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EFFECTIVE PROCEDURE OF VERIFYING STOCHASTIC ORDERING OF SYSTEM LIFETIMES

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Abstract

The Samaniego signature is a relevant tool for studying the performance of a system whose component lifetimes are exchangeable. It is well known that the stochastic ordering of the signatures of two systems implies the same for the respective system lifetimes. We prove that the reverse claim is not true when the component lifetimes are independent and identically distributed. There exist small proportions of systems with stochastically ordered lifetimes whose signatures are not ordered. We present a simple procedure in order to check whether the system lifetimes are stochastically ordered even if their signatures are not comparable.

Keywords: Coherent system; mixed system; Samaniego signature; stochastic order

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Secondary 62G30

1. Introduction

The framework of a coherent system composed of n elements is described by its structure function $\varphi\colon \{0,1\}^n \mapsto \{0,1\}$. It has the Samaniego structural signature $\boldsymbol{s} = (s_1, \dots, s_n) \in \mathcal{S}^n = \{\boldsymbol{s} = (s_1, \dots, s_n) \in \mathbb{R}^n\colon s_i \geq 0,\ i = 1, \dots, n, \sum_{i=1}^n s_i = 1\}$, and the signature coordinates are determined as follows:

$$s_i = \frac{1}{\binom{n}{i-1}} \sum_{\sum_{j=1}^n x_j = n-i+1} \varphi(x_1, \dots x_n) - \frac{1}{\binom{n}{i}} \sum_{\sum_{j=1}^n x_j = n-i} \varphi(x_1, \dots x_n), \qquad i = 1, \dots, n. \tag{1.1}$$

The concept of signature was introduced by Samaniego [11], and (1.1) was presented in Boland [1]. If the component lifetimes $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) with a common continuous distribution function F, supported on some subset of nonnegative half-axis, and $X_{1:n}, \dots, X_{n:n}$ and T denote the respective order statistics and system lifetime, then the Samaniego signature has a simple probabilistic interpretation $s_i = \mathbb{P}(T = X_{i:n})$, $i = 1, \dots, n$. The system lifetime distribution function can be written as

$$F_T(t) = \sum_{i=1}^n s_i \mathbb{P}(X_{i:n} \leq t) = \sum_{i=1}^n s_i G_{i:n}(F(t)), \tag{1.2}$$

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where $G_{i:n}(u) = \sum_{j=i}^{n} B_{j,n}(u)$ denotes the distribution function of the ith order statistic based on i.i.d. standard uniform random variables, and

$$B_{j,n}(u) = \binom{n}{j} u^j (1-u)^{n-j}, \qquad 0 \le u \le 1,\ j = 0, \dots, n,$$

denotes the Bernstein polynomials of degree n. Equation (1.2) was proved by Samaniego [11] in the case of independent and identically continuously distributed component lifetimes. Navarro *et al.* [5] showed that its first part is also valid for dependent exchangeable component lifetimes. In this paper we merely focus on the i.i.d. case. Though, we admit that the component lifetime distribution can be discontinuous. Under the assumption that the lifetimes are i.i.d. possibly discontinuous, (1.2) remains valid.

The system whose lifetime is $X_{k:n}$ is called the k-out-of-n system (more precisely, k-out-of-n: F system, but later on we drop F from the system name). Due to (1.2), the distribution of T is identical with the distribution of randomly chosen k-out-of-n system for $k = 1, \dots, n$, when the choice probability is s_k. Motivated by this observation, Boland and Samaniego [2] introduced the notion of a mixed system of size n, which is just the randomly selected k-out-of-n system with probability s_k when the probability vector $\boldsymbol{s} = (s_1, \dots, s_n)$ is arbitrarily chosen from the simplex $\mathcal{S}^n$.

It is well known (see Kochar *et al.* [3]) that stochastic ordering of Samaniego signatures of mixed (or coherent) systems $\boldsymbol{s}^1 = (s_1^1, \dots, s_n^1) \preceq_{\mathrm{st}} \boldsymbol{s}^2 = (s_1^2, \dots, s_n^2)$, which is defined as

$$\sum_{j=1}^{i} s_j^1 \ge \sum_{j=1}^{i} s_j^2, \qquad i = 1, \dots, n-1,$$

implies the st-ordering for the respective system lifetimes $T_1 \preceq_{\mathrm{st}} T_2$ which means that

$$F_{T_1}(t) = \sum_{i=1}^{n} s_i^1 G_{i:n}(F(t)) \ge F_{T_2}(t) = \sum_{i=1}^{n} s_i^2 G_{i:n}(F(t)), \qquad t \ge 0,$$

under the condition that component lifetimes $X_1^1, \dots, X_n^1$ and $X_1^2, \dots, X_n^2$ are i.i.d. with arbitrary common distribution function F (in fact, the respective implication is true for exchangeable lifetimes as well; see [5]).

