Combinatorics of one-dimensional simple Toeplitz subshifts

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Abstract. This paper provides a systematic study of fundamental combinatorial properties of one-dimensional, two-sided infinite simple Toeplitz subshifts. Explicit formulas for the complexity function, the palindrome complexity function and the repetitivity function are proved. Moreover, a complete description of the de Bruijn graphs of the subshifts is given. Finally, the Boshernitzan condition is characterized in terms of combinatorial quantities, based on a recent result of Liu and Qu [Uniform convergence of Schrödinger cocycles over simple Toeplitz subshift. *Ann. Henri Poincaré* **12**(1) (2011), 153–172]. Particular simple characterizations are provided for simple Toeplitz subshifts that correspond to the orbital Schreier graphs of the family of Grigorchuk's groups, a class of subshifts that serves as the main example throughout the paper.

Contents

1. *Introduction*

The investigation of Toeplitz subshifts has a long history. In fact, Toeplitz subshifts have been rediscovered in various contexts and serve as a prime source of (counter)examples (see, for instance, [29, 44] or [27]). Our investigation of Toeplitz subshifts is motivated by their utilization in two seemingly unrelated fields. First, for Grigorchuk's group it has been shown in [26] (see [25] as well) that the Laplacian on the two-sided Schreier graphs of the group is unitarily equivalent to the Jacobi operator on a Toeplitz subshift. Thus the investigation of Toeplitz subshifts can help to improve the understanding of self-similar groups (see [43] as well). Second, Toeplitz words have also become more popular as a model for quasicrystals over the last decade (see, for instance, [2]). In particular, spectral properties of Schrödinger operators with a potential that is given by an element of a Toeplitz subshift have received quite some attention and are studied, for example, in [15, 35, 36]. The present work addresses fundamental combinatorial properties of so-called simple Toeplitz words and their associated subshifts.

1.1. *Simple Toeplitz words.*   Toeplitz words were introduced in [29] as elements in $\{0, 1\}^{\mathbb{Z}}$ with a certain regularity. The regularity stems from the construction of Toeplitz words via so called 'partial words'. These are periodic words with letters in $\{0, 1\}$ and some undetermined positions ('holes'). The holes are then successively filled with other partial words. It is required in [29] that no undetermined part remains in the limit of the hole-filling process. Thus, in the limit word $\omega \in \{0, 1\}^{\mathbb{Z}}$, every letter $\omega(j)$ is repeated periodically, but the period depends on the position j, i.e.,

$$\text{for all } j \in \mathbb{Z} \quad \text{there exists } p \in \mathbb{N} \text{ such that for all } k \in \mathbb{Z} : \omega_j = \omega_{j+kp}.$$

In the following, we will consider the subclass of so-called simple Toeplitz words. In their construction, every partial words consists of the repetition of a single letter and there is exactly one hole in every partial word. Note that the details of the notion of simple Toeplitz words in the literature differ corresponding to the different settings that are considered. For example, simple Toeplitz words are defined in [30] in the context
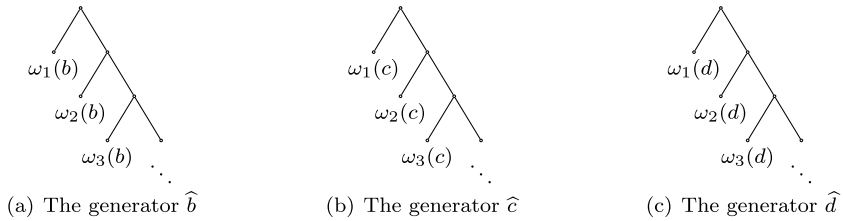
(a) The generator $\widehat{b}$ (b) The generator $\widehat{c}$ (c) The generator $\widehat{d}$

FIGURE 1. The generators $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$ of the group G_ω.

of one-sided infinite words over a set of cardinality two without an undetermined part. In [**40**], simple Toeplitz words are two-sided infinite 'words' in $\mathbb{Z}^d$ that must not have an undetermined part. In this work, we will use the definition from [**35**] and define simple Toeplitz words as one-dimensional, two-sided infinite words $\omega \in \mathcal{A}^{\mathbb{Z}}$ over a finite set $\mathcal{A}$. Words with an undetermined part are allowed as well, after filling the undetermined position appropriately.

