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Direct and inverse problems of wave propagation through a one-dimensional inhomogeneous medium†

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This paper deals with a wave process induced by an acoustic impulse in a vertically inhomogeneous half-space. The solvability of the corresponding direct initial boundary value problem is proved. We also consider a class of one-dimensional inverse problems including, in particular, the classical inverse problem of the scattering theory, the transmission and reflection problems of seismology, etc. It is shown that all these inverse problems are equivalent in a certain sense, and therefore can be solved by identical methods. Theoretical results are illustrated by numerical examples.

1 Introduction

Let us consider propagation of plane SH-waves through an elastic half-space $z \geqslant 0$ of the Euclidean space R^3 with mechanical properties which depend only on the depth z. It is assumed that the waves are polarized along some axis parallel to the plane $z = 0$. Under these conditions, a displacement w of the medium depends only on time t and z, and satisfies the acoustic equation

$$(\mu w_z)_z = \rho w_{tt}. \tag{1.1}$$

Here $\rho(z)$ is the density of the medium and $\mu(z)$ is the shear modulus at depth z. It is assumed that ρ and μ are strictly positive, twice continuously differentiable, and that, for $z \geqslant z_0 > 0$, they are constant and equal to some known values ρ_0 and μ_0.

The wave process is caused by external sources either at the surface or inside the medium. The initial and boundary conditions for equation (1.1) depend on the location of these sources and their physical nature. We consider a few typical boundary value problems.

Let the normal stresses at the surface $z = 0$ be zero, i.e.

$$w_z\big|_{z=0} = 0, \tag{1.2}$$

and suppose that oscillations are caused by an incident wave of the type $\varphi(tv_0 + z)$ which propagates from the region $z > z_0$ at the velocity $v_0 = \sqrt{\mu_0/\rho_0}$. The function $\varphi(z)$ is

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considered to be zero outside some interval (a, ∞). This means that the wave has a leading front which moves along the z-axis towards the boundary $z = 0$. Obviously, the full wave field in the medium coincides with the incident wave until its leading front reaches the domain $z \leqslant z_0$. In other words, there exists a moment t_0 such that $w(z, t) \equiv \varphi(tv_0 + z)$ for all $t \leqslant t_0$, $z \geqslant 0$. Without loss of generality we can assume that $t_0 = 0$. Then the support of $\varphi(z)$ lies in the domain $z \geqslant z_0$, and the condition

$$w\big|_{t\leqslant 0} = \varphi(tv_0 + z) \tag{1.3}$$

is valid. The system (1.1)–(1.3) is called a problem with internal sources or simply an internal problem. We denote it by (i).

If the sources of oscillations are outside the half-space or at its boundary, condition (1.2) should be replaced by the inhomogeneous equation

$$2\mu w_z\big|_{z=0} = \psi(t), \tag{1.4}$$

where $\psi(t)$ is zero for $t \leqslant 0$. Since the medium must be at rest before the action of (1.4), we have

$$w\big|_{t\leqslant 0} \equiv 0. \tag{1.5}$$

The system (1.1), (1.4), (1.5) is called an external problem and we denote it by (e).

Equations (1.1)–(1.5) are conveniently rewritten in a canonical form. Denote the velocity of wave propagation at the depth z by $v(z) = \sqrt{\mu(z)/\rho(z)}$, and assume that

$$x(z) = \int_0^z \frac{dy}{v(y)}, \quad \sigma(x) = \rho(z(x)) \cdot v(z(x)).$$

Physically $x(z)$ is the travel time from the surface to the depth z, and $\sigma(x)$ is called the *acoustic impedance*. Let us turn from z to the new independent variable x, and from $w(z, t)$ to $u(x, t) = \sqrt{\sigma(x)}w(z(x), t)$. After a simple transformation, equation (1.1) leads to

$$u_{xx} - q(x)u = u_{tt}, \quad x > 0,\ t \in R; \quad q(x) = (\ln\sqrt{\sigma})'' + \left[(\ln\sqrt{\sigma})'\right]^2. \tag{1.6}$$

Conditions (1.2)–(1.3) become

$$(u_x - ku)\big|_{x=0} = 0, \quad u\big|_{t\leqslant 0} = f(t + x); \quad f(x) = \sqrt{\sigma_0}\varphi(v_0(x - x_0) + z_0). \tag{1.7}$$

Here $2k = \sigma'(0)/\sigma(0)$, $x_0 = x(z_0)$, $\sigma_0 = \rho_0 v_0$. Recall that ρ, μ and, consequently, v, σ are twice continuously differentiable and $\sigma(x) \equiv \sigma_0$ for $x \geqslant x_0$. Hence, the coefficient $q(x)$ is continuous on the semi-axis $x \geqslant 0$ and vanishes for $x \geqslant x_0$. The support of the function $f(x)$ lies in the segment $[x_0, \infty)$.

Using the same transformation, we can reduce the problem (e) to equation (1.6) with the initial and boundary conditions

$$(u_x - ku)\big|_{x=0} = g(t), \quad u\big|_{t\leqslant 0} \equiv 0; \quad 2\sqrt{\sigma(0)}g(t) = \psi(t). \tag{1.8}$$

A popular geophysical inverse problem is to recover the acoustic impedance σ using measurements of wave fields for the processes (i) or (e). Such problems may also be of interest in scattering theory, biomedical tomography, nondestructive control, and other applications. Solutions of the corresponding systems (i) or (e) measured at some fixed point $z = \zeta$ for all $t > 0$, together with the right-hand sides of the conditions (1.3) or (1.4),

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serve as data for reconstruction. The point of measurement can be either at the surface of the medium ($\zeta = 0$) or inside it ($\zeta > 0$). In the latter case, we assume that this point lies in the region $z \geqslant z_0$, where the parameter values of the medium are known.

These inverse problems can be formulated equivalently for equations (1.6)–(1.8). It is necessary to determine the potential $q(x)$ in equation (1.6) and the coefficient k in the boundary condition (1.7) or (1.8) using one of the following pieces of information:

(ie) the form $f(x)$ of the incident wave and the corresponding response $u(0,t)$ at the point $\xi = 0$ for system (1.6), (1.7);

(ii) the function $f(x)$ and the corresponding solution $u(\xi,t)$ of system (1.6), (1.7) at some point $\xi \geqslant x_0$;

(ee) the excitation $g(t)$ and the corresponding response $u(0,t)$ at the point $\xi = 0$ for system (1.6), (1.8);

(ei) the function $g(t)$ and the corresponding solution $u(\xi,t)$ of system (1.6), (1.8) at some point $\xi \geqslant x_0$.

The first symbol in the abbreviations (ie), (ii), etc., characterizes the location of the source, and the second symbol is the location of the receiver of perturbations (i denotes an internal position and e is an external position). In accordance with the geophysical terminology, (ie) and (ei) are called the transmission inverse problems, and (ii) and (ee) the reflection inverse problems.

Most of these problems have already arisen earlier in different areas of mathematical physics. The reflection problem (ii) is equivalent to the classical inverse problem of the scattering theory. It was solved long ago with the help of spectral methods [1]–[7]. The scattering technique was first applied by Alekseev [8, 9] to the other reflection problem (ee). Various aspects of this application were studied in more detail by Alekseev & Megrabov [10], Balanis [11], Sabatier [12], Symes [13], Carroll & Santosa [14], Bube & Burridge [15], and other authors. The relation between the transmission problem (ei) and the reflection problem (ee) was considered by Carroll & Santosa [16]. Only the problem (ie) has not been studied in detail.