Similar properties hold for the hazard rate (hr) and likelihood ratio (lr) orders; see [3]. Navarro *et al.* [5] proved that these properties can be extended to systems with exchangeable components when we assume that the respective order statistics are hr or lr ordered. In addition, Navarro and Rubio [7] showed that the ordering property of the signatures is a necessary and sufficient condition for obtaining the respective ordering of the system lifetimes for any joint exchangeable distribution of the component lifetimes such that the order statistics are accordingly ordered. Recently, Navarro [4] proved that in the case of coherent systems composed of items with i.i.d. lifetimes, there exist systems with hr-ordered (lr-ordered) lifetimes whose signatures are not hr-ordered (lr-ordered) for any component lifetime distribution. Although it was shown there that all the st-ordered coherent systems built of at most four components with i.i.d. lifetimes have st-ordered signatures, we prove in this paper that the above nonequivalence of the system signature and lifetime orderings occurs for the stochastic order as well. Namely, we show that there are pairs of mixed systems with at least four components and those of coherent ones with at least five components whose lifetimes are stochastically ordered in the case of i.i.d. component lifetimes, but the respective signatures are not ordered.

In order to explain the problem more clearly, we formulate three claims:

(A) $s_1 \leq_{\text{st}} s_2$;

(B) $T_1 \leq_{\text{st}} T_2$ for all product joint distributions $\boldsymbol{F}$ with identical factors (which means that the component lifetimes are i.i.d.);

(C) $T_1 \leq_{\text{st}} T_2$ for all exchangeable joint distribution $\boldsymbol{F}$ (which means that the component lifetimes are exchangeable),

where T_1 and T_2 denote the lifetimes of two coherent systems with an identical joint distribution $\boldsymbol{F}$ of their components, and $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ denote the respective signatures.

It is obvious that (C) is a stronger condition than (B), i.e. (C) implies (B), since relation $T_1 \leq_{\text{st}} T_2$ has to be satisfied by a larger class of joint distributions in (C) than in (B). It is also known that (A) implies (B) (see [3]) and that (A) implies (C) (see [5]). Moreover, in [7] it was also proved that (C) implies (A), which means that claims (A) and (C) are equivalent. In this paper we show that (B) does not imply (A), i.e. there are systems such that $T_1 \leq_{\text{st}} T_2$ for the i.i.d. case (claim (B)), but such that s_1 and s_2 are not ordered (not (A)). It follows that (B) does not imply (C) either, i.e. there are pairs of systems whose lifetimes are stochastically ordered when the component lifetimes are i.i.d., but there are exchangeable component lifetime distributions $\boldsymbol{F}$ such that the system lifetimes are not ordered. Moreover, we present a simple method of verifying stochastic ordering of system lifetimes in the i.i.d. case even if the respective signatures are not ordered.

The rest of the paper is organized as follows. In Section 2 we first prove that for every $n \geq 4$ there exist pairs of n-component mixed systems with st-ordered lifetimes whose component lifetimes are i.i.d., and whose signatures are not st-ordered. We present all 24 such pairs of coherent systems with five components. An example of two coherent six-component systems which share this property is also presented. Finally, we describe an effective procedure of checking if two systems composed of elements with i.i.d. lifetimes are st-ordered for any parent distributions of component lifetimes and for any system structures. Some conclusions and open questions for future research are presented in Section 3. The proof of our main result can be found in Appendix A and the tables containing all the pairs of ordered coherent systems with five components having unordered signatures are presented in Section 2.

2. Main results

The main result of this paper is contained in the following proposition.

Proposition 2.1. *For every $n \geq 4$, there exist signatures $\boldsymbol{s}_n^1, \boldsymbol{s}_n^2 \in \mathcal{S}^n$ such that $\boldsymbol{s}_n^1 \not\leq_{\text{st}} \boldsymbol{s}_n^2$, but we have $T_1 \preceq_{\text{st}} T_2$ for the respective mixed systems with i.i.d. components and any arbitrary baseline distribution function F.*

Proof. The proof can be found in Appendix A. □

Remark 2.1. Choosing arbitrary $\boldsymbol{c}_n \in \mathbb{R}_+^n$ and $0 < \varepsilon \leq \varepsilon_n$ for various $n \geq 4$ in the proof of Proposition 2.1, we can construct large families of stochastically unordered signatures $\boldsymbol{s}_n^1, \boldsymbol{s}_n^2 \in \mathcal{S}^n$ which generate ordered system lifetimes. If we assume that vectors $\boldsymbol{c}_n$, $n \geq 4$, are symmetric, i.e. relations $c_{i,n} = c_{n+1-i,n}$, $i = 1, \ldots, n$, hold, then the resulting signatures satisfy $s_{i,n}^2 = s_{n+1-i,n}^1$, $i = 1, \ldots, n$, which means that they represent mutually dual systems.