1.2. *Schreier graphs of self-similar groups.*

Let X be a finite set and consider $\bigcup_{k=0}^{\infty} X^k$. This union can be thought of as the vertex set of a regular rooted tree, where two vertices are connected if and only if they are of the form u and ua with $u \in \bigcup_{k=0}^{\infty} X^k$ and $a \in X$. A group G of automorphisms on such a tree is called self similar if, for every $g \in G$ and every $a \in X$, there exist elements $h \in G$ and $b \in X$ such that $g(au) = bh(u)$ holds for all $u \in \bigcup_{k=0}^{\infty} X^k$. In other words, for every group element g and every $a \in X$, there is a group element h such that g acts on the subtree below a in the same way as h acts on the whole tree. Such groups have been studied during recent decades since they provide examples of groups with interesting properties (see, for example, [**3**, **4**, **38**] and the references therein). For instance, Grigorchuk's group, introduced in [**21**], is a self-similar group of automorphisms on the binary tree and was the first example of a group with intermediate growth [**22**]. With X chosen as $X = \{0, 1\}$, Grigorchuk's group is generated by the four elements a, b, c and d that satisfy

$$a(0u) = 1u, \quad b(0u) = 0a(u), \quad c(0u) = 0a(u), \quad d(0u) = 0u,$$
$$a(1u) = 0u, \quad b(1u) = 1c(u), \quad c(1u) = 1d(u), \quad d(1u) = 1b(u).$$

In fact, the above definition was generalized in [**22**] to a whole family $(G_\omega)_\omega$ of groups: let a be defined as above. Moreover, let $\omega \in \{\pi_b, \pi_c, \pi_d\}^{\mathbb{N}}$ be a sequence of the maps

$$\pi_b : b \mapsto \mathrm{id}, \ c \mapsto a, \ d \mapsto a, \quad \pi_c : b \mapsto a, \ c \mapsto \mathrm{id}, \ d \mapsto a,$$
$$\pi_d : b \mapsto a, \ c \mapsto a, \ d \mapsto \mathrm{id}.$$

We can now consider the automorphism $\widehat{b}$ that acts like $\omega_1(b)$ below the vertex 0, like $\omega_2(b)$ below the vertex 10, like $\omega_3(b)$ below the vertex 110, etc. Similarly, we define the automorphism $\widehat{c}$ by the action of $(\omega_1(c), \omega_2(c), \omega_3(c), \ldots)$ and the the automorphism $\widehat{d}$ by the action of $(\omega_1(d), \omega_2(d), \omega_3(d), \ldots)$ (see Figure 1).

For every such sequence $\omega \in \{\pi_b, \pi_c, \pi_d\}^{\mathbb{N}}$, we define G_ω as the group that is generated by a, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$. Note that the periodic sequence $\omega = (\pi_d, \pi_c, \pi_b, \ldots)$ yields precisely Grigorchuk's group. The family $\{G_\omega : \omega \in \{\pi_b, \pi_c, \pi_d\}^{\mathbb{N}}\}$ has been studied heavily in the