The main goal of this paper is to establish the complete equivalence of all the inverse problems stated above. We show that any problem can be reduced to any other by explicit formulae and the Fourier transform. Therefore, it is possible to formulate a unified approach to their solution. These results were announced by Alekseev & Belonosov [17] and partially published earlier [18].

2 Direct problems

The direct problems (i) and (e) have been much studied. Nevertheless, some review of these results is desirable. We obtain a special representation of solutions, which is convenient for further application to the corresponding inverse problems.

Let us consider the spectral problem

$$U_{xx} - q(x)U = -\omega^2 U, \quad x > 0; \tag{2.1}$$

$$U_x - kU = 0, \quad x = 0 \tag{2.2}$$

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obtained from equation (1.6) and the first condition from (1.7) by the formal Fourier transform

$$U(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\, u(x,t)\, dt.$$

Since the coefficient $q(x)$ vanishes on the semiaxis $x \geqslant x_0$, equation (2.1) has a special solution $e(x,\omega)$, which coincides with $\exp(i\omega x)$ for $x \geqslant x_0$. The function e is called the Jost solution in scattering theory. It is holomorphic with respect to ω in the whole complex plane and, together with the x-derivatives e' and e'', can be expressed in the form

$$\begin{aligned} e^{(k)}(x,\omega) &= e^{i\omega x}\left[(i\omega)^k + \psi_k(x,\omega)\right], \quad k = 0,\ 1,\ 2; \\ |\psi_k(x,\omega)| &\leqslant \begin{cases} C_k(|\omega|+1)^{k-1}\exp\left[\alpha(|\operatorname{Im}\omega| - \operatorname{Im}\omega)\right], & x < x_0, \\ 0, & x \geqslant x_0. \end{cases} \end{aligned} \tag{2.3}$$

Here C_k and α are some positive numbers.

For any $\omega \neq 0$ the functions $e(x,\omega)$ and $e(x,-\omega)$ form a pair of linearly independent solutions of equation (2.1) which are complex conjugates for real ω, i.e. $e(x,-\omega) = \overline{e(x,\omega)}$. Their Wronskian $e'(x,\omega)e(x,-\omega) - e'(x,-\omega)e(x,\omega)$ is independent of x and equals $2i\omega$.

Substituting $e(x,\omega)$ into the boundary condition (2.2), we obtain the entire analytic function $j(\omega) = e'(0,\omega) - k\, e(0,\omega)$. It is called the Jost function, and is important for further consideration.

Lemma 1 *The Jost function $j(\omega)$ does not vanish for $\operatorname{Im}\omega \geqslant 0$, $\omega \neq 0$, but it has a simple zero at the point $\omega = 0$.*

Proof The fact that the Jost function has no zeros for $\operatorname{Im}\omega \geqslant 0$, $\omega \neq 0$ is well-known from scattering theory [1]–[6]. It remains to consider the exceptional value $\omega = 0$. In this case, $e(x,0)$ and its ω-derivative $e'_\omega(x,0)$ satisfy the same equation $U'' - q(x)U = 0$. By the transformation $U = \sqrt{\sigma(x)}V$ this equation reduces to $V'' + [\ln\sigma(x)]'V' = 0$, and its general solution can be found easily:

$$U(x) = \sqrt{\sigma(x)}\left[C_1 \int_0^x \frac{dy}{\sigma(y)} + C_2\right].$$

For $x \geqslant x_0$ the identities $e(x,0) = 1$ and $e'_\omega(x,0) = ix$ are valid. Therefore

$$e(x,0) = \sqrt{\frac{\sigma(x)}{\sigma_0}}, \quad e'_\omega(x,0) = i\sqrt{\sigma_0\sigma(x)}\left[\int_0^x \frac{dy}{\sigma(y)} - \int_0^{x_0} \frac{dy}{\sigma(y)}\right] + ix_0 e(x,0).$$

Hence $j(0) = 0$, $j'(0) = i\sqrt{\sigma_0/\sigma(0)} \neq 0$. □

Besides the auxiliary spectral problem (2.1)–(2.2), we introduce some functional spaces that are required. Let X and T be two intervals (probably infinite) on the axes x and t, and $Q = X \times T$ be a rectangle in the plane (x,t). We denote by $V(Q)$ the set of all functions $u(x,t)$ that are continuous in Q and such that

(a) both mappings $x \mapsto u(x,\cdot) : X \mapsto L_2(T)$ and $t \mapsto u(\cdot,t) : T \mapsto L_2(X)$ are continuous;

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(b) there exists a finite norm of the form

$$\|u\|_Q^2 = \sup_x \int_T |u(x,t)|^2\, dt + \sup_t \int_X |u(x,t)|^2\, dx.$$

The closure of $V(Q)$ with respect to the norm $\|\cdot\|_Q$ is called the space $V_2(Q)$.

The symbol $V^n(Q)$ denotes the set of functions $u(x,t)$ that have all the derivatives $u^{(k,m)} = D_x^k D_t^m u$ of the orders $0 \leqslant k+m \leqslant n$ belonging to $V(Q)$. The closure of $V^n(Q)$ in the norm

$$\|u\|_{n,Q}^2 = \sum_{k+m=0}^{n} \|u^{(k,m)}\|_Q^2$$

is called the space $V_2^n(Q)$. The elements of $V_2^n(Q)$ have generalized derivatives in Q up to the order n. The continuous mappings $x \mapsto u^{(k,m)}(x,\cdot)$ and $t \mapsto u^{(k,m)}(\cdot,t)$ for each such derivative $u^{(k,m)}$ are determined with values in the Sobolev spaces $W_2^{n-k-m}(T)$ and $W_2^{n-k-m}(X)$, respectively. For brevity, the space V_2^0 is identified with V_2, and the norm $\|\cdot\|_{0,Q}$ with the norm $\|\cdot\|_Q$. The norm in the Sobolev space W_2^n is denoted by $|\cdot|_n$.

Let us consider the improper integral

$$v(x,t) = \int_{-\infty}^{\infty} e^{i\omega t}\Phi(\omega)\psi(x,\omega)\, d\omega, \tag{2.4}$$

which is understood as a limit of the proper integrals

$$v_n(x,t) = \int_{P_n} e^{i\omega t}\Phi(\omega)\psi(x,\omega)\, d\omega.$$

Here $\{P_n\}$ is an arbitrary sequence of expanding finite intervals, the union of which is equal to $(-\infty,\infty)$.

Lemma 2 *Assume that $\Phi(\omega) \in L_2(R)$; X is some interval on the x-axis, $Q = X \times R$; $\psi(x,\omega)$ is continuous in Q, belongs to $L_2(Q)$, and $|\psi(x,\omega)| \leqslant M$ for some $M < \infty$. Then the sequence $v_n(x,t)$ is convergent in $V_2(Q)$. The corresponding value $v(x,t)$ of the integral* (2.4) *also belongs to $V_2(Q)$ and satisfies the inequality $\|v\|_Q \leqslant C \cdot |\Phi|_0$, where C depends only on M and $|\psi|_0$.*

This statement directly follows from the Plancherel theorem and the Hölder inequality. Therefore, the details are left to the reader.

Now we can formulate and prove the main result of the present section.