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The simplest choice of this type is the vector $\boldsymbol{c}_n = (0, \dots, 0)$ which generates signatures

$$\boldsymbol{s}^1_{2m} = \Bigg(\frac{1}{2+\varepsilon}, 0, \dots, 0, \underbrace{\frac{1+\varepsilon}{2+\varepsilon}}_{m+1}, 0, \dots, 0\Bigg), \qquad \boldsymbol{s}^2_{2m} = \Bigg(0, \dots, 0, \underbrace{\frac{1+\varepsilon}{2+\varepsilon}}_{m}, 0, \dots, 0, \frac{1}{2+\varepsilon}\Bigg),$$

and

$$\boldsymbol{s}^1_{2m+1} = \Bigg(\frac{1}{2+\varepsilon}, 0, \dots, 0, \underbrace{\frac{1+\varepsilon}{2+\varepsilon}}_{m+2}, 0, \dots, 0\Bigg), \qquad \boldsymbol{s}^2_{2m+1} = \Bigg(0, \dots, \underbrace{\frac{1+\varepsilon}{2+\varepsilon}}_{m}, 0, \dots, 0, \frac{1}{2+\varepsilon}\Bigg)$$

for $m = 2, 3, \dots$, where the subscript under the fraction $(1+\varepsilon)/(2+\varepsilon)$ describes its location in the respective sequences. Taking maximal $\varepsilon = \varepsilon_n$ together with $\boldsymbol{c}_n = (0, \dots, 0)$, we obtain maximal violation of the signature ordering.

Remark 2.2. For the mixed systems with two and three components, relations $T_1 \preceq_{\mathrm{st}} T_2$ and $\boldsymbol{s}^1 \preceq_{\mathrm{st}} \boldsymbol{s}^2$ are equivalent in the i.i.d. case. For $n = 2$, the first relation holds if and only if $(s^1_1 - s^2_1)B_{1,2}(u) \ge 0$, $0 \le u \le 1$, i.e. when $s^1_1 \ge s^2_1$. For $n = 3$, $T_1 \preceq_{\mathrm{st}} T_2$ means that $(s^1_1 - s^2_1)B_{1,3}(u) + (s^1_1 + s^1_2 - s^2_1 - s^2_2)B_{2,3}(u) \ge 0$, $0 \le u \le 1$, which simplifies to $(s^1_1 - s^2_1)(1-u) + (s^1_1 + s^1_2 - s^2_1 - s^2_2)u \ge 0$. A linear function is nonnegative on an interval if and only if it is so at the interval end-points. This is equivalent to $s^1_1 \ge s^2_1$ and $s^1_1 + s^1_2 \ge s^2_1 + s^2_2$ which just defines the ordering of the signatures.

Remark 2.3. Note that the signatures presented in Remark 2.1 cannot correspond to coherent systems since they have internal zeros; see [8] and [9]. An important question is whether there exist coherent systems which possess the properties described in Proposition 2.1. Navarro [4] showed that this is impossible for the systems of sizes $n \le 4$. We study the $n = 5$ case in the next example.

Example 2.1. Navarro and Rubio [6] proved that there exist 180 coherent systems with five components and computed their signatures. So there are 16,110 pairs of coherent systems with five components. Among these pairs there exist 24 pairs for which the system lifetimes in the i.i.d. case are stochastically ordered, and the corresponding signatures are not. This amounts to 0.149% of the total number. In Tables 1 and 2 we present these exceptional pairs with their corresponding signatures. The pairs are constructed of 24 systems. Note that system numbered by $i + 12$ is dual to the system with number i, $i = 1, \dots, 12$. The systems are represented by their minimal path sets. For brevity, the set $\{i_1, \dots, i_m\}$ is written as $i_1 \cdots i_m$. In both tables, the pairs of systems with unordered signatures and ordered lifetimes are gathered in two columns which are separated by a horizontal line. The system in the ith row above the line has an st-smaller lifetime than that in the ith row below the line. For example, $T_1 \preceq_{\mathrm{st}} T_6$ (see Table 1) and $T_{18} \preceq_{\mathrm{st}} T_{13}$ (see Table 2). We easily see that $\boldsymbol{s}_1 \not\preceq_{\mathrm{st}} \boldsymbol{s}_6$ and $\boldsymbol{s}_{18} \preceq_{\mathrm{st}} \boldsymbol{s}_{13}$."

The lack of order between the signatures is apparent. In Figure 1, we present the graphs of functions $F_{T_i} - F_{T_j}$ for the systems with numbers $i = 1, \dots, 5$ and $j = 6, \dots 12$ appearing in Table 1. We assume here that the single component lifetime distribution is standard uniform.

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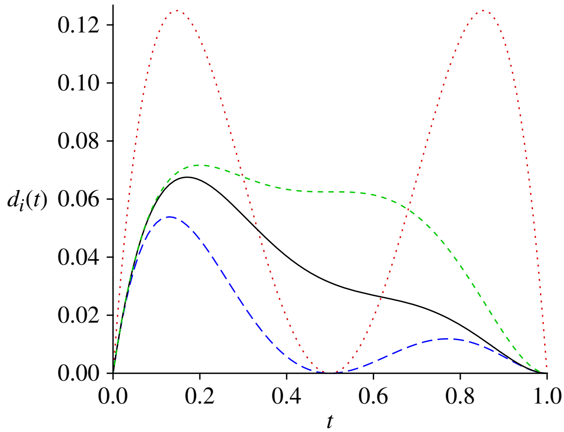

FIGURE 1: Plots of the difference $d_i(t)$ for $i = 1, 2, 3, 4$ (*solid*, *long dash*, *short dash*, *dotted*, respectively) for the systems studied in Example 2.1.