past. Of particular relevance to our investigation are the recent works [7, 26], since they connect self-similar groups and Toeplitz subshifts. More precisely, it was shown in [7] that every group G_ω can be embedded into the topological full group of a minimal subshift. This subshift is constructed from so-called Schreier graphs and, although not explicitly mentioned in [7], can be shown to be a Toeplitz subshift. In [26], the Laplacians associated to the Schreier graphs of Grigorchuk's group are treated. A key insight is that these operators are unitarily equivalent to certain Jacobi operators associated to a subshift. This subshift is noted to be a Toeplitz subshift and turns out to be the same as the one that was described in [7]. However, in [26] the subshift is obtained by a different construction, which is based on the fact that the group action on the tree induces an action on the boundary $X^\mathbb{N}$ as well. Hence the following directed, labelled graph can be defined: the set of vertices is given by $(\bigcup_{k \in \mathbb{N}} X^k) \cup X^\mathbb{N}$. There is an edge from vertex u to vertex v labelled $s \in \{a, b, c, d\}$ if and only if $s(u) = v$ holds. The connected components of this graph are called Schreier graphs. For a vertex $u \in X^k$, the connected component of u corresponds to the kth level of the tree. The Schreier graph of the level $k + 1$ can be obtained from two copies of the level-k-Schreier graph by connecting them in a certain way that is specified by the element ω_k in the sequence $\omega = (\pi_d, \pi_c, \pi_b, \ldots)$. This yields a structure for the Schreier graphs that is similar to the one of a Toeplitz word (see [26] for details): roughly speaking, every second connection in the graph corresponds to the action of the generator a. Of the remaining 'holes' in the graph, every second connection corresponds to the action defined by $\omega_1 = \pi_d$. Of the connections that are still missing after that, every second one corresponds to the action defined by $\omega_2 = \pi_c$, etc. In fact, this construction of the subshift can be generalized to the whole family of groups [23].

1.3. *Schrödinger operators on quasicrystals.* Aperiodic words can serve as mathematical models for quasicrystals. The quantum mechanical properties of the quasicrystal are then described by the spectrum of the Schrödinger operator whose potential is given by the aperiodic word. For the relevance of quasicrystals, see, for example, the recent books [1, 32]. One much studied class of models are Sturmian words, that is, words that have exactly $L + 1$ subwords of length L for every $L \in \mathbb{N}$. By the famous Morse–Hedlund theorem [37], this is the least possible number of subwords that an aperiodic word can exhibit. Accordingly, these models have played a major role in the investigation of quasicrystals and, in particular, the associated Schrödinger operators have attracted a lot of attention [6, 9, 11–13, 33, 39, 41, 42]. By [6], the spectrum of a discrete Schrödinger operator with Sturmian potential is always a Cantor set of Lebesgue measure zero. Moreover, the absence of eigenvalues for all Sturmian potentials was shown in [11]. Together, these two results imply a purely singular continuous spectrum. In recent years, pattern Sturmian models, which are a generalization of Sturmian models, have become a focus of research. Pattern Sturmian words (in the sense of [31]) have exactly $2L$ different subsets of L elements for every $L \in \mathbb{N}$. This is the least possible number of subwords, that an aperiodic word can exhibit and every Sturmian word is pattern Sturmian as well (see [31]). In [15], the above mentioned results of [6, 11] could be generalized to these words: the spectrum of a discrete Schrödinger operator with a potential that is given by a pattern Sturmian Toeplitz sequence has zero Lebesgue measure and is purely singular

continuous. Moreover, it was shown in [**20**] that one-sided infinite simple Toeplitz words over a 2-letter alphabet without an undetermined part are pattern Sturmian.

1.4. *Content of the paper.* Because of their importance for quasicrystals and self-similar groups, the focus of the research has often been on the spectral properties of Schrödinger operators. Here, we give a systematic discussion of fundamental combinatorial properties of simple Toeplitz subshifts instead. The complexity function and the repetitivity function of simple Toeplitz subshifts are explicitly computed in full generality, i.e., over any finite alphabet $\mathcal{A}$ and with arbitrary coding sequences $(a_k) \in \mathcal{A}^{\mathbb{N}_0}$ and $(n_k) \in (\mathbb{N}\backslash\{1\})^{\mathbb{N}_0}$. This substantially generalizes and extends earlier pieces of work dealing with special cases: in [**24**], the complexity was determined for the subshift that is associated to Grigorchuk's group, i.e., $\mathcal{A} = \{a, x, y, z\}$, $(a_k) = (a, x, y, z, x, y, z, \ldots)$ and $(n_k) = (2, 2, 2, 2, \ldots)$. In [**17**], simple Toeplitz subshifts of the form $\mathcal{A} = \{a, x, y, z\}$, $(a_k) = (a, x, y, z, x, y, z, \ldots)$ and $(n_k) = (2, 2^{l_1}, 2^{l_2}, 2^{l_3}, \ldots)$ with $l_1, l_2, l_3, \ldots \in \mathbb{N}$ are considered. For them, among other things, a formula for the complexity, estimates for the repetitivity and a characterization of α-repetitivity were obtained.

