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NEW GENERALISATIONS OF VAN HAMME'S (G.2) SUPERCONGRUENCE

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Abstract

Swisher ['On the supercongruence conjectures of van Hamme', *Res. Math. Sci.* **2** (2015), Article no. 18] and He ['Supercongruences on truncated hypergeometric series', *Results Math.* **72** (2017), 303–317] independently proved that Van Hamme's (G.2) supercongruence holds modulo p^4 for any prime $p \equiv 1 \pmod 4$. Swisher also obtained an extension of Van Hamme's (G.2) supercongruence for $p \equiv 3 \pmod 4$ and $p > 3$. In this note, we give new one-parameter generalisations of Van Hamme's (G.2) supercongruence modulo p^3 for any odd prime p. Our proof uses the method of 'creative microscoping' introduced by Guo and Zudilin ['A q-microscope for supercongruences', *Adv. Math.* **346** (2019), 329–358].

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1. Introduction

In his first letter to Hardy in 1913, Ramanujan mentioned the following formula (see [1, page 25, (2)]):

$$\sum_{k=0}^{\infty}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4} = \frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma(\frac{3}{4})^2}. \tag{1.1}$$

Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the rising factorial and $\Gamma(x)$ is the Gamma function. Ramanujan did not give a proof of (1.1) and the first proof was given by Hardy [7]. In 1997, Van Hamme [12] conjectured 13 p-adic analogues of Ramanujan's series, including

$$\sum_{k=0}^{(p-1)/4}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4} \equiv p\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})} \pmod{p^3} \quad \text{for } p \equiv 1 \pmod 4 \tag{1.2}$$

(marked (G.2) in Van Hamme's list). Here and throughout the paper, p always denotes an odd prime and $\Gamma_p(x)$ stands for Morita's p-adic Gamma function [10]. Swisher [11] and He [8] independently showed that (1.2) holds modulo the stronger power p^4.

We shall give a generalisation of (1.2): for $p \equiv 1 \pmod 4$ and $0 \leqslant s \leqslant (p-1)/4$,

$$\sum_{k=s}^{(p-1)/4} (8k+1)\frac{(\frac{1}{4})_{k-s}(\frac{1}{4})_{k+s}(\frac{1}{4})_k^2}{(k-s)!\,(k+s)!\,k!^2} \equiv (p+4s)\frac{(\frac{1}{4})_s^2(\frac{1}{4})_{(p-1)/4+s}(\frac{1}{2})_{(p-1)/4-s}}{s!^2\,(1)_{(p-1)/4+s}(\frac{1}{4})_{(p-1)/4-s}} \pmod{p^3}. \tag{1.3}$$

When $s = 0$, the right-hand side of (1.3) reduces to $p(\frac{1}{2})_{(p-1)/4}/(1)_{(p-1)/4}$, which is congruent to the right-hand side of (1.2) modulo p^3 (see [9]). Thus, the supercongruence (1.3) is indeed a generalisation of (1.2). A similar extension of the (A.2) supercongruence of Van Hamme was recently given by Guo [3].

We shall prove (1.3) by establishing the following q-supercongruence.

THEOREM 1.1. *Let $n \equiv 1 \pmod 4$ be an integer greater than 1 and let $0 \leqslant s \leqslant (n-1)/4$. Then*

$$\sum_{k=s}^{(n-1)/4} [8k+1]\frac{(q;q^4)_{k-s}(q;q^4)_{k+s}(q;q^4)_k^2}{(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k^2}q^{2k}$$
$$\equiv [n+4s]\frac{(q;q^4)_s^2(q;q^4)_{(n-1)/4+s}(q^2;q^4)_{(n-1)/4-s}}{(q^4;q^4)_s^2(q^4;q^4)_{(n-1)/4+s}(q;q^4)_{(n-1)/4-s}}q^{3s+(1-n)/4} \pmod{\Phi_n(q)^3}. \tag{1.4}$$

Here we need to be familiar with the standard q-notation. The q-shifted factorial is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for any positive integer n. The q-integer is defined as $[n] = (1-q^n)/(1-q)$ and $\Phi_n(q)$ denotes the nth *cyclotomic polynomial*, which can be written as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is a primitive nth root of unity.