It appears that the functions are represented by three curves only, i.e.

$$F_{T_i}(t) - F_{T_j}(t) = \begin{cases} d_1(t), & (i, j) = (1, 6), (2, 6), (3, 7), (3, 8), (4, 9), (4, 10), (4, 11), (5, 12), \\ d_2(t), & (i, j) = (3, 6), (4, 7), (4, 8), \\ d_3(t), & (i, j) = (4, 12), \end{cases}$$

where

$$d_1(t) = 0.2B_{1,5}(t) - 0.1B_{2,5}(t) + 0.1B_{3,5}(t),$$
$$d_2(t) = 0.2B_{1,5}(t) - 0.2B_{2,5}(t) + 0.1B_{3,5}(t),$$
$$d_3(t) = 0.2B_{1,5}(t) - 0.1B_{2,5}(t) + 0.2B_{3,5}(t).$$

They are compared with the differences of lifetime distribution functions for mutually dual mixed systems with signatures $\boldsymbol{s}_5 = (0.4, 0, 0, 0.6, 0)$ and $\boldsymbol{s}_5^{\mathrm{D}} = (0, 0.6, 0, 0, 0.4)$ constructed in Remark 2.1 with $\boldsymbol{c}_5 = (0, \dots, 0)$ and $\varepsilon = \varepsilon_5 = 0.5$ determined in the proof of Proposition 2.1. The function has the form

$$F_T(t) - F_{T^{\mathrm{D}}}(t) = d_4(t) = 0.4B_{1,5}(t) = -0.2B_{2,5}(t) - 0.2B_{3,5}(t) + 0.4B_{4,5}(t).$$

We see that all $d_i(t)$, $i = 1, \dots, 4$, are nonnegative on the standard unit interval.

For the systems with numbers $18 \le i \le 24$ and $13 \le j \le 17$, we have

$$F_{T_i}(t) - F_{T_j}(t) = F_{T_{i-12}}(1 - t) - F_{T_{j-12}}(1 - t),$$

which means that the graphs for the systems from Table 2 arise from the graphs of their dual counterparts by folding them about the straight line $t = 0.5$. These functions are obviously nonnegative on $[0, 1]$.

Example 2.1 justifies our strong belief that a small proportion of the pairs of coherent systems with ordered lifetimes in the i.i.d. case and unordered signatures can be found in higher dimensions. It is a challenging task to detect them, though. In Example 2.2 we present a pair of coherent systems of size 6 which possess these properties.

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TABLE 1: Systems composed of five components with st-unordered signatures and st-ordered lifetimes when component lifetimes are i.i.d.

	Number	System	Signature
	1	123, 124, 135, 145	(0.2,0.4,0.4,0.0,0.0)
	2	123, 124, 125, 134	(0.2,0.4,0.4,0.0,0.0)
	3	123, 124, 125, 134, 135	(0.2,0.3,0.5,0.0,0.0)
	3	123, 124, 125, 134, 135	(0.2,0.3,0.5,0.0,0.0)
	3	123, 124, 125, 134, 135	(0.2,0.3,0.5,0.0,0.0)
i	4	123, 124, 125, 134, 135, 145	(0.2,0.2,0.6,0.0,0.0)
	4	123, 124, 125, 134, 135, 145	(0.2,0.2,0.6,0.0,0.0)
	4	123, 124, 125, 134, 135, 145	(0.2,0.2,0.6,0.0,0.0)
	4	123, 124, 125, 134, 135, 145	(0.2,0.2,0.6,0.0,0.0)
	4	123, 124, 125, 134, 135, 145	(0.2,0.2,0.6,0.0,0.0)
	4	123, 124, 125, 134, 135, 145	(0.2,0.2,0.6,0.0,0.0)
	5	12, 134, 135, 145	(0.2,0.2,0.5,0.1,0.0)
	6	12, 1345, 2345	(0.0,0.7,0.2,0.1,0.0)
	6	12, 1345, 2345	(0.0,0.7,0.2,0.1,0.0)
	6	12, 1345, 2345	(0.0,0.7,0.2,0.1,0.0)
	7	12, 134, 2345	(0.0,0.6,0.3,0.1,0.0)
	8	12, 345	(0.0,0.6,0.3,0.1,0.0)
j	7	12, 134, 2345	(0.0,0.6,0.3,0.1,0.0)
	8	12, 345	(0.0,0.6,0.3,0.1,0.0)
	9	12, 134, 135, 2345	(0.0,0.5,0.4,0.1,0.0)
	10	12, 134, 345	(0.0,0.5,0.4,0.1,0.0)
	11	12, 134, 235	(0.0,0.5,0.4,0.1,0.0)
	12	12, 13, 2345	(0.0,0.5,0.3,0.2,0.0)
	12	12, 13, 2345	(0.0,0.5,0.3,0.2,0.0)

