

as in (1.5).

Now define the quantum Hölder double sine function:

$$F_q(x) = \begin{cases} \displaystyle\prod_{n=1}^{\infty}\biggl(\frac{1-q^{-n}\mathrm{e}^{2\pi\mathrm{i}x}}{1-q^{-n}\mathrm{e}^{-2\pi\mathrm{i}x}}\biggr)^{q-1} & \text{for } q>1, \\ (F(x)(2\sin\pi x)^{-x})^{-4\pi\mathrm{i}} & \text{for } q=1, \\ \displaystyle\prod_{n=0}^{\infty}\biggl(\frac{1-q^{n}\mathrm{e}^{2\pi\mathrm{i}x}}{1-q^{n}\mathrm{e}^{-2\pi\mathrm{i}x}}\biggr)^{1-q} & \text{for } 0<q<1, \\ -\mathrm{e}^{2\pi\mathrm{i}x} & \text{for } q=0. \end{cases}$$

Theorem 4.1.

(1) $F_q(x) = \exp\biggl(-2\mathrm{i}\displaystyle\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m[m]_q}\biggr) = \exp(\mathrm{Li}_{2,q}(\mathrm{e}^{-2\pi\mathrm{i}x}) - \mathrm{Li}_{2,q}(\mathrm{e}^{2\pi\mathrm{i}x}))$.

(2) $G_q \equiv \frac{1}{2}\mathrm{i}\log F_q(\frac{1}{4}) \bmod \pi\mathbb{Z}$.

Proof. We prove the case $0 < q < 1$. The case $q > 1$ is similar.

(1) From the definition of $F_q(x)$,

$$\begin{aligned}\log F_q(x) &= (1-q)\sum_{n=0}^{\infty}(\log(1-q^n\mathrm{e}^{2\pi\mathrm{i}x}) - \log(1-q^n\mathrm{e}^{-2\pi\mathrm{i}x}))\\ &= (1-q)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}mx}-\mathrm{e}^{2\pi\mathrm{i}mx}}{m}q^{nm}\\ &= (1-q)\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}mx}-\mathrm{e}^{2\pi\mathrm{i}mx}}{m(1-q^m)}\\ &= \sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}mx}-\mathrm{e}^{2\pi\mathrm{i}mx}}{m[m]_q}\\ &= \mathrm{Li}_{2,q}(\mathrm{e}^{-2\pi\mathrm{i}x}) - \mathrm{Li}_{2,q}(\mathrm{e}^{2\pi\mathrm{i}x})\\ &= -2\mathrm{i}\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m[m]_q}.\end{aligned}$$

(2) This follows from (1) by putting $x = \frac{1}{4}$.

□

Theorem 4.2. *The function*

$$\frac{\log F_q(x) - \log F_0(x)}{\sqrt{q}}$$

is invariant under $q \mapsto q^{-1}$.

Proof. This follows from Theorem 4.1 (1) with some calculation. Alternatively, we may proceed as follows. For a suitable series $\{a(n) \mid n = 1, 2, \dots\}$ and $0 \leqslant q \leqslant \infty$ define

$$A_q = \sum_{n=1}^{\infty}\frac{a(n)}{[n]_q}$$

with

$$[n]_q = \begin{cases}\dfrac{1-q^n}{1-q} & \text{for } q \neq 1, \infty,\\ n & \text{for } q = 1,\\ \delta_{n,1} & \text{for } q = \infty.\end{cases}$$

Then $(A_q - A_0)/\sqrt{q}$ is invariant under $q \mapsto q^{-1}$. Actually, the proof is simple. Let $0 < q < 1$. Then

$$\begin{aligned}A_{q^{-1}} &= \sum_{n=1}^{\infty}\frac{a(n)}{[n]_{q^{-1}}}\\ &= q^{-1}\sum_{n=1}^{\infty}\frac{q^n a(n)}{[n]_q},\end{aligned}$$

so

$$\begin{aligned} A_q - qA_{q^{-1}} &= \sum_{n=1}^{\infty} \frac{(1-q^n)a(n)}{[n]_q} \\ &= (1-q)\sum_{n=1}^{\infty} a(n) \\ &= (1-q)A_0. \end{aligned}$$