It is easy to see that (1.3) follows from (1.4) by taking $n = p$ and $q \to 1$. The $s = 0$ case of (1.4) was given by Liu and Wang [9] and can also be deduced from [5, Theorem 4.3].

Swisher [11, (3)] gave the following extension of Van Hamme's (G.2) supercongruence: for $p \equiv 3 \pmod 4$ and $p > 3$,

$$\sum_{k=0}^{(3p-1)/4} (8k+1)\frac{(\frac{1}{4})_k^4}{k!^4} \equiv -\frac{3p^2\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{2\Gamma_p(\frac{3}{4})} \pmod{p^4}. \tag{1.5}$$

(The negative sign was missing in Swisher's original supercongruence.)

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We shall give a new generalisation of (1.5) modulo p^3 as follows: for $p \equiv 3 \pmod 4$ and $0 \leqslant s \leqslant (p-3)/4$,

$$\sum_{k=s}^{(3p-1)/4} (8k+1) \frac{(\frac{1}{4})_{k-s}(\frac{1}{4})_{k+s}(\frac{1}{4})_k^2}{(k-s)!\,(k+s)!\,k!^2} \equiv (3p+4s) \frac{(\frac{1}{4})_s^2(\frac{1}{4})_{(3p-1)/4+s}(\frac{1}{2})_{(3p-1)/4-s}}{s!^2\,(1)_{(3p-1)/4+s}(\frac{1}{4})_{(3p-1)/4-s}} \pmod{p^3}. \tag{1.6}$$

When $s = 0$, the right-hand side of (1.6) reduces to $3p(\frac{1}{2})_{(3p-1)/4}/(1)_{(3p-1)/4}$, which is easily seen to be congruent to the right-hand side of (1.5) modulo p^3. Thus, the supercongruence (1.6) can be deemed a generalisation of the modulo p^3 case of (1.5). A result of Guo and Schlosser [5, Corollary 1.2 with $d = 4$ and $q \to 1$] implies that (1.6) is even true modulo p^4 for $s = 0$. However, numerical evaluation indicates that (1.6) is not true modulo p^4 for general s.

In the same way as before, we shall prove (1.6) via the following q-supercongruence.

Theorem 1.2. *Let $n \equiv 3 \pmod 4$ be a positive integer and let $0 \leqslant s \leqslant (n-3)/4$. Then*

$$\sum_{k=s}^{(3n-1)/4} [8k+1] \frac{(q;q^4)_{k-s}(q;q^4)_{k+s}(q;q^4)_k^2}{(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}(q^4;q^4)_k^2} q^{2k}$$
$$\equiv [3n+4s] \frac{(q;q^4)_s^2(q;q^4)_{(3n-1)/4+s}(q^2;q^4)_{(3n-1)/4-s}}{(q^4;q^4)_s^2(q^4;q^4)_{(3n-1)/4+s}(q;q^4)_{(3n-1)/4-s}} q^{3s+(1-3n)/4} \pmod{\Phi_n(q)^3}. \tag{1.7}$$

Our proof of Theorems 1.1 and 1.2 will use the powerful method of 'creative microscoping', which was devised by Guo and Zudilin [6].

2. Proof of Theorem 1.1

We require the following easily proved q-congruence, which was first given by Guo and Schlosser [4, Lemma 3].

Lemma 2.1. *Let d, m and n be positive integers with $m \leqslant n-1$ and $dm \equiv -1 \pmod n$. Then, for $0 \leqslant k \leqslant m$,*

$$\frac{(aq;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} \equiv (-a)^{m-2k} \frac{(aq;q^d)_k}{(q^d/a;q^d)_k} q^{m(dm-d+2)/2+(d-1)k} \pmod{\Phi_n(q)}.$$

Following Gasper and Rahman's monograph [2], the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k \cdots (a_{r+1};q)_k}{(q;q)_k(b_1;q)_k \cdots (b_r;q)_k} z^k.$$

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Jackson's ${}_6\phi_5$ summation (see [2, Appendix (II.21)]) can be stated as follows:

$$ {}_6\phi_5\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ c,\ q^{-n} \\ a^{1/2},\ -a^{1/2},\ aq/b,\ aq/c,\ aq^{n+1} \end{matrix};q,\ \frac{aq^{n+1}}{bc}\right] = \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}. \tag{2.1} $$

To prove Theorem 1.1, we first establish the following result.