Example 2.2. Consider the coherent system with six components and the following minimal path sets: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3, 4\}$, $\{2, 5, 6\}$, and $\{1, 4, 5, 6\}$. Its signature is $\boldsymbol{s} = (0, \frac{4}{60}, \frac{29}{60}, \frac{19}{60}, \frac{8}{60}, 0)$. The dual system has signature $\boldsymbol{s}^{\mathrm{D}} = (0, \frac{8}{60}, \frac{19}{60}, \frac{29}{60}, \frac{4}{60}, 0)$. The signatures are obviously not st-ordered, but $T^{\mathrm{D}} \preceq_{\mathrm{st}} T$ for all F, since $F_{T^{\mathrm{D}}}(t) - F_T(t) = F^2(t)[1-F(t)]^2[2F(t)-1]^2 \geq 0$, $t \geq 0$. Calculations verifying the results are left to the reader.

Simple observations of the following proposition allow us to compare system lifetimes in the st-order when the respective signatures are not stochastically ordered.

Proposition 2.2. *Consider two systems with i.i.d. component lifetimes. If the distribution functions G_1, G_2 of the system lifetimes in the case of standard uniform component lifetimes satisfy $G_1(u) \geq G_2(u)$ for every $u \in \mathcal{U} \subseteq (0, 1)$, then $F_1(t) \geq F_2(t)$ for all $t > 0$ when all the values of the common component distribution F are contained in $\mathcal{U}$. If, in particular, $\mathcal{U} = (0, 1)$ then $F_1(t) \geq F_2(t)$ for arbitrary F and all $t > 0$.*

Proof. Suppose that the systems have signatures $\boldsymbol{s}^1 = (s_1^1, \ldots, s_n^1)$ and $\boldsymbol{s}^2 = (s_1^2, \ldots, s_n^2)$. From (1.2), we have

$$F_j(t) = \sum_{i=1}^{n} s_i^j G_{i:n}(F(t)) = G_j(F(t)), \qquad j = 1, 2.$$

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TABLE 2: Systems composed of five components with st-unordered signatures and st-ordered lifetimes when component lifetimes are i.i.d.

	Number	System	Signature
i	18	12, 13, 14, 15, 23, 24, 25	(0.0,0.1,0.2,0.7,0.0)
	18	12, 13, 14, 15, 23, 24, 25	(0.0,0.1,0.2,0.7,0.0)
	18	12, 13, 14, 15, 23, 24, 25	(0.0,0.1,0.2,0.7,0.0)
	19	12, 13, 14, 23, 24, 25	(0.0,0.1,0.3,0.6,0.0)
	20	13, 14, 15, 23, 24, 25	(0.0,0.1,0.3,0.6,0.0)
	19	12, 13, 14, 23, 24, 25	(0.0,0.1,0.3,0.6,0.0)
	20	13, 14, 15, 23, 24, 25	(0.0,0.1,0.3,0.6,0.0)
	21	12, 13, 14, 15, 23, 245	(0.0,0.1,0.4,0.5,0.0)
	22	13, 14, 15, 23, 24	(0.0,0.1,0.4,0.5,0.0)
	23	12, 13, 23, 24, 15	(0.0,0.1,0.4,0.5,0.0)
	24	12, 13, 14, 15, 23	(0.0,0.2,0.3,0.5,0.0)
	24	12, 13, 14, 15, 23	(0.0,0.2,0.3,0.5,0.0)
j	13	1, 25, 34	(0.0,0.0,0.4,0.4,0.2)
	14	1, 23, 24, 345	(0.0,0.0,0.4,0.4,0.2)
	15	1, 23, 245, 345	(0.0,0.0,0.5,0.3,0.2)
	15	1, 23, 245, 345	(0.0,0.0,0.5,0.3,0.2)
	15	1, 23, 245, 345	(0.0,0.0,0.5,0.3,0.2)
	16	1, 234, 245, 345	(0.0,0.0,0.6,0.2,0.2)
	16	1, 234, 245, 345	(0.0,0.0,0.6,0.2,0.2)
	16	1, 234, 245, 345	(0.0,0.0,0.6,0.2,0.2)
	16	1, 234, 245, 345	(0.0,0.0,0.6,0.2,0.2)
	16	1, 234, 245, 345	(0.0,0.0,0.6,0.2,0.2)
	16	1, 234, 245, 345	(0.0,0.0,0.6,0.2,0.2)
	17	1, 234, 235, 245	(0.0,0.1,0.5,0.2,0.2)

If relation $G_1(F(t)) \geq G_2(F(t))$ holds for all possible values of F then clearly $F_1(t) \geq F_2(t)$ for all t. For the latter claim, it suffices to note that the standard unit interval contains all the possible values of any distribution function. □

Remark 2.4. Using the preceding proposition we can construct a procedure which with minimal calculations allows us to stochastically compare lifetimes of arbitrary pairs of systems composed of elements whose lifetimes are i.i.d. with any parent distribution function F. Let T_1 and T_2 denote the lifetimes of the systems, and let $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ denote their respective signatures.