Before discussing the combinatorial properties, §2 gives the definition of simple Toeplitz words that will be used throughout this work. Moreover, the associated simple Toeplitz subshifts and the main examples are introduced. We call them the Grigorchuk subshift and the generalized Grigorchuk subshifts. In §3, an explicit formula for the subword complexity of a simple Toeplitz subshift is derived. For this, complexity bounds at certain word lengths are proved and the growth rate of the complexity is estimated. Based on these results, the sequence of de Bruijn graphs is investigated in §4. A particular reflection symmetry in the graphs yields an explicit formula for the palindrome complexity. The analysis of combinatorial properties continues in §5 with the deduction of a formula for the repetitivity function, followed by a short discussion of α-repetitivity. Finally, the Boshernitzan condition for simple Toeplitz subshifts is studied in §6 and its implication for the spectrum of Jacobi operators is briefly reviewed.

2. *Simple Toeplitz subshifts*

In this section, our objects of interest are defined. As in [**35**], we consider simple Toeplitz words which are one-dimensional, two-sided infinite words over a finite alphabet and may have an undetermined position. Some basic properties of simple Toeplitz words are given in the first subsection. In the second part, different ways of associating subshifts to simple Toeplitz words are discussed. Our main examples, the Grigorchuk subshift and what we call the generalized Grigorchuk subshifts, are defined.

2.1. *Simple Toeplitz words.* Let $\mathcal{A}$ be a finite set, called the alphabet. Its elements are referred to as letters. A word of length $L \in \mathbb{N}_0$ is an element $u = u(1) \ldots u(L)$ of $\mathcal{A}^{\{1,\ldots,L\}}$. For a given (finite) word u, we use $|u| \in \mathbb{N}_0$ to denote its length, where the word of length zero is called the empty word. A two-sided infinite word is an element $\alpha = \ldots \alpha(-1)\alpha(0)\alpha(1) \ldots$ of $\mathcal{A}^{\mathbb{Z}}$. We consider the discrete topology on $\mathcal{A}$ and equip $\mathcal{A}^{\mathbb{Z}}$ with the product topology. By $C_j(u)$, we denote the cylinder set $C_j(u) := \{\alpha \in \mathcal{A}^{\mathbb{Z}} : \alpha(j) = u(1), \ldots, \alpha(j + |u| - 1) = u(|u|)\}$ of those two-sided infinite words, in which

the finite word u appears at position j. The cylinder sets form a base of open sets in $\mathcal{A}^{\mathbb{Z}}$. Actually, the topological space $\mathcal{A}^{\mathbb{Z}}$ is metrizable and one metric is defined by

$$d(\alpha_1, \alpha_2) := \sum_{j=-\infty}^{\infty} \frac{\delta(\alpha_1(j), \alpha_2(j))}{2^{|j|}} \quad \text{where } \delta(a_1, a_2) := \begin{cases} 0 & \text{if } a_1 = a_2, \\ 1 & \text{if } a_1 \neq a_2, \end{cases}$$

for two-sided infinite words $\alpha_1, \alpha_2 \in \mathcal{A}^{\mathbb{Z}}$ and letters $a_1, a_2 \in \mathcal{A}$. Two words are close in this topology if they agree on a large interval around the origin. Thus a sequence of two-sided infinite words (α_k) converges to a word α if, for every interval, there exists a number k_0 such that all α_k, for $k \geq k_0$, agree with α on this interval.

We turn now to the construction of simple Toeplitz words. As mentioned in the introduction, they are obtained as the limit of a sequence of two-sided infinite periodic words with 'holes' that are successively filled. To make this precise, we introduce an additional letter $? \notin \mathcal{A}$, which represents the hole. Let $\alpha_1 \in (\mathcal{A} \cup \{?\})^{\mathbb{Z}}$ be periodic with period n and only a single occurrence of $?$ per period. Then there exists an integer r with $0 \leq r < n$ such that $n\mathbb{Z} + r$ are the positions of the holes