THEOREM 2.2. *Let $n \equiv 1 \pmod 4$ be an integer greater than 1. Let $0 \leqslant s \leqslant (n-1)/4$ and let a be an indeterminate. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$ \sum_{k=s}^{(n-1)/4} [8k+1] \frac{(aq;q^4)_k(q/a;q^4)_k(q;q^4)_{k-s}(q;q^4)_{k+s}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}} q^{2k} $$

$$ \equiv [n+4s] \frac{(aq;q^4)_s(q/a;q^4)_s(q;q^4)_{(n-1)/4+s}(q^2;q^4)_{(n-1)/4-s}}{(aq^4;q^4)_s(q^4/a;q^4)_s(q^4;q^4)_{(n-1)/4+s}(q;q^4)_{(n-1)/4-s}} q^{3s+(1-n)/4}. \tag{2.2} $$

PROOF. For $a = q^{-n}$ or $a = q^n$, the left-hand side of (2.2) is equal to

$$ \sum_{k=s}^{(n-1)/4} [8k+1] \frac{(q^{1-n};q^4)_k(q^{1+n};q^4)_k(q;q^4)_{k-s}(q;q^4)_{k+s}}{(q^{4-n};q^4)_k(q^{4+n};q^4)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}} q^{2k} $$

$$ = \sum_{k=0}^{(n-1)/4-s} [8k+8s+1] \frac{(q^{1-n};q^4)_{k+s}(q^{1+n};q^4)_{k+s}(q;q^4)_k(q;q^4)_{k+2s}}{(q^{4-n};q^4)_{k+s}(q^{4+n};q^4)_{k+s}(q^4;q^4)_k(q^4;q^4)_{k+2s}} q^{2k+2s} $$

$$ = [8s+1] \frac{(q^{1-n};q^4)_s(q^{1+n};q^4)_s(q;q^4)_{2s}}{(q^{4-n};q^4)_s(q^{4+n};q^4)_s(q^4;q^4)_{2s}} q^{2s} $$

$$ \times {}_6\phi_5\left[\begin{matrix} q^{1+8s}, & q^{\frac{9}{2}+4s}, & -q^{\frac{9}{2}+4s}, & q, & q^{1+n+4s}, & q^{1-n+4s} \\ & q^{\frac{1}{2}+4s}, & -q^{\frac{1}{2}+4s}, & q^{4+8s}, & q^{4-n+4s}, & q^{4+n+4s} \end{matrix};q^4,q^2\right]. \tag{2.3} $$

Letting $q \mapsto q^4$, $a = q^{1+8s}$, $b = q$, $c = q^{1+n+4s}$ and $n \mapsto (n-1)/4 - s$ in (2.1), one sees that the right-hand side of (2.3) can be simplified as

$$ q^{2s}[8s+1] \frac{(q^{1-n};q^4)_s(q^{1+n};q^4)_s(q;q^4)_{2s}}{(q^{4-n};q^4)_s(q^{4+n};q^4)_s(q^4;q^4)_{2s}} \frac{(q^{5+8s};q^4)_{(n-1)/4-s}(q^{3-n+4s};q^4)_{(n-1)/4-s}}{(q^{4+8s};q^4)_{(n-1)/4-s}(q^{4-n+4s};q^4)_{(n-1)/4-s}} $$

$$ = [n+4s] \frac{(q^{1-n};q^4)_s(q^{1+n};q^4)_s}{(q^{4-n};q^4)_s(q^{4+n};q^4)_s} \frac{(q;q^4)_{(n-1)/4+s}(q^{3-n+4s};q^4)_{(n-1)/4-s}}{(q^4;q^4)_{(n-1)/4+s}(q^{4-n+4s};q^4)_{(n-1)/4-s}} q^{2s} $$

$$ = [n+4s] \frac{(q^{1-n};q^4)_s(q^{1+n};q^4)_s}{(q^{4-n};q^4)_s(q^{4+n};q^4)_s} \frac{(q;q^4)_{(n-1)/4+s}(q^2;q^4)_{(n-1)/4-s}}{(q^4;q^4)_{(n-1)/4+s}(q;q^4)_{(n-1)/4-s}} q^{3s+(1-n)/4}. $$

Thus, we have proved that (2.2) is true modulo $1 - aq^n$ and $a - q^n$.