(i) If $\boldsymbol{s}^1 \preceq_{\mathrm{st}} \boldsymbol{s}^2$ ($\boldsymbol{s}^1 \succeq_{\mathrm{st}} \boldsymbol{s}^2$) then $T_1 \preceq_{\mathrm{st}} T_2$ ($T_1 \succeq_{st} T_2$, respectively) for all F.

(ii) If the signatures are not ordered, but $G_1 \preceq_{\mathrm{st}} G_2$ ($G_1 \succeq_{\mathrm{st}} G_2$), then $T_1 \preceq_{\mathrm{st}} T_2$ ($T_1 \succeq_{\mathrm{st}} T_2$, respectively) for all F.

(iii) If G_1 and G_2 are not stochastically ordered then $T_1 \preceq_{\mathrm{st}} T_2$ ($T_1 \succeq_{\mathrm{st}} T_2$, respectively) for all F which all have values in $\mathcal{U} = \{0 \leq u \leq 1 \colon G_1(u) \geq G_2(u)\}$ ($\{0 \leq u \leq 1 \colon G_1(u) \leq G_2(u)\}$, respectively).

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Note that for point (i) knowledge of signatures is merely needed. For (ii) we also calculate G_1 and G_2. In (iii) the component lifetimes are analyzed. Obviously, we may perform an analogous algorithm if we know the system lifetime distributions when the component lifetime distribution is fixed and continuous. However, restricting ourselves to the uniform component lifetime is the most convenient, and consists in comparing two polynomials on (0, 1).

3. Conclusions

This paper was devoted to relations between stochastic ordering of system Samaniego signatures and stochastic ordering of system lifetimes when the component lifetimes are i.i.d. It is commonly known that stochastic ordering of system signatures implies the same for system lifetimes. However, we showed that in order to verify the latter relation it does not suffice to check whether the respective signatures are ordered. We proved that there exist many mixed systems with four or more components which have ordered lifetimes and unordered signatures.

Analysis of purely coherent systems is much more difficult. Navarro [4] checked that all the ordered systems with at most four i.i.d. components have ordered signatures. However, in the present paper we have found 24 pairs of coherent systems of size $n = 5$ whose lifetimes are stochastically ordered, but the system signatures are not.

The procedure described in Remark 2.4 can be used to detect all the ordering relationships between coherent systems with i.i.d. components. This procedure completes the theory of signatures showing how to approach two systems with unordered signatures.

The findings in the paper strongly support the hypothesis that in the classes of coherent systems with five and more components, the families of system pairs with stochastically ordered lifetimes in the i.i.d. case are essentially greater than those with ordered signatures. A challenging open question for future research is how to effectively detect the systems which belong to the difference of these families. The same problem can be considered for the hazard rate and likelihood ratio orders.

Appendix A.

In the proof of Proposition 2.1, we use the well-known variation diminishing property of Bernstein polynomials described in the following lemma.

Lemma A.1. (Rychlik [10, p. 66].) *The number of zeros of a given nonzero linear combination of Bernstein polynomials of a fixed degree*

$$B(u) = \sum_{i=0}^{n} a_i B_{i,n}(u), \qquad 0 < u < 1,$$

is not greater than the number of sign changes in the sequence $a_0, \dots, a_n$. Moreover, the signs of B in the right neighborhood of 0 and the left neighborhood of 1 coincide with the signs of the first and last nonzero elements among $a_0, \dots, a_n$, respectively.

Proof of Proposition 2.1. First we analyze the systems with even sizes $n = 2m$ for $m \geq 2$. For fixed m and some $\varepsilon > 0$, define function $d_{2m,\varepsilon}\colon [0, 1] \mapsto \mathbb{R}$ by means of

$$d_{2m,\varepsilon}(u) = 1 - B_{0,2m}(u) - B_{2m,2m}(u) - (1+\varepsilon)B_{m,2m}(u). \tag{A.1}$$

Note that $d_{2m,\varepsilon}(0) = d_{2m,\varepsilon}(1) = 0$, and

$$d'_{2m,\varepsilon}(u) = 2m[B_{0,2m-1}(u) - B_{2m-1,\,2m-1}(u) - (1+\varepsilon)B_{m-1,\,2m-1}(u) + (1+\varepsilon)B_{m,2m-1}(u)].$$