Theorem 1 *For any $f \in W_2^2(R)$ vanishing outside the interval $[x_0,\infty)$, there exists a unique solution of the problem* (1.6), (1.7) *that belongs to V_2^2 in the half-plane $Q = \{(x,t) : x \geqslant 0\}$. This solution satisfies the inequality*

$$\|u\|_{2,Q} \leqslant C \cdot |f|_2. \tag{2.5}$$

The Fourier image $U(x,\omega)$ of the solution can be expressed for any $\omega \neq 0$, $\operatorname{Im}\omega \leqslant 0$ in the form

$$U(x,\omega) = F(\omega)\left[e(x,\omega) - \frac{j(\omega)}{j(-\omega)}e(x,-\omega)\right], \tag{2.6}$$

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where $F(\omega)$ is the Fourier image of the function $f(t)$, $e(x,\omega)$ is the Jost solution, and $j(\omega)$ is the corresponding Jost function.

Proof The uniqueness of the solution belonging to V_2^2 follows from the general theory of hyperbolic equations. Suppose that such a solution exists, and deduce the representation (2.6).

By D'Alembert's formula, $u(x,t)$ equals to $f(t+x)+g(t-x)$ in the domain $x > x_0$. Here f is the right-hand side of the condition (1.7). The functions $u(x,t)$, $f(t+x)$ and, hence, $g(t-x)$ belong to V_2^2 in the half-plane $\{x > x_0,\ t \in R\}$. Consequently, it is possible to apply the Fourier transform in t to each of these functions. As a result, we obtain

$$U(x,\omega) = F(\omega)\, e^{i\omega x} + G(\omega)\, e^{-i\omega x}, \qquad x > x_0, \tag{2.7}$$

where U, F and G are the Fourier images of u, f and g, respectively.

On the other hand, the function U is a solution of the spectral problem (2.1), (2.2), and decomposes into the linear combination $U(x,\omega) = a(\omega)e(x,\omega) + b(\omega)e(x,-\omega)$ for all $x \geqslant 0$, $\omega \neq 0$. Comparing this decomposition with the equation (2.7) and taking into account that $e(x,\pm\omega) = \exp(\pm i\omega x)$ for $x \geqslant x_0$, we find

$$U(x,\omega) = F(\omega)\, e(x,\omega) + G(\omega)\, e(x,-\omega), \qquad x \geqslant 0, \quad \omega \neq 0.$$

Substituting this expression into the boundary condition (2.2), we obtain the equation

$$U_x(0,\omega) - kU(0,\omega) = F(\omega)j(\omega) + G(\omega)j(-\omega) = 0$$

for the coefficient $G(\omega)$. By Lemma 1, the Jost function $j(-\omega)$ does not have zeroes for $\operatorname{Im}\omega \leqslant 0$, $\omega \neq 0$. Therefore $G(\omega)$ can be determined uniquely, and (2.6) is valid.

It remains to prove that the Fourier original $u(x,t)$ of the function $U(x,\omega)$ presented by formula (2.6) belongs to V_2^2 and satisfies the system (1.6), (1.7). For this purpose, consider the auxiliary integral

$$u_n(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{i\omega t} U(x,\omega)\, d\omega.$$

The function u_n is twice continuously differentiable. Besides, differentiation can be made under the integral. It follows from this and (2.1)–(2.2) that u_n satisfies equation (1.6) and the first condition from (1.7). If it can be proved that every derivative $D_x^k D_t^m u_n$ of the order $k+m \leqslant 2$ tends to some limit in the space V_2 as $n \to \infty$, the function $u(x,t)$ is an element of V_2^2 and also satisfies the same equations.

Using (2.6) and (2.3), the corresponding derivatives can be easily expressed by the formula

$$D_x^k D_t^m u_n = J_{1n} + J_{2n} + J_{3n}, \tag{2.8}$$

where

$$J_{1n} = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{i\omega(t+x)} (i\omega)^{k+m} F(\omega)\, d\omega,$$
$$J_{2n} = \frac{-1}{\sqrt{2\pi}} \int_{-n}^{n} e^{i\omega(t-x)} (i\omega)^{k+m} F(\omega) \frac{j(\omega)}{j(-\omega)}\, d\omega,$$
$$J_{3n} = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{i\omega t} (i\omega)^{m} F(\omega) \left[e^{i\omega x}\psi_k(x,\omega) - e^{-i\omega x}\psi_k(x,-\omega)\frac{j(\omega)}{j(-\omega)} \right] d\omega.$$

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The expression $(i\omega)^q F(\omega)$ is the Fourier transform of the derivative $f^{(q)}(t)$. For $q \leqslant 2$ all these derivatives, together with the functions $(i\omega)^q F(\omega)$, belong to the space L_2. In accordance with the properties of the Jost function, we have

$$\left| \frac{j(\omega)}{j(-\omega)} \right| = \begin{cases} O(\exp(2\alpha|\operatorname{Im}\omega|)), & \operatorname{Im}\omega < 0, \\ 1, & \operatorname{Im}\omega = 0. \end{cases} \tag{2.9}$$

By the Plancherel theorem, J_{1n} and J_{2n} have limits in the space V_2, and the norms of the limiting functions are dominated by the value

$$C\,|(i\omega)^{k+m}F(\omega)|_0 = C\,|D_t^{k+m}f(t)|_0 \leqslant C\,|f|_2.$$

The convergence of J_{3n}, as well as a similar estimate of the norm of the limiting function, follow from formulas (2.3), (2.9) and Lemma 2.

Thus, the function $u(x,t)$ satisfies the equation (1.6) and the first condition from (1.7); the inequality (2.5) also holds. It remains to verify the second condition from (1.7). On account of the uniqueness theorem, it is sufficient to show that $u(x,t) = f(x+t)$ in some quadrant $\{\, x \geqslant 0,\ \ t \leqslant \tau \leqslant 0 \,\}$.

This can be easily done when f is finite and infinitely differentiable. Indeed, by the Paley–Wiener theorem [19], the Fourier image $F(\omega)$ of such a function is analytic in the whole complex plane, and for any $p \geqslant 0$ it allows the estimate

$$|F(\omega)| \leqslant A_p(|\omega|+1)^{-p}\, \exp\,(a \operatorname{Im}\omega), \tag{2.10}$$

where a is the left bound of $\operatorname{supp} f$.

Let us choose $k = m = 0$ in (2.8). Because of the inversion formula of the Fourier transform, the integral J_{1n} tends to $f(x+t)$. To find the limit of J_{2n}, we use the analyticity of the integrand and modify J_{2n} to the integral over the semicircle $\Gamma_n = \{\, |\omega| = n,\ \ \operatorname{Im}\omega \leqslant 0 \,\}$. On the basis of (2.9) and (2.10), we have for $\omega \in \Gamma_n$

$$\left| e^{-i\omega x}F(\omega)\frac{j(\omega)}{j(-\omega)} \right| = O\left((|\omega|+1)^{-p}\exp[(2\alpha - a)|\operatorname{Im}\omega|]\right).$$

But then, in accordance with the Jordan lemma, the integral J_{2n} tends to zero for all $t < a - 2\alpha$. Similarly, using (2.3), (2.9), (2.10) and the Jordan lemma, we see that J_{3n} tends to zero for $t < a - 2\alpha - x_0$.

In the general case, there exists a sequence $f_n(t)$ of infinitely differentiable finite functions converging to f in the space W_2^2. This sequence can always be chosen so that the supports of all functions lie in some interval $[a-\varepsilon, \infty)$, $\varepsilon > 0$. Let $v_n(x,t)$ denote the solution of the problem (1.6), (1.7) with the incident wave $f_n(t)$. We have shown that $v_n(x,t) = f_n(x+t)$ for $t < a - \varepsilon - 2\alpha - x_0$. On the other hand, the inequality (2.5) is valid for the difference $u - v_n$. Thus,

$$\|u - v_n\|_2 \leqslant C\,|f - f_n|_2 \to 0 \quad (n \to \infty).$$

Hence, for all $t < a - \varepsilon - 2\alpha - x_0$,

$$u(x,t) = \lim_{n\to\infty} v_n(x,t) = \lim_{n\to\infty} f_n(x+t) = f(x+t).$$

The theorem is proved. □

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Finally, we consider the external problem (1.6), (1.8), which is different from the internal problem studied above. The differences are clear from the following statement:

Theorem 2 *Fix* $g(t) \in W_2^1(R)$ *vanishing for* $t < 0$. *Then there exists a unique solution of the problem* (1.6), (1.8) *that belongs to the space* $V_2^2(Q_\tau)$, $Q_\tau = \{ x \geqslant 0,\ \ t \leqslant \tau \}$ *for any finite* τ. *The derivative* u_t *of this solution belongs to* V_2^1 *in the half-plane* $Q = \{ (x,t) : x \geqslant 0 \}$, *and the following estimates hold:*

$$\|u\|_{2,Q_\tau} \leqslant C_\tau \cdot |g|_1, \quad \|u_t\|_{1,Q} \leqslant C \cdot |g|_1. \tag{2.11}$$

The proof of this theorem is similar to the previous one. Therefore, we give only a sketch. First, let g be a smooth finite function and $v(x,t)$ the solution of the problem (1.6), (1.8), where the function g is replaced by its derivative g_t. The existence of the solution of this auxiliary problem in the space $V_2^2(Q)$ and the estimate $\|v\|_{1,Q} \leqslant C \cdot |g|_1$ are proved as in Theorem 1. It is clear that the function

$$u(x,t) = \int_0^t v(x,\theta)\, d\theta,$$

for which the both inequalities (2.11) are valid, satisfies the original system (1.6), (1.8).

For the general case, g should be approximated by the smooth finite functions g_n. Then the corresponding solutions u_n are found, and we pass to the limit using the estimates (2.11), which have already been proved for u_n.

Generally speaking, the solution $u(x,t)$ of the problem (e) does not belong to $L_p(R)$ in the variable t for any $p < \infty$. Therefore, its Fourier transform exists only in the distributional sense. Nevertheless, it follows from Theorem 2 that the Fourier image of the derivative u_t can be understood in the ordinary sense. Denoting it by $V(x,\omega)$ and modifying, correspondingly, the reasoning given above, we obtain

$$V(x,\omega) = G(\omega) \left[\frac{i\omega}{j(-\omega)} \right] e(x,-\omega). \tag{2.12}$$

Here, as before, $\{ \operatorname{Im}\omega \leqslant 0,\ \ \omega \neq 0 \}$; $G(\omega)$ is the Fourier image of the data $g(t)$ in the condition (1.8).

3 Inverse problems

Let us return to the inverse problems stated in Section 1. Suppose that the functions f or g satisfy the conditions of Theorems 1 or 2, respectively. Substitute the observation point $x = \xi$ into formulas (2.6) or (2.12). Recall that $\xi = 0$ for the inverse problems (ie), (ee), and $\xi \geqslant x_0$ for the problems (ii), (ei). As a result, we obtain four relations:

$$\begin{aligned}
U(\xi,\omega) &= P_{ii}(\xi,\omega)F(\omega), & P_{ii} &= e^{i\omega\xi} - e^{-i\omega\xi}\left[j(\omega)/j(-\omega) \right];\\
U(0,\omega) &= P_{ie}(\omega)F(\omega), & P_{ie} &= -2i\omega/j(-\omega);\\
V(\xi,\omega) &= P_{ei}(\xi,\omega)G(\omega), & P_{ei} &= e^{-i\omega\xi} i\omega/j(-\omega);\\
V(0,\omega) &= P_{ee}(\omega)G(\omega), & P_{ee} &= \left[i\omega/j(-\omega) \right] e(0,\omega).
\end{aligned}$$

The factors P_{ii}, P_{ie}, etc., are called the transition functions. These functions can be

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determined from the data of the inverse problems: all one needs to do is to divide the both parts of the corresponding equality by $F(\omega)$ or $G(\omega)$. The only possible difficulty is the presence of zeroes of $F(\omega)$ or $G(\omega)$, which can be regularized in the following way.

All data of the inverse problems vanish on some rays going to $-\infty$. By the Paley–Wiener theorem, the Fourier images of these data are analytic and have only isolated zeros in the half-plane $\operatorname{Im}\omega < 0$. It follows from the above formula that the quotients U/F or V/G are also determined and are holomorphic for $\operatorname{Im}\omega < 0$. Since the transition functions are continuous in the closed half-plane $\operatorname{Im}\omega \leqslant 0$, their values on the real axis can be interpreted as the limits of the corresponding quotients as $\operatorname{Im}\omega \to -0$. Moreover, we can state

Theorem 3 *On the real axis the Jost function can be expressed by any transition function, and vice versa.*

Proof This statement is evident for P_{ie} and P_{ei}. We consider P_{ee}, which is the most complicated case of the two remaining transition functions.

The relation between $P_{ee}(\omega)$ and $j(-\omega)$ includes one more unknown value $e(0,-\omega)$. Let us pass to $\operatorname{Re} P_{ee}$ to exclude $e(0,-\omega)$. The values of P_{ee} at the real points ω and $-\omega$ are complex conjugates and therefore

$$\begin{aligned} 2\operatorname{Re} P_{ee}(\omega) &= P_{ee}(\omega) + P_{ee}(-\omega) \\ &= \frac{i\omega}{|j(\omega)|^2}\left[e'(0,\omega)e(0,-\omega) - e'(0,-\omega)e(0,\omega)\right] = -\frac{2\omega^2}{|j(\omega)|^2}. \end{aligned}$$

Consequently, the function $\operatorname{Re} P_{ee}(\omega)$ for all real $\omega \neq 0$ is expressed by $|j(\omega)|$, and vice versa. All one needs to do is to show that P_{ee} and $j(\omega)$ are uniquely determined, respectively, by the values of $\operatorname{Re} P_{ee}(\omega)$ and $|j(\omega)|$ on the real axis.

For this purpose, consider the function $P_{ee}(\omega) + 1$. It is analytic in the half-plane $\operatorname{Im}\omega \leqslant 0$ and, owing to (2.3), has the asymptotics $P_{ee}(\omega) + 1 = O(1/|\omega|)$ as $|\omega| \to \infty$. According to the Paley–Wiener theorem [19], we obtain the integral representation

$$P_{ee}(\omega) + 1 = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\omega t} p(t)\, dt, \quad \operatorname{Im}\omega \leqslant 0, \tag{3.1}$$

where $p(t)$ is a real-valued function belonging to $L_2(0,\infty)$. Hence,

$$\operatorname{Re} P_{ee}(\omega) + 1 = \frac{1}{\sqrt{2\pi}} \int_0^\infty \cos\omega t \cdot p(t)\, dt, \quad \operatorname{Im}\omega = 0. \tag{3.2}$$

If the values of $\operatorname{Re} P_{ee}$ on the real axis are known, $p(t)$ can be found by inversion of the cosine transform in (3.2). After this, P_{ee} can be easily reconstructed by (3.1).

The function $\log(-i\omega/j(-\omega))$ possesses similar properties: it is analytical in the half-plane $\operatorname{Im}\omega \leqslant 0$ and has the asymptotics $\log(-i\omega/j(-\omega)) = \log\left(1 + O(1/|\omega|)\right) = O(1/|\omega|)$ as $|\omega| \to \infty$. Consequently, for $\operatorname{Im}\omega = 0$ this function can be uniquely reconstructed using $\operatorname{Re}\log(-i\omega/j(-\omega)) = \ln(|\omega|/|j(\omega)|)$. Then $j(\omega)$ can also be found, if the values of $|j(\omega)|$ on the real axis are known.

Similarly, the transition function P_{ii} makes it possible to find $\arg j(\omega)$ for real ω. Then we can reconstruct all values of the Jost function [1]–[7]. □

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Thus, all the inverse problems stated in the previous section are equivalent in the following sense: each of the four transition functions determines the Jost function uniquely and, consequently, the remaining three transition functions. The Jost function, in turn, makes it possible to reconstruct the coefficients q and k in the original equations (1.6)–(1.8). We next prove this last statement, reducing it to the well-studied inverse spectral problem for the Sturm–Liouville equation.

First, we recall the necessary information from the spectral theory of differential operators. Let A denote the operator

$$u(x) \to -u''(x) + q(x)\,u(x)$$

that acts in $L_2(0,\infty)$. It is defined for all functions that belong to $W_2^2(0,\infty)$ and satisfy the boundary condition (2.2), i.e., $u'(0) - k\,u(0) = 0$.

This operator is evidently self-adjoint, and therefore its spectrum lies on the real axis $\operatorname{Im}\lambda = 0$ of the complex plane. Moreover, the finiteness of $q(x)$ guarantees (see [6]) that the whole half-line $\lambda > 0$ belongs to the continuous spectrum, but the spectrum in the half-line $\lambda < 0$ can be only discrete. In our case operator A cannot have any eigenvalues in the half-line $\lambda \leqslant 0$. Otherwise, if we return to the initial physical variables, such as density, velocity, depth, and displacement, we find non-positive eigenvalues for the spectral problem

$$-(\mu w_z)_z = \lambda \rho w, \quad z > 0; \qquad w_z\big|_{z=0} = 0,$$

which is impossible.

Consider the solution of the Cauchy problem

$$-\theta''(x,\lambda) + q(x)\,\theta(x,\lambda) = \lambda\theta(x,\lambda), \qquad \theta(0,\lambda) = 1, \quad \theta'(0,\lambda) = k,$$

which is determined for all $x \geqslant 0$ and any complex λ. The function $\theta(x,\lambda)$ generates the integral transform

$$v(\lambda) = \int_0^\infty u(x)\theta(x,\lambda)\,dx. \tag{3.3}$$

It is known [1]–[7] that this transform maps $L_2(0,\infty)$ isometrically onto some weight space $L_{2,r}(-\infty,\infty)$ that consists of all r-measurable functions $v(\lambda)$ such that

$$\int_{-\infty}^\infty |v(\lambda)|^2\,dr(\lambda) < \infty.$$

Operator A is transformed into multiplication by λ using (3.3), and the mapping inverse to (3.3) has the form

$$u(x) = \int_{-\infty}^\infty v(\lambda)\theta(x,\lambda)\,dr(\lambda). \tag{3.4}$$

Here $r(\lambda)$ is a non-decreasing real-valued function that is continuous on the right. It is called the spectral distribution function of the operator A. The points of increase for $r(\lambda)$ coincide with the spectral points of A, and a continuous function $r(\lambda)$ corresponds to an operator with a continuous spectrum. The properties of the operator A make it possible to integrate over the interval $(0,\infty)$ in the inversion formula (3.4).

The classical Sturm–Liouville inverse problem consists of determinating $q(x)$ and k by the spectral distribution function $r(\lambda)$, which corresponds to the operator A. There

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exist several effective methods for solving this problem. They were developed by V. A. Marčenko, M. G. Kreĭn, I. M. Gelfand and B. M. Levitan. We shall not dwell on the description of these well-known results (see the review in Levitan [5]), but present a theorem linking $j(\omega)$ and $r(\lambda)$.

Theorem 4 *The spectral function $r(\lambda)$ of operator A vanishes in the domain $\lambda \leqslant 0$. For $\lambda > 0$ we have*

$$r'(\lambda) = \frac{\sqrt{\lambda}}{\pi |j(\sqrt{\lambda})|^2}. \tag{3.5}$$

This formula was first proved in scattering theory [2]–[7] for the Sturm–Liouville operator with the Dirichlet boundary conditions. A different case of conditions (2.2) is considered in earlier work [8] using some results of Kreĭn [20]. Unfortunately, the proof of these results was not published. Now it is possible to obtain formula (3.5) from the latest achievements of the spectral theory [4, 5, 14]. Nevertheless, to make this discussion complete we present a new simple proof, which is based on the statements of § 2.

Proof Operator A has no negative spectral points. Therefore, $r(\lambda) \equiv 0$ for $\lambda \leqslant 0$, and we have to consider only the positive values of λ.

Note that the function $\theta(x, \lambda)$ mentioned above satisfies equation (2.1) for $\omega = \sqrt{\lambda}$ and decomposes into a linear combination of $e(x, \omega)$ and $e(x, -\omega)$. The coefficients of this decomposition are easily found from the initial conditions $\theta(0, \lambda) = 1$ and $\theta'(0, \lambda) = k$:

$$\theta(x, \lambda) = -\frac{j(-\omega)}{2i\omega}\, e(x, \omega) + \frac{j(\omega)}{2i\omega}\, e(x, -\omega). \tag{3.6}$$

Let us choose some finite infinitely differentiable function $f(x)$ that vanishes for $x < x_0$ and consider the corresponding solution $u(x, t)$ of the direct problem (1.6), (1.7). In accordance with Theorem 1, the function u belongs to the space V_2^2. Consequently, for any fixed t it lies in the domain of definition of operator A. This makes it possible to apply the transform (3.3) to both parts of equation (1.6). Assuming that

$$v(\lambda, t) = \int_0^\infty u(x, t)\theta(x, \lambda)\, dx, \quad \lambda > 0, \tag{3.7}$$

and taking into account that operator A becomes the multiplication by λ, we obtain the elementary equation $v_{tt} = -\lambda v$, which has the general solution of the form

$$v(\lambda, t) = \alpha(\lambda) \exp(i\omega t) + \beta(\lambda) \exp(-i\omega t).$$

The coefficients α and β can be determined from the condition (1.7). Let $t < 0$; then $u(x, t) = f(x + t)$ and the support of the function $f(x + t)$ lies in the domain $x > x_0$. In the same domain, $e(x, \pm\omega)$ coincides with $\exp(\pm i\omega x)$. Consequently, substituting (3.6) into (3.7) and taking $F(\omega)$ to denote the Fourier image of the function $f(x)$, we obtain, for negative t,

$$\begin{aligned} v(\lambda, t) &= \frac{1}{2i\omega} \int_0^\infty f(x + t) \left[j(\omega) e^{-i\omega x} - j(-\omega) e^{i\omega x} \right] dx \\ &= \frac{\sqrt{2\pi}\, j(\omega)}{2i\omega} F(\omega)\, e^{i\omega t} - \frac{\sqrt{2\pi}\, j(-\omega)}{2i\omega} F(-\omega)\, e^{-i\omega t}. \end{aligned}$$

This relation also holds for $t \geqslant 0$ by the analyticity of $v(\lambda, t)$ in t.

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Now let us substitute the found decomposition into (3.4) and assume that $x = 0$. Taking into account that $\theta(0, \lambda) = 1$, we obtain

$$u(0,t) = \int_0^\infty v(\lambda, t)\theta(0,\lambda)\, dr(\lambda)$$
$$= \sqrt{2\pi}\int_0^\infty \frac{j(\omega)}{2i\omega} F(\omega)\, e^{i\omega t}\, dr(\lambda) - \sqrt{2\pi}\int_0^\infty \frac{j(-\omega)}{2i\omega} F(-\omega)\, e^{-i\omega t}\, dr(\lambda).$$

Changing the sign of ω in the second integral, we find

$$u(0,t) = \sqrt{2\pi}\int_{-\infty}^\infty \frac{j(\omega)}{2i|\omega|} F(\omega)\, e^{i\omega t}\, dr(\omega^2).$$

On the other hand, the Fourier image of the function $u(0, t)$ is related to $F(\omega)$ by the equality (2.6). Hence,

$$u(0,t) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{2i\omega}{j(-\omega)} F(\omega)\, e^{i\omega t}\, d\omega.$$

These different representations of the function $u(0, t)$ may be interpreted as the Fourier transforms of two measures. Each measure is uniquely determined by its Fourier transform. Therefore

$$\frac{\pi j(\omega)}{|\omega|}\, dr(\omega^2) = \frac{2\omega}{j(-\omega)}\, d\omega.$$

This equality implies that for $\omega > 0$

$$\frac{d}{d\omega}\, r(\omega^2) = \frac{2\omega^2}{\pi\, |j(\omega)|^2},$$

which is equivalent to (3.5). The theorem is completely proved. □

4 Numerical examples

To summarize the previous discussion, the spectral function $r(\lambda)$ can be recovered from the data of any inverse problem stated above. Now the corresponding differential operator can be reconstructed in many ways. One of the most effective methods developed by Kreĭn [21] leads to the integral equation

$$y(x, 2\xi) + \int_0^\xi K(x-\zeta)y(\zeta,\xi)\, d\zeta = 1, \quad 0 \leqslant x \leqslant 2\xi,$$

where

$$K(x) = \frac{1}{2}\int_0^\infty \cos(\omega x)\, d\left\{ r(\omega^2) - \frac{2\omega}{\pi} \right\}.$$

This equation has a unique solution $y(x, \xi)$ [6, 21]. The required coefficients $q(x)$, k and the impedance profile $\sigma(x)$ defined in § 1 can be expressed by y using the explicit formulae

$$q(x) = \frac{1}{y(2x,2x)}\frac{d^2}{dx^2}\, y(2x,2x), \quad k = \frac{d}{dx}\ln y(2x,2x)\Big|_{x=0},$$
$$\sigma(x) = \sigma(0)\cdot y^2(2x,2x).$$

We have constructed a numerical algorithm for typical inverse problems of geophysics

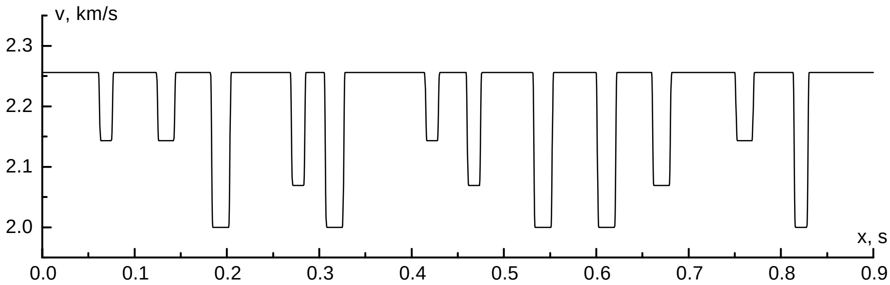

FIGURE 1. Original velocity.

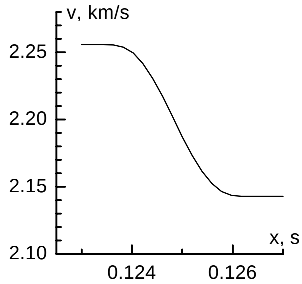

FIGURE 2. A scaled fragment.

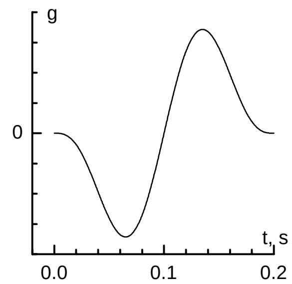

FIGURE 3. The boundary influence.

and some other applications. This algorithm demonstrates a sufficiently high efficiency and accuracy for the test examples. One of our numerical experiments is presented below.

We consider a model of a medium described by the profile in Figure 1. As before, x denotes the travel time from the surface to the current point, and v is the wave propagation velocity. We use realistic units to indicate the scale: seconds (s) and kilometres (km). The velocity $v(x)$ is assumed to have a constant value for $x \geqslant 0.9$.

It should be noted that the function represented in Figure 1 seems to be discontinuous, but the basic conditions of § 1 admit only smooth medium parameters. Actually, we suppose that $v(x)$ is twice continuously differentiable. This is clear from Figure 2, where a part of the initial profile is shown on a more convenient scale.

Finally, we assume that the density of the medium does not depend on x. Then the transmission problem (1.6), (1.8) becomes

$$u_{xx} - q(x)u = u_{tt}, \quad x > 0, \ t \in R; \qquad q(x) = (\ln \sqrt{v})'' + \left[(\ln \sqrt{v})'\right]^2;$$
$$u_x\big|_{x=0} = g(t), \qquad u\big|_{t \leqslant 0} \equiv 0.$$

The corresponding inverse problem (ei) is to recover the velocity $v(x)$, if the boundary influence $g(t)$ and the values of $u(\xi, t)$ at some point $\xi \geqslant 0.9$ are known for all $t > 0$.

We consider the boundary impulse $g(t)$ of the form shown in Figure 3 and calculate the corresponding synthetic seismogram $u(x, t)$. Since the problem is linear, the amplitude of g is of no importance, and the values of g and u are shown on a representative scale.

To find $u(x, t)$ we pass on to the frequency domain, calculate the solution of the obtained boundary value problem for the ordinary differential equation (2.1), and return to the

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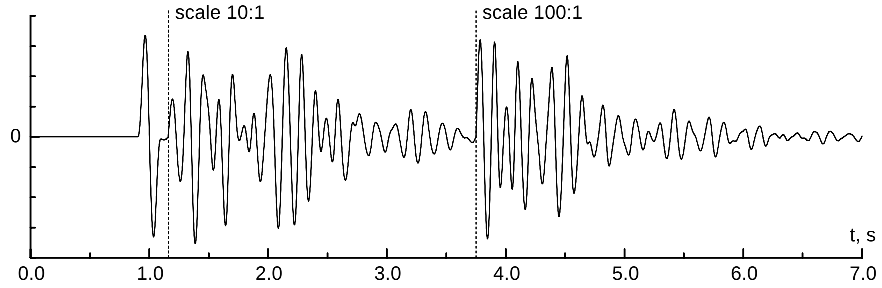

FIGURE 4. Calculated signal.

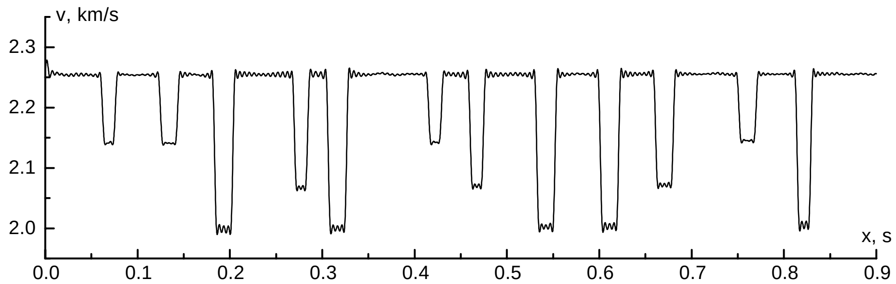

FIGURE 5. The restored profile.

time domain. This approach makes it possible to achieve a sufficient accuracy for each frequency and, consequently, for all values of time. Figure 4 illustrates the computed derivative $u_t(\xi, t)$ at the point $\xi = 0.9$.

Now the function $g(t)$ and the calculated synthetic seismogram are used as the data for the inverse problem. We assume that $u_t(\xi, t)$ coincides with the profile in Figure 4 for $t \leqslant 7$ and equals zero otherwise. The result is represented in Figure 5.

The most unstable step of the above algorithm is the division $V(\omega) : G(\omega)$ to determine the transition function. It can be interpreted as deconvolution in the time domain. There exist a few effective deconvolution methods. In our case, however, a simpler procedure can be used. Before computing $G(\omega)$, we estimate an absolute error ε of the numerical Fourier transform for $g(t)$. Then the transition function is calculated for all ω such that $|G(\omega)| > \varepsilon$ by division. Interpolation is applied in the intervals where $|G(\omega)| < \varepsilon$. All other steps of the recovering algorithm are rather stable.

Our final experiment shows the sensitivity of the algorithm to random noise. We add 5% of uniformly distributed random noise to the accurate profile in Figure 4, and try to repeat the medium's reconstruction. This attempt is unsuccessful, because distortion of the calculated transition function becomes too large for high frequency values. Nevertheless, in the low frequency range the transition function changes only slightly, and can be used for satisfactory medium's recovery.

It follows from (2.3) that for $|\omega| \to \infty$ the function $P_{ei}(\omega)$ should have the asymptotics

$$P_{ei}(\omega) = e^{-i\omega\xi} i\omega / j(-\omega) = -e^{-i\omega\xi} + O(|\omega|^{-1}).$$

We modify the calculated transition function, assuming that it is equal to $-e^{-i\omega\xi}$ in the

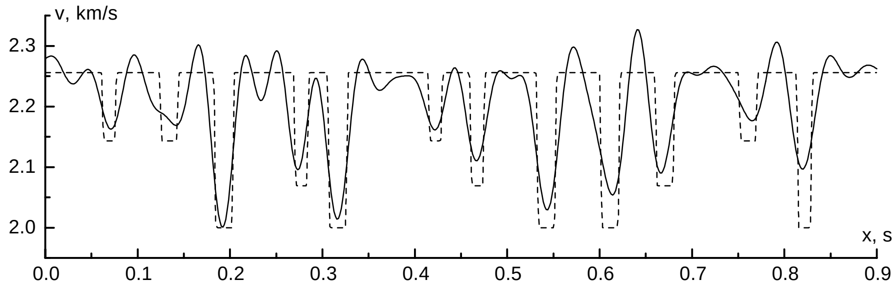

FIGURE 6. The influence of random noise.

frequency range $f = \omega/2\pi > 12{,}8\,\text{Hz}$. After this the recovery procedure is successfully completed and leads to the profile shown in Figure 6.

5 Final observations

The internal inverse problems (ii) and (ie) are typical for applications, where the form of an incident wave $f(t)$ is known. Information on this form is naturally absent in the problems of geophysics. Here the values of the full wave field (i.e. the sum of incoming and scattered waves) at one or several points deep inside the medium and on its surface are the usually observed characteristics of the wave process. In the present section the algorithms described above are adapted to the needs of seismic prospecting.

We first find the information about the medium's structure and the wave process that is sufficient to recover an unknown incident wave. Let us return to formula (2.6) which relates the Fourier images of $u(x,t)$ and $f(t)$. Differentiating (2.6) with respect to x and assuming that $J(\omega) = j(\omega)/j(-\omega)$, we come to the system

$$\left.\begin{aligned} U(x,\omega) &= F(\omega)\,e(x,\omega) - J(\omega)F(\omega)\,e(x,-\omega),\\ U_x(x,\omega) &= F(\omega)\,e'(x,\omega) - J(\omega)F(\omega)\,e'(x,-\omega). \end{aligned}\right\} \tag{5.1}$$

Hence, taking into account the expression for the Wronskian of the Jost solutions, we get

$$F(\omega) = \frac{1}{2i\omega}[U_x(x,\omega)\,e(x,-\omega) - U(x,\omega)\,e'(x,-\omega)]. \tag{5.2}$$

Consequently, $F(\omega)$ can be found if the values of $e(x,-\omega)$, $e'(x,-\omega)$ and $U(x,\omega)$, $U_x(x,\omega)$ at some point x are known. Note that $e(x,\omega)$ coincides with $\exp(i\omega x)$ in the domain $x \geqslant x_0$. Therefore it is sufficient to measure the full wave field $u(x,t)$ and its x-derivative at some point $x \geqslant x_0$ to recover $F(\omega)$ and its Fourier original $f(t)$.

This brings up the following question: to what extent are the medium's parameters in the internal problem (ie) determined by $u(\xi,t)$ and $u_\xi(\xi,t)$ at some $\xi < x_0$? We will show that these data, together with $u(0,t)$, uniquely determine the coefficient $q(x)$ of equation (1.6) only in the segment $0 \leqslant x \leqslant \xi$. For $x > \xi$, the values of $q(x)$ can be arbitrary.

Let $u(x,t)$ be a solution of the problem (1.6)–(1.7) with $f(t)$ that satisfies the conditions of Theorem 1. We choose some ξ from the interval $(0, x_0)$ and change arbitrarily the impedance $\sigma(x)$ outside the segment $[0, \xi]$ for it to remain positive, twice continuously differentiable, and constant for all x large enough. The coefficient $q(x)$ also changes.

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These modified functions σ and q are denoted by $\tilde{\sigma}$ and $\tilde{q}$. The identities $\tilde{\sigma}(x) \equiv \sigma(x)$ and $\tilde{q}(x) \equiv q(x)$ are evidently valid in the segment $[0, \xi]$.

Theorem 5 *There exists a function $\tilde{f}(t) \in W_2^2$ that vanishes in some ray $t < \alpha$ and such that a solution $\tilde{u}(x,t)$ of the problem* (1.6)–(1.7) *with the coefficient $\tilde{q}(x)$ and the incident wave $\tilde{f}(t)$ coincides with $u(x,t)$ for all $x \leqslant \xi$, $-\infty < t < \infty$.*

Proof On the basis of Theorem 1 we pass from evolutional problems to the corresponding spectral problems. The symbol $\tilde{e}(x,\omega)$ denotes the lost solution for the equation

$$\tilde{U}_{xx} - \tilde{q}(x)\tilde{U} = -\omega^2\tilde{U}. \tag{5.3}$$

We need only find a solution of equation (5.3) that coincides with $U(x,\omega)$ in the interval $0 \leqslant x \leqslant \xi$ and decomposes into the linear combination

$$\tilde{U}(x,\omega) = \tilde{F}(\omega)\,\tilde{e}(x,\omega) + \tilde{G}(\omega)\,\tilde{e}(x,-\omega). \tag{5.4}$$

Here $\tilde{F}(\omega)$ and $\tilde{G}(\omega)$ are the Fourier transforms of some functions $\tilde{f}(t)$ and $\tilde{g}(t)$ from the class W_2^2 with the supports bounded from the left. Let us show that the solution of the Cauchy problem for equation (5.3) with the initial conditions

$$\tilde{U}(\xi,\omega) = U(\xi,\omega), \quad \tilde{U}_x(\xi,\omega) = U_x(\xi,\omega) \tag{5.5}$$

satisfies these requirements.

Since $q(x)$ equals $\tilde{q}(x)$ in the interval $[0, \xi]$, the function $U(x,\omega)$ also equals $\tilde{U}(x,\omega)$ for $x \leqslant \xi$ and all ω due to uniqueness of the solution of the Cauchy problem. In particular, $\tilde{U}(x,\omega)$, together with $U(x,\omega)$, satisfy the boundary condition (2.2).

Let us find the coefficients of the decomposition (5.4). Substituting (5.4) into the boundary conditions (5.5) and applying (5.1), we obtain a system of linear equations for $\tilde{F}$ and $\tilde{G}$, which implies that

$$\tilde{F}(\omega) = \frac{F(\omega)}{2i\omega}\left[E'(\xi,\omega)\tilde{e}(\xi,-\omega) - E(\xi,\omega)\tilde{e}'(\xi,-\omega)\right],$$
$$\tilde{G}(\omega) = \frac{F(\omega)}{2i\omega}\left[E(\xi,\omega)\tilde{e}'(\xi,\omega) - E'(\xi,\omega)\tilde{e}(\xi,\omega)\right],$$

where

$$E(\xi,\omega) = e(\xi,\omega) - J(\omega)\,e(\xi,-\omega), \quad E'(\xi,\omega) = e'(\xi,\omega) - J(\omega)\,e'(\xi,-\omega).$$

Note that $\tilde{F}$ and $\tilde{G}$ have removable singularities at $\omega = 0$. Actually, on the basis of Lemma 1 we have $j(0) = 0$, $j'(0) \neq 0$ and, therefore, $J(0) = -1$, $E(\xi,0) = 2e(\xi,0)$, $E'(\xi,0) = 2e'(\xi,0)$. From this, using the representations

$$e(\xi,0) = [\sigma(\xi)/\sigma_0]^{1/2}, \quad \tilde{e}(\xi,0) = [\tilde{\sigma}(\xi)/\tilde{\sigma}_0]^{1/2}$$

found in Lemma 1, we obtain

$$E'(\xi,\omega)\tilde{e}(\xi,-\omega) - E(\xi,\omega)\tilde{e}'(\xi,-\omega)\,\Big|_{\omega=0}$$
$$= [\sigma_0\tilde{\sigma}_0]^{-1/2}[\sigma'(\xi)(\tilde{\sigma}(\xi)/\sigma(\xi))^{1/2} - \tilde{\sigma}'(\xi)(\sigma(\xi)/\tilde{\sigma}(\xi))^{1/2}].$$

This expression is equal to zero, because σ and $\tilde{\sigma}$ coincide (together with their derivatives)

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at the point ξ. Similarly,

$$E(\xi,\omega)\tilde{e}'(\xi,\omega) - E'(\xi,\omega)\tilde{e}(\xi,\omega)\Big|_{\omega=0} = 0.$$

Thus, $\tilde{F}$ and $\tilde{G}$ are analytic in the half-plane $\operatorname{Im}\omega \leqslant 0$ and, due to (2.3), their modules are majorized by the value $C\exp(-\gamma \operatorname{Im}\omega)$, where C and γ are some positive constants. All the necessary properties of the functions $\tilde{F}$ and $\tilde{G}$ follow from this statement. For example, let us consider $\tilde{F}$.

First, $f(t)$ belongs to W_2^2. Hence, the product $F(\omega)(1+\omega^2)$ belongs to $L_2(-\infty,\infty)$. But then $\tilde{F}(\omega)(1+\omega^2)$ is also in $L_2(-\infty,\infty)$. Consequently, $\tilde{F}(\omega)$ is the Fourier transform of some function $\tilde{f}(t)$ from the class W_2^2.

Second, the support of the function $f(t)$ lies in some ray $t \geqslant a$. By the Paley–Wiener theorem, this is equivalent to the analyticity of $F(\omega)$ in the half-plane $\operatorname{Im}\omega < 0$ and the uniform estimate

$$\int_{-\infty}^{\infty} |F(\lambda + i\mu)|^2 \, d\lambda \leqslant \text{const} \cdot e^{2a\mu}, \quad \mu < 0.$$

On replacement of a by $a-\gamma$, the function $\tilde{F}$ also satisfies the same estimate. Therefore, $\operatorname{supp}\tilde{f}$ lies in the domain $t \geqslant a-\gamma$. The theorem is proved. □

Thus, if the form of an incoming wave is unknown, one cannot say anything definite about the medium's structure outside the observation interval. For any parameters of the medium in the domain $x > \xi$, the form of an incident wave can always be chosen so that the response caused by it coincides with the observed wave field in the whole interval $0 \leqslant x \leqslant \xi$ for all time moments.

Nevertheless, the result of Theorem 5 is positive, because it allows us to recover the coefficients $\sigma(x)$ and $q(x)$ in the interval $0 \leqslant x \leqslant \xi$ using the given values

$$u(\xi,t) = \varphi(t), \quad u_x(\xi,t) = \psi(t), \quad u(0,t) = \chi(t).$$

Indeed, let the values of the function σ and its first and second derivatives at the point $x = \xi$ be known additionally. We construct a new function $\tilde{\sigma}(x)$ coinciding in the interval $[0, \xi]$ with the function $\sigma(x)$ which is already available, but yet unknown. We determine this function for $x > \xi$ in any known way according to the requirements of Theorem 5. By the conclusion of Theorem 5, there exists such an incident wave $\tilde{f}(t)$ that the corresponding solution $\tilde{u}(x,t)$ satisfies the conditions

$$\tilde{u}(\xi,t) = \varphi(t), \quad \tilde{u}_x(\xi,t) = \psi(t), \quad \tilde{u}(0,t) = \chi(t). \tag{5.6}$$

Besides, we know the function $\tilde{\sigma}(x)$ and the coefficient $\tilde{q}(x)$ for $x \geqslant \xi$. Hence, we can find the Jost solution $\tilde{e}(x,\omega)$ of equation (5.4) and its derivative $\tilde{e}'(x,\omega)$ in the domain $x \geqslant \xi$ and, in particular, at the point ξ itself. Then the Fourier image $\tilde{F}(\omega)$ of a hypothetical incident wave $\tilde{f}(t)$ is easily determined from the conditions (5.6) using formula (5.2). Thus, we obtain the inverse problem (ie) on reconstruction of the function $\tilde{\sigma}(x)$ using the given $\tilde{f}(t)$ and $\tilde{u}(0,t)$. The solution of this problem provides full information about the initial function $\sigma(x)$ in the interval $[0, \xi]$.

Conclusion

We have an effective theory for a wide class of one-dimensional inverse problems, in particular, for both the transmission and reflection problems of seismology. Nevertheless, some principal questions are open. For example, the spectral methods are not justified for operators with piecewise continuous coefficients. Also, stable algorithms of deconvolution pose a challenge in numerical analysis. Finally, new correct statements of inverse problems with unknown incident waves are of great interest in geophysics. All these aspects call for more detailed research.

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