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Since $n \equiv 1 \pmod 4$, letting $d = 4$ and $m = (n-1)/4$ in Lemma 2.1, we obtain

$$\frac{(aq;q^4)_{m-k}}{(q^4/a;q^d)_{m-k}} \equiv (-a)^{m-2k} \frac{(aq;q^4)_k}{(q^4/a;q^d)_k} q^{m(2m-1)+3k} \pmod{\Phi_n(q)} \tag{2.4}$$

for $0 \leqslant k \leqslant m$. Using this q-congruence, we can easily verify the following congruence, for $m = (n-1)/4$ and $s \leqslant k \leqslant m-s$,

$$[8(m-k)+1] \frac{(aq;q^4)_{m-k}(q/a;q^4)_{m-k}(q;q^4)_{m-k-s}(q;q^4)_{m-k+s}}{(aq^4;q^4)_{m-k}(q^4/a;q^4)_{m-k}(q^4;q^4)_{m-k-s}(q^4;q^4)_{m-k+s}} q^{2m-2k}$$
$$\equiv -[8k+1] \frac{(aq;q^4)_k(q/a;q^4)_k(q;q^4)_{k-s}(q;q^4)_{k+s}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}} q^{2k} \pmod{\Phi_n(q)}. \tag{2.5}$$

Moreover, for $(n-1)/4 - s < k \leqslant (n-1)/4$, the summand indexed k on the left-hand side of (2.2) is congruent to 0 modulo $\Phi_n(q)$ because $k + s > (n-1)/4$ and $(q;q^4)_{k+s}$ in the numerator incorporates the factor $1 - q^n$. This means that the left-hand side of (2.2) is congruent to 0 modulo $\Phi_n(q)$. Since

$$[n+4s](q;q^4)_{(n-1)/4+s} = [n](q;q^4)_{(n-1)/4}(q^{n+4};q^4)_s \equiv 0 \pmod{\Phi_n(q)}$$

for $n > 1$, we conclude that (2.2) is also true modulo $\Phi_n(q)$. Noting that the polynomials $1 - aq^n$, $a - q^n$ and $\Phi_n(q)$ are pairwise relatively prime, we complete the proof of the theorem. □

Proof of Theorem 1.1. For $a = 1$, the denominators on both sides of (2.2) are relatively prime to $\Phi_n(q)$. Moreover, when $a = 1$ the polynomial $(1-aq^n)(a-q^n) = (1-q^n)^2$ contains the factor $\Phi_n(q)^2$. Therefore, putting $a = 1$ in (2.2), we obtain the desired q-supercongruence (1.4). □

3. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We first establish the following parametric generalisation of Theorem 1.2.

Theorem 3.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Let $0 \leqslant s \leqslant (n-3)/4$ and let a be an indeterminate. Then, modulo $\Phi_n(q)(1-aq^{3n})(a-q^{3n})$,*

$$\sum_{k=s}^{(3n-1)/4} [8k+1] \frac{(aq;q^4)_k(q/a;q^4)_k(q;q^4)_{k-s}(q;q^4)_{k+s}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_{k-s}(q^4;q^4)_{k+s}} q^{2k}$$
$$\equiv [n+4s] \frac{(aq;q^4)_s(q/a;q^4)_s(q;q^4)_{(3n-1)/4+s}(q^2;q^4)_{(3n-1)/4-s}}{(aq^4;q^4)_s(q^4/a;q^4)_s(q^4;q^4)_{(3n-1)/4+s}(q;q^4)_{(3n-1)/4-s}} q^{3s+(1-3n)/4}. \tag{3.1}$$