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From Lemma A.1, $d'_{2m,\varepsilon}$ is either first positive and then negative or consecutively positive, negative, positive, and eventually negative. Observe that (A.1) is symmetric about $\frac{1}{2}$, i.e. $d_{2m,\varepsilon}(u) = d_{2m,\varepsilon}(1-u)$. This has the following consequences. If $d'_{2m,\varepsilon}$ is positive–negative then $d_{2m,\varepsilon}$ is positive on (0,1), and attains its maximum at $\frac{1}{2}$. Otherwise, $d_{2m,\varepsilon}$ has a local minimum at $\frac{1}{2}$, and this may be either nonnegative or negative. Accordingly, the function is either nonnegative on the whole interval [0, 1] or it is nonnegative on some neighborhoods of 0 and 1, and negative in between. The former case holds if and only if

$$d_{2m,\varepsilon}\left(\frac{1}{2}\right) = 1 - 2^{1-2m} - (1+\varepsilon)\binom{2m}{m}2^{-2m} \geq 0,$$

i.e. when $\varepsilon \leq \varepsilon_{2m} = (2^{2m}-2)/\binom{2m}{m} - 1$. We need to check if $\varepsilon_{2m} > 0$ for all $m \geq 2$. We immediately obtain $\varepsilon_4 = \frac{4}{3} > 0$, and further show that $(\varepsilon_{2m})_{m=2}^{\infty}$ is increasing. Indeed, we have

$$\varepsilon_{2m+2} - \varepsilon_{2m} = \frac{4^{m+1}-2}{\binom{2m+2}{m+1}} - \frac{4^m-2}{\binom{2m}{m}} = \frac{4^m+3m+1}{(2m+1)\binom{2m}{m}} > 0,$$

as claimed. Note that due to the Stirling approximation $n! \simeq \sqrt{2\pi n}(n/\mathrm{e})^n$, the sequence tends to ∞ at rate $\varepsilon_{2m} \simeq \sqrt{\pi m}$.

Now take $d_{2m,\varepsilon}(u) \geq 0,\ 0 \leq u \leq 1$, for some $0 < \varepsilon \leq \varepsilon_{2m}$. It has the representation

$$\begin{aligned} d_{2m,\varepsilon}(u) &= \sum_{i=1}^{m-1} B_{i,2m}(u) - \varepsilon B_{m,2m}(u) + \sum_{i=m+1}^{2m-1} B_{i,2m}(u) \\ &= G_{1:2m}(u) - (1+\varepsilon)G_{m:2m}(u) + (1+\varepsilon)G_{m+1:2m}(u) - G_{2m:2m}(u). \end{aligned}$$

For an arbitrarily chosen vector $\boldsymbol{c}_{2m} = (c_{1,2m}, \dots, c_{2m,2m}) \in \mathbb{R}^{2m}$ with nonnegative coordinates, we define vectors $\boldsymbol{s}^1_{2m} = (s^1_{1,2m}, \dots, s^1_{2m,2m})$ and $\boldsymbol{s}^2_{2m} = (s^2_{1,2m}, \dots, s^2_{2m,2m})$ as

$$s^1_{i,2m} = \frac{1}{\sum_{j=1}^{2m} c_{j,2m} + 2 + \varepsilon} \times \begin{cases} c_{i,2m} + 1, & i = 1, \\ c_{i,2m} + 1 + \varepsilon, & i = m+1, \\ c_{i,2m} & \text{otherwise}, \end{cases} \qquad 1 \leq i \leq 2m,$$

and

$$s^2_{i,2m} = \frac{1}{\sum_{j=1}^{2m} c_{j,2m} + 2 + \varepsilon} \times \begin{cases} c_{i,2m} + 1 + \varepsilon, & i = m, \\ c_{i,2m} + 1, & i = 2m, \\ c_{i,2m} & \text{otherwise}, \end{cases} \qquad 1 \leq i \leq 2m.$$

It is easy to verify that $\boldsymbol{s}^1_{2m}, \boldsymbol{s}^2_{2m} \in \mathcal{S}^{2m}$.

Let T_1 and T_2 denote the lifetimes of mixed systems with signatures $\boldsymbol{s}^1_{2m}$ and $\boldsymbol{s}^2_{2m}$, respectively, whose component lifetimes are i.i.d. with some common distribution function F. Their distribution functions amount to $F_{T_j}(t) = \sum_{i=1}^{2m} s^j_{i,2m} G_{i:2m}(F(t)),\ j = 1, 2$. By the definition,

$$F_{T_1}(t) - F_{T_2}(t) = \frac{d_{2m,\varepsilon}(F(t))}{\sum_{j=1}^{2m} c_{j,2m} + 2 + \varepsilon} \geq 0 \quad \text{for all } t \geq 0,$$

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which means that $T_1 \preceq_{\rm st} T_2$. However, $\boldsymbol{s}^1_{2m} \not\preceq_{\rm st} \boldsymbol{s}^2_{2m}$ since

$$\sum_{i=1}^m s^1_{i,2m} = \frac{\sum_{i=1}^m c_{i,2m} + 1}{\sum_{j=1}^{2m} c_{j,2m} + 2 + \varepsilon} < \sum_{i=1}^m s^2_{i,2m} = \frac{\sum_{i=1}^m c_{i,2m} + 1 + \varepsilon}{\sum_{j=1}^{2m} c_{j,2m} + 2 + \varepsilon}.$$

In a similar way, we treat the odd cases with $n = 2m + 1$ for $m \ge 2$. For positive ε, set

$$d_{2m+1,\varepsilon}(u) = 1 - B_{0,2m+1}(u) - B_{2m+1,\,2m+1}(u) - (1+\varepsilon)[B_{m,2m+1}(u) + B_{m+1,\,2m+1}(u)].$$

The function is symmetric about $\frac{1}{2}$, vanishes at 0 and 1, and has derivative

$$d'_{2m+1,\varepsilon}(u) = (2m+1)[B_{0,2m}(u) - B_{2m,2m}(u) - (1+\varepsilon)B_{m-1,2m}(u) + (1+\varepsilon)B_{m+1,2m}(u)].$$

Mimicking the arguments of the previous part of the proof, we arrive at the conclusion that $d_{2m+1,\varepsilon}$ is nonnegative on [0, 1] if and only if

$$d_{2m+1,\varepsilon}\Big(\frac{1}{2}\Big) = 1 - 2^{-2m} - (1+\varepsilon)\binom{2m+1}{m}2^{-2m} \ge 0$$

or, equivalently, if and only if $\varepsilon \le \varepsilon_{2m+1} = (2^{2m} - 1)/\binom{2m+1}{m} - 1$. The upper restrictions are positive for all $m \ge 2$, since $\varepsilon_5 = \frac{1}{2}$ and the sequence $(\varepsilon_{2m+1})_{m=2}^{\infty}$ is increasing. We check its monotonicity by means of

$$\varepsilon_{2m+3} - \varepsilon_{2m+1} = \frac{4^{m+1} - 1}{\binom{2m+3}{m+1}} - \frac{4^m - 1}{\binom{2m+1}{m}} = \frac{2 \cdot 4^m + 3m + 4}{(4m+6)\binom{2m+1}{m}} > 0.$$

By the Stirling formula, the sequence approaches ∞ at rate $\varepsilon_{2m+1} \simeq \frac{1}{2}\sqrt{\pi m}$.

For fixed $0 < \varepsilon \le \varepsilon_{2m+1}$, the nonnegative function $d_{2m+1,\varepsilon}$ has the form

$$d_{2m+1,\varepsilon}(u) = G_{1:2m+1}(u) - (1+\varepsilon)G_{m:2m+1}(u) + (1+\varepsilon)G_{m+2:2m+1}(u) - G_{2m+1:2m+1}(u).$$

Given a nonnegative sequence $\boldsymbol{c}_{2m+1} = (c_{1,2m+1}, \dots, c_{2m+1,\,2m+1})$, we construct signatures $\boldsymbol{s}^1_{2m+1}, \boldsymbol{s}^2_{2m+1} \in \mathcal{S}^{2m+1}$ in the following way, i.e.

$$s^1_{i,2m+1} = \frac{1}{\sum_{j=1}^{2m+1} c_{j,2m+1} + 2 + \varepsilon} \times \begin{cases} c_{i,2m+1} + 1, & i = 1, \\ c_{i,2m+1} + 1 + \varepsilon, & i = m+2, \\ c_{i,2m+1} & \text{otherwise,} \end{cases} \qquad 1 \le i \le 2m+1,$$

and

$$s^2_{i,2m+1} = \frac{1}{\sum_{j=1}^{2m+1} c_{j,2m+1} + 2 + \varepsilon} \times \begin{cases} c_{i,2m+1} + 1 + \varepsilon, & i = m, \\ c_{i,2m+1} + 1, & i = 2m+1, \\ c_{i,2m+1} & \text{otherwise,} \end{cases} \qquad 1 \le i \le 2m+1.$$

It is easy to check that the distribution functions $F_{T_j}(t) = \sum_{i=1}^{2m+1} s^j_{i,2m+1} G_{i:2m+1}(F(t))$, $j = 1, 2$, for arbitrary F satisfy

$$F_{T_1}(t) - F_{T_2}(t) = \frac{d_{2m+1,\varepsilon}(F(t))}{\sum_{j=1}^{2m+1} c_{j,2m+1} + 2 + \varepsilon} \ge 0, \qquad t \ge 0,$$

which implies the stochastic ordering $T_1 \preceq_{\rm st} T_2$. On the other hand, either of the easily verifiable inequalities $\sum_{i=1}^m s^1_{i,2m+1} < \sum_{i=1}^m s^2_{i,2m+1}$ and $\sum_{i=1}^{m+1} s^1_{i,2m+1} < \sum_{i=1}^{m+1} s^2_{i,2m+1}$ contradicts an analogous relation $\boldsymbol{s}^1_{2m+1} \preceq_{\rm st} \boldsymbol{s}^2_{2m+1}$ for the respective signatures. □

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