□

There is another quantization of $F(x)$ using the regularized double sine function

$$F(x,(\omega_1,\omega_2)) = \frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}$$

due to Shintani [**21**], where $\Gamma_2(x,(\omega_1,\omega_2))$ is the (regularized) double gamma function of Barnes [**1**]; we refer to [**11**] for a general treatment of regularized multiple sine functions $S_r(x,(\omega_1,\dots,\omega_r))$ generalizing $S_2(x,(\omega_1,\omega_2)) = F(x,(\omega_1,\omega_2))$ and $S_1(x,\omega) = 2\sin(\pi x/\omega)$.

Theorem 4.3. *Let* $0<q<1$ *and put* $\tau = \log q/2\pi\mathrm{i}$. *Then*

$$G_q = \tfrac{1}{2}\mathrm{i}(1-q)\log\biggl(\frac{F(\frac{1}{4},(1,\tau))F(\frac{1}{4},(1,-\tau))}{(1+\mathrm{i})\sin(\pi/4\tau)}\biggr).$$

Proof. From [**21**, Proposition 5] we have

$$\begin{aligned} \log F(x,(1,\tau)) = {}&-\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}mx}}{m(1-q^m)} + \sum_{m=1}^{\infty}\frac{q'^m\mathrm{e}^{2\pi\mathrm{i}mx/\tau}}{m(1-q^m)} \\ &+ \tfrac{1}{2}\pi\mathrm{i}\biggl(\frac{x^2}{\tau} - \biggl(\frac{1}{\tau}+1\biggr)x\biggr) + \tfrac{1}{4}\pi\mathrm{i} + \tfrac{1}{2}\pi\mathrm{i}\biggl(\frac{1}{\tau}+\tau\biggr), \end{aligned}$$

where

$$q = \mathrm{e}^{2\pi\mathrm{i}\tau} \quad \text{and} \quad q' = \mathrm{e}^{-2\pi\mathrm{i}/\tau}.$$

Similarly,

$$\begin{aligned} \log F\biggl(y,\biggl(1,-\frac{1}{\tau}\biggr)\biggr) = {}&-\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}my}}{m(1-q'^m)} + \sum_{m=1}^{\infty}\frac{q^m\mathrm{e}^{-2\pi\mathrm{i}my}}{m(1-q^m)} \\ &+ \tfrac{1}{2}\pi\mathrm{i}(-\tau y^2 - (-\tau+1)y) + \tfrac{1}{4}\pi\mathrm{i} + \tfrac{1}{2}\pi\mathrm{i}\biggl(-\tau-\frac{1}{\tau}\biggr). \end{aligned}$$

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

Hence,

$$\begin{aligned}\log F(x,(1,\tau)) + \log F\biggl(\frac{x}{\tau},\biggl(1,-\frac{1}{\tau}\biggr)\biggr) &= -\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}mx}-q^m\mathrm{e}^{-2\pi\mathrm{i}mx}}{m(1-q^m)} - \sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}mx/\tau}}{m} - \frac{\pi\mathrm{i}x}{\tau} + \tfrac{1}{2}\pi\mathrm{i} \\ &= -\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}mx}-\mathrm{e}^{-2\pi\mathrm{i}mx}}{m(1-q^m)} - \sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}mx}}{m} - \sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}mx/\tau}}{m} - \frac{\pi\mathrm{i}x}{\tau} + \tfrac{1}{2}\pi\mathrm{i} \\ &= -2\mathrm{i}\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m(1-q^m)} + \log(1-\mathrm{e}^{-2\pi\mathrm{i}x}) + \log(1-\mathrm{e}^{2\pi\mathrm{i}x/\tau}) - \frac{\pi\mathrm{i}x}{\tau} + \tfrac{1}{2}\pi\mathrm{i}.\end{aligned}$$

Then letting $x = \frac{1}{4}$ and multiplying by $\frac{1}{2}\mathrm{i}(1-q)$ we get

$$\begin{aligned}\tfrac{1}{2}\mathrm{i}(1-q)\log\biggl(F(\tfrac{1}{4},(1,\tau))F\biggl(\frac{1}{4\tau},\biggl(1,-\frac{1}{\tau}\biggr)\biggr)\biggr) \\ = G_q + \tfrac{1}{2}\mathrm{i}(1-q)\biggl(\tfrac{1}{2}\log 2 + \tfrac{3}{4}\pi\mathrm{i} - \frac{\pi\mathrm{i}}{4\tau} + \log(1-\mathrm{e}^{\pi\mathrm{i}/2\tau})\biggr).\end{aligned}$$

Then, as the homogeneity

$$F(cx,(c\omega_1,c\omega_2)) = F(x,(\omega_1,\omega_2))$$

proved in [**11**] implies that

$$\begin{aligned}F\biggl(\frac{1}{4\tau},\biggl(1,-\frac{1}{\tau}\biggr)\biggr) &= F(\tfrac{1}{4},(\tau,-1)) \\ &= F(\tfrac{1}{4},(-1,\tau)),\end{aligned}$$

we obtain Theorem 4.3. □

References

1. E. W. Barnes, On the theory of the multiple gamma function, *Trans. Camb. Phil. Soc.* **19** (1904), 374–425.
2. D. W. Boyd, Mahler's measure and special values of L-functions, *Exp. Math.* **7** (1998), 37–82.
3. E. Catalan, *Recherches sur la constante G, et sur les intégrales eulériennes*, Mémoires de l'Académie de Saint-Pétersbourg (7), Volume 31, No. 3 (l'Académie de Saint-Pétersbourg, St Petersburg, 1883).
4. C. Deninger, Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions, *J. Am. Math. Soc.* **10** (1997), 259–281.
5. G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics* (Springer, 1999).
6. L. D. Faddeev and R. M. Kashaev, Quantum dilogarithm, *Mod. Phys. Lett.* A **9** (1994), 427–434.
7. O. Hölder, Ueber eine transcendente Function, *Nachr. Ges. Wiss. Göttingen* **1886-16** (1886), 514–522.

https://doi.org/10.1017/S0013091505000015 Published online by Cambridge University Press

8. A. N. Kirillov, Dilogarithm identities, *Prog. Theor. Phys. Suppl.* **118** (1995), 61–142.
9. N. Kurokawa, Multiple zeta functions: an example, *Adv. Stud. Pure Math.* **21** (1992), 219–226.
10. N. Kurokawa, A q-Mahler measure, *Proc. Jpn Acad.* A **80** (2004), 70–73.
11. N. Kurokawa and S. Koyama, Multiple sine functions, *Forum Math.* **15** (2003), 839–876.
12. N. Kurokawa and H. Ochiai, Mahler measures via the crystalization, *Comment. Math. Univ. Sancti Pauli* **54** (2005), 121–137.
13. N. Kurokawa, H. Ochiai and M. Wakayama, Multiple trigonometry and zeta functions, *J. Ramanujan Math. Soc.* **17** (2002), 101–113.
14. L. Lewin, *Polylogarithms and associated functions* (North-Holland, Amsterdam, 1981).
15. D. Lind, K. Schmidt and T. Ward, Mahler measure and entropy for commuting automorphisms of compact groups, *Invent. Math.* **101** (1990), 593–629.
16. K. Mahler, An application of Jensen's formula to polynomials, *Mathematika* **7** (1960), 98–100.
17. Yu. I. Manin, Lectures on zeta functions and motives (according to Deninger and Kurokawa), *Astérisque* **228** (1995), 121–163.
18. J. Milnor, On polylogarithms, Hurwitz zeta functions, and the Kubert identities, *Enseign. Math.* **29** (1983), 281–322.
19. T. Rivoal and W. Zudilin, Diophantine properties of numbers related to Catalan's constant, *Math. Ann.* **326** (2003), 705–721.
20. C. J. Smyth, On measures of polynomials in several variables, *Bull. Austral. Math. Soc.* **23** (1981), 49–63 (G. Myerson and C. J. Smyth, Corrigendum, *Bull. Austral. Math. Soc.* **26** (1982), 317–319).
21. T. Shintani, On a Kronecker limit formula for real quadratic fields, *J. Fac. Sci. Univ. Tokyo* 1A **24** (1977), 167–199.