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PROOF. For $a = q^{-3n}$ or $a = q^{3n}$, the left-hand side of (3.1) is equal to

$$\sum_{k=s}^{(3n-1)/4} [8k+1] \frac{(q^{1-3n};q^4)_k (q^{1+3n};q^4)_k (q;q^4)_{k-s} (q;q^4)_{k+s}}{(q^{4-3n};q^4)_k (q^{4+3n};q^4)_k (q^4;q^4)_{k-s} (q^4;q^4)_{k+s}} q^{2k}$$

$$= [8s+1] \frac{(q^{1-3n};q^4)_s (q^{1+3n};q^4)_s (q;q^4)_{2s}}{(q^{4-3n};q^4)_s (q^{4+3n};q^4)_s (q^4;q^4)_{2s}} q^{2s}$$

$$\times {}_6\phi_5 \left[\begin{matrix} q^{1+8s}, & q^{\frac{9}{2}+4s}, & -q^{\frac{9}{2}+4s}, & q, & q^{1+3n+4s}, & q^{1-3n+4s} \\ & q^{\frac{1}{2}+4s}, & -q^{\frac{1}{2}+4s}, & q^{4+8s}, & q^{4-3n+4s}, & q^{4+3n+4s} \end{matrix} ; q^4, q^2 \right]. \tag{3.2}$$

Letting $q \mapsto q^4$, $a = q^{1+8s}$, $b = q$, $c = q^{1+3n+4s}$ and $n \mapsto (3n-1)/4 - s$ in (2.1), one sees that the right-hand side of (3.2) may be written as

$$q^{2s}[8s+1] \frac{(q^{1-3n};q^4)_s (q^{1+3n};q^4)_s (q;q^4)_{2s}}{(q^{4-3n};q^4)_s (q^{4+3n};q^4)_s (q^4;q^4)_{2s}} \frac{(q^{5+8s};q^4)_{(3n-1)/4-s} (q^{3-3n+4s};q^4)_{(3n-1)/4-s}}{(q^{4+8s};q^4)_{(3n-1)/4-s} (q^{4-3n+4s};q^4)_{(3n-1)/4-s}}$$

$$= [3n+4s] \frac{(q^{1-3n};q^4)_s (q^{1+3n};q^4)_s}{(q^{4-3n};q^4)_s (q^{4+3n};q^4)_s} \frac{(q;q^4)_{(3n-1)/4+s} (q^2;q^4)_{(3n-1)/4-s}}{(q^4;q^4)_{(3n-1)/4+s} (q;q^4)_{(3n-1)/4-s}} q^{3s+(1-3n)/4}.$$

This proves that (2.2) holds modulo $1 - aq^{3n}$ and $a - q^{3n}$.

Since $n \equiv 3 \pmod 4$, letting $d = 4$ and $m = (3n-1)/4$ in Lemma 2.1, we again obtain (2.4) for $0 \leqslant k \leqslant m$. Applying this q-congruence, we can check (2.5) for $m = (3n-1)/4$ and $s \leqslant k \leqslant m - s$. For $(3n-1)/4 - s < k \leqslant (3n-1)/4$, the summand indexed k on the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q)$ because $k + s > (3n-1)/4$ and $(q;q^4)_{k+s}$ has the factor $1 - q^{3n}$. This implies that the left-hand side of (3.1) is congruent to 0 modulo $\Phi_n(q)$. Since

$$[3n+4s](q;q^4)_{(3n-1)/4+s} = [3n](q;q^4)_{(3n-1)/4}(q^{3n+4};q^4)_s \equiv 0 \pmod{\Phi_n(q)},$$

we conclude that (3.1) is also true modulo $\Phi_n(q)$. □

PROOF OF THEOREM 1.1. When $a = 1$, the polynomial $(1 - aq^{3n})(a - q^{3n}) = (1 - q^{3n})^2$ contains the factor $\Phi_n(q)^2$. Thus, letting $a = 1$ in (2.2), we get the q-supercongruence (1.7). □

4. An open problem

We believe that the following conjecture is true.

CONJECTURE 4.1. The q-supercongruences (1.4) and (1.7) are also true modulo $[n]\Phi_n(q)^2$.

The above conjecture is clearly true for $s = 0$ (see [5, 9]). Numerical computation indicates that both sides of (1.4) (or (1.7)) should be congruent to 0 modulo $[n]$. However, it seems difficult to confirm this. The technique of proving a q-congruence

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modulo $[n]$ introduced in [6] does not work here, because of the additional parameter s.

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