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Well-posedness of the Muskat problem in subcritical L_p-Sobolev spaces

H. ABELS and B.-V. MATIOC

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
emails: helmut.abels@ur.de; bogdan.matioc@ur.de

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We study the Muskat problem describing the vertical motion of two immiscible fluids in a two-dimensional homogeneous porous medium in an L_p-setting with $p \in (1, \infty)$. The Sobolev space $W^s_p(\mathbb{R})$ with $s = 1 + 1/p$ is a critical space for this problem. We prove, for each $s \in (1 + 1/p, 2)$, that the Rayleigh–Taylor condition identifies an open subset of $W^s_p(\mathbb{R})$ within which the Muskat problem is of parabolic type. This enables us to establish the local well-posedness of the problem in all these subcritical spaces together with a parabolic smoothing property.

Key words: Muskat problem, Rayleigh–Taylor condition, subcritical space, singular integral

2020 Mathematics Subject Classification: 35R37, 35K55 (Primary); 35Q35, 42B20 (Secondary)

1 Introduction

In this paper, we study the following system of nonlinear and nonlocal equations:

$$\begin{aligned} \partial_t f(t,x) &= \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{y + \partial_x f(t,x)(f(t,x) - f(t,x-y))}{y^2 + (f(t,x) - f(t,x-y))^2}\,\overline{\omega}(t,x-y)\,dy, \\ -C_\Theta \partial_x f(t,x) &= \overline{\omega}(t,x) + \frac{a_\mu}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{y\partial_x f(t,x) - (f(t,x) - f(t,x-y))}{y^2 + (f(t,x) - f(t,x-y))^2}\,\overline{\omega}(t,x-y)\,dy, \end{aligned} \tag{1.1a}$$

for $t \geqslant 0$ and $x \in \mathbb{R}$ that describes the motion of two immiscible Newtonian fluids with viscosities μ_- and μ_+ and densities ρ_- and ρ_+ in a vertical two-dimensional porous medium with constant permeability k that we identify with $\mathbb{R}^2$. The fluid located below is denoted by $-$. The unknown function f parameterises the sharp interface between the fluids, and $2(1 + (\partial_x f)^2)^{-1/2}\overline{\omega}$ is the jump of the velocity field in tangential direction at the interface, cf. [36, equation (2.6)]. For the Muskat problem (1.1a), we consider the general scenario when

$$\mu_- - \mu_+ \in \mathbb{R} \qquad \text{and} \qquad \Theta := g(\rho_- - \rho_+) + \frac{\mu_- - \mu_+}{k} V \in \mathbb{R}.$$

The constant g is the Earth's gravity, $|V| \in \mathbb{R}$ is the velocity at which the fluid system moves vertically upwards if $V > 0$ or downwards if $V < 0$, and

$$a_\mu := \frac{\mu_- - \mu_+}{\mu_- + \mu_+} \in (-1, 1) \qquad \text{and} \qquad C_\Theta := \frac{k\Theta}{\mu_- + \mu_+},$$

where a_μ is called Atwood number. Moreover, PV denotes the principal value and is taken at zero and/or at infinity. The system (1.1a) is supplemented with the initial condition:

$$f(0,\cdot)=f_0. \tag{1.1b}$$

Critical spaces for (1.1). It can be verified that, if f is a solution to (1.1a), then, given $\lambda>0$, the function f_λ with:

$$f_\lambda(t,x):=\lambda^{-1}f(\lambda t,\lambda x)$$

also solves (1.1a). Moreover, given $p\in(1,\infty)$ and $r\in(0,1)$, it holds that

$$\left[\partial_x^{(k)}f(\lambda t)\right]_{W_p^r}=\left[\partial_x^{(k)}f_\lambda(t)\right]_{W_p^r}$$

exactly for $k=1$ and $r=1/p$. This property identifies the space $W_p^s(\mathbb{R})$ with $s=1+1/p$ as a critical space for (1.1a). We recall that, given $0<s\notin\mathbb{N}$ with $s=[s]+\{s\}$, where $[s]\in\mathbb{N}$ and $\{s\}\in(0,1)$, $W_p^s(\mathbb{R})$ is a Banach space with the norm:

$$\|f\|_{W_p^s}:=\left(\|f\|_{W_p^{[s]}}^p+[f]_{W_p^s}^p\right)^{1/p},$$

where

$$[f]_{W_p^s}^p:=\int_{\mathbb{R}^2}\frac{|f^{([s])}(x)-f^{([s])}(y)|^p}{|x-y|^{1+\{s\}p}}\,d(x,y)=\int_{\mathbb{R}}\frac{\|f^{([s])}-\tau_\xi f^{([s])}\|_p^p}{|\xi|^{1+\{s\}p}}\,d\xi.$$

Here, $\{\tau_\xi\}_{\xi\in\mathbb{R}}$ denotes the group of right translations and $\|\cdot\|_p:=\|\cdot\|_{L_p(\mathbb{R})}$. We study the problem (1.1) in all subcritical spaces $W_p^s(\mathbb{R})$ with $s\in(1+1/p,2)$.

Reformulation of (1.1). In a compact form, the problem (1.1) can be formulated as:

$$\begin{cases} \dfrac{df}{dt}=\mathbb{B}(f)[\overline{\omega}], & t\geqslant 0,\\ -C_\Theta f'=(1+a_\mu\mathbb{A}(f))[\overline{\omega}], & t\geqslant 0,\\ f(0)=f_0. \end{cases} \tag{1.2}$$

The first two equations of (1.2) should hold in $W_p^{s-1}(\mathbb{R})$ and $f'(t):=d(f(t))/dx$. Moreover, $\mathbb{A}(f)$ and $\mathbb{B}(f)$ are the singular integral operators defined by:

$$\mathbb{A}(f)[\overline{\omega}]:=\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{yf'(x)-(f(x)-f(x-y))}{y^2+(f(x)-f(x-y))^2}\overline{\omega}(x-y)\,dy, \tag{1.3}$$

$$\mathbb{B}(f)[\overline{\omega}]:=\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{y+f'(x)(f(x)-f(x-y))}{y^2+(f(x)-f(x-y))^2}\overline{\omega}(x-y)\,dy. \tag{1.4}$$

Summary of known results. The Muskat problem was introduced in [43], but the reformulation (1.1) and many of the results on this classical problem are very recent. It is important to stress out that most of the results pertaining to (1.1) are established in L_2-based Sobolev spaces. The main reasons are

- The L_2-continuity of singular integral operators is an important problem in the harmonic analysis and many results are available in this context;

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- Plancherel's theorem can be used;
- When $a_\mu \neq 0$, the equation (1.1a)$_2$ (see also (1.2)$_2$) is a linear equation for $\overline{\omega}$. In the L_2-setting, this equation can be solved using an integral identity, known as the Rellich formula. An L_p-version, $p \neq 2$, of the Rellich formula is not available.

In the particular case when the Atwood number satisfies $a_\mu = 0$, the equation (1.1a)$_2$ identifies $\overline{\omega}$ as a function of f and (1.1) can be recast as a quasilinear equation for f which is parabolic when the fluid located below is denser, that is, when $\rho_- > \rho_+$, cf., for example, [37]. The well-posedness of the resulting equation in L_2-based Sobolev spaces was established in [18] in $H^3(\mathbb{R})$ and in [37] for $H^{3/2+\varepsilon}(\mathbb{R})$-data, $\varepsilon \in (0, 1/2)$, while [16] addressed this issue in $W^2_p(\mathbb{R}) \cap L_2(\mathbb{R})$ with $1 < p \leqslant \infty$. Solutions corresponding to medium-size data in $H^{3/2+\varepsilon}(\mathbb{R})$ exists globally, cf. [9, 14, 15, 37, 46], while the solutions determined by certain initial data with steeper slope break down in finite time [10, 11, 12]. Exponential stability results of the (flat) equilibria for the periodic counterpart of (1.1) were established in [35, 39]. For well-posedness results in homogeneous L_2-Sobolev spaces, we refer to [2, 21]. Moreover, the papers [8, 30] studied the inhomogeneous Muskat problem with nonconstant permeability, while [20, 31] consider (1.1) in a confined geometry.

The general case when $a_\mu \neq 0$ is more involved as additionally, the equation (1.1a)$_2$ needs to be solved. In this context, the quasilinear character is lost and the Muskat problem has to be treated as a fully nonlinear and nonlocal problem which is of parabolic type in the open subset of the phase space identified by the Rayleigh–Taylor condition, cf. for example, [36]. The Rayleigh–Taylor condition is a restriction imposed in the classical formulation of the Muskat problem on the sign of the jump of the normal derivative of the pressure at the interface between the fluids. The normal is taken to point into the upper region occupied by the fluid +. To be more precise, the jump of the normal derivative of the pressure has to have positive sign at each point of the interface when passing from the region occupied by the fluid − into the region of the fluid +. Local existence for the periodic counterpart of (1.1) was first established in [17] in the phase space $H^3(\mathbb{S})$. Later on in [13], the authors proved a well-posedness result for H^2-data with small $H^{3/2+\varepsilon}$-norm, with $\varepsilon << 1$. More recently, it was shown in [36, 38] that (1.1) is well posed in $H^2(\mathbb{R})$ and $H^2(\mathbb{S})$ without any smallness conditions. Well-posedness in the subcritical spaces $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$ was only recently established in [45] by using a paradifferential approach. This is the first local well-posedness result that covers all L_2-subcritical spaces in all dimensions. The existence of global weak solutions for medium-size initial data in critical spaces together with sharp algebraic decay estimates for the 3D counterpart of (1.1) was addressed in [28]. Finally, we point out that the exponential stability of the (flat) equilibria is established in the periodic setting in [38].

Other papers consider the Muskat problem in other geometries or settings, cf. [4, 5, 7, 19, 22, 23, 24, 26, 33, 50, 52]. Besides, there are also many studies which address the Muskat problem with surface tension effects, cf. [5, 22, 23, 24, 26, 29, 44, 48, 49, 51], see also the review articles [27, 32]. A particular feature of the Muskat problem with surface tension is that in the case when the less viscous fluid penetrates the region occupied by the fluid with a larger viscosity (or when the denser fluid is located above), there may exist finger-shaped equilibria. The finger-shaped equilibria with small amplitude are unstable, cf. [22, 23, 38]. Moreover, it is shown in [29] in the context of the one-phase Muskat problem that surface tension prevents, for fluid interfaces smaller than an explicit constant, the formation of fluid drops in finite time, and the corresponding solutions are global in time. Furthermore, the solutions become instantly analytic.

Main results and strategy of proof. The main goal of this paper is to establish a well-posedness theory for (1.1) that covers all subcritical spaces $W^s_p(\mathbb{R})$ with

$$s \in (1+1/p, 2) \quad \text{and} \quad 1 < p < \infty.$$

This setting has been previously considered only in [16] in the special case $a_\mu = 0$. We point out that in [16], not all subcritical spaces were covered and additional L_2-integrability of the data was required. Our strategy is to formulate (1.1) as an abstract evolution problem, cf. (4.3), and to prove that this problem is parabolic in the set where the Rayleigh–Talyor condition holds. In this setting, the Rayleigh–Taylor condition can be formulated as:

$$C_\Theta + a_\mu \mathbb{B}(f)[\overline{\omega}(f)] > 0, \tag{1.5}$$

see Section 4, where $\mathbb{B}(f)$ is the operator introduced in (1.4). Moreover, given $f \in W^s_p(\mathbb{R})$, the function $\overline{\omega} = \overline{\omega}(f)$ is identified as the unique solution to $(1.2)_2$, cf. (4.1). Our analysis shows that $\mathbb{B}(f)[\overline{\omega}(f)] \in W^{s-1}_p(\mathbb{R})$, and therefore (1.5) implies that $\Theta \geqslant 0$. For $\Theta > 0$ (the case $\Theta = 0$ is not interesting, see Section 4), we prove that

$$\mathcal{O} := \{f \in W^s_p(\mathbb{R}) \,:\, C_\Theta + a_\mu \mathbb{B}(f)[\overline{\omega}(f)] > 0\}$$

defines an open subset of $W^s_p(\mathbb{R})$ and the problem (1.1) is parabolic within $\mathcal{O}$[1].

An important tool in our analysis is the following result.

Theorem 1 *Let $a : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with bounded and Hölder-continuous first derivative. For $f \in C^\infty_0(\mathbb{R})$ let*

$$\begin{aligned} T_a[f](x) &:= \mathrm{PV}\int_{\mathbb{R}} \frac{f(x-y)}{y} \exp\left(i\frac{a(x)-a(x-y)}{y}\right) dy \\ &:= \lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} \frac{f(x-y)}{y} \exp\left(i\frac{a(x)-a(x-y)}{y}\right) dy. \end{aligned}$$

Given $p \in (1,\infty)$, the operator T_a has an extension $T_a \in \mathcal{L}(L_p(\mathbb{R}))$ and it holds that

$$\|T_a\|_{\mathcal{L}(L_p(\mathbb{R}))} \leqslant C_p(1 + \|a'\|_\infty).$$

The constant C_p depends only on p.

The result of Theorem 1 also holds for a merely Lipschitz continuous. Then, the operator T_a has to be defined by a suitable series as in [40, Section 9.6]. In the canonical case $p = 2$, this result has already been established in [42] (see also [40, Chapter 9, Rel. (6.7)] and [41] for a weaker version of this result). Theorem 1 extends the result from [42] to the L_p-setting with $p \in (1,\infty)$.

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[1] Given $f \in W^s_p(\mathbb{R})$, the operators $\mathbb{A}(f)$, $\mathbb{B}(f)$ are linear and $\overline{\omega}(f) = -C_\Theta(1 + a_\mu \mathbb{A}(f))^{-1}[f']$. Hence, the Rayleigh–Taylor condition is equivalent to the relations:

$$\Theta > 0 \quad \text{and} \quad 1 - a_\mu \mathbb{B}(f)[(1 + a_\mu \mathbb{A}(f))^{-1}[f']] > 0.$$

The first condition is imposed on the constants only, while the second one relates the Atwood number a_μ to f.

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Actually, having established Theorem 1 for $p \in (1, 2)$, the case $p > 2$ follows by duality since the adjoint T_a^* of $T_a \in \mathcal{L}(L_2(\mathbb{R}))$ is given by the formula:

$$T_a^* = -T_{-a}.$$

Theorem 1 follows in the case $p \in (1, 2)$ from well-known results of the theory of singular integral operators, for example, [1, Theorem 5.5], once the so-called Hörmander condition is established, which is done in Lemma 1 below. We note that the estimate of the operator norm by a multiple of $1 + \|a'\|_\infty$ follows by a simple scaling argument and an inspection of the proof in the same way as, for example, in [1, Proposition 4.28]. Here, one uses that the constant in the Hörmander condition and the operator norm on $L_2(\mathbb{R})$ can be bounded by a multiple of $1 + \|a'\|_\infty$.

A further issue that we had to consider was to solve the equation $(1.1a)_2$ (or equivalently $(1.2)_2$) for $\overline{\omega}$, as the Rellich formula is not available for $p \neq 2$. The arguments use quite technical localisation procedures. Moreover, the proof in the case $p \in (1, 2)$ is different from that for $p \in (2, \infty)$, see Theorems 3 and 4 below.

The analysis becomes quite involved also when showing that the evolution problem (4.3) below (which is a compact reformulation of (1.2)) is parabolic in $\mathcal{O}$. With respect to this goal, we establish in Lemma 12 a commutator estimate which is used several times in the paper (especially, in the proof of the lemmas in the Appendix A, Theorem 7 and Proposition 1).

The main result of this paper is the following theorem.

Theorem 2 *Let $p \in (1, \infty)$, $s \in (1 + 1/p, 2)$ and assume that $\Theta > 0$. Then, the following hold true:*

(i) (Well-posedness) *Given $f_0 \in \mathcal{O}$, there exists a unique maximal solution:*

$$f = f(\cdot; f_0) \in \mathrm{C}([0, T_+), \mathcal{O}) \cap \mathrm{C}^1([0, T_+), W_p^{s-1}(\mathbb{R})),$$

where $T_+ = T_+(f_0) \in (0, \infty]$, to (1.1). Moreover, $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $\mathcal{O}$.

(ii) (Parabolic smoothing)

(iia) $[(t, x) \mapsto f(t, x)] : (0, T_+) \times \mathbb{R} \to \mathbb{R}$ *is a real analytic function;*

(iib) *Given $k \in \mathbb{N}$, it holds that $f \in \mathrm{C}^\omega((0, T_+), W_p^k(\mathbb{R}))$.*[2]

The proof of Theorem 2 and of Remark 1 below is postponed to the end of Section 4.

Remark 1 *Given $\alpha \in (0, 1)$, $T > 0$, and a Banach space X, let $B((0, T], X)$ denote the Banach space of all bounded functions from $(0, T]$ into X and set*

$$\mathrm{C}_\alpha^\alpha((0, T], X) := \left\{ f \in B((0, T], X) \,:\, [f]_{C_\alpha^\alpha} := \sup_{s \neq t} \frac{\|t^\alpha f(t) - s^\alpha f(s)\|_X}{|t - s|^\alpha} < \infty \right\}.$$

Then, for each $f_0 \in \mathcal{O}$, the solution $f = f(\cdot; f_0)$ found in Theorem 2 also satisfies

$$f \in \bigcap_{\alpha \in (0,1)} \mathrm{C}_\alpha^\alpha((0, T], W_p^s(\mathbb{R})) \qquad \forall\, T \in (0, T_+(f_0)).$$

[2]Here, C^ω denotes real analyticity.

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Organisation of the paper. In Section 2, we establish the boundedness of certain multilinear singular operators which is then used to derive some useful mapping properties for the operators $\mathbb{A}$ and $\mathbb{B}$ in (1.2). Section 3 is devoted to the solvability issue for the equation $(1.2)_2$. Finally, in Section 4, we formulate (1.1) as an evolution equation for f and show that this equation is parabolic in $\mathcal{O}$. We conclude this section with the proof of Theorem 2. In Appendix A, we prove some technical results that are used in Section 4.

2 Preliminaries

We first clarify the notation used in this paper. Then, we check the Hörmander condition for the kernel of the operator T_a in Theorem 1. This condition builds the fundament of the proof of Theorem 1. The bulk of this section addresses the boundedness of certain multilinear singular operators and culminates with the proof of Lemma 8 where mapping properties for the operators $\mathbb{A}$ and $\mathbb{B}$ from (1.3) and (1.4) are established.

Notation. Given $k \in \mathbb{N}$, we let $\mathrm{C}^k(\mathbb{R})$ denote the Banach space of k-times continuously differentiable functions having bounded derivatives. Given $\alpha \in (0, 1)$, the Hölder space $\mathrm{C}^{k+\alpha}(\mathbb{R})$ is the subspace of $\mathrm{C}^k(\mathbb{R})$ that consists of functions with kth derivative having finite Hölder seminorm, that is,

$$[f^{(k)}]_\alpha := \sup_{x\neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} < \infty.$$

Sobolev's embedding states that $W^r_p(\mathbb{R}) \hookrightarrow \mathrm{C}^{r-1/p}(\mathbb{R})$ provided that $r > 1/p$. Besides, given $k \in \mathbb{N}$ with $k < r - 1/p$, since the smooth function with compact support are dense in $W^r_p(\mathbb{R})$, for $f \in W^r_p(\mathbb{R})$ it holds that $f^{(k)}(x) \to 0$ for $|x| \to \infty$. Furthermore, the following estimate finds several times application in the analysis:

$$\|gh\|_{W^r_p} \leqslant 2(\|g\|_\infty \|h\|_{W^r_p} + \|h\|_\infty \|g\|_{W^r_p}), \quad g,\ h \in W^r_p(\mathbb{R}), \tag{2.1}$$

where $r \in (1/p, 1)$ and $p \in [1, \infty)$. We also write C^{1-} to denote local Lipschitz continuity.

The Hörmander condition. Defining the singular kernel:

$$k(x, y) := \frac{1}{y} \exp\left(i\frac{a(x) - a(x-y)}{y}\right), \quad x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\}, \tag{2.2}$$

for $f \in \mathrm{C}^\infty_0(\mathbb{R})$ it holds that:

$$T_a[f](x) = \mathrm{PV}\int_{\mathbb{R}} f(y)k(x, x-y)\,dy.$$

A simple computation reveals that:

$$|\partial_y k(x, y)| \leqslant 2(1 + \|a'\|_\infty)y^{-2}, \quad x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\}, \tag{2.3}$$

and the Hörmander condition can be now established.

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Lemma 1 (The Hörmander condition) *Let $a:\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function and let k be the kernel defined in (2.2). Given $x_0\in\mathbb{R}$ and $y\in\mathbb{R}\setminus\{0\}$, it then holds*

$$\int_{[|x|>2|y|]} |k(x+x_0,x-y)-k(x+x_0,x)|\,dx \leqslant 8(1+\|a'\|_\infty).$$

Proof It follows from (2.3) and the mean value theorem that:

$$\begin{aligned}\int_{[|x|>2|y|]} |k(x+x_0,x-y)-k(x+x_0,x)|\,dx &= \int_{[|x|>2|y|]} |\partial_y k(x+x_0,\xi_y)y|\,dx\\ &\leqslant 2(1+\|a'\|_\infty)\int_{[|x|>2|y|]} |y\xi_y^{-2}|\,dx \leqslant 8(1+\|a'\|_\infty),\end{aligned}$$

where we used that $\xi_y=x-ty$, with $t\in[0,1]$, satisfies $|\xi_y|\geqslant |x|/2$. □

Boundedness of some multilinear singular integral operators. The first goal of this subsection is to show that, for any $s\in(1+1/p,2)$, with $p\in(1,\infty)$, it holds

$$\mathbb{A},\ \mathbb{B}\in \mathrm{C}^\omega(W^s_p(\mathbb{R}),\mathcal{L}(L_p(\mathbb{R})))\cap \mathrm{C}^\omega(W^s_p(\mathbb{R}),\mathcal{L}(W^{s-1}_p(\mathbb{R}))). \tag{2.4}$$

Theorem 1 is essential for this purpose. In the following, we set

$$\delta_{[x,y]}f := f(x)-f(x-y)=(f-\tau_y f)(x) \qquad \text{for } x,\ y\in\mathbb{R}.$$

In order to establish (2.4), but also for later purposes, we provide the following lemma.

Lemma 2 *Let $p\in(1,\infty)$ and $n,\ m\in\mathbb{N}$ be given.*

(i) *Given Lipschitz continuous functions $a_1,\ldots,a_m,\ b_1,\ldots,b_n:\mathbb{R}\to\mathbb{R}$, the singular integral operator $B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\ \cdot\,]$ defined by:*

$$B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}](x) := \mathrm{PV}\int_{\mathbb{R}} \frac{\overline{\omega}(x-y)}{y}\,\frac{\prod_{i=1}^n\big(\delta_{[x,y]}b_i/y\big)}{\prod_{i=1}^m\big[1+\big(\delta_{[x,y]}a_i/y\big)^2\big]}\,dy,$$

belongs to $\mathcal{L}(L_p(\mathbb{R}))$ and $\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\ \cdot\,]\|_{\mathcal{L}(L_p(\mathbb{R}))}\leqslant C\prod_{i=1}^n\|b_i'\|_\infty$, where C is a constant depending only on $n,\ m$, and $\max_{i=1,\ldots,m}\|a_i'\|_\infty$.

Moreover, $B_{n,m}\in \mathrm{C}^{1-}((W^1_\infty(\mathbb{R}))^m,\mathcal{L}_{n+1}((W^1_\infty(\mathbb{R}))^n\times L_p(\mathbb{R}),L_p(\mathbb{R})))$.

(ii) *Given $r\in(1+1/p,2)$ and $\tau\in(1/p,1)$, it holds*

$$\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}]\|_\infty \leqslant C\|\overline{\omega}\|_{W^\tau_p}\prod_{i=1}^n\|b_i\|_{W^r_p}$$

for all $a_1,\ldots,a_m,b_1,\ldots,b_n\in W^r_p(\mathbb{R})$ and $\overline{\omega}\in W^\tau_p(\mathbb{R})$, where C is a constant depending only on $\tau,\ r,\ n,\ m$, and $\max_{i=1,\ldots,m}\|a_i\|_{W^r_p}$.

Moreover, $B_{n,m}\in \mathrm{C}^{1-}((W^r_p(\mathbb{R}))^m,\mathcal{L}_{n+1}((W^r_p(\mathbb{R}))^n\times W^\tau_p(\mathbb{R}),L_\infty(\mathbb{R})))$.

Proof The proof of (i) is similar to that in the case $p=2$, cf. [37, Lemma 3.3], and relies to a large extent on Theorem 1. The proof of (ii) uses similar arguments as that in the case $p=2$, cf. [36, Lemma 3.1]. □

The next lemma collects some properties of the operators $B_{n,m}$.

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Lemma 3 *Let $p \in (1,\infty)$ and $n,\ m \in \mathbb{N}$. Let further $a_1,\dots,a_m,\ b_1,\dots,b_n : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and $\overline{\omega} \in L_p(\mathbb{R})$.*

(i) *If $n \geqslant 1$ and additionally $b_1,\ \varphi \in W^1_\infty(\mathbb{R})$, then*

$$\begin{aligned} &\varphi B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\overline{\omega}] - B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\varphi\overline{\omega}] \\ &\quad = b_1 B_{n,m}(a_1,\dots,a_m)[b_2,\dots,b_n,\varphi,\overline{\omega}] - B_{n,m}(a_1,\dots,a_m)[b_2,\dots,b_n,\varphi,b_1\overline{\omega}]. \end{aligned} \tag{2.5}$$

(ii) *If $\widetilde{a}_1,\dots,\widetilde{a}_m$ are Lipschitz continuous, then*

$$\begin{aligned} &B_{n,m}(\widetilde{a}_1,\dots,\widetilde{a}_m)[b_1,\dots,b_n,\overline{\omega}] - B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\overline{\omega}] \\ &\quad = \sum_{i=1}^m B_{n+2,m+1}(\widetilde{a}_1,\dots,\widetilde{a}_i,a_i,\dots\dots,a_m)[b_1,\dots,b_n,a_i+\widetilde{a}_i,a_i-\widetilde{a}_i,\overline{\omega}]. \end{aligned} \tag{2.6}$$

Proof The proof is elementary. □

The importance of the operators $B_{n,m}$ becomes clear when considering the relations:

$$\pi\mathbb{A}(f)[\overline{\omega}] = f' B_{0,1}(f)[\overline{\omega}] - B_{1,1}(f)[f,\overline{\omega}], \tag{2.7}$$

$$\pi\mathbb{B}(f)[\overline{\omega}] = B_{0,1}(f)[\overline{\omega}] + f' B_{1,1}(f)[f,\overline{\omega}]. \tag{2.8}$$

These relations together with Lemma 2 (i) show that, given $f \in W^s_p(\mathbb{R})$, $s \in (1+1/p, 2)$, it holds that $\mathbb{A}(f),\ \mathbb{B}(f) \in \mathcal{L}(L_p(\mathbb{R}))$. Arguing as in [37, Section 5], it actually holds

$$\mathbb{A},\ \mathbb{B} \in \mathrm{C}^\omega(W^s_p(\mathbb{R}), \mathcal{L}(L_p(\mathbb{R}))).$$

In order to establish the second mapping property in (2.4), some further analysis of the operators $B_{n,m}$ is needed. To this end, we establish in Lemma 4 new estimates. The estimate (2.9) is used in Lemma 5 below (which is the main ingredient in the proof of (2.4)), while (2.10) provides a commutator type L_p-estimate which is essential when estimating the W^{r-1}_p-norm of this commutator, cf. Lemma 6. Lemma 6 is used in the proof of Theorem 7.

Lemma 4 *Let $n, m \in \mathbb{N}$ with $n \geqslant 1$, $r \in (1+1/p, 2)$ and $\tau \in (2-r+1/p, 1)$ be given. Given $a_1,\dots,a_m \in W^r_p(\mathbb{R})$, there exists a constant C, depending only on n, m, r, and $\max_{1\leqslant i\leqslant m}\|a_i\|_{W^r_p}$ (and on τ in (2.10)), such that*

$$\|B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\overline{\omega}]\|_p \leqslant C\|b_1'\|_p \|\overline{\omega}\|_{W^{r-1}_p} \prod_{i=2}^n \|b_i'\|_{W^{r-1}_p} \tag{2.9}$$

and

$$\begin{aligned} &\|B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\overline{\omega}] - \overline{\omega}B_{n-1,m}(a_1,\dots,a_m)[b_2,\dots,b_n,b_1']\|_p \\ &\quad \leqslant C\|b_1\|_{W^\tau_p}\|\overline{\omega}\|_{W^{r-1}_p} \prod_{i=2}^n \|b_i'\|_{W^{r-1}_p} \end{aligned} \tag{2.10}$$

for all $b_1,\dots,b_n \in W^r_p(\mathbb{R})$ and $\overline{\omega} \in W^{r-1}_p(\mathbb{R})$.

Moreover, $B_{n,m} \in \mathrm{C}^{1-}((W^r_p(\mathbb{R}))^m, \mathcal{L}_{n+1}(W^1_p(\mathbb{R}) \times (W^r_p(\mathbb{R}))^{n-1} \times W^{r-1}_p(\mathbb{R}), L_p(\mathbb{R})))$.

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Proof Without loss of generality, we may assume $\overline{\omega} \in W^r_p(\mathbb{R})$. Using the identities:

$$\frac{\partial}{\partial y}\left(\frac{\delta_{[x,y]}b_1}{y}\right) = \frac{b_1'(x-y)}{y} - \frac{\delta_{[x,y]}b_1}{y^2} \quad \text{and} \quad \overline{\omega}'(x-y) = \frac{\partial}{\partial y}(\overline{\omega}(x) - \overline{\omega}(x-y))$$

and integration by parts (as in the proof of [36, Lemma 3.2]), we arrive at

$$\begin{aligned} &B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}](x)\\ &\quad = \overline{\omega}(x)B_{n-1,m}(a_1,\ldots,a_m)[b_2,\ldots,b_n,b_1'](x)\\ &\qquad - \sum_{j=2}^{n}\int_{\mathbb{R}} K_{1,j}(x,y)\overline{\omega}(x-y)\,dy + \sum_{j=1}^{m}\int_{\mathbb{R}} K_{2,j}(x,y)\,\overline{\omega}(x-y)\,dy\\ &\qquad - \int_{\mathbb{R}} K(x,y)\,dy - \sum_{j=2}^{n}\int_{\mathbb{R}} K_{3,j}(x,y)\,dy + 2\sum_{j=1}^{m}\int_{\mathbb{R}} K_{4,j}(x,y)\,dy, \end{aligned}$$

where, given $x \in \mathbb{R}$ and $y \neq 0$, we have set

$$\begin{aligned} K(x,y) &:= \frac{\prod_{i=1}^{n}\delta_{[x,y]}b_i/y}{\prod_{i=1}^{m}\left[1+\left(\delta_{[x,y]}a_i/y\right)^2\right]}\,\frac{\delta_{[x,y]}\overline{\omega}}{y},\\ K_{1,j}(x,y) &:= \frac{\prod_{i=1,i\neq j}^{n}\left(\delta_{[x,y]}b_i/y\right)}{\prod_{i=1}^{m}\left[1+\left(\delta_{[x,y]}a_i/y\right)^2\right]}\,\frac{\delta_{[x,y]}b_j - yb_j'(x-y)}{y^2},\\ K_{2,j}(x,y) &:= 2\frac{\prod_{i=1}^{n}\left(\delta_{[x,y]}b_i/y\right)}{\left[1+\left(\delta_{[x,y]}a_j/y\right)^2\right]\prod_{i=1}^{m}\left[1+\left(\delta_{[x,y]}a_i/y\right)^2\right]}\,\frac{\delta_{[x,y]}a_j - ya_j'(x-y)}{y^2}\,\frac{\delta_{[x,y]}a_j}{y},\\ K_{3,j}(x,y) &:= \frac{\prod_{i=1,i\neq j}^{n}\delta_{[x,y]}b_i/y}{\prod_{i=1}^{m}\left[1+\left(\delta_{[x,y]}a_i/y\right)^2\right]}\,\frac{\delta_{[x,y]}\overline{\omega}}{y}\left(\frac{\delta_{[x,y]}b_j}{y} - b_j'(x-y)\right),\\ K_{4,j}(x,y) &:= \frac{\left(\delta_{[x,y]}a_j/y\right)\prod_{i=1}^{n}\delta_{[x,y]}b_i/y}{\left[1+\left(\delta_{[x,y]}a_j/y\right)^2\right]\prod_{i=1}^{m}\left[1+\left(\delta_{[x,y]}a_i/y\right)^2\right]}\,\frac{\delta_{[x,y]}\overline{\omega}}{y}\left(\frac{\delta_{[x,y]}a_j}{y} - a_j'(x-y)\right). \end{aligned}$$

Recalling Lemma 2 (i), we get, with respect to (2.9), that

$$\|\overline{\omega}B_{n-1,m}(a_1,\ldots,a_m)[b_2,\ldots,b_n,b_1']\|_p \leqslant C\|\overline{\omega}\|_\infty\|b_1'\|_p\prod_{i=2}^{n}\|b_1'\|_\infty. \tag{2.11}$$

Let now $\alpha \in \{\tau, 1\}$. Since $\alpha > 1/p$ we have $W^\alpha_p(\mathbb{R}) \hookrightarrow \mathrm{C}^{\alpha-1/p}(\mathbb{R})$ and together with Minkowski's integral inequality we obtain that

$$\begin{aligned} &\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{1,j}(x,y)\overline{\omega}(x-y)\,dy\right|^p dx\right)^{1/p}\\ &\quad \leqslant \|\overline{\omega}\|_\infty \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |K_{1,j}(x,y)|^p\,dx\right)^{1/p} dy\\ &\quad \leqslant \|\overline{\omega}\|_\infty[b_1]_{\alpha-1/p}\left(\prod_{i=2,i\neq j}^{n}\|b_i'\|_\infty\right)\int_{\mathbb{R}}\frac{\|b_j - \tau_y b_j - y\tau_y b_j'\|_p}{|y|^{3-\alpha+1/p}}\,dy \end{aligned}$$

for $2 \leqslant j \leqslant n$. Fubini's theorem, Minkowski's integral inequality, Hölder's inequality and a change of variables now yield

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\|b_j - \tau_y b_j - y\tau_y b_j'\|_p}{|y|^{3-\alpha+1/p}}\, dy \\
&\quad = \int_{\mathbb{R}} \frac{1}{|y|^{2-\alpha+1/p}} \left(\int_{\mathbb{R}} \left| \int_0^1 [b_j'(x-(1-s)y) - b_j'(x-y)]\, ds \right|^p dx \right)^{1/p} dy \\
&\quad \leqslant \int_0^1 \left[\int_{\mathbb{R}} \frac{1}{|y|^{2-\alpha+1/p}} \left(\int_{\mathbb{R}} |b_j'(x-(1-s)y) - b_j'(x-y)|^p\, dx \right)^{1/p} dy \right] ds \\
&\quad \leqslant 2\|b_j'\|_p \int_{[|y|\geqslant 1]} \frac{1}{|y|^{2-\alpha+1/p}}\, dy + \int_0^1 \int_{[|y|<1]} \frac{\|b_j' - \tau_{-sy} b_j'\|_p}{|y|^{2-\alpha+1/p}} dy\, ds \\
&\quad \leqslant C\|b_j'\|_p + C\left(\int_{[|y|<1]} \frac{1}{|y|^{(3-\alpha-r)p/(p-1)}}\, dy \right)^{(p-1)/p} \|b_j'\|_{W_p^{r-1}} \\
&\quad \leqslant C\|b_j'\|_{W_p^{r-1}}.
\end{aligned}$$

Consequently, given $2 \leqslant j \leqslant n$, we get

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{1,j}(x,y)\overline{\omega}(x-y)\, dy \right|^p dx \right)^{1/p} \leqslant C\|\overline{\omega}\|_\infty [b_1]_{\alpha-1/p} \left(\prod_{i=2}^n \|b_i'\|_{W_p^{r-1}} \right), \tag{2.12}$$

and by similar arguments:

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{2,j}(x,y)\overline{\omega}(x-y)\, dy \right|^p dx \right)^{1/p} \leqslant C\|\overline{\omega}\|_\infty [b_1]_{\alpha-1/p} \left(\prod_{i=2}^n \|b_i'\|_\infty \right) \tag{2.13}$$

Furthermore, given $2 \leqslant j \leqslant n$, Hölder's inequality, Minkowski's integral inequality and the Sobolev embedding $W_p^\alpha(\mathbb{R}) \hookrightarrow \mathrm{C}^{\alpha-1/p}(\mathbb{R})$ yield

$$\begin{aligned}
&\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{3,j}(x,y)\, dy \right|^p dx \right)^{1/p} \\
&\quad \leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K_{3,j}(x,y)|^p\, dx \right)^{1/p} dy \\
&\quad \leqslant 2[b_1]_{\alpha-1/p} \left(\prod_{i=2}^n \|b_i'\|_\infty \right) \int_{\mathbb{R}} \frac{1}{|y|^{2-\alpha+1/p}} \left(\int_{\mathbb{R}} |\overline{\omega} - \tau_y \overline{\omega}|^p\, dx \right)^{1/p} dy
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{|y|^{2-\alpha+1/p}} \left(\int_{\mathbb{R}} |\overline{\omega} - \tau_y \overline{\omega}|^p\, dx \right)^{1/p} dy \\
&\quad \leqslant C\|\overline{\omega}\|_p + \int_{[|y|<1]} \frac{1}{|y|^{2-\alpha+1/p}} \left(\int_{\mathbb{R}} |\overline{\omega} - \tau_y \overline{\omega}|^p\, dx \right)^{1/p} dy \\
&\quad \leqslant C\|\overline{\omega}\|_p + \|\overline{\omega}\|_{W_p^{r-1}} \left(\int_{[|y|<1]} \frac{1}{|y|^{(3-\alpha-r)p/(p-1)}}\, dy \right)^{(p-1)/p} \\
&\quad \leqslant C\|\overline{\omega}\|_{W_p^{r-1}}.
\end{aligned}$$

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We arrive at

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{3,j}(x,y)\,dy\right|^p dx\right)^{1/p} \leqslant C\|\overline{\omega}\|_{W_p^{r-1}}[b_1]_{\alpha-1/p}\left(\prod_{i=2}^{n}\|b_i'\|_\infty\right), \qquad 2\leqslant j\leqslant n. \tag{2.14}$$

The same arguments show that

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K(x,y)\,dy\right|^p dx\right)^{1/p}+\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{4,j}(x,y)\,dy\right|^p dx\right)^{1/p}$$

$$\leqslant C\|\overline{\omega}\|_{W_p^{r-1}}[b_1]_{\alpha-1/p}\left(\prod_{i=2}^{n}\|b_i'\|_\infty\right), \qquad 1\leqslant j\leqslant m.$$

Choosing $\alpha=1$, (2.9) follows from (2.11) to (2.15) and the relation $[b_1]_{\alpha-1/p}\leqslant\|b_1'\|_p$. Moreover, (2.10) follows from (2.12) to (2.15) for $\alpha=\tau$. Finally, the local Lipschitz continuity property is a consequence of (2.6). □

Lemma 4 enables us to establish estimates in suitable fractional Sobolev spaces for the multilinear operators $B_{n,m}$ considered above.

Lemma 5 *Let $n,m\in\mathbb{N}$ and $r\in(1+1/p,2)$ be given. Given $a_1,\ldots,a_m\in W_p^r(\mathbb{R})$, there exists a constant C, depending only on n, m, r and $\max_{1\leqslant i\leqslant m}\|a_i\|_{W_p^r}$, such that*

$$\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}]\|_{W_p^{r-1}}\leqslant C\|\overline{\omega}\|_{W_p^{r-1}}\prod_{i=1}^{n}\|b_i'\|_{W_p^{r-1}} \tag{2.15}$$

for all $b_1,\ldots,b_n\in W_p^r(\mathbb{R})$ and $\overline{\omega}\in W_p^{r-1}(\mathbb{R})$.

Moreover, $B_{n,m}\in C^{1-}((W_p^r(\mathbb{R}))^m,\mathcal{L}_{n+1}((W_p^r(\mathbb{R}))^n\times W_p^{r-1}(\mathbb{R}),W_p^{r-1}(\mathbb{R})))$.

Proof Set $B_{n,m}:=B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}]$. Recalling Lemma 2 (i), it holds that

$$\|B_{n,m}\|_p\leqslant C\|\overline{\omega}\|_p\prod_{i=1}^{n}\|b_i\|_{W_p^r}.$$

It thus remains to consider the W_p^{r-1}-seminorm of $B_{n,m}$. To this end we observe that

$$[B_{n,m}]^p_{W_p^{r-1}}=\int_{\mathbb{R}}\frac{\|B_{n,m}-\tau_\xi B_{n,m}\|_p^p}{|\xi|^{1+(r-1)p}}\,d\xi,$$

where, using (2.6), we write

$$(B_{n,m}-\tau_\xi B_{n,m})(x)=T_1(x,\xi)+T_2(x,\xi)-T_3(x,\xi), \qquad x\in\mathbb{R},\ \xi\neq 0,$$

with

$$T_1(x,\xi):=B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}-\tau_\xi\overline{\omega}](x),$$

$$T_2(x,\xi):=\sum_{i=1}^{n}B_{n,m}(a_1,\ldots,a_m)[\tau_\xi b_1,\ldots,\tau_\xi b_{i-1},b_i-\tau_\xi b_i,b_{i+1},\ldots b_n,\tau_\xi\overline{\omega}](x),$$

$$T_3(x,\xi):=\sum_{i=1}^{m}B^i_{n+2,m+1}[\tau_\xi b_1,\ldots\tau_\xi b_n,a_i+\tau_\xi a_i,a_i-\tau_\xi a_i,\tau_\xi\overline{\omega}](x)$$

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and $B^i_{n+2,m+1} := B_{n+2,m+1}(a_1, \ldots, a_i, \tau_\xi a_i, \ldots, \tau_\xi a_m)$. Hence,

$$[B_{n,m}]^p_{W^{r-1}_p} \leqslant 3^p \sum_{\ell=1}^{3} \int_{\mathbb{R}} \frac{\|T_\ell(\cdot, \xi)\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi, \tag{2.16}$$

and, recalling Lemma 2 (i), it holds

$$\int_{\mathbb{R}} \frac{\|T_1(\cdot, \xi)\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi \leqslant C^p \left(\prod_{i=1}^{n} \|b'_i\|^p_\infty\right) \int_{\mathbb{R}} \frac{\|\overline{\omega} - \tau_\xi \overline{\omega}\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi \leqslant \left(C\|\overline{\omega}\|_{W^{r-1}_p} \prod_{i=1}^{n} \|b'_i\|_\infty\right)^p. \tag{2.17}$$

Furthermore, in virtue of (2.9), we get that

$$\begin{aligned}\int_{\mathbb{R}} \frac{\|T_2(\cdot, \xi)\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi &\leqslant C^p \|\overline{\omega}\|^p_{W^{r-1}_p} \sum_{i=1}^{n} \left(\int_{\mathbb{R}} \frac{\|b'_i - \tau_\xi b'_i\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi \prod_{j\neq i} \|b'_j\|^p_{W^{r-1}_p}\right)\\ &\leqslant \left(C\|\overline{\omega}\|_{W^{r-1}_p} \prod_{i=1}^{n} \|b'_i\|_{W^{r-1}_p}\right)^p\end{aligned} \tag{2.18}$$

and, by similar arguments:

$$\int_{\mathbb{R}} \frac{\|T_3(\cdot, \xi)\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi \leqslant \left(C\|\overline{\omega}\|_{W^{r-1}_p} \prod_{i=1}^{n} \|b'_i\|_{W^{r-1}_p}\right)^p. \tag{2.19}$$

The estimates (2.16)–(2.19) lead to the desired estimate. Finally, the local Lipschitz continuity follows from (2.6) and (2.15). □

We now estimate the commutator type operator from (2.10) in the $\|\cdot\|_{W^{r-1}_p}$-norm.

Lemma 6 *Let $n, m \in \mathbb{N}$, $n \geqslant 1$, $r \in (1 + 1/p, 2)$ and $r' \in (1 + 1/p, r)$ be given. Given $a_1, \ldots, a_m \in W^r_p(\mathbb{R})$, there exists a constant C that depends only on n, m, r, r' and $\max_{1\leqslant i\leqslant m} \|a_i\|_{W^r_p}$, such that*

$$\begin{aligned}&\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \overline{\omega}] - \overline{\omega} B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b'_1]\|_{W^{r-1}_p}\\ &\qquad\leqslant C\|b_1\|_{W^{r'}_p} \|\overline{\omega}\|_{W^{r-1}_p} \prod_{i=2}^{n} \|b_i\|_{W^r_p}\end{aligned} \tag{2.20}$$

for all $b_1, \ldots, b_n \in W^r_p(\mathbb{R})$ and $\overline{\omega} \in W^{r-1}_p(\mathbb{R})$.

Proof Let $T := B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \overline{\omega}] - \overline{\omega} B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b'_1]$. In view of (2.10), $\|T\|_p$ can be estimated as in (2.20). It remains to consider the term:

$$[T]^p_{W^{r-1}_p} = \int_{\mathbb{R}} \frac{\|T - \tau_\xi T\|^p_p}{|\xi|^{1+(r-1)p}}\, d\xi,$$

for which it is convenient to write

$$(T - \tau_\xi T)(x) = T_1(x, \xi) + T_2(x, \xi) + T_3(x, \xi) + T_4(x, \xi), \qquad x \in \mathbb{R},\ \xi \in \mathbb{R},$$

where,

$$
\begin{aligned}
T_1(\cdot,\xi) &:= B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\overline{\omega}-\tau_\xi\overline{\omega}]\\
&\quad -(\overline{\omega}-\tau_\xi\overline{\omega})B_{n-1,m}(a_1,\ldots,a_m)[b_2,\ldots,b_n,b_1'],\\
T_2(\cdot,\xi) &:= B_{n,m}(a_1,\ldots,a_m)[b_1-\tau_\xi b_1,b_2,\ldots,b_n,\tau_\xi\overline{\omega}]\\
&\quad -\tau_\xi\overline{\omega}B_{n-1,m}(a_1,\ldots,a_m)[b_2,\ldots,b_n,b_1'-\tau_\xi b_1'],\\
T_3(\cdot,\xi) &:= B_{n,m}(a_1,\ldots,a_m)[\tau_\xi b_1,b_2,\ldots,b_n,\tau_\xi\overline{\omega}]\\
&\quad -B_{n,m}(\tau_\xi a_1,\ldots,\tau_\xi a_m)[\tau_\xi b_1,\ldots,\tau_\xi b_n,\tau_\xi\overline{\omega}],\\
T_4(\cdot,\xi) &:= \tau_\xi\overline{\omega}B_{n-1,m}(\tau_\xi a_1,\ldots,\tau_\xi a_m)[\tau_\xi b_2,\ldots,\tau_\xi b_n,\tau_\xi b_1']\\
&\quad -\tau_\xi\overline{\omega}B_{n-1,m}(a_1,\ldots,a_m)[b_2,\ldots,b_n,\tau_\xi b_1'].
\end{aligned}
$$

Lemma 2 (i) and (ii) (with $\tau=r'-1$) implies that

$$\|T_1(\cdot,\xi)\|_p\leqslant C\|\overline{\omega}-\tau_\xi\overline{\omega}\|_p\|b_1\|_{W_p^{r'}}\prod_{i=2}^n\|b_i\|_{W_p^r},$$

while (2.10) (with $\tau=r'-r+1\in(2-r+1/p,1)$) yields

$$\|T_2(\cdot,\xi)\|_p\leqslant C\|b_1-\tau_\xi b_1\|_{W_p^{r'-r+1}}\|\overline{\omega}\|_{W_p^{r-1}}\prod_{i=2}^n\|b_i\|_{W_p^r}.$$

Finally, recalling (2.6), it holds that

$$
\begin{aligned}
T_3(\cdot,\xi) &= \sum_{i=2}^n B_{n,m}(a_1,\ldots,a_m)[\tau_\xi b_1,\ldots,\tau_\xi b_{i-1},b_i-\tau_\xi b_i,b_{i+1},\ldots,b_n,\tau_\xi\overline{\omega}]\\
&\quad -\sum_{i=1}^m B^i_{n+2,m+1}[\tau_\xi b_1,\ldots,\tau_\xi b_n,a_i+\tau_\xi a_i,a_i-\tau_\xi a_i,\tau_\xi\overline{\omega}],\\
T_4(\cdot,\xi) &= \tau_\xi\overline{\omega}\sum_{i=2}^n B_{n,m}(a_1,\ldots,a_m)[b_2,\ldots,b_{i-1},\tau_\xi b_i-b_i,\tau_\xi b_{i+1},\ldots,\tau_\xi b_n,\tau_\xi b_1']\\
&\quad +\tau_\xi\overline{\omega}\sum_{i=1}^m B^i_{n+1,m+1}[\tau_\xi b_2,\ldots,\tau_\xi b_n,a_i+\tau_\xi a_i,a_i-\tau_\xi a_i,\tau_\xi b_1'],
\end{aligned}
$$

where, given $1\leqslant i\leqslant m$, we set

$$
\begin{aligned}
B^i_{n+1,m+1} &:= B_{n+1,m+1}(\tau_\xi a_1,\ldots,\tau_\xi a_i,a_i,\ldots a_m),\\
B^i_{n+2,m+1} &:= B_{n+2,m+1}(a_1,\ldots,a_i,\tau_\xi a_i,\ldots\tau_\xi a_m).
\end{aligned}
$$

Repeated use of Lemma 4 (with $r=r'$) yields

$$
\begin{aligned}
\|T_3(\cdot,\xi)\|_p &\leqslant C\|\overline{\omega}\|_{W_p^{r-1}}\|b_1\|_{W_p^{r'}}\Bigg[\sum_{i=2}^n\Bigg(\prod_{j=2,j\neq i}^n\|b_j\|_{W_p^r}\Bigg)\|b_i-\tau_\xi b_i\|_{W_p^1}\\
&\quad +\Bigg(\prod_{j=2}^n\|b_j\|_{W_p^r}\Bigg)\sum_{i=1}^m\|a_i-\tau_\xi a_i\|_{W_p^1}\Bigg]
\end{aligned}
$$

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and

$$\|T_4(\cdot,\xi)\|_p \leqslant C\|\overline{\omega}\|_{W_p^{r-1}}\|b_1\|_{W_p^{r'}}\Bigg[\sum_{i=2}^{n}\Bigg(\prod_{j=2, j\neq i}^{n}\|b_j\|_{W_p^r}\Bigg)\|b_i-\tau_\xi b_i\|_{W_p^1} + \Bigg(\prod_{j=2}^{n}\|b_j\|_{W_p^r}\Bigg)\sum_{i=1}^{m}\|a_i-\tau_\xi a_i\|_{W_p^1}\Bigg].$$

Gathering these estimates, we conclude that

$$[T]_{W_p^{r-1}} \leqslant C\|\overline{\omega}\|_{W_p^{r-1}}\prod_{i=2}^{n}\|b_i\|_{W_p^r}\left[\|b_1\|_{W_p^{r'}} + \left(\int_{\mathbb{R}}\frac{\|b_1-\tau_\xi b_1\|^p_{W_p^{r'-r+1}}}{|\xi|^{1+(r-1)p}}\,d\xi\right)^{1/p}\right],$$

which together with Lemma 7 below proves the claim. □

In the proof of Lemma 6, we have used the following result.

Lemma 7 *Let $p\in(1,\infty)$ and $1<r'<r<2$. Then, there exists a constant $C>0$ such that*

$$\int_{\mathbb{R}}\frac{\|b-\tau_\xi b\|^p_{W_p^{r'-r+1}}}{|\xi|^{1+(r-1)p}}\,d\xi \leqslant C\|b\|^p_{W_p^{r'}} \qquad \textit{for all } b\in W_p^{r'}(\mathbb{R}).$$

Proof The claim follows using the mean value theorem and the definition of the Sobolev norm. We omit the details. □

We are now in a position to prove the second claim in (2.4).

Lemma 8 *Given $s\in(1+1/p,2)$, it holds that*

$$\mathbb{A},\ \mathbb{B}\in \mathrm{C}^\omega(W_p^s(\mathbb{R}),\mathcal{L}(W_p^{s-1}(\mathbb{R}))). \tag{2.21}$$

Proof Combining (2.7)–(2.8), Lemma 5, and the algebra property of $W_p^{s-1}(\mathbb{R})$, it follows that

$$[f\mapsto \mathbb{A}(f)],\ [f\mapsto\mathbb{B}(f)]\in \mathrm{C}^{1-}(W_p^s(\mathbb{R}),\mathcal{L}(W_p^{s-1}(\mathbb{R}))).$$

Moreover, arguing as in [37, Section 5], it can be shown that $\mathbb{A}$ and $\mathbb{B}$ depend analytically on $f\in W_p^s(\mathbb{R})$. □

3 On the resolvent set of $\mathbb{A}(f)$

We now fix $s\in(1+1/p,2)$ and $f\in W_p^s(\mathbb{R})$. The main goal of this section is to show that the equation $(1.2)_2$ has a unique solution $\overline{\omega}\in W_p^{s-1}(\mathbb{R})$. Compared to the canonical case $p=2$, where the Rellich formula, see [36, equation (3.24)], can be used to solve $(1.2)_2$, for $p\neq 2$ we need to find a new approach as the Rellich formula does not apply directly.

To start, we infer from the arguments in [37, Theorem 3.5] that, given $\lambda\in\mathbb{R}$ with $|\lambda|\geqslant 1$, the operator $\lambda-\mathbb{A}(f)$ is an $L_2(\mathbb{R})$-isomorphism, that is, it belongs to $\mathrm{Isom}(L_2(\mathbb{R}))$. Moreover, the

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L_2-adjoint $(\mathbb{A}(f))^*$ of $\mathbb{A}(f)$ is given by:

$$(\mathbb{A}(f))^*[\overline{\omega}] = \pi^{-1}(B_{1,1}(f)[f,\overline{\omega}] - B_{0,1}(f)[f'\overline{\omega}]).$$

Letting $p' = p/(p-1)$ denote the dual exponent to p, it follows from Lemma 2 (i) that

$$(\mathbb{A}(f))^* \in \mathcal{L}(L_{p'}(\mathbb{R})). \tag{3.1}$$

The main step towards our goal is to prove the invertibility of $\lambda - \mathbb{A}(f)$ in $\mathcal{L}(L_p(\mathbb{R}))$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, see Theorems 3 and 4 below. These results are then used to establish the invertibility of $\lambda - \mathbb{A}(f)$ in $\mathcal{L}(W_p^{s-1}(\mathbb{R}))$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, see Theorem 5. This necessitates the introduction of suitable partitions of unity. To be more precise, we choose for each $\varepsilon \in (0,1)$, a finite ε-localisation family, that is, a family:

$$\{\pi_j^\varepsilon \,:\, -N+1 \leqslant j \leqslant N\} \subset \mathrm{C}^\infty(\mathbb{R}, [0,1]),$$

with $N = N(\varepsilon) \in \mathbb{N}$ sufficiently large, such that

- $\operatorname{supp} \pi_j^\varepsilon$ is an interval of length ε for all $|j| \leqslant N-1$, $\operatorname{supp} \pi_N^\varepsilon \subset [|x| \geqslant 1/\varepsilon]$;
- $\pi_j^\varepsilon \cdot \pi_l^\varepsilon = 0$ if $[|j-l| \geqslant 2, \max\{|j|,|l|\} \leqslant N-1]$ or $[|l| \leqslant N-2, j = N]$;
- $\sum_{j=-N+1}^{N} (\pi_j^\varepsilon)^2 = 1$;
- $\|(\pi_j^\varepsilon)^{(k)}\|_\infty \leqslant C\varepsilon^{-k}$ for all $k \in \mathbb{N}$, $-N+1 \leqslant j \leqslant N$.

To each finite ε-localisation family we associate the second family:

$$\{\chi_j^\varepsilon \,:\, -N+1 \leqslant j \leqslant N\} \subset \mathrm{C}^\infty(\mathbb{R}, [0,1])$$

with the following properties:

- $\chi_j^\varepsilon = 1$ on $\operatorname{supp} \pi_j^\varepsilon$;
- $\operatorname{supp} \chi_j^\varepsilon$ is an interval of length 3ε and with the same midpoint as $\operatorname{supp} \pi_j^\varepsilon$, $|j| < N$;
- $\operatorname{supp} \chi_N^\varepsilon \subset [|x| \geqslant 1/\varepsilon - \varepsilon]$ and $\xi + \operatorname{supp} \pi_N^\varepsilon \subset \operatorname{supp} \chi_N^\varepsilon$ for $|\xi| < \varepsilon$.

Each ε-localisation family induces on $W_p^r(\mathbb{R})$ a norm equivalent to the standard norm.

Lemma 9 *Let $\varepsilon > 0$ and let $\{\pi_j^\varepsilon \,:\, -N+1 \leqslant j \leqslant N\}$ be a finite ε-localisation family. Given $p \in (1,\infty)$ and $r \in [0,\infty)$, there exists $c = c(\varepsilon, r, p) \in (0,1)$ such that*

$$c\|f\|_{W_p^r} \leqslant \sum_{j=-N+1}^{N} \|\pi_j^\varepsilon f\|_{W_p^r} \leqslant c^{-1}\|f\|_{W_p^r}, \qquad f \in W_p^r(\mathbb{R}).$$

Proof We omit the elementary proof. □

The result established in the next lemma is used in an essential way in the proof of Theorems 3 and 4 below.

Lemma 10 *Let $\varepsilon \in (0,1)$ be arbitrary (but fixed) and let $\{\pi_j^\varepsilon \,:\, -N+1 \leqslant j \leqslant N\}$ and $\{\chi_j^\varepsilon \,:\, -N+1 \leqslant j \leqslant N\}$ be as described above. Furthermore, fix $f \in W_p^s(\mathbb{R})$, where $s \in (1+1/p, 2)$.*

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Given $-N+1 \leqslant j \leqslant N$, the operator $K_j : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ defined by

$$K_j[\overline{\omega}] := \chi_j^\varepsilon(\pi_j^\varepsilon \mathbb{A}(f)[\overline{\omega}] - \mathbb{A}(f)[\pi_j^\varepsilon \overline{\omega}]), \qquad \overline{\omega} \in L_p(\mathbb{R}),$$

is compact.

Proof According to the Riesz–Fréchet–Kolmogorov theorem, it suffices to show that

$$\sup_{\|\overline{\omega}\|_p \leqslant 1} \int_{[|x|>R]} |K_j[\overline{\omega}](x)|^p \, dx \to 0 \qquad \text{for } R \to \infty, \tag{3.2}$$

$$\sup_{\|\overline{\omega}\|_p \leqslant 1} \|\tau_\xi(K_j[\overline{\omega}]) - K_j[\overline{\omega}]\|_p \to 0 \qquad \text{for } \xi \to 0. \tag{3.3}$$

Step 1. For $|j| \leqslant N-1$, the assertion (3.2) is obvious. Let now $j = N$. Then, it holds

$$\begin{aligned}
&\left(\int_{[|x|>R]} |K_j[\overline{\omega}](x)|^p \, dx\right)^{1/p} \\
&\leqslant \left(\int_{[|x|>R]} \left|\int_{\mathbb{R}} \frac{yf'(x) - \delta_{[x,y]}f}{1 + \left(\delta_{[x,y]}f/y\right)^2} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} \overline{\omega}(x-y)\, dy\right|^p dx\right)^{1/p} \\
&\leqslant \left(\int_{[|x|>R]} \left|\int_{[|y|\leqslant 1]} \frac{yf'(x) - \delta_{[x,y]}f}{1 + \left(\delta_{[x,y]}f/y\right)^2} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} \overline{\omega}(x-y)\, dy\right|^p dx\right)^{1/p} \\
&\quad + \left(\int_{[|x|>R]} \left|\int_{[|y|>1]} \frac{yf'(x) - \delta_{[x,y]}f}{1 + \left(\delta_{[x,y]}f/y\right)^2} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} \overline{\omega}(x-y)\, dy\right|^p dx\right)^{1/p} \\
&=: T_1 + T_2.
\end{aligned}$$

If R is sufficiently large, then $\pi_j^\varepsilon(x) = 1 = \pi_j^\varepsilon(x-y)$ for all $|x| > R$ and $|y| \leqslant 1$; hence, $T_1 = 0$. Concerning T_2, it holds $T_2 \leqslant T_{2a} + T_{2b} + T_{2c}$, where,

$$\begin{aligned}
T_{2a} &:= \left(\int_{[|x|>R]} |f'(x)|^p \left|\int_{[|y|>1]} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y} \overline{\omega}(x-y)\, dy\right|^p dx\right)^{1/p} \leqslant C\|f'\|_{L_p([|x|>R])}, \\
T_{2b} &:= \left(\int_{[|x|>R]} |f(x)|^p \left|\int_{[|y|>1]} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} \overline{\omega}(x-y)\, dy\right|^p dx\right)^{1/p} \leqslant C\|f\|_{L_p([|x|>R])}, \\
T_{2c} &:= \left(\int_{[|x|>R]} \left|\int_{[|y|>1]} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} (f\overline{\omega})(x-y)\, dy\right|^p dx\right)^{1/p}.
\end{aligned}$$

If R is sufficiently large, then $\pi_j^\varepsilon(x) = 1$ for all $|x| > R$. In view of $\pi_j^\varepsilon(x-y) = 1$ for all $|x-y| > 1/\varepsilon + \varepsilon$, it follows that $\delta_{[x,y]}\pi_j^\varepsilon = 0$ for $|x| > R$ and $|x-y| > 1/\varepsilon + \varepsilon$, hence,

$$\begin{aligned}
T_{2c} &= \left(\int_{[|x|>R]} \left|\int_{[|y|>R-1/\varepsilon-\varepsilon]} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} (f\overline{\omega})(x-y)\, dy\right|^p dx\right)^{1/p} \\
&\leqslant \int_{[|y|>R-1/\varepsilon-\varepsilon]} \frac{2}{y^2} \left(\int_{[|x|>R]} |(f\overline{\omega})(x-y)|^p \, dx\right)^{1/p} dy \leqslant \frac{C}{R - 1/\varepsilon - \varepsilon} \|f\|_\infty.
\end{aligned}$$

These arguments show that (3.2) holds for all $-N+1 \leqslant j \leqslant N$.

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Step 2. With respect to (3.3), we note that $\|\tau_\xi(K_j[\overline{\omega}])-K_j[\overline{\omega}]\|_p \leqslant T_1+T_2+T_3$ for all $-N+1\leqslant j\leqslant N$, where

$$T_1 := \|\tau_\xi \chi_j^\varepsilon - \chi_j^\varepsilon\|_\infty \|\pi_j^\varepsilon \mathbb{A}(f)[\overline{\omega}] - \mathbb{A}(f)[\pi_j^\varepsilon \overline{\omega}]\|_p \leqslant C\|\tau_\xi \chi_j^\varepsilon - \chi_j^\varepsilon\|_\infty \leqslant C|\xi|,$$

$$T_2 := \Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\Big[\frac{\tau_\xi f'(x)}{1+\big(\delta_{[x-\xi,y-\xi]}f/(y-\xi)\big)^2}\frac{\delta_{[x-\xi,y-\xi]}\pi_j^\varepsilon}{y-\xi} - \frac{f'(x)}{1+\big(\delta_{[x,y]}f/y\big)^2}\frac{\delta_{[x,y]}\pi_j^\varepsilon}{y}\Big]\overline{\omega}(x-y)\,dy\Big|^p\,dx\Big)^{1/p},$$

$$T_3 := \Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\Big[\frac{\delta_{[x-\xi,y-\xi]}f}{1+\big(\delta_{[x-\xi,y-\xi]}f/(y-\xi)\big)^2}\frac{\delta_{[x-\xi,y-\xi]}\pi_j^\varepsilon}{(y-\xi)^2} - \frac{\delta_{[x,y]}f}{1+\big(\delta_{[x,y]}f/y\big)^2}\frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2}\Big]\overline{\omega}(x-y)\,dy\Big|^p\,dx\Big)^{1/p}.$$

Moreover, $T_2\leqslant T_{2a}+T_{2b}$, where, using Hölder's inequality, we have

$$T_{2a} := \left(\int_{\mathbb{R}}|\tau_\xi f'(x)-f'(x)|^p\left(\int_{\mathbb{R}}\Big|\frac{\delta_{[x-\xi,y-\xi]}\pi_j^\varepsilon}{y-\xi}\overline{\omega}(x-y)\Big|\,dy\right)^p\,dx\right)^{1/p} \leqslant C\|\tau_\xi f'-f'\|_p,$$

uniformly for $|\xi|<1/2$. Moreover, T_{2b} is defined as the expression:

$$\left(\int_{\mathbb{R}}|f'(x)|^p\Big|\int_{\mathbb{R}}\Big[\frac{\delta_{[x-\xi,y-\xi]}\pi_j^\varepsilon/(y-\xi)}{1+\big(\delta_{[x-\xi,y-\xi]}f/(y-\xi)\big)^2} - \frac{\delta_{[x,y]}\pi_j^\varepsilon/y}{1+\big(\delta_{[x,y]}f/y\big)^2}\Big]\overline{\omega}(x-y)\,dy\Big|^p dx\right)^{1/p}.$$

Taking into account that for $|\xi|<1/2$ it holds that

$$\Big|\frac{\delta_{[x-\xi,y-\xi]}\pi_j^\varepsilon/(y-\xi)}{1+\big(\delta_{[x-\xi,y-\xi]}f/(y-\xi)\big)^2} - \frac{\delta_{[x,y]}\pi_j^\varepsilon/y}{1+\big(\delta_{[x,y]}f/y\big)^2}\Big| \leqslant C|\xi|^{s-1-1/p}\left(\mathbf{1}_{[|y|\leqslant 1]}(y)+\mathbf{1}_{[|y|>1]}(y)\frac{1}{|y|}\right),$$

Hölder's inequality leads, for $|\xi|<1/2$, to

$$T_{2b}\leqslant C|\xi|^{s-1-1/p}.$$

It remains to show that $T_3\to 0$ for $\xi\to 0$. Since,

$$\Big|\frac{\delta_{[x-\xi,y-\xi]}f}{1+\big(\delta_{[x-\xi,y-\xi]}f/(y-\xi)\big)^2}\frac{\delta_{[x-\xi,y-\xi]}\pi_j^\varepsilon}{(y-\xi)^2} - \frac{\delta_{[x,y]}f}{1+\big(\delta_{[x,y]}f/y\big)^2}\frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2}\Big| \leqslant C|\xi|^{s-1-1/p}\left(\mathbf{1}_{[|y|\leqslant 1]}(y)+\mathbf{1}_{[|y|>1]}(y)\frac{1}{|y|^2}\right)$$

for all $|\xi|<1/2$, Minkowski's inequality yields

$$T_2\leqslant C|\xi|^{s-1-1/p}.$$

Hence, (3.3) holds true for all $-N+1\leqslant j\leqslant N$ and the proof is complete. □

The next result is a key ingredient in the proof of Theorems 3 and 4 below.

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Lemma 11 *Let $\varepsilon\in(0,1)$ and $\{\pi_j^\varepsilon : -N+1\leqslant j\leqslant N\}$ and $\{\chi_j^\varepsilon : -N+1\leqslant j\leqslant N\}$ be as described above. Let further $f\in W_p^s(\mathbb{R})$, $s\in(1+1/p,2)$,and pick $q\in[p,\infty)$. Then, for each $-N+1\leqslant j\leqslant N$, it holds that*

$$\|\chi_j^\varepsilon\mathbb{A}(f)\chi_j^\varepsilon\|_{\mathcal{L}(L_q(\mathbb{R}))}<1,$$

provided that ε is sufficiently small.

Proof We first establish the claim for $|j|\leqslant N-1$. Using Minkowski's inequality and the embedding $W_p^s(\mathbb{R})\hookrightarrow \mathrm{C}^{s-1/p}(\mathbb{R})$, it follows that

$$\begin{aligned}
\|\chi_j^\varepsilon\mathbb{A}(f)[\chi_j^\varepsilon\overline{\omega}]\|_q &= \left(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\chi_j^\varepsilon(x)\frac{yf'(x)-\delta_{[x,y]}f}{1+\big(\delta_{[x,y]}f/y\big)^2}\frac{(\chi_j^\varepsilon\overline{\omega})(x-y)}{y^2}\,dy\Big|^q\,dx\right)^{1/q}\\
&\leqslant [f']_{s-1-1/p}\int_{\mathbb{R}}|y|^{s-2-1/p}\left(\int_{\operatorname{supp}\chi_j^\varepsilon}|(\chi_j^\varepsilon\overline{\omega})(x-y)|^q\,dx\right)^{1/q}dy\\
&\leqslant [f']_{s-1-1/p}\|\overline{\omega}\|_q\int_{\operatorname{supp}\chi_j^\varepsilon-\operatorname{supp}\chi_j^\varepsilon}|y|^{s-2-1/p}\,dy\\
&\leqslant C\|f\|_{W_p^s}\varepsilon^{s-1-1/p}\|\overline{\omega}\|_q.
\end{aligned}$$

In the case $j=N$, we have $\|\chi_j^\varepsilon\mathbb{A}(f)[\chi_j^\varepsilon\overline{\omega}]\|_q\leqslant T_1+T_2$, where

$$\begin{aligned}
T_1 &:= \left(\int_{\mathbb{R}}\Big|\int_{[|y|\leqslant 1]}\chi_j^\varepsilon(x)\frac{yf'(x)-\delta_{[x,y]}f}{1+\big(\delta_{[x,y]}f/y\big)^2}\frac{(\chi_j^\varepsilon\overline{\omega})(x-y)}{y^2}\,dy\Big|^q\,dx\right)^{1/q},\\
T_2 &:= \left(\int_{\mathbb{R}}\Big|\int_{[|y|>1]}\chi_j^\varepsilon(x)\frac{yf'(x)-\delta_{[x,y]}f}{1+\big(\delta_{[x,y]}f/y\big)^2}\frac{(\chi_j^\varepsilon\overline{\omega})(x-y)}{y^2}\,dy\Big|^q\,dx\right)^{1/q}.
\end{aligned}$$

Using the mean value theorem, we have

$$|yf'(x)-\delta_{[x,y]}f|\leqslant 2\|f'\|^{1/2}_{L_\infty([|x|>1/\varepsilon-2])}[f']^{1/2}_{s-1-1/p}|y|^{s/2+1/2-1/2p},$$

and herewith the term T_1 can be estimated as follows:

$$\begin{aligned}
T_1 &\leqslant 2\|f'\|^{1/2}_{L_\infty([|x|>1/\varepsilon-2])}[f']^{1/2}_{s-1-1/p}\left(\int_{\mathbb{R}}\Big|\int_{[|y|\leqslant 1]}\chi_j^\varepsilon(x)\frac{(\chi_j^\varepsilon\overline{\omega})(x-y)}{y^{3/2+1/2p-s/2}}\,dy\Big|^q\,dx\right)^{1/q}\\
&\leqslant 2\|f'\|^{1/2}_{L_\infty([|x|>1/\varepsilon-2])}[f']^{1/2}_{s-1-1/p}\int_{[|y|\leqslant 1]}\frac{1}{y^{3/2+1/2p-s/2}}\left(\int_{\mathbb{R}}|(\chi_j^\varepsilon\overline{\omega})(x-y)|^q\,dx\right)^{1/q}dy\\
&\leqslant C\|f'\|^{1/2}_{L_\infty([|x|>1/\varepsilon-2])}\|f\|^{1/2}_{W_p^s}\|\overline{\omega}\|_q.
\end{aligned}$$

Furthermore, $T_2\leqslant T_{2a}+T_{2b}+T_{2c}$ where

$$\begin{aligned}
T_{2a} &:= \left(\int_{\mathbb{R}}|(f'\chi_j^\varepsilon)(x)|^q\Big|\int_{[|y|>1]}\frac{(\chi_j^\varepsilon\overline{\omega})(x-y)}{y}\,dy\Big|^q\,dx\right)^{1/q}\\
&\leqslant C\|f'\|^{p/q}_{L_p([|x|>1/\varepsilon-1])}\|f'\|^{(q-p)/q}_{L_\infty([|x|>1/\varepsilon-1])}\|\overline{\omega}\|_q,
\end{aligned}$$

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and

$$T_{2b} := \left(\int_{\mathbb{R}} \Big| \int_{[|y|>1]} (f\chi_j^\varepsilon)(x) \frac{(\chi_j^\varepsilon \overline{\omega})(x-y)}{y^2}\, dy \Big|^q dx \right)^{1/q}$$
$$\leqslant \int_{[|y|>1]} \frac{1}{y^2} \left(\int_{\mathbb{R}} |(f\chi_j^\varepsilon)(x)(\chi_j^\varepsilon\overline{\omega})(x-y)|^q\, dx \right)^{1/q} dy$$
$$\leqslant C\|f\|_{L_\infty([|x|>1/\varepsilon-1])}\|\overline{\omega}\|_q,$$

and finally,

$$T_{2c} := \left(\int_{\mathbb{R}} \Big| \int_{[|y|>1]} \chi_j^\varepsilon(x) \frac{(f\chi_j^\varepsilon \overline{\omega})(x-y)}{y^2}\, dy \Big|^q dx \right)^{1/q}$$
$$\leqslant \int_{[|y|>1]} \frac{1}{y^2} \left(\int_{\mathbb{R}} |(f\chi_j^\varepsilon\overline{\omega})(x-y)|^q\, dx \right)^{1/q} dy$$
$$\leqslant C\|f\|_{L_\infty([|x|>1/\varepsilon-1])}\|\overline{\omega}\|_q.$$

Gathering these estimates and observing that $f^{(k)}(x) \to 0$ for $|x| \to \infty$ and $k = 0,\ 1$, we conclude that the claim holds true. □

We are now in a position to establish the aforementioned invertibility result in $\mathcal{L}(L_p(\mathbb{R}))$ for $p \in (1, 2]$.

Theorem 3 *Let* $p \in (1, 2]$ *and* $s \in (1 + 1/p, 2)$*. Given* $f \in W_p^s(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| \geqslant 1$, *it holds*

$$\lambda - \mathbb{A}(f) \in \mathrm{Isom}\,(L_p(\mathbb{R})).$$

Proof The claim in the particular case $p = 2$ has been established in [36, Theorem 3.5]. Let now $p \in (1, 2), f \in W_p^s(\mathbb{R})$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$ be given.

Step 1. We first prove that $\lambda - \mathbb{A}(f)$ is injective. Let $\overline{\omega} \in L_p(\mathbb{R})$ satisfy $(\lambda - \mathbb{A}(f))[\overline{\omega}] = 0$. Given $\varepsilon > 0$, this equation is equivalent to the following system of equations:

$$(\lambda - \chi_j^\varepsilon \mathbb{A}(f)\chi_j^\varepsilon)[\pi_j^\varepsilon \overline{\omega}] = K_j[\overline{\omega}] \qquad \text{for } -N+1 \leqslant j \leqslant N, \tag{3.4}$$

where K_j, $-N + 1 \leqslant j \leqslant N$, are the operators introduced in Lemma 10. Since $p \in (1, 2)$, in view of Lemma 11, we may choose $\varepsilon > 0$ such that

$$(\lambda - \chi_j^\varepsilon \mathbb{A}(f)\chi_j^\varepsilon) \in \mathrm{Isom}(L_2(\mathbb{R})) \cap \mathrm{Isom}(L_p(\mathbb{R})) \qquad \text{for all } -N+1 \leqslant j \leqslant N.$$

As $\overline{\omega} \in L_p(\mathbb{R})$, the right-hand side $K_j[\overline{\omega}]$ of (3.4) belongs to $L_p(\mathbb{R})$. Moreover, using once more the fact that $p \in (1, 2)$ together with the L_∞-bound:

$$|K_j[\overline{\omega}](x)| \leqslant \int_{\mathbb{R}} \Big| \frac{yf'(x) - \delta_{[x,y]}f}{1 + \left(\delta_{[x,y]}f/y\right)^2} \frac{\delta_{[x,y]}\pi_j^\varepsilon}{y^2} \overline{\omega}(x-y) \Big|\, dy$$
$$\leqslant C \int_{\mathbb{R}} \Big| \left(\mathbf{1}_{[|y|\leqslant 1]}(y) + \mathbf{1}_{[|y|>1]}(y) \frac{1}{|y|} \right) \overline{\omega}(x-y) \Big|\, dy$$
$$\leqslant C\|\overline{\omega}\|_p, \quad x \in \mathbb{R},$$

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it follows that $K_j[\overline{\omega}] \in L_2(\mathbb{R})$ for all $-N+1 \leqslant j \leqslant N$. Invoking (3.4), we then get

$$\pi_j^\varepsilon \overline{\omega} = (\lambda - \chi_j^\varepsilon \mathbb{A}(f)\chi_j^\varepsilon)^{-1}[K_j[\overline{\omega}]] \in L_p(\mathbb{R}) \cap L_2(\mathbb{R}), \qquad -N+1 \leqslant j \leqslant N,$$

and therewith $\overline{\omega} \in L_2(\mathbb{R})$. Since $\lambda - \mathbb{A}(f) \in \mathrm{Isom}(L_2(\mathbb{R}))$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, we conclude that the equation $(\lambda - \mathbb{A}(f))[\overline{\omega}] = 0$ in $L_p(\mathbb{R})$ has only the trivial solution.

Step 2. We now prove there exists $C > 0$ with the property:

$$\|(\lambda - \mathbb{A}(f))[\overline{\omega}]\|_p \geqslant C\|\overline{\omega}\|_p \qquad \text{for all } \overline{\omega} \in L_p(\mathbb{R}) \text{ and } \lambda \in \mathbb{R} \text{ with } |\lambda| \geqslant 1. \tag{3.5}$$

Indeed, assuming the claim is false, we may find a sequence $(\overline{\omega}_n)_n \subset L_p(\mathbb{R})$ and a bounded sequence $(\lambda_n)_n \subset \mathbb{R}$ with the properties $|\lambda_n| \geqslant 1$, $\|\overline{\omega}_n\|_p = 1$ for all $n \in \mathbb{N}$, and such that $(\lambda_n - \mathbb{A}(f))[\overline{\omega}_n] =: \varphi_n \to 0$ in $L_p(\mathbb{R})$. After possibly extracting a subsequence, we may assume that $\lambda_n \to \lambda$ in $\mathbb{R}$ and $\overline{\omega}_n \rightharpoonup \overline{\omega}$ in $L_p(\mathbb{R})$. In virtue of (3.1), it holds that

$$\langle(\lambda_n - \mathbb{A}(f))[\overline{\omega}_n]|h\rangle_{L_2} = \langle\overline{\omega}_n|(\lambda_n - (\mathbb{A}(f))^*)[h]\rangle_{L_2}$$

for all $n \in \mathbb{N}$ and $h \in L_{p'}(\mathbb{R})$. Passing to the limit $n \to \infty$ in the previous equation, it results that $\langle(\lambda - \mathbb{A}(f))[\overline{\omega}]|h\rangle_{L_2} = 0$ for all $h \in L_{p'}(\mathbb{R})$. Since $\lambda - \mathbb{A}(f)$ is injective, we get $\overline{\omega} = 0$.

Let now $\varepsilon > 0$ be chosen such that $(\lambda_n - \chi_j^\varepsilon \mathbb{A}(f)\chi_j^\varepsilon)$, $(\lambda - \chi_j^\varepsilon \mathbb{A}(f)\chi_j^\varepsilon) \in \mathrm{Isom}(L_p(\mathbb{R}))$ for all $-N+1 \leqslant j \leqslant N$ and all $n \in \mathbb{N}$. Such an ε exists in virtue of Lemma 11 and of the fact that $|\lambda_n| \geqslant 1$ for all $n \in \mathbb{N}$. Because of

$$1 = \|\overline{\omega}_n\|_p \leqslant \sum_{j=-N+1}^{N} \|\pi_j^\varepsilon \overline{\omega}_n\|_p,$$

there exists $-N+1 \leqslant j_* \leqslant N$ and a subsequence of $(\overline{\omega}_n)_n$ (not relabelled) such that

$$\|\pi_{j_*}^\varepsilon \overline{\omega}_n\|_p \geqslant (2N)^{-1} \qquad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

From $(\lambda_n - \mathbb{A}(f))[\overline{\omega}_n] = \varphi_n$, it then follows

$$\pi_{j_*}^\varepsilon \overline{\omega}_n = (\lambda_n - \chi_{j_*}^\varepsilon \mathbb{A}(f)\chi_{j_*}^\varepsilon)^{-1}[K_{j_*}[\overline{\omega}_n]] + (\lambda_n - \chi_{j_*}^\varepsilon \mathbb{A}(f)\chi_{j_*}^\varepsilon)^{-1}[\pi_{j_*}^\varepsilon \varphi_n] \tag{3.7}$$

for all $n \in \mathbb{N}$. Recalling Lemma 10, we obtain $K_{j^*}[\overline{\omega}_n] \to 0$ in $L_p(\mathbb{R})$. Furthermore, taking into account that $\lambda_n - \chi_{j_*}^\varepsilon \mathbb{A}(f)\chi_{j_*}^\varepsilon \to \lambda - \chi_{j_*}^\varepsilon \mathbb{A}(f)\chi_{j_*}^\varepsilon$ in $\mathcal{L}(L_p(\mathbb{R}))$, we deduce from (3.7) that $\pi_{j_*}^\varepsilon \overline{\omega}_n \to 0$ in $L_p(\mathbb{R})$, which contradicts (3.6).

We have thus established the validity of (3.5). Since $\lambda - \mathbb{A}(f) \in \mathrm{Isom}(L_p(\mathbb{R}))$ for $|\lambda|$ sufficiently large, the method of continuity, cf. [3, Proposition I.1.1.1], leads us to the conclusion that $\lambda - \mathbb{A}(f) \in \mathrm{Isom}(L_p(\mathbb{R}))$ for all $|\lambda| \geqslant 1$. This completes the proof. ☐

We now consider the case $p \in (2, \infty)$. From the proof of Theorem 3, we may infer that if $\lambda - \mathbb{A}(f) \in \mathcal{L}(L_p(\mathbb{R}))$ is injective for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, then $\lambda - \mathbb{A}(f)$ is an isomorphism for all such λ. The arguments used to establish the injectivity property of $\lambda - \mathbb{A}(f)$ in the case $p \in (1, 2]$, however, do not work for $p \in (2, \infty)$, and therefore a new strategy is needed.

Theorem 4 *Let $p \in (2, \infty)$ and $s \in (1 + 1/p, 2)$. Given $f \in W_p^s(\mathbb{R})$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, it holds*

$$\lambda - \mathbb{A}(f) \in \mathrm{Isom}(L_p(\mathbb{R})).$$

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Proof To each $\varepsilon \in (0,1)$, we associate a function $a_\varepsilon \in C^\infty(\mathbb{R},[0,1])$ with the properties that $a_\varepsilon(x)=0$ for $|x|<\varepsilon^{-1}$, $a_\varepsilon(x)=1$ for $|x|>\varepsilon^{-1}+1$ and $|a_\varepsilon'|\leqslant 2$. Recalling (2.7), it is suitable to write

$$\mathbb{A}(f)=\mathbb{A}_{1,\varepsilon}+\mathbb{A}_{2,\varepsilon},$$

where,

$$\pi\mathbb{A}_{1,\varepsilon} := (a_\varepsilon f)'B_{0,1}(f)-B_{1,1}(f)[a_\varepsilon f,\cdot],$$
$$\pi\mathbb{A}_{2,\varepsilon} := [(1-a_\varepsilon)f]'B_{0,1}(f)-B_{1,1}(f)[(1-a_\varepsilon)f,\cdot].$$

According to Lemma 2 (i), it holds that $\mathbb{A}_{j,\varepsilon}\in\mathcal{L}(L_q(\mathbb{R}))$ for all $1<q<\infty$. Since f and f' vanish both at infinity, we obtain for $\varepsilon\to 0$ that

$$\|\mathbb{A}_{1,\varepsilon}\|_{\mathcal{L}(L_q(\mathbb{R}))}\leqslant C\|(a_\varepsilon f)'\|_\infty\leqslant C\|f\|_{W^1_\infty([|x|>\varepsilon^{-1}])}\to 0.$$

Hence, if ε is sufficiently small, then

$$\lambda-\mathbb{A}_{1,\varepsilon}\in \mathrm{Isom}(L_p(\mathbb{R}))\cap\mathrm{Isom}(L_2(\mathbb{R}))$$

for all $|\lambda|\geqslant 1$.

Let now $\lambda\in\mathbb{R}$ with $|\lambda|\geqslant 1$ and $\overline{\omega}\in L_p(\mathbb{R})$ satisfy $(\lambda-\mathbb{A}(f))[\overline{\omega}]=0$, or equivalently

$$(\lambda-\mathbb{A}_{1,\varepsilon})[\overline{\omega}]=\mathbb{A}_{2,\varepsilon}[\overline{\omega}].$$

Assuming that

$$\mathbb{A}_{2,\varepsilon}[\overline{\omega}]\in L_2(\mathbb{R})\cap L_p(\mathbb{R})\qquad \text{for all } \varepsilon \text{ that are sufficiently small,} \tag{3.8}$$

the previous equality together with $\lambda-\mathbb{A}_{1,\varepsilon}\in \mathrm{Isom}(L_p(\mathbb{R}))\cap\mathrm{Isom}(L_2(\mathbb{R}))$ leads us to $w\in L_2(\mathbb{R})$. Recalling that $\lambda-\mathbb{A}(f)$ is an $L_2(\mathbb{R})$-isomorphism, we may then conclude that $\overline{\omega}=0$.

It thus remains to establish (3.8). To this end, we write

$$\mathbb{A}_{2,\varepsilon}[\overline{\omega}]=\mathbb{A}_{2,\varepsilon}[a_{\varepsilon^2}\overline{\omega}]+\mathbb{A}_{2,\varepsilon}[(1-a_{\varepsilon^2})\overline{\omega}].$$

In view of $p\in(2,\infty)$, it holds that $(1-a_{\varepsilon^2})\overline{\omega}\in L_2(\mathbb{R})\cap L_p(\mathbb{R})$ and therefore we obtain that $\mathbb{A}_{2,\varepsilon}[(1-a_{\varepsilon^2})\overline{\omega}]\in L_2(\mathbb{R})\cap L_p(\mathbb{R})$. Letting $g_\varepsilon:=(1-a_\varepsilon)f$, it holds that

$$\|\mathbb{A}_{2,\varepsilon}[a_{\varepsilon^2}\overline{\omega}]\|_2\leqslant\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\frac{yg_\varepsilon'(x)-\delta_{[x,y]}g_\varepsilon}{y^2+(\delta_{[x,y]}f)^2}(a_{\varepsilon^2}\overline{\omega})(x-y)\,dy\right|^2dx\right)^{1/2}\leqslant T_1+T_2$$

where,

$$T_1:=\left(\int_{\mathbb{R}}\left|\int_{[|y|\leqslant 1]}\frac{yg_\varepsilon'(x)-\delta_{[x,y]}g_\varepsilon}{y^2+(\delta_{[x,y]}f)^2}(a_{\varepsilon^2}\overline{\omega})(x-y)\,dy\right|^2dx\right)^{1/2},$$
$$T_2:=\left(\int_{\mathbb{R}}\left|\int_{[|y|>1]}\frac{yg_\varepsilon'(x)-\delta_{[x,y]}g_\varepsilon}{y^2+(\delta_{[x,y]}f)^2}(a_{\varepsilon^2}\overline{\omega})(x-y)\,dy\right|^2dx\right)^{1/2}.$$

If ε is sufficiently small, then $T_1=0$. Indeed, let $x\in\mathbb{R}$. We distinguish two cases, namely when $|x|<\varepsilon^{-2}-1$ or $|x|>\varepsilon^{-1}+2$. If $|x|<\varepsilon^{-2}-1$, it follows that $|x-y|<\varepsilon^{-2}$ and therewith $a_{\varepsilon^2}(x-y)=0$. In the case when $|x|>\varepsilon^{-1}+2$, it holds $|x-y|>\varepsilon^{-1}+1$ and $g_\varepsilon'(x)=g_\varepsilon(x)=g_\varepsilon(x-y)=0$.

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Concerning T_2, we note that $g_\varepsilon a_{\varepsilon^2} = 0$. Given $|x| > \varepsilon^{-1} + 1$, we get $g'_\varepsilon(x) = g_\varepsilon(x) = 0$, and Hölder's inequality leads to:

$$\begin{aligned} T_2 &= \left(\int_{[|x| \leqslant \varepsilon^{-1}+1]} \left| \int_{[|y|>1]} \frac{y g'_\varepsilon(x) - \delta_{[x,y]} g_\varepsilon}{y^2 + (\delta_{[x,y]} f)^2} (a_{\varepsilon^2} \overline{\omega})(x-y)\, dy \right|^2 dx \right)^{1/2} \\ &\leqslant C \|g'_\varepsilon\|_\infty \left(\int_{[|x| \leqslant \varepsilon^{-1}+1]} \left(\int_{[|y|>1]} \frac{1}{|y|} |a_{\varepsilon^2} \overline{\omega}|(x-y)\, dy \right)^2 dx \right)^{1/2} \\ &\leqslant C(\varepsilon) \|g'_\varepsilon\|_\infty \|\overline{\omega}\|_{L_p}. \end{aligned}$$

This proves (3.8) and the injectivity of $\lambda - \mathbb{A}(f) \in \mathcal{L}(L_p(\mathbb{R}))$. □

Finally, we establish the invertibility of $\lambda - \mathbb{A}(f)$, $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, in $\mathcal{L}(W_p^{s-1}(\mathbb{R}))$.

Theorem 5 *Let $p \in (1, \infty)$ and $s \in (1 + 1/p, 2)$. Given $f \in W_p^s(\mathbb{R})$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, it holds*

$$\lambda - \mathbb{A}(f) \in \mathrm{Isom}(W_p^{s-1}(\mathbb{R})).$$

Proof Given $\overline{\omega} \in W_p^{s-1}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant 1$, let $\varphi := (\lambda - \mathbb{A}(f))[\overline{\omega}]$. Lemma 8 then yields $\varphi \in W_p^{s-1}(\mathbb{R})$. Recalling Theorems 3 and 4, it holds that

$$\begin{aligned} [\overline{\omega}]^p_{W_p^{s-1}} &= \int_{\mathbb{R}} \frac{\|\overline{\omega} - \tau_\xi \overline{\omega}\|_p^p}{|\xi|^{1+(s-1)p}}\, d\xi \leqslant C \int_{\mathbb{R}} \frac{\|(\lambda - \mathbb{A}(\tau_\xi f))[\overline{\omega} - \tau_\xi \overline{\omega}]\|_p^p}{|\xi|^{1+(s-1)p}}\, d\xi \\ &\leqslant 2^p C \int_{\mathbb{R}} \frac{\|\varphi - \tau_\xi \varphi\|_p^p}{|\xi|^{1+(s-1)p}}\, d\xi + 2^p C \int_{\mathbb{R}} \frac{\|(\mathbb{A}(f) - \mathbb{A}(\tau_\xi f))[\overline{\omega}])\|_p^p}{|\xi|^{1+(s-1)p}}\, d\xi \\ &\leqslant 2^p C [\varphi]^p_{W_p^{s-1}} + 2^p C \int_{\mathbb{R}} \frac{\|(\mathbb{A}(f) - \mathbb{A}(\tau_\xi f))[\overline{\omega}])\|_p^p}{|\xi|^{1+(s-1)p}}\, d\xi, \end{aligned}$$

with $C \geqslant \max_{[|\lambda| \geqslant 1] \cap \mathbb{R}} \|(\lambda - \mathbb{A}(f))^{-1}\|^p_{\mathcal{L}(L_p(\mathbb{R}))}$. Recalling (2.7), we further compute

$$\begin{aligned} \|\mathbb{A}(f) - \mathbb{A}(\tau_\xi f))[\overline{\omega}]\|_p \leqslant{}& \|f' B_{0,1}(f)[\overline{\omega}] - \tau_\xi f' B_{0,1}(\tau_\xi f)[\overline{\omega}]\|_p \\ &+ \|B_{1,1}(f)[f, \overline{\omega}] - B_{1,1}(\tau_\xi f)[\tau_\xi f, \overline{\omega}]\|_p, \end{aligned}$$

and the relation (2.6), Lemma 2(ii) (with $\tau = s/2 - 1/2 + 1/2p \in (1/p, 1)$ and $r = s$), and (2.9) (with $r = s/2 + 1/2 + 1/2p \in (1 + 1/p, 2)$) yield

$$\begin{aligned} &\|f' B_{0,1}(f)[\overline{\omega}] - \tau_\xi f' B_{0,1}(\tau_\xi f)[\overline{\omega}]\|_p \\ &\quad \leqslant \|f' - \tau_\xi f'\|_p \|B_{0,1}(f)[\overline{\omega}]\|_\infty + \|f'\|_\infty \|B_{2,2}(f, \tau_\xi f)[f + \tau_\xi f, f - \tau_\xi f, \overline{\omega}]\|_p \\ &\quad \leqslant C \|f' - \tau_\xi f'\|_p \|\overline{\omega}\|_{W_p^{s/2-1/2+1/2p}}, \end{aligned}$$

where $C = C(\|f\|_{W_p^s})$, and by similar arguments:

$$\begin{aligned} &\|B_{1,1}(f)[f, \overline{\omega}] - B_{1,1}(\tau_\xi f)[\tau_\xi f, \overline{\omega}]\|_p \\ &\quad \leqslant \|B_{1,1}(f)[f - \tau_\xi f, \overline{\omega}]\|_p + \|B_{3,2}(f, \tau_\xi f)[f + \tau_\xi f, f - \tau_\xi f, \tau_\xi f, \overline{\omega}]\|_p \\ &\quad \leqslant C \|f' - \tau_\xi f'\|_p \|\overline{\omega}\|_{W_p^{s/2-1/2+1/2p}}. \end{aligned}$$

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The latter estimates together with Theorems 3 and 4 imply there exists a constant C_0 (which depends only on f) such that

$$\|\overline{\omega}\|_{W_p^{s-1}} \leqslant C_0(\|(\lambda - \mathbb{A}(f))[\overline{\omega}]\|_{W_p^{s-1}} + \|\overline{\omega}\|_{W_p^{s/2-1/2+1/2p}}), \qquad \overline{\omega} \in W_p^{s-1}(\mathbb{R}). \tag{3.9}$$

Using the interpolation property:

$$W_p^{(1-\theta)s_1+\theta s_2}(\mathbb{R}) = (W_p^{s_1}(\mathbb{R}), W_p^{s_2}(\mathbb{R}))_{\theta,p}, \quad 0 \leqslant s_1 < s_1 < \infty,\ (1-\theta)s_1 + \theta s_2 \notin \mathbb{N}, \tag{3.10}$$

where $(\cdot,\cdot)_{\theta,p}$, $\theta \in (0,1)$, denotes the real interpolation functor of exponent θ and parameter p, in the particular case $s_1 = 0$, $s_2 = s-1$ and $\theta := (s-1)^{-1}(s/2 - 1/2 + 1/2p)$, it follows from Young's inequality that

$$\|\overline{\omega}\|_{W_p^{s/2-1/2+1/2p}} \leqslant \frac{1}{2C_0}\|\overline{\omega}\|_{W_p^{s-1}} + C\|\overline{\omega}\|_p, \qquad \overline{\omega} \in W_p^{s-1}(\mathbb{R}).$$

This property combined with (3.9), Theorems 3, and 4 yields

$$\|\overline{\omega}\|_{W_p^{s-1}} \leqslant C\|(\lambda - \mathbb{A}(f))[\overline{\omega}]\|_{W_p^{s-1}}, \quad \overline{\omega} \in W_p^{s-1}(\mathbb{R}) \text{ and } \lambda \in \mathbb{R} \text{ with } |\lambda| \geqslant 1.$$

The method of continuity [3, Proposition I.1.1.1] leads now to the desired conclusion. □

4 The abstract evolution problem

In this section, we first use the results of Section 3 to formulate (1.2) as an evolution problem in $W_p^{s-1}(\mathbb{R})$ with f as the only unknown (see (4.3)). Subsequently, we show that the Rayleigh–Taylor condition identifies a domain of parabolicity for (4.3), cf. Theorem 6.

Observing that the Atwood number a_μ satisfies $|a_\mu| < 1$, Theorem 5 ensures that, for each $f \in W_p^s(\mathbb{R})$, the equation $(1.2)_2$ has a unique solution:

$$\overline{\omega}(f) := -C_\Theta(1 + a_\mu\mathbb{A}(f))^{-1}[f']. \tag{4.1}$$

Moreover, Lemma 8 yields

$$[f \mapsto \overline{\omega}(f)] \in C^\omega(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R})). \tag{4.2}$$

We can thus reformulate the system (1.2) as the abstract evolution problem:

$$\frac{df}{dt} = \Phi(f), \quad t \geqslant 0, \quad f(0) = f_0, \tag{4.3}$$

where the (fully) nonlinear and nonlocal operator $\Phi : W_p^s(\mathbb{R}) \to W_p^{s-1}(\mathbb{R})$ is defined by

$$\Phi(f) := \mathbb{B}(f)[\overline{\omega}(f)].$$

In virtue of (2.21) and (4.2), it holds

$$\Phi \in C^\omega(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R})). \tag{4.4}$$

It is important to point out that the operator Φ is fully nonlinear as the definition of the function $\overline{\omega}(f) = -C_\Theta(1 + a_\mu\mathbb{A}(f))^{-1}[f'] \in W_p^{s-1}(\mathbb{R})$ requires that $f' \in W_p^{s-1}(\mathbb{R})$, but also the 'nonlinear argument' f in $(1 + a_\mu\mathbb{A}(f))^{-1}$is required to belong to $W_p^s(\mathbb{R})$. This differs of course if $a_\mu = 0$ and in this setting Φ has (in a suitable setting) a quasilinear structure, cf. [37].

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The Rayleigh–Taylor condition can be simply formulated in our notation as:

$$C_\Theta + a_\mu \Phi(f) > 0, \tag{4.5}$$

cf., e.g., [36]. Since $\Phi(f) \in W_p^{s-1}(\mathbb{R})$, this condition implies, under the assumption[3] $\Theta \neq 0$, that $\Theta > 0$. Restricting to the setting when $\Theta > 0$, it follows from (4.4) that the set

$$\mathcal{O} := \{f \in W_p^s(\mathbb{R}) \, : \, C_\Theta + a_\mu \Phi(f) > 0\} \tag{4.6}$$

is an open subset of $W_p^s(\mathbb{R})$. The analysis below is devoted to showing that the Fréchet derivative $\partial\Phi(f_0)$ of Φ at $f_0 \in \mathcal{O}$ generates an analytic semigroup in $\mathcal{L}(W_p^{s-1}(\mathbb{R}))$, which in the notation from [3] writes $-\partial\Phi(f_0) \in \mathcal{H}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R}))$. This property identifies (4.3) as a parabolic evolution equation in $\mathcal{O}$ and facilitates us the use of abstract parabolic theory from [34] when solving it.

Theorem 6 *Let $f_0 \in \mathcal{O}$. It then holds*

$$-\partial\Phi(f_0) \in \mathcal{H}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R})).$$

The proof of Theorem 6 (which is postponed to the end of the section) requires some preparation. In the following, we set

$$\overline{\omega}_0 := \overline{\omega}(f_0) \in W_p^{s-1}(\mathbb{R}).$$

By the chain rule,

$$\partial\Phi(f_0)[f] = \partial\mathbb{B}(f_0)[f][\overline{\omega}_0] + \mathbb{B}(f_0)[\partial\overline{\omega}(f_0)[f]], \qquad f \in W_p^s(\mathbb{R}),$$

where

$$\begin{aligned}\pi\,\partial\mathbb{B}(f_0)[f][\overline{\omega}_0] = &-2B_{2,2}(f_0,f_0)[f_0,f,\overline{\omega}_0] + f'B_{1,1}(f_0)[f_0,\overline{\omega}_0]\\ &+f_0'B_{1,1}(f_0)[f,\overline{\omega}_0] - 2f_0'B_{3,2}(f_0,f_0)[f_0,f_0,f,\overline{\omega}_0].\end{aligned}$$

Furthermore, differentiation of (4.1) with respect to f at f_0 yields

$$(1 + a_\mu\mathbb{A}(f_0))[\partial\overline{\omega}(f_0)[f]] = -a_\mu\partial\mathbb{A}(f_0)[f][\overline{\omega}_0] - C_\Theta f', \quad f \in W_p^s(\mathbb{R}),$$

where,

$$\begin{aligned}\pi\,\partial\mathbb{A}(f_0)[f][\overline{\omega}_0] = &f'B_{0,1}(f_0)[\overline{\omega}_0] - 2f_0'B_{2,2}(f_0,f_0)[f_0,f,\overline{\omega}_0]\\ &-B_{1,1}(f_0)[f,\overline{\omega}_0] + 2B_{3,2}(f_0,f_0)[f_0,f_0,f,\overline{\omega}_0], \quad f \in W_p^s(\mathbb{R}).\end{aligned} \tag{4.7}$$

In the derivation of (4.7), we have several times made use of the formula:

$$\begin{aligned}\partial(B_{n,m}(f,\ldots,f)[f,\ldots,f,\cdot])\big|_{f_0}[f] = &\,nB_{n,m}(f_0,\ldots,f_0)[f_0,f_0,\ldots,f,\cdot]\\ &-2mB_{n+2,m+1}(f_0,\ldots,f_0)[f_0,f_0,\ldots,f,\cdot].\end{aligned}$$

[3]The case when $\Theta = 0$ is trivial as the function $\overline{\omega}(f)$ defined in (4.1) (hence also $\Phi(f)$) is identically zero. Consequently, the initial surface is transported vertically with velocity V (and the Rayleigh–Taylor condition needs not to be imposed) and the velocities of the fluids are zero.

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In order to establish Theorem 6, we consider a continuous path in $\mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))$ which is related to $\partial\Phi(f_0)$, that is we define $\Psi : [0, 1] \to \mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))$ by setting:

$$\Psi(\tau)[f] := \tau\,\partial\mathbb{B}(f_0)[f][\overline{\omega}_0] + \mathbb{B}(\tau f_0)[w(\tau)[f]],$$

where $w \in \mathrm{C}([0, 1], \mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R})))$ is defined as the solution to:

$$(1 + a_\mu\mathbb{A}(\tau f_0))[w(\tau)[f]] = -\tau a_\mu\partial\mathbb{A}(f_0)[f][\overline{\omega}_0] - C_\Theta f' - (1-\tau)a_\mu f'\Phi(f_0) \tag{4.8}$$

for $\tau \in [0, 1]$ and $f \in W^s_p(\mathbb{R})$.

Remark 2 *(i) If $\tau = 1$, then $w(1) = \partial\overline{\omega}(f_0)$ and therewith $\Psi(1) = \partial\Phi(f_0)$.*

(ii) Letting H denote the Hilbert transform, it holds $\Psi(0) = H \circ w(0)$. Moreover, noticing that $\mathbb{A}(0) = 0$, it holds

$$w(0) = -[C_\Theta + a_\mu\Phi(f_0)]\frac{d}{dx}.$$

It is important to point out that the function of the right-hand side of the latter relation is exactly the function in the Rayleigh–Taylor condition (4.5). This is one of the reasons why we artificially introduced the term $(1-\tau)a_\mu f'\Phi(f_0)$ in the definition (4.8). This construction is essential for our purpose because it provides on one hand some useful cancellations in the proof of Theorem 7 and on the other hand it facilitates us to establish the invertibility of $\lambda - \Psi(0) \in \mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))$ for sufficiently large and positive λ, cf. Proposition 1. The latter point is important when establishing the invertibility of $\lambda - \partial\Phi(f_0)$ for such λ.

(iii) H is the Fourier multiplier with symbol $[\xi \mapsto -i\,\mathrm{sign}(\xi)]$ and

$$H \circ (d/dx) = (-d^2/dx^2)^{1/2}.$$

(iv) Given $s' \in (1 + 1/p, s)$, Lemma 2 (i), Theorems 3 and 4 imply there exists a constant $C = C(f_0)$ such that

$$\|w(\tau)[f]\|_p \leqslant C\|f\|_{W^{s'}_p}, \quad f \in W^s_p(\mathbb{R}),\ \tau \in [0, 1]. \tag{4.9}$$

Besides, Lemma 5 (with $r = s'$) and Theorem 5 (with $s = s'$) yield

$$\|w(\tau)[f]\|_{W^{s'-1}_p} \leqslant C\|f\|_{W^{s'}_p}, \quad \tau \in [0, 1], f \in W^s_p(\mathbb{R}). \tag{4.10}$$

Theorem 7 is the main step in the proof of Theorem 6. In Theorem 7, it is shown that the operator $\Psi(\tau)$ can be locally approximated by certain Fourier multipliers $\mathbb{A}_{j,\tau}$. Theorem 7 also reveals the importance of the Rayleigh–Taylor condition which ensures in this context the positivity of the coefficient $\alpha_\tau(x^\varepsilon_j)$ in the definition of $\mathbb{A}_{j,\tau}$ below.

Theorem 7 *Let $\mu > 0$ be given and fix $s' \in (1 + 1/p, s)$. Then, there exist $\varepsilon \in (0, 1)$, an ε-localisation family $\{\pi^\varepsilon_j : -N + 1 \leqslant j \leqslant N\}$, a constant $K = K(\varepsilon, f_0)$ and bounded operators:*

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R})), \qquad j \in \{-N+1, \dots, N\} \text{ and } \tau \in [0, 1],$$

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such that

$$\|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{W_p^{s-1}} \leqslant \mu \|\pi_j^\varepsilon f\|_{W_p^s} + K\|f\|_{W_p^{s'}} \tag{4.11}$$

for all $j \in \{-N+1, \ldots, N\}$, $\tau \in [0,1]$, *and* $f \in W_p^s(\mathbb{R})$. *The operators* $\mathbb{A}_{j,\tau}$ *are defined by*

$$\mathbb{A}_{j,\tau} := -\alpha_\tau(x_j^\varepsilon)\left(-\frac{d^2}{dx^2}\right)^{1/2} + \beta_\tau(x_j^\varepsilon)\frac{d}{dx}, \quad |j| \leqslant N-1, \qquad \mathbb{A}_{N,\tau} := -C_\Theta\left(-\frac{d^2}{dx^2}\right)^{1/2},$$

where $x_j^\varepsilon \in \operatorname{supp} \pi_j^\varepsilon$, $|j| \leqslant N-1$, *and with functions* α_τ, β_τ *given by*

$$\alpha_\tau := \left(1 - \frac{\tau f_0'^2}{1+f_0'^2}\right)[C_\Theta + a_\mu \Phi(f_0)], \qquad \beta_\tau := \frac{\tau}{\pi} B_{1,1}(f_0)[f_0, \overline{\omega}_0] - \tau a_\mu \frac{\overline{\omega}_0}{1+f_0'^2}.$$

Before proving Theorem 7, we first present some lemmas (which are proved in the Appendix A) which are used in an essential way when establishing Theorem 7. The following commutator estimate is used several times in the paper.

Lemma 12 *Let* $n, m \in \mathbb{N}$, $p \in (1,\infty)$, $s \in (1+1/p, 2)$, $f \in W_p^s(\mathbb{R})$ *and* $\varphi \in \mathrm{C}^1(\mathbb{R})$ *with uniformly continuous derivative* φ' *be given. Then, there exist a constant K that depends only on n, m,* $\|\varphi'\|_\infty$, *and* $\|f\|_{W_p^s}$ *such that*

$$\|\varphi B_{n,m}(f,\ldots,f)[f,\ldots,f,h] - B_{n,m}(f,\ldots,f)[f,\ldots,f,\varphi h]\|_{W_p^1} \leqslant K\|h\|_p \tag{4.12}$$

for all $h \in L_p(\mathbb{R})$.

The next lemma is used in the proof of Theorem 7.

Lemma 13 *Let* $n, m \in \mathbb{N}$, $p \in (1,\infty)$, $1+1/p < s' < s < 2$ *and* $\nu \in (0,\infty)$ *be given. Let further* $f \in W_p^s(\mathbb{R})$ *and* $\overline{\omega} \in \{1\} \cup W_p^{s-1}(\mathbb{R})$. *For sufficiently small* $\varepsilon \in (0,1)$, *there exists a constant* $K = K(\varepsilon, n, m, \|f\|_{W_p^s}, \|\overline{\omega}\|_{W_p^{s-1}})$ *such that*

$$\begin{aligned}\Big\|\pi_j^\varepsilon \overline{\omega} B_{n,m}(f,\ldots,f)[f,\ldots,f,h] - \frac{\overline{\omega}(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m} B_{0,0}[\pi_j^\varepsilon h]\Big\|_{W_p^{s-1}} \\ \leqslant \nu \|\pi_j^\varepsilon h\|_{W_p^{s-1}} + K\|h\|_{W_p^{s'-1}}\end{aligned} \tag{4.13}$$

for all $|j| \leqslant N-1$ *and* $h \in W_p^{s-1}(\mathbb{R})$ *(with* $x_j^\varepsilon \in \operatorname{supp} \pi_j^\varepsilon$*).*

The next two lemmas are analogues of Lemma 13 and deal with the case $j = N$.

Lemma 14 *Let* $n, m \in \mathbb{N}$, $p \in (1,\infty)$, $1+1/p < s' < s < 2$ *and* $\nu \in (0,\infty)$ *be given. Let further* $f \in W_p^s(\mathbb{R})$ *and* $\overline{\omega} \in W_p^{s-1}(\mathbb{R})$. *For sufficiently small* $\varepsilon \in (0,1)$, *there exists a constant* $K = K(\varepsilon, n, m, \|f\|_{W_p^s}, \|\overline{\omega}\|_{W_p^{s-1}})$ *such that*

$$\|\pi_j^\varepsilon \overline{\omega} B_{n,m}(f,\ldots,f)[f,\ldots,f,h]\|_{W_p^{s-1}} \leqslant \nu\|\pi_j^\varepsilon h\|_{W_p^{s-1}} + K\|h\|_{W_p^{s'-1}} \tag{4.14}$$

for $j = N$ *and* $h \in W_p^{s-1}(\mathbb{R})$.

We now establish the counterpart of (4.14) in the case when $\overline{\omega} = 1$.

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Lemma 15 *Let n, $m \in \mathbb{N}$, $p \in (1,\infty)$, $1+1/p < s' < s < 2$ and $\nu \in (0,\infty)$ be given. Let further $f \in W^s_p(\mathbb{R})$. For sufficiently small $\varepsilon \in (0,1)$, there exist a constant $K = K(\varepsilon, n, m, \|f\|_{W^s_p})$ such that*

$$\|\pi_j^\varepsilon B_{0,m}(f,\ldots,f)[h] - B_{0,0}[\pi_j^\varepsilon h]\|_{W^{s-1}_p} \leqslant \nu\|\pi_j^\varepsilon h\|_{W^{s-1}_p} + K\|h\|_{W^{s'-1}_p} \tag{4.15}$$

and

$$\|\pi_j^\varepsilon B_{n,m}(f,\ldots,f)[f,\ldots,f,h]\|_{W^{s-1}_p} \leqslant \nu\|\pi_j^\varepsilon h\|_{W^{s-1}_p} + K\|h\|_{W^{s'-1}_p}, \qquad n \geqslant 1, \tag{4.16}$$

for $j = N$ and all $h \in W^{s-1}_p(\mathbb{R})$.

We are now in a position to prove Theorem 7.

Proof of Theorem 7 Let $\{\pi_j^\varepsilon : -N+1 \leqslant j \leqslant N\}$ be an ε-localisation family and let $\{\chi_j^\varepsilon : -N+1 \leqslant j \leqslant N\}$ be an associated family, with $\varepsilon \in (0,1)$ to be fixed later on. In this proof, we denote by C constants that depend only on f_0. Constants denoted by K may depend only on ε and f_0.

Step 1: The terms $\partial\mathbb{B}(f_0)[f][\overline{\omega}_0]$. In virtue of Lemma 6 (with $r = s$ and $r' = s'$), it holds

$$\begin{aligned}
&\big\| -2B_{2,2}(f_0,f_0)[f_0,f,\overline{\omega}_0] + f_0'B_{1,1}(f_0)[f,\overline{\omega}_0] - 2f_0'B_{3,2}(f_0,f_0)[f,f_0,f_0,\overline{\omega}_0] \\
&\quad -\overline{\omega}_0\big(-2B_{1,2}(f_0,f_0)[f_0,f'] + f_0'B_{0,1}(f_0)[f'] - 2f_0'B_{2,2}(f_0,f_0)[f_0,f_0,f']\big)\big\|_{W^{s-1}_p} \\
&\quad\leqslant C\|f\|_{W^{s'}_p}.
\end{aligned} \tag{4.17}$$

Moreover, invoking Lemma 13, if ε is sufficiently small, then

$$\begin{aligned}
&\big\|\pi_j^\varepsilon\overline{\omega}_0\big(-2B_{1,2}(f_0,f_0)[f_0,f'] + f_0'B_{0,1}(f_0)[f'] - 2f_0'B_{2,2}(f_0,f_0)[f_0,f_0,f']\big) \\
&\quad + \frac{\overline{\omega}_0(x_j^\varepsilon)f_0'(x_j^\varepsilon)}{1+(f_0'(x_j^\varepsilon))^2}B_{0,0}[\pi_j^\varepsilon f']\big\|_{W^{s-1}_p} \leqslant \frac{\mu}{4}\|\pi_j^\varepsilon f\|_{W^s_p} + K\|f\|_{W^{s'}_p}
\end{aligned}$$

for all $|j| \leqslant N-1$. Besides, for $j = N$, Lemma 14 yields

$$\begin{aligned}
&\big\|\pi_j^\varepsilon\overline{\omega}_0\big(-2B_{1,2}(f_0,f_0)[f_0,f'] + f_0'B_{0,1}(f_0)[f'] - 2f_0'B_{2,2}(f_0,f_0)[f_0,f_0,f']\big)\big\|_{W^{s-1}_p} \\
&\quad \leqslant \frac{\mu}{4}\|\pi_j^\varepsilon f\|_{W^s_p} + K\|f\|_{W^{s'}_p}.
\end{aligned}$$

Finally, in view of (2.1) and of $B_{1,1}(f_0)[f_0,\overline{\omega}_0] \in \mathrm{C}^{s-1-1/p}(\mathbb{R})$, it holds

$$\begin{aligned}
&\|\pi_j^\varepsilon f' B_{1,1}(f_0)[f_0,\overline{\omega}_0] - (\pi_j^\varepsilon f)'B_{1,1}(f_0)[f_0,\overline{\omega}_0](x_j)\|_{W^{s-1}_p} \\
&\quad\leqslant \|\chi_j^\varepsilon(B_{1,1}(f_0)[f_0,\overline{\omega}_0] - B_{1,1}(f_0)[f_0,\overline{\omega}_0](x_j))(\pi_j^\varepsilon f)'\|_{W^{s-1}_p} + K\|f\|_{W^{s-1}_p} \\
&\quad\leqslant \frac{\mu}{4}\|\pi_j^\varepsilon f\|_{W^s_p} + K\|f\|_{W^{s'}_p}
\end{aligned} \tag{4.18}$$

for $|j| \leqslant N-1$. Moreover, since $B_{1,1}(f_0)[f_0,\overline{\omega}_0] \to 0$ for $|x| \to \infty$, (2.1) yields

$$\|\pi_j^\varepsilon f' B_{1,1}(f_0)[f_0,\overline{\omega}_0]\|_{W^{s-1}_p} \leqslant \frac{\mu}{4}\|\pi_j^\varepsilon f\|_{W^s_p} + K\|f\|_{W^{s'}_p}$$

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for $j=N$. Hence, if ε is sufficiently small, then

$$\begin{aligned}\Big\|\pi_j^\varepsilon\tau\partial\mathbb{B}(f_0)[f][\overline{\omega}_0]+\frac{\tau\overline{\omega}_0(x_j^\varepsilon)f_0'(x_j^\varepsilon)}{\pi(1+(f_0'(x_j^\varepsilon))^2)}B_{0,0}[(\pi_j^\varepsilon f)']\\ -\tfrac{\tau}{\pi}B_{1,1}(f_0)[f_0,\overline{\omega}_0](x_j)(\pi_j^\varepsilon f)'\Big\|_{W_p^{s-1}}\leqslant\frac{\mu}{2}\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}\end{aligned}\tag{4.19}$$

for all $|j|\leqslant N-1$ and $f\in W_p^s(\mathbb{R})$, and

$$\|\pi_j^\varepsilon\tau\partial\mathbb{B}(f_0)[f][\overline{\omega}_0]\|_{W_p^{s-1}}\leqslant\frac{\mu}{2}\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}\tag{4.20}$$

for $j=N$ and all $f\in W_p^s(\mathbb{R})$.

Step 2: The terms $\mathbb{B}(\tau f_0)[w(\tau)[f]]$. We first estimate $\|\pi_j^\varepsilon w(\tau)[f]\|_{W_p^{s-1}}$, $-N+1\leqslant j\leqslant N$. Recalling Lemma 12 and (4.9), it holds

$$\|\pi_j^\varepsilon\mathbb{A}(\tau f_0))[w(\tau)[f])]-\mathbb{A}(\tau f_0))[\pi_j^\varepsilon(w(\tau)[f]]\|_{W_p^{s-1}}\leqslant K\|f\|_{W_p^{s'}}.$$

Besides, multiplying (4.8) by π_j^ε, it follows from Lemmas 5, 6, and 12 that

$$\begin{aligned}\|\pi_j^\varepsilon(1+a_\mu\mathbb{A}(\tau f_0))[w(\tau)[f]]\|_{W_p^{s-1}}&\leqslant C\|\pi_j^\varepsilon\partial\mathbb{A}(f_0)[f][\overline{\omega}_0]\|_{W_p^{s-1}}+C\|\pi_j^\varepsilon f'\|_{W_p^{s-1}}\\&\leqslant C\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}.\end{aligned}$$

In order to estimate the last three terms of $\pi_j^\varepsilon\partial\mathbb{A}(f_0)[f][\overline{\omega}_0]$, we used Lemma 6 in a similar manner as in the derivation of (4.17), and afterwards the commutator estimate in Lemma 12 to write in the end π_j^ε as a multiplying factor of f'.

Combining the last two estimates, we arrive at

$$\|(1+a_\mu\mathbb{A}(\tau f_0))[\pi_j^\varepsilon w(\tau)[f]]\|_{W_p^{s-1}}\leqslant C\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}.$$

Finally, Theorem 5 ensures there exists a constant $C_0=C_0(f_0)>0$ with

$$\|\pi_j^\varepsilon w(\tau)[f]\|_{W_p^{s-1}}\leqslant C_0\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}.\tag{4.21}$$

In virtue of Lemma 13 (with $\nu=\mu/(8C_0)$, where C_0 is the constant in (4.21)), (4.10) and (4.21) for ε sufficiently small and $|j|\leqslant N-1$, it holds that

$$\begin{aligned}&\|\pi_j^\varepsilon\mathbb{B}(\tau f_0)[w(\tau)[f]]-\pi^{-1}B_{0,0}[\pi_j^\varepsilon w(\tau)[f]]\|_{W_p^{s-1}}\\&\leqslant\frac{\mu}{4C_0}\|\pi_j^\varepsilon w(\tau)[f]\|_{W_p^{s-1}}+K\|w(\tau)[f]\|_{W_p^{s'-1}}\\&\leqslant\frac{\mu}{4}\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}.\end{aligned}\tag{4.22}$$

Lemmas 14, 15, (4.10) and (4.21) show that (4.22) stays true also for $j=N$.

We now set

$$\varphi_\tau:=C_\Theta+(1-\tau)a_\mu\Phi(f_0)+\frac{\tau a_\mu}{\pi}B_{0,1}(f_0)[\overline{\omega}_0],\qquad\tau\in[0,1],$$

and prove that

$$\begin{aligned}\Big\|B_{0,0}[\pi_j^\varepsilon w(\tau)[f]]+\varphi_\tau(x_j^\varepsilon)B_{0,0}[(\pi_j^\varepsilon f)']+\tau a_\mu\pi\frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}(\pi_j^\varepsilon f)'\Big\|_{W_p^{s-1}}\\\leqslant\frac{\mu}{4}\|\pi_j^\varepsilon f\|_{W_p^s}+K\|f\|_{W_p^{s'}}\end{aligned}\tag{4.23}$$

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for all $|j| \leqslant N-1$, provided ε is small. Indeed, since $B_{0,0}^2 = \pi^2 H^2 = -\pi^2\, \mathrm{id}_{L_p(\mathbb{R})}$, it holds

$$\begin{aligned}&\Big\|B_{0,0}[\pi_j^\varepsilon w(\tau)[f]] + \varphi_\tau(x_j^\varepsilon)B_{0,0}[(\pi_j^\varepsilon f)'] + \tau a_\mu \pi \frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}(\pi_j^\varepsilon f)'\Big\|_{W_p^{s-1}}\\ &\quad\leqslant C_1\Big\|\pi_j^\varepsilon w(\tau)[f] + \varphi_\tau(x_j^\varepsilon)\pi_j^\varepsilon f' - \frac{\tau a_\mu}{\pi}\frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon f']\Big\|_{W_p^{s-1}} + K\|f\|_{W_p^{s-1}}.\end{aligned} \tag{4.24}$$

Besides, multiplying (4.8) by π_j^ε and using the definition of φ_τ, we arrive at

$$\pi_j^\varepsilon w(\tau)[f] + \varphi_\tau(x_j^\varepsilon)\pi_j^\varepsilon f' - \frac{\tau a_\mu}{\pi}\frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon f'] = T_1 + T_2 + T_3, \tag{4.25}$$

where

$$\begin{aligned}T_1 &:= (1-\tau)a_\mu(\Phi(f_0)(x_j^\varepsilon) - \Phi(f_0))\pi_j^\varepsilon f',\\ T_2 &:= -\frac{\tau a_\mu}{\pi}\left(\pi\pi_j^\varepsilon \partial\mathbb{A}(f_0)[f][\overline{\omega}_0] - B_{0,1}(f_0)[\overline{\omega}_0](x_j^\varepsilon)\pi_j^\varepsilon f' + \frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon f']\right),\\ T_3 &:= -a_\mu \pi_j^\varepsilon \mathbb{A}(\tau f_0)[w(\tau)[f]].\end{aligned}$$

The term T_1 can be estimated, in view of $\Phi(f_0) \in \mathrm{C}^{s-1-1/p}(\mathbb{R})$, by arguing as in (4.18). Concerning T_3, we infer from (2.7) that

$$\begin{aligned}T_3 = -\frac{\tau a_\mu}{\pi}\Bigg[&\left(\pi_j^\varepsilon f_0' B_{0,1}(\tau f_0)[w(\tau)[f]] - \frac{f_0'(x_j^\varepsilon)}{1+\tau^2 f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon w(\tau)[f]]\right)\\ &- \left(\pi_j^\varepsilon B_{1,1}(\tau f_0)[f_0, w(\tau)[f]] - \frac{f_0'(x_j^\varepsilon)}{1+\tau^2 f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon w(\tau)[f]]\right)\Bigg]\end{aligned}$$

and both terms can be estimated using (4.10), Lemma 13 and (4.21). Finally, using (4.7), it holds that

$$\begin{aligned}T_2 = -\frac{\tau a_\mu}{\pi}\Bigg[&\pi_j^\varepsilon f'\big(B_{0,1}(f_0)[\overline{\omega}_0] - B_{0,1}(f_0)[\overline{\omega}_0](x_j^\varepsilon)\big) + T_{\mathrm{LOT}}[f]\\ &-2\left(\pi_j^\varepsilon f_0'\overline{\omega}_0 B_{1,2}(f_0,f_0)[f_0,f'] - \frac{f_0'^2(x_j^\varepsilon)\overline{\omega}_0(x_j^\varepsilon)}{[1+f_0'^2(x_j^\varepsilon)]^2}B_{0,0}[\pi_j^\varepsilon f']\right)\\ &-\left(\pi_j^\varepsilon \overline{\omega}_0 B_{0,1}(f_0)[f'] - \frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon f']\right)\\ &+2\left(\pi_j^\varepsilon \overline{\omega}_0 B_{2,2}(f_0,f_0)[f_0,f_0,f'] - \frac{f_0'^2(x_j^\varepsilon)\overline{\omega}_0(x_j^\varepsilon)}{[1+f_0'^2(x_j^\varepsilon)]^2}B_{0,0}[\pi_j^\varepsilon f']\right)\Bigg],\end{aligned}$$

where

$$\begin{aligned}T_{\mathrm{LOT}}[f] := &-2(f_0' B_{2,2}(f_0,f_0)[f_0,f,\overline{\omega}_0] - f_0'\overline{\omega}_0 B_{1,2}(f_0,f_0)[f_0,f'])\\ &-(B_{1,1}(f_0)[f,\overline{\omega}_0] - \overline{\omega}_0 B_{0,1}(f_0)[f'])\\ &+2(B_{3,2}(f_0,f_0)[f_0,f_0,f,\overline{\omega}_0] - \overline{\omega}_0 B_{2,2}(f_0,f_0)[f_0,f_0,f']).\end{aligned}$$

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Lemma 6 yields

$$\|\pi_j^\varepsilon T_{\mathrm{LOT}}[f]\|_{W_p^{s-1}} \leqslant K\|f\|_{W_p^{s'}}.$$

The first term in the decomposition of T_2 is estimated, since $B_{0,1}(f_0)[\overline{\omega}_0] \in \mathrm{C}^{s-1-1/p}(\mathbb{R})$, by arguing as in (4.18). For the last three terms, we rely on Lemma 13, (4.10) and (4.21). Altogether, we conclude that if ε is sufficiently small, then

$$\left\|\pi_j^\varepsilon w(\tau)[f] + \varphi_\tau(x_j^\varepsilon)\pi_j^\varepsilon f' - \frac{\tau a_\mu}{\pi}\frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}B_{0,0}[\pi_j^\varepsilon f']\right\|_{W_p^{s-1}} \leqslant \frac{\mu}{4C_1}\|\pi_j^\varepsilon f\|_{W_p^s} + K\|f\|_{W_p^{s'}}$$

and together with (4.24) we have proven (4.23). From (4.22) and (4.23), we finally conclude

$$\begin{aligned}&\left\|\pi_j^\varepsilon \mathbb{B}(\tau f_0)[w(\tau)[f]] + \pi^{-1}\varphi_\tau(x_j^\varepsilon)B_{0,0}[(\pi_j^\varepsilon f)'] + \tau a_\mu\frac{\overline{\omega}_0(x_j^\varepsilon)}{1+f_0'^2(x_j^\varepsilon)}(\pi_j^\varepsilon f)'\right\|_{W_p^{s-1}} \\ &\quad\leqslant \frac{\mu}{2}\|\pi_j^\varepsilon f\|_{W_p^s} + K\|f\|_{W_p^{s'}}\end{aligned} \tag{4.26}$$

for all $|j| \leqslant N-1$, provided that ε is small. Using also Lemmas 14 and 15, it is not difficult to infer from the latter relations that

$$\left\|\pi_j^\varepsilon \mathbb{B}(\tau f_0)[w_1(\tau)[f]] + \pi^{-1}C_\Theta B_{0,0}[(\pi_j^\varepsilon f)']\right\|_{W_p^{s-1}} \leqslant \frac{\mu}{2}\|\pi_j^\varepsilon f\|_{W_p^s} + K\|f\|_{W_p^{s'}} \tag{4.27}$$

for $j = N$, provided that ε is small.

Combining the relation $\overline{\omega}_0 = -C_\Theta f_0' - a_\mu \mathbb{A}(f_0)[\overline{\omega}_0]$ with the estimates (4.19) and (4.26) (and recalling also Remark 2 (iii) and the identity $B_{0,0} = \pi H$), we conclude that (4.11) holds true in the case when $|j| \leqslant N-1$. For $j = N$, the desired claim (4.11) follows from (4.20) and (4.27). □

We now consider the Fourier multipliers from Theorem 7 more closely.

Lemma 16 *There exists a constant $\eta = \eta(f_0) \in (0,1)$ with the property that*

$$\eta \leqslant \alpha_\tau \leqslant \frac{1}{\eta} \qquad \text{and} \qquad \|\beta_\tau\|_\infty \leqslant \frac{1}{\eta} \tag{4.28}$$

for all $\tau \in [0,1]$. Moreover, given $\alpha \in [\eta, 1/\eta]$ and $|\beta| \leqslant 1/\eta$, there exists a constant $\kappa_0 \geqslant 1$ such that the Fourier multiplier:

$$\mathbb{A}_{\alpha,\beta} := -\alpha\left(-\frac{d^2}{dx^2}\right)^{1/2} + \beta\frac{d}{dx},$$

satisfies

- $\lambda - \mathbb{A}_{\alpha,\beta} \in \mathrm{Isom}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R})), \qquad \forall\ \mathrm{Re}\,\lambda \geqslant 1,$ (4.29)
- $\kappa_0\|(\lambda - \mathbb{A}_{\alpha,\beta})[f]\|_{W_p^{s-1}} \geqslant |\lambda|\cdot\|f\|_{W_p^{s-1}} + \|f\|_{W_p^s} \quad \forall f \in W_p^s(\mathbb{R}), \mathrm{Re}\,\lambda \geqslant 1.$ (4.30)

Proof The bounds (4.28) are a consequence of $f_0 \in \mathcal{O}$. Finally, in order to prove the properties (4.29)–(4.30), we first consider the realisations:

$$\mathbb{A}_{\alpha,\beta} \in \mathcal{L}(W_p^1(\mathbb{R}), L_p(\mathbb{R})) \quad \text{and} \quad \mathbb{A}_{\alpha,\beta} \in \mathcal{L}(W_p^2(\mathbb{R}), W_p^1(\mathbb{R}))$$

for which the properties (4.29)–(4.30) (in the appropriate spaces) can be established in view of the identification $W^k_p(\mathbb{R}) = H^k_p(\mathbb{R})$, $k \in \mathbb{N}$ (using Fourier analysis and, in particular, Mikhlin's multiplier theorem, cf. e.g. [1, Theorem 4.23]). Then, using the interpolation property (3.10), we conclude that (4.29)–(4.30) hold true. □

We next exploit for a second time the Rayleigh–Taylor condition to show that $\lambda - \Psi(0)$ is an isomorphism provided that $\lambda \in \mathbb{R}$ is sufficiently large.

Proposition 1 *Let* $\delta > 0$ *and* $\phi \in W^{s-1}_p(\mathbb{R})$ *satisfy* $a := \delta + \phi > 0$. *Then, there exists a constant* $\omega_0 > 0$ *such that*

$$\lambda + H \circ \left(a\frac{d}{dx}\right) \in \mathrm{Isom}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R})) \qquad \text{for } \lambda \in [\omega_0, \infty). \tag{4.31}$$

Proof Let $[\tau \mapsto B(\tau)] : [0, 1] \to \mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))$ be the continuous path given by

$$B(\tau) := H \circ \left(a_\tau \frac{d}{dx}\right),$$

where $a_\tau := (1-\tau)\delta + \tau a = \delta + \tau\phi$, $\tau \in [0, 1]$. We show below that there exist constants $\omega_0 > 0$ and $C > 0$ such that

$$\|(\lambda + B(\tau))[f]\|_{W^{s-1}_p} \geqslant C\|f\|_{W^s_p}, \qquad \forall\, \tau \in [0, 1], \lambda \in [\omega_0, \infty), f \in W^s_p(\mathbb{R}). \tag{4.32}$$

Since $\lambda + B(0)$ is the Fourier multiplier with symbol $m_\lambda(\xi) := \lambda + \delta|\xi|$, $\xi \in \mathbb{R}$, it then holds that $\lambda + B(0) \in \mathrm{Isom}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R}))$ for all $\lambda > 0$. The method of continuity and (4.32) imply then that (4.31) holds true.

Step 1. Let $s' \in (1 + 1/p, s)$. Given $\mu > 0$, we find below $\varepsilon \in (0, 1)$, an ε-localisation family $\{\pi^\varepsilon_j : -N+1 \leqslant j \leqslant N\}$, a constant $K = K(\varepsilon)$ and bounded operators:

$$\mathbb{B}_{j,\tau} \in \mathcal{L}(W^s_p(\mathbb{R}), W^{s-1}_p(\mathbb{R})), \qquad j \in \{-N+1, \ldots, N\} \text{ and } \tau \in [0, 1],$$

such that

$$\|\pi^\varepsilon_j B(\tau)[f] - \mathbb{B}_{j,\tau}[\pi^\varepsilon_j f]\|_{W^{s-1}_p} \leqslant \mu\|\pi^\varepsilon_j f\|_{W^s_p} + K\|f\|_{W^{s'}_p} \tag{4.33}$$

for all $j \in \{-N+1, \ldots, N\}$, $\tau \in [0, 1]$ and $f \in W^s_p(\mathbb{R})$. The operators $\mathbb{B}_{j,\tau}$ are defined by

$$\mathbb{B}_{N,\tau} := \delta\left(-\frac{d^2}{dx^2}\right)^{1/2} \qquad \text{and} \qquad \mathbb{B}_{j,\tau} := a_\tau(x^\varepsilon_j)\left(-\frac{d^2}{dx^2}\right)^{1/2}, \quad |j| \leqslant N-1,$$

where $x^\varepsilon_j \in \operatorname{supp} \pi^\varepsilon_j$.

For $-N+1 \leqslant j \leqslant N$ it holds that

$$\begin{aligned}\|\pi^\varepsilon_j B(\tau)[f] - \mathbb{B}_{j,\tau}[\pi^\varepsilon_j f]\|_{W^{s-1}_p} \leqslant\ & \|\pi^\varepsilon_j B(\tau)[f] - B(\tau)[\pi^\varepsilon_j f]\|_{W^{s-1}_p} \\ & + \|B(\tau)[\pi^\varepsilon_j f] - \mathbb{B}_{j,\tau}[\pi^\varepsilon_j f]\|_{W^{s-1}_p}.\end{aligned}$$

Lemma 12 yields

$$\|\pi^\varepsilon_j B(\tau)[f] - B(\tau)[\pi^\varepsilon_j f]\|_{W^{s-1}_p} \leqslant K\|f\|_{W^1_p}, \qquad -N+1 \leqslant j \leqslant N.$$

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Moreover, for $|j| \leqslant N-1$, we use the identity $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$ together with (2.1), Lemma 2 (ii) (with $r = s$ and $\tau = s' - 1$), and Lemma 5 to derive that

$$\begin{aligned} \|B(\tau)[\pi_j^\varepsilon f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{W_p^{s-1}} &= \|H[(a_\tau - a_\tau(x_j^\varepsilon))\chi_j^\varepsilon(\pi_j^\varepsilon f)']\|_{W_p^{s-1}} \\ &\leqslant C\|(\phi - \phi(x_j^\varepsilon))\chi_j^\varepsilon(\pi_j^\varepsilon f)']\|_{W_p^{s-1}} \\ &\leqslant C\|(\phi - \phi(x_j^\varepsilon))\chi_j^\varepsilon\|_\infty \|(\pi_j^\varepsilon f)'\|_{W_p^{s-1}} + K\|f\|_{W_p^{s'}} \\ &\leqslant \mu\|\pi_j^\varepsilon f\|_{W_p^s} + K\|f\|_{W_p^{s'}}, \end{aligned}$$

provided that ε is sufficiently small. This proves (4.33) for $|j| \leqslant N-1$. Finally, for $j = N$, we have

$$\begin{aligned} \|B(\tau)[\pi_j^\varepsilon f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{W_p^{s-1}} &= \|H[(a_\tau - \delta)\chi_j^\varepsilon(\pi_j^\varepsilon f)']\|_{W_p^{s-1}} \\ &\leqslant C\|\phi\chi_j^\varepsilon\|_\infty \|(\pi_j^\varepsilon f)'\|_{W_p^{s-1}} + K\|f\|_{W_p^{s'}} \\ &\leqslant \mu\|\pi_j^\varepsilon f\|_{W_p^s} + K\|f\|_{W_p^{s'}}, \end{aligned}$$

provided that ε is sufficiently small, and (4.33) holds also for $j = N$.

Step 2. Let $\eta \in (0, 1)$ be chosen such that the function a_τ satisfies

$$\eta \leqslant a_\tau \leqslant 1/\eta, \qquad \tau \in [0, 1].$$

Lemma 16 implies there exists a constant $\kappa = \kappa(\eta) \geqslant 1$ such that the Fourier multipliers:

$$\mathbb{B}_\alpha := \alpha\left(-\frac{d^2}{dx^2}\right)^{1/2}, \qquad \alpha \in [\eta, \eta^{-1}],$$

satisfy

$$\kappa\|(\lambda + \mathbb{B}_\alpha)[f]\|_{W_p^{s-1}} \geqslant |\lambda| \cdot \|f\|_{W_p^{s-1}} + \|f\|_{W_p^s}, \qquad f \in W_p^s(\mathbb{R}),\ \lambda \geqslant 1. \tag{4.34}$$

Let $\varepsilon > 0$ be determined in the previous step for $\mu := (2\kappa)^{-1}$. It then holds

$$\begin{aligned} 2\kappa\|\pi_j^\varepsilon(\lambda + B(\tau))[f]\|_{W_p^{s-1}} &\geqslant 2\kappa\|(\lambda + \mathbb{B}_{j,\tau})[\pi_j^\varepsilon f]\|_{W_p^{s-1}} - 2\kappa\|\pi_j^\varepsilon B(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{W_p^{s-1}} \\ &\geqslant \|\pi_j^\varepsilon f\|_{W_p^s} + 2\lambda\|\pi_j^\varepsilon f\|_{W_p^{s-1}} - 2\kappa K\|f\|_{W_p^{s'}} \end{aligned}$$

for $-N+1 \leqslant j \leqslant N$, $\tau \in [0, 1]$, and $\lambda \geqslant 1$. Summing up over j, we conclude together with Lemma 9, (3.10) and Young's inequality there are constants $\kappa_0 \geqslant 1$ and $\omega_0 > 0$ such that

$$\kappa_0\|(\lambda + B(\tau))[f]\|_{W_p^{s-1}} \geqslant \|f\|_{W_p^s} + \lambda\|f\|_{W_p^{s-1}}$$

for all $\tau \in [0, 1]$, $\lambda \geqslant \omega_0$ and $f \in W_p^s(\mathbb{R})$. This proves (4.32) and the proof is complete. □

We are now in a position to prove Theorem 6.

Proof of Theorem 6 Let $s' \in (1 + 1/p, s)$. Let further $\kappa_0 \geqslant 1$ be the constant determined in Lemma 16 and set $\mu := 1/2\kappa_0$. Theorem 7 implies there exist $\varepsilon \in (0, 1)$, an ε-localisation family $\{\pi_j^\varepsilon : -N+1 \leqslant j \leqslant N\}$, a constant $K = K(\varepsilon, f_0) > 0$ and bounded operators $\mathbb{A}_{j,\tau} \in$

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$\mathcal{L}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R}))$, $-N+1 \leqslant j \leqslant N$ and $\tau \in [0,1]$, satisfying:

$$2\kappa_0 \|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{W_p^{s-1}} \leqslant \|\pi_j^\varepsilon f\|_{W_p^s} + 2\kappa_0 K \|f\|_{W_p^{s'}}, \qquad f \in W_p^s(\mathbb{R}).$$

Furthermore, Lemma 16 yields

$$2\kappa_0 \|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\|_{W_p^{s-1}} \geqslant 2|\lambda| \cdot \|\pi_j^\varepsilon f\|_{W_p^{s-1}} + 2\|\pi_j^\varepsilon f\|_{W_p^s}$$

for all $-N+1 \leqslant j \leqslant N$, $\tau \in [0,1]$, $\mathrm{Re}\,\lambda \geqslant 1$, and $f \in W_p^s(\mathbb{R})$. These two inequalities yield

$$\begin{aligned} &2\kappa_0 \|\pi_j^\varepsilon(\lambda - \Psi(\tau))[f]\|_{W_p^{s-1}} \\ &\quad \geqslant 2\kappa_0 \|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\|_{W_p^{s-1}} - 2\kappa_0 \|\pi_j^\varepsilon \Psi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\|_{W_p^{s-1}} \\ &\quad \geqslant 2|\lambda| \cdot \|\pi_j^\varepsilon f\|_{W_p^{s-1}} + \|\pi_j^\varepsilon f\|_{W_p^s} - 2\kappa_0 K \|f\|_{W_p^{s'}} \end{aligned}$$

for all $-N+1 \leqslant j \leqslant N$, $\tau \in [0,1]$, $\mathrm{Re}\,\lambda \geqslant 1$ and $f \in W_p^s(\mathbb{R})$. Summing up over j, Lemma 9, (3.10) and Young's inequality imply there exist constants $\kappa = \kappa(f_0) \geqslant 1$ and $\omega_1 = \omega_1(f_0) > 0$ such that

$$\kappa \|(\lambda - \Psi(\tau))[f]\|_{W_p^{s-1}} \geqslant |\lambda| \cdot \|f\|_{W_p^{s-1}} + \|f\|_{W_p^s} \tag{4.35}$$

for all $\tau \in [0,1]$, $\mathrm{Re}\,\lambda \geqslant \omega_1$ and $f \in W_p^s(\mathbb{R})$.

Let $\omega_0 > 0$ denote the constant from Proposition 1 found for $\delta := C_\Theta$ and $\phi := a_\mu \Phi(f_0)$. Setting $\omega := \max\{\omega_0, \omega_1\}$, it holds $\omega - \Psi(0) \in \mathrm{Isom}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R}))$, cf. Lemma 16. The method of continuity together with (4.35) yields that

$$\omega - \Psi(1) \in \mathrm{Isom}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R})). \tag{4.36}$$

Gathering (4.35) (with $\tau = 1$) and (4.36), it follows that $-\partial\Phi(f_0) \in \mathcal{H}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R}))$, cf. [3, Chapter I] and the proof is complete. □

We conclude this section with the proof of our main result. The well-posedness result follows by applying abstracts result for fully nonlinear parabolic problems from [34]. It is important to point out that in fact we can establish the uniqueness of solutions in the setting of strict solutions (as stated in Theorem 2), which is an improvement compared to the theory in [34]. This feature is essential when proving the claim (ii) of Theorem 2, as it enables us to use a parameter trick which was successfully applied also to other problems, cf., for example, [6, 25, 37, 47].

Proof of Theorem 2 *Well-posedness:* In view of (4.4) and Theorem 6, we find that the assumptions of [34, Theorem 8.1.1] are satisfied in the context of the evolution problem (4.3) when restricting Φ to the open set $\mathcal{O}$. Hence, given $f_0 \in \mathcal{O}$, there exists $T > 0$ and a solution $f(\cdot; f_0)$ to (4.3) that satisfies

$$f \in \mathrm{C}([0,T], \mathcal{O}) \cap \mathrm{C}^1([0,T], W_p^{s-1}(\mathbb{R})) \cap \mathrm{C}^\alpha_\alpha((0,T], W_p^s(\mathbb{R}))$$

for some $\alpha \in (0,1)$ (actually, since the problem is autonomous, for all $\alpha \in (0,1)$). Moreover, the solution is unique within the class:

$$\bigcup_{\alpha \in (0,1)} \mathrm{C}^\alpha_\alpha((0,T], W_p^s(\mathbb{R})) \cap \mathrm{C}([0,T], \mathcal{O}) \cap \mathrm{C}^1([0,T], W_p^{s-1}(\mathbb{R})).$$

In fact, the solution is unique in $\mathrm{C}([0,T], \mathcal{O}) \cap \mathrm{C}^1([0,T], W_p^{s-1}(\mathbb{R}))$. Indeed, assuming there are two solutions $f, \widetilde{f} : [0,T] \to \mathcal{O}$ corresponding to the same initial data $f_0 \in \mathcal{O}$, since the problem

(4.3) is autonomous, we can assume $f(t) \neq \widetilde{f}(t)$ for $t \in (0, T]$. Let now $s' \in (1 + 1/p, s)$ and set $\alpha := s - s' \in (0, 1)$. In virtue of (3.10), there exists a positive constant $C > 0$ such that

$$\|f(t_1) - f(t_2)\|_{W_p^{s'}} + \|\widetilde{f}(t_1) - \widetilde{f}(t_2)\|_{W_p^{s'}} \leqslant C|t_1 - t_2|^{\alpha}, \qquad t_1,\ t_2 \in [0, T], \tag{4.37}$$

and therefore $f, \widetilde{f} \in \mathrm{C}^{\alpha}([0, T], W_p^{s'}(\mathbb{R})) \hookrightarrow \mathrm{C}_{\alpha}^{\alpha}((0, T], W_p^{s'}(\mathbb{R}))$. We may now apply the abstract result [34, Theorem 8.1.1] in the context of (4.3) with $\Phi \in \mathrm{C}^{\omega}(\widetilde{\mathcal{O}}, W_p^{s'-1}(\mathbb{R}))$, where

$$\widetilde{\mathcal{O}} := \{f \in W_p^{s'}(\mathbb{R}) \,:\, C_{\Theta} + a_{\mu}\Phi(f) > 0\}.$$

Since $f_0 \in \widetilde{\mathcal{O}}$, we get in virtue of (4.37), that $f = \widetilde{f}$ on $[0, T]$; hence, our assumption was false.

Finally, the unique solution can be extended up to a maximal existence time $T_+(f_0)$, see [34, Section 8.2]. In virtue of [34, Proposition 8.2.3], the solution map also defines a semiflow on $\mathcal{O}$, and it remains to establish (ii).

Parabolic smoothing: Given $\lambda := (\lambda_1, \lambda_2) \in (0, \infty) \times \mathbb{R}$ and a maximal solution $f = f(\cdot; f_0)$ with maximal existence time $T_+ = T_+(f_0)$ to (4.3), let

$$f_{\lambda}(t, x) := f(\lambda_1 t, x + \lambda_2 t), \qquad x \in \mathbb{R},\ 0 \leqslant t \leqslant T_{+,\lambda} := T_+/\lambda_1.$$

Straightforward calculations show that $f_{\lambda} \in \mathrm{C}([0, T_{+,\lambda}), \mathcal{O}) \cap \mathrm{C}^1([0, T_{+,\lambda}), W_p^{s-1}(\mathbb{R}))$ is a solution to the evolution problem:

$$\frac{df}{dt} = \Psi(f, \lambda), \quad t \geqslant 0, \qquad f(0) = f_0, \tag{4.38}$$

where $\Psi : \mathcal{O} \times (0, \infty) \times \mathbb{R} \subset W_p^s(\mathbb{R}) \times \mathbb{R}^2 \to W_p^{s-1}(\mathbb{R})$ is defined by

$$\Psi(f, \lambda) := \lambda_1 \Phi(f) + \lambda_2 \frac{df}{dx}.$$

Using (4.4), we get $\Psi \in \mathrm{C}^{\omega}(\mathcal{O} \times (0, \infty) \times \mathbb{R}, W_p^{s-1}(\mathbb{R}))$. Also, given $(f_0, \lambda) \in \mathcal{O} \times (0, \infty) \times \mathbb{R}$, the partial derivative of Ψ with respect to f is

$$\partial_f \Psi(f_0, \lambda) = \lambda_1 \partial\Phi(f_0) + \lambda_2 \frac{d}{dx}.$$

Since d/dx is a Fourier multiplier with symbol $m(\xi) = i\xi$, $\xi \in \mathbb{R}$, the results leading to Theorem 6 can be easily adapted to deduce that the operator $-\partial_f \Psi(f_0, \lambda)$ belongs to $\mathcal{H}(W_p^s(\mathbb{R}), W_p^{s-1}(\mathbb{R}))$ for all $(f_0, \lambda) \in \mathcal{O} \times (0, \infty) \times \mathbb{R}$. According to [34, Theorem 8.1.1 and Theorem 8.3.9] and arguing as in the proof of (i), it follows that (4.38) has for each $(f_0, \lambda) \in \mathcal{O} \times (0, \infty) \times \mathbb{R}$ a unique strict solution:

$$f = f(\,\cdot\,; f_0, \lambda) \in \mathrm{C}([0, \widetilde{T}_+), \mathcal{O}) \cap \mathrm{C}^1([0, \widetilde{T}_+), W_p^{s-1}(\mathbb{R})),$$

where $\widetilde{T}_+ = T_+(f_0, \lambda) \in (0, \infty]$ is the maximal existence time. Moreover, the set

$$\Omega := \{(t, f_0, \lambda) \,:\, (f_0, \lambda) \in \mathcal{O} \times (0, \infty) \times \mathbb{R},\ 0 < t < T_+(f_0, \lambda)\}$$

is open and

$$[(t, f_0, \lambda) \mapsto f(t; f_0, \lambda)] \in \mathrm{C}^{\omega}(\Omega, \mathcal{O}).$$

Hence, given $f_0 \in \mathcal{O}$, we may conclude that

$$T_+(f_0, \lambda) = \frac{T_+(f_0)}{\lambda_1} \qquad \text{and} \qquad f(t; f_0, \lambda) = f_\lambda(t), \quad 0 \leqslant t < \frac{T_+(f_0)}{\lambda_1}.$$

In particular, given $t_0 < T_+(f_0)$, we may choose $\delta > 0$ such that $t_0 < T_+(f_0, \lambda)$ for all λ belonging to the disc $D_\delta((1, 0))$, and therewith

$$[\lambda \mapsto f_\lambda(t_0)] : D_\delta((1, 0)) \to W^s_p(\mathbb{R}) \tag{4.39}$$

is also a real analytic map. Repeated differentiation with respect to λ_2 immediately yields (iib). Let now $x_0 \in \mathbb{R}$. Since $[h \mapsto h(x_0)] : W^s_p(\mathbb{R}) \to \mathbb{R}$ is real analytic, then so is

$$[\lambda \mapsto f(\lambda_1 t_0, x_0 + \lambda_2 t_0)] : D_\delta((1, 0)) \to \mathbb{R}.$$

Besides, if $\varepsilon > 0$ small, the mapping $\varphi : D_\varepsilon((t_0, x_0)) \to D_\delta((1, 0))$ with

$$\varphi(t, x) := \left(\frac{t}{t_0}, \frac{x - x_0}{t_0}\right)$$

is well defined and real analytic, and composing it with the previous function shows that

$$[(t, x) \mapsto f(t, x)] : D_\varepsilon((t_0, x_0)) \to \mathbb{R},$$

is also real analytic. This proves (iia). □

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Conflict of interest

None.

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Appendix A Preparatory results used in Section 4

In this section, we present the proofs of Lemmas 12–15.

Proof of Lemma 12 We may assume that $h \in C_0^\infty(\mathbb{R})$. Setting

$$T := \varphi B_{n,m}(f,\dots,f)[f,\dots,f,h] - B_{n,m}(f,\dots,f)[f,\dots,f,\varphi h],$$

it follows from Lemma 2 (i) that

$$\|T\|_p \leqslant K\|h\|_p. \tag{A.1}$$

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Moreover, given $0 \neq \xi \in \mathbb{R}$, it holds that

$$\frac{\tau_\xi T - T}{\xi} = T_1 + T_2 + T_3 + T_4 + T_5,$$

where, using (2.5) and (2.6), it holds

$$\begin{aligned}
T_1 &:= \frac{\tau_\xi \varphi - \varphi}{\xi} B_{n,m}(\tau_\xi f, \ldots, \tau_\xi f)[\tau_\xi f, \ldots, \tau_\xi f, \tau_\xi h],\\
T_2 &:= \varphi B_{n,m}(\tau_\xi f, \ldots, \tau_\xi f)\Big[\tau_\xi f, \ldots, \tau_\xi f, \frac{\tau_\xi h - h}{\xi}\Big],\\
T_3 &:= -B_{n,m}(\tau_\xi f, \ldots, \tau_\xi f)\Big[\tau_\xi f, \ldots, \tau_\xi f, \frac{\tau_\xi(\varphi h) - \varphi h}{\xi}\Big],\\
T_4 &:= \frac{\tau_\xi f - f}{\xi} \sum_{i=1}^{n} B_{n,m}(\tau_\xi f, \ldots, \tau_\xi f)[f_1, \ldots, f_i, \varphi, \tau_\xi f, \ldots \tau_\xi f, h]\\
&\quad - \sum_{i=1}^{n} B_{n,m}(\tau_\xi f, \ldots, \tau_\xi f)\Big[f_1, \ldots, f_i, \varphi, \tau_\xi f, \ldots \tau_\xi f, \frac{\tau_\xi f - f}{\xi} h\Big],\\
T_5 &:= -\frac{\tau_\xi f - f}{\xi} \sum_{i=1}^{m} B_{n+2,m+1}(\tau_\xi f_1, \ldots, \tau_\xi f_i, f, \ldots, f)\Big[\varphi, \tau_\xi f + f, f, \ldots, f, h\Big]\\
&\quad + \sum_{i=1}^{m} B_{n+2,m+1}(\tau_\xi f_1, \ldots, \tau_\xi f_i, f, \ldots, f)\Big[\varphi, \tau_\xi f + f, f, \ldots, f, \frac{\tau_\xi f - f}{\xi} h\Big],
\end{aligned}$$

where $f_j := f$ for $1 \leqslant j \leqslant i$. Lemma 2 (i) implies the limit $\lim_{\xi \to 0}(\tau_\xi T - T)/\xi$ exists in $L_p(\mathbb{R})$. Hence, $T \in W^1_p(\mathbb{R})$ and

$$\begin{aligned}
T' = &\ \varphi' B_{n,m}(f, \ldots, f)[f, \ldots, f, h] + \varphi B_{n,m}(f, \ldots, f)[f, \ldots, f, h']\\
&- B_{n,m}(f, \ldots, f)[f, \ldots, f, \varphi h'] - B_{n,m}(f, \ldots, f)[f, \ldots, f, \varphi' h]\\
&+ nf' B_{n,m}(f, \ldots, f)[\varphi, f, \ldots, f, h] - nB_{n,m}(f, \ldots, f)[\varphi, f, \ldots, f, f'h]\Big]\\
&- 2mf' B_{n+2,m+1}(f, \ldots, f)[\varphi, f, \ldots, f, h] + 2mB_{n+2,m+1}(f, \ldots, f)[\varphi, f, \ldots, f, f'h].
\end{aligned}$$

Using again Lemma 2 (i), we get

$$\|T' - \varphi B_{n,m}(f, \ldots, f)[f, \ldots, f, h'] + B_{n,m}(f, \ldots, f)[f, \ldots, f, \varphi h']\|_p \leqslant K\|h\|_p. \tag{A.2}$$

It remains to estimate the term:

$$T_6 := \varphi B_{n,m}(f, \ldots, f)[f, \ldots, f, h'] - B_{n,m}(f, \ldots, f)[f, \ldots, f, \varphi h'].$$

Since,

$$T_6(x) = \int_{\mathbb{R}} \frac{(\delta_{[x,y]} f/y)^n}{(1 + (\delta_{[x,y]} f/y)^2)^m} \frac{\delta_{[x,y]}\varphi}{y} \frac{d}{dy}(-h(x-y))\, dy, \qquad x \in \mathbb{R},$$

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integration by parts leads to the following representation:

$$\begin{aligned}T_6 &= B_{n,m}(f,\ldots,f)[f,\ldots,f,\varphi' h] - B_{n+1,m}(f,\ldots,f)[f,\ldots,f,\varphi,h]\\&\quad+nB_{n,m}(f,\ldots,f)[f,\ldots,f,\varphi,f'h] - nB_{n+1,m}(f,\ldots,f)[f,\ldots,f,\varphi,h]\\&\quad-2mB_{n+2,m+1}(f,\ldots,f)[f,\ldots,f,\varphi,f'h] + 2mB_{n+3,m+1}(f,\ldots,f)[f,\ldots,f,\varphi,h]\end{aligned}$$

and Lemma 2 (i) yields

$$\|T_6\|_p \leqslant K\|h\|_p. \tag{A.3}$$

Gathering (A.1)–(A.3), we arrive at (4.12) and the proof is complete. □

We now establish Lemma 13.

Proof of Lemma 13 We first deal with the case $|j| \leqslant N-1$ and write

$$\pi_j^\varepsilon \overline{\omega} B_{n,m}(f,\ldots,f)[f,\ldots,f,h] - \frac{\overline{\omega}(x_j^\varepsilon)(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m} B_{0,0}[\pi_j^\varepsilon h] = T_1 + \overline{\omega}(x_j^\varepsilon)T_2,$$

with

$$\begin{aligned}T_1 &:= \pi_j^\varepsilon \overline{\omega} B_{n,m}(f,\ldots,f)[f,\ldots,f,h] - \overline{\omega}(x_j^\varepsilon) B_{n,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h],\\T_2 &:= B_{n,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h] - \frac{(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m} B_{0,0}[\pi_j^\varepsilon h].\end{aligned}$$

The term T_1. In view of $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, we decompose $T_1 = T_{1a} + \overline{\omega}(x_j^\varepsilon)T_{1b}$, where

$$\begin{aligned}T_{1a} &:= \chi_j^\varepsilon(\overline{\omega} - \overline{\omega}(x_j^\varepsilon))\pi_j^\varepsilon B_{n,m}(f,\ldots,f)[f,\ldots,f,h],\\T_{1b} &:= \pi_j^\varepsilon B_{n,m}(f,\ldots,f)[f,\ldots,f,h] - B_{n,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h].\end{aligned}$$

Applying Lemma 12, we get

$$\|T_{1b}\|_{W_p^{s-1}} \leqslant K\|h\|_p, \qquad -N+1 \leqslant j \leqslant N. \tag{A.4}$$

Moreover, recalling (2.1), it follows from (A.4), Lemma 5 (with $r=s$) and Lemma 2 (ii) (with $r=s$ and $\tau = s'-1$) that

$$\begin{aligned}\|T_{1a}\|_{W_p^{s-1}} &\leqslant 2\|\chi_j^\varepsilon(\overline{\omega} - \overline{\omega}(x_j^\varepsilon))\|_\infty \|\pi_j^\varepsilon B_{n,m}(f,\ldots,f)[f,\ldots,f,h]\|_{W_p^{s-1}}\\&\quad+K\|B_{n,m}(f,\ldots,f)[f,\ldots,f,h]\|_\infty\\&\leqslant 2\|\chi_j^\varepsilon(\overline{\omega} - \overline{\omega}(x_j^\varepsilon))\|_\infty \|B_{n,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h]\|_{W_p^{s-1}} + K\|h\|_{W_p^{s'-1}}\\&\leqslant \frac{\nu}{2}\|\pi_j^\varepsilon h\|_{W_p^{s-1}} + K\|h\|_{W_p^{s'-1}},\end{aligned}$$

provided ε is sufficiently small, and therewith

$$\|T_1\|_{W_p^{s-1}} \leqslant \frac{\nu}{2}\|\pi_j^\varepsilon h\|_{W_p^{s-1}} + K\|h\|_{W_p^{s'-1}}. \tag{A.5}$$

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The term T_2. We use again the identity $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$ and write $T_2 = T_{2a} + T_{2b}$, where

$$\begin{aligned} T_{2a} &:= \frac{(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m}(\chi_j^\varepsilon B_{0,0}[\pi_j^\varepsilon h] - B_{0,0}[\chi_j^\varepsilon(\pi_j^\varepsilon h)]) \\ &\quad -(\chi_j^\varepsilon B_{n,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h] - B_{n,m}(f,\ldots,f)[f,\ldots,f,\chi_j^\varepsilon(\pi_j^\varepsilon h)]), \\ T_{2b} &:= \chi_j^\varepsilon B_{n,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h] - \frac{(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^m}\chi_j^\varepsilon B_{0,0}[\pi_j^\varepsilon h]. \end{aligned}$$

Lemma 12 yields

$$\|T_{2a}\|_{W_p^{s-1}} \leqslant K\|h\|_p. \tag{A.6}$$

It remains to estimate T_{2b}. We first use Lemma 2 (i) to deduce that

$$\|T_{2b}\|_p \leqslant K\|h\|_p. \tag{A.7}$$

Moreover, noticing that $f'(x_j^\varepsilon) = \delta_{[x,y]}(f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}})/y$ and recalling (2.6), we write

$$\begin{aligned} T_{2b} &= \sum_{k=0}^{n-1}(f'(x_j^\varepsilon))^{n-k-1}\chi_j^\varepsilon B_{k+1,m}(f,\ldots,f)[f,\ldots,f,f-f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon h] \\ &\quad -\sum_{k=0}^{m-1}\frac{(f'(x_j^\varepsilon))^n}{[1+(f'(x_j^\varepsilon))^2]^{m-k}}\chi_j^\varepsilon B_{2,k+1}(f,\ldots,f)[f-f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},f+f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon h]. \end{aligned}$$

Let $T_k := \chi_j^\varepsilon B_{k+1,m}(f,\ldots,f)[f,\ldots,f,f-f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon h]$ for $0 \leqslant k \leqslant n-1$. In order to estimate the W_p^{s-1}-seminorm of T_k, we write for $\xi \in \mathbb{R}$

$$T_k - \tau_\xi T_k = T_{kA} + T_{kB} + \chi_j^\varepsilon T_{kC},$$

where, appealing again to (2.6), it holds

$$\begin{aligned} T_{kA} &:= (\chi_j^\varepsilon - \tau_\xi\chi_j^\varepsilon)\tau_\xi B_{k+1,m}(f,\ldots,f)[f,\ldots,f,f-f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon h], \\ T_{kB} &:= \chi_j^\varepsilon B_{k+1,m}(f,\ldots,f)[f,\ldots,f,f-f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)], \\ T_{kC} &:= \sum_{j=1}^{k} B_{k+1,m}(f,\ldots,f)[\tau_\xi f_1,\ldots,\tau_\xi f_{j-1},f-\tau_\xi f,f,\ldots,f,f-f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\tau_\xi(\pi_j^\varepsilon h)] \\ &\quad + B_{k+1,m}(f,\ldots,f)[\tau_\xi f,\ldots,\tau_\xi f,f-\tau_\xi f,\tau_\xi(\pi_j^\varepsilon h)] \\ &\quad + \sum_{j=1}^{m} B_{k+3,m+1}^j[\tau_\xi f,\ldots,\tau_\xi f,\tau_\xi f - f'(x_j^\varepsilon)\,\mathrm{id}_{\mathbb{R}},\tau_\xi f+f,\tau_\xi f-f,\tau_\xi(\pi_j^\varepsilon h)], \end{aligned}$$

where $f_i := f$ for $1 \leqslant i \leqslant j-1$ and

$$B_{k+3,m+1}^j := B_{k+3,m+1}(f,f_1,\ldots,f_{j-1},\tau_\xi f,\ldots,\tau_\xi f).$$

Observing that

$$T_{kA} = (\chi_j^\varepsilon - \tau_\xi\chi_j^\varepsilon)\tau_\xi\big(B_{k+1,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h] - f'(x_j^\varepsilon)B_{k,m}(f,\ldots,f)[f,\ldots,f,\pi_j^\varepsilon h]\big),$$

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Lemma 2 (ii) (with $r = s$ and $\tau = s' - 1$) yields

$$\|T_{kA}\|_p \leqslant K\|\chi_j^\varepsilon - \tau_\xi \chi_j^\varepsilon\|_p \|\pi_j^\varepsilon h\|_{W_p^{s'-1}} \leqslant K\|\chi_j^\varepsilon - \tau_\xi \chi_j^\varepsilon\|_p \|h\|_{W_p^{s'-1}}. \tag{A.8}$$

In order to estimate T_{kB}, let F denote the Lipschitz continuous function defined by $F = f$ on $\operatorname{supp} \chi_j^\varepsilon$ and $F' = f'(x_j^\varepsilon)$ on $\mathbb{R} \setminus \operatorname{supp} \chi_j^\varepsilon$. If $|\xi| \geqslant \varepsilon$, we infer from Lemma 2 (i) that

$$\|T_{kB}\|_p \leqslant K\|h\|_p. \tag{A.9}$$

If $|\xi| < \varepsilon$, then $\xi + \operatorname{supp} \pi_j^\varepsilon \subset \operatorname{supp} \chi_j^\varepsilon$, and Lemma 2 (i) and the properties defining F lead to

$$\begin{aligned} \|T_{kB}\|_p &= \|\chi_j^\varepsilon B_{k+1,m}(f, \ldots, f)[f, \ldots, f, F - f'(x_j^\varepsilon)\, \mathrm{id}_{\mathbb{R}}, \pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)]\|_p \\ &\leqslant \|f' - f'(x_j^\varepsilon)\|_{L_\infty(\operatorname{supp} \chi_j^\varepsilon)} \|\pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)\|_p \\ &\leqslant \frac{\nu}{12(n+1)C_0^n} \|\pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)\|_p, \end{aligned} \tag{A.10}$$

provided that ε is sufficiently small, where $C_0 := 1 + \|\overline{\omega}\|_\infty + \|f'\|_\infty$.

Finally, Lemma 4 (with $r = s'$) yields

$$\|\chi_j^\varepsilon T_{kC}\|_p \leqslant K\|f' - \tau_\xi f'\|_p \|\pi_j^\varepsilon h\|_{W_p^{s'-1}} \leqslant K\|f' - \tau_\xi f'\|_p \|h\|_{W_p^{s'-1}}. \tag{A.11}$$

The estimates (A.8)–(A.10) combined imply that

$$[T_k]_{W_p^{s-1}} \leqslant \frac{\nu}{4(n+1)C_0^n} \|\pi_j^\varepsilon h\|_{W_p^{s-1}} + \|h\|_{W_p^{s'-1}}.$$

The arguments used to estimate T_k show also that

$$\begin{aligned} &\big[\chi_j^\varepsilon B_{2,k+1}(f, \ldots, f)[f - f'(x_j^\varepsilon)\, \mathrm{id}_{\mathbb{R}}, f + f'(x_j^\varepsilon)\, \mathrm{id}_{\mathbb{R}}, \pi_j^\varepsilon h]\big]_{W_p^{s-1}} \\ &\qquad \leqslant \frac{\nu}{4(m+1)C_0^{n+1}} \|\pi_j^\varepsilon h\|_{W_p^{s-1}} + \|h\|_{W_p^{s'-1}}, \end{aligned}$$

provided that ε is chosen sufficiently small. Recalling also (A.7), we obtain for such ε that

$$\|T_{2b}\|_{W_p^{s-1}} \leqslant \frac{\nu}{2C_0} \|\pi_j^\varepsilon h\|_{W_p^{s-1}} + \|h\|_{W_p^{s'-1}},$$

and together with (A.5) and (A.6), we have established (4.13). □

We continue with the proof of Lemma 14.

Proof of Lemma 14 Let now $j = N$. Since $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, it holds that

$$\pi_j^\varepsilon \overline{\omega} B_{n,m}(f, \ldots, f)[f, \ldots, f, h] = T_1 + T_2,$$

where

$$\begin{aligned} T_1 &:= \chi_j^\varepsilon \overline{\omega}\big(\pi_j^\varepsilon B_{n,m}(f, \ldots, f)[f, \ldots, f, h] - B_{n,m}(f, \ldots, f)[f, \ldots, f, \pi_j^\varepsilon h]\big), \\ T_2 &:= \chi_j^\varepsilon \overline{\omega} B_{n,m}(f, \ldots, f)[f, \ldots, f, \pi_j^\varepsilon h]. \end{aligned}$$

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Lemma 5 (with $r=s$) together with Lemma 2 (i) (with $r=s$ and $\tau=s'-1$) and (2.1) yields

$$\begin{aligned}\|T_2\|_{W_p^{s-1}} &\leqslant 2\|\chi_j^\varepsilon\overline{\omega}\|_\infty\|B_{n,m}(f,\dots,f)[f,\dots,f,\pi_j^\varepsilon h]\|_{W_p^{s-1}}\\ &\quad+K\|B_{n,m}(f,\dots,f)[f,\dots,f,\pi_j^\varepsilon h]\|_\infty\\ &\leqslant C\|\chi_j^\varepsilon\overline{\omega}\|_\infty\|\pi_j^\varepsilon h\|_{W_p^{s-1}}+K\|\pi_j^\varepsilon h\|_{W_p^{s'-1}}\\ &\leqslant \nu[\pi_j^\varepsilon h]_{W_p^{s-1}}+K\|h\|_{W_p^{s'-1}},\end{aligned}$$

provided that ε is sufficiently small. We have made use here of the fact that if ε is sufficiently small, then $\|\overline{\omega}\|_{L_\infty(\operatorname{supp}\chi_j^\varepsilon)}<\nu$. Furthermore, Lemma 12 shows that

$$\|T_1\|_{W_p^{s-1}}\leqslant K\|h\|_p$$

and (4.14) follows. □

We conclude this appendix with the proof of Lemma 15.

Proof of Lemma 15 We first address the case $n=0$. Then,

$$\pi_j^\varepsilon B_{0,m}(f,\dots,f)[h]-B_{0,0}[\pi_j^\varepsilon h]=T_a+T_b+T_c,$$

where

$$\begin{aligned}T_a &:= \pi_j^\varepsilon B_{0,m}(f,\dots,f)[h]-B_{0,m}(f,\dots,f)[\pi_j^\varepsilon h],\\ T_b &:= \chi_j^\varepsilon B_{0,0}[\pi_j^\varepsilon h]-B_{0,0}[\chi_j^\varepsilon(\pi_j^\varepsilon h)]-(\chi_j^\varepsilon B_{0,m}(f,\dots,f)[\pi_j^\varepsilon h]-B_{0,m}(f,\dots,f)[\chi_j^\varepsilon(\pi_j^\varepsilon h)]),\\ T_c &:= \chi_j^\varepsilon(B_{0,m}(f,\dots,f)[\pi_j^\varepsilon h]-B_{0,0}[\pi_j^\varepsilon h]).\end{aligned}$$

Lemma 12 yields

$$\|T_a\|_{W_p^{s-1}}+\|T_b\|_{W_p^{s-1}}\leqslant K\|h\|_p. \tag{A.12}$$

It remains to estimate the term:

$$T_c=-\sum_{k=0}^{m-1}\chi_j^\varepsilon B_{2,m-k}(f,\dots,f)[f,f,\pi_j^\varepsilon h].$$

Using Lemma 2 (i), we get

$$\|T_c\|_p\leqslant K\|h\|_p. \tag{A.13}$$

Let $T_k:=\chi_j^\varepsilon B_{2,m-k}(f,\dots,f)[f,f,\pi_j^\varepsilon h]$, $0\leqslant k\leqslant m-1$. To estimate the W_p^{s-1}-seminorm of T_k, we write for $\xi\in\mathbb{R}$

$$T_k-\tau_\xi T_k=T_{kA}+T_{kB}+\chi_j^\varepsilon T_{kC},$$

where, using (2.6), we get

$$\begin{aligned}T_{kA} &:= (\chi_j^\varepsilon-\tau_\xi\chi_j^\varepsilon)\tau_\xi B_{2,m-k}(f,\dots,f)[f,f,\pi_j^\varepsilon h],\\ T_{kB} &:= \chi_j^\varepsilon B_{2,m-k}(f,\dots,f)[f,f,\pi_j^\varepsilon h-\tau_\xi(\pi_j^\varepsilon h)],\end{aligned}$$

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$$T_{kC} := B_{2,m-k}(f,\dots,f)[f-\tau_\xi f, f, \tau_\xi(\pi_j^\varepsilon h')] + B_{2,m-k}(f,\dots,f)[\tau_\xi f, f-\tau_\xi f, \tau_\xi(\pi_j^\varepsilon h)]$$
$$+ \sum_{\ell=1}^{m-k} B_{4,m-k+1}(f_1,\dots,f_\ell,\tau_\xi f,\dots,\tau_\xi f)[\tau_\xi f,\tau_\xi f,\tau_\xi f+f,\tau_\xi f-f,\tau_\xi(\pi_j^\varepsilon h)],$$

where $f_j := f$ for $1 \leqslant j \leqslant \ell$. Lemma 2 (ii) (with $r = s$ and $\tau = s' - 1$) yields

$$\|T_{kA}\|_p \leqslant K\|\chi_j^\varepsilon - \tau_\xi \chi_j^\varepsilon\|_p \|\pi_j^\varepsilon h\|_{W_p^{s'-1}} \leqslant K\|(1-\chi_j^\varepsilon) - \tau_\xi(1-\chi_j^\varepsilon)\|_p \|h\|_{W_p^{s'-1}}. \tag{A.14}$$

Let F denote the Lipschitz continuous function defined by $F = f$ on $[|x| \geqslant 1/\varepsilon - \varepsilon]$ and which is linear in $[|x| \leq 1/\varepsilon - \varepsilon]$. If $|\xi| \geqslant \varepsilon$, we infer from Lemma 2 (i) that

$$\|T_{kB}\|_p \leqslant K\|h\|_p. \tag{A.15}$$

If $|\xi| < \varepsilon$, then $\xi + \operatorname{supp} \pi_j^\varepsilon \subset \operatorname{supp} \chi_j^\varepsilon$, and using Lemma 2 (i) we get

$$\begin{aligned} \|T_{kB}\|_p &= \|\chi_j^\varepsilon B_{2,m-k}(f,\dots,f)[F,F,\pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)]\|_p \\ &\leqslant C\|F'\|_\infty^2 \|\pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)\|_p \\ &\leqslant \frac{\nu}{3(m+1)} \|\pi_j^\varepsilon h - \tau_\xi(\pi_j^\varepsilon h)\|_p, \end{aligned} \tag{A.16}$$

provided that ε is sufficiently small. The arguments in (A.16) rely on the fact that $\|F'\|_\infty \to 0$ for $\varepsilon \to 0$. Finally, Lemma 4 (with $r = s'$) yields

$$\|\chi_j^\varepsilon T_{kC}\|_p \leqslant K\|f' - \tau_\xi f'\|_p \|\pi_j^\varepsilon h\|_{W_p^{s'-1}} \leqslant K\|f' - \tau_\xi f'\|_p \|h\|_{W_p^{s'-1}}. \tag{A.17}$$

The estimates (A.12)–(A.17) lead us to (4.15).

The estimate (4.16) can be derived using the same arguments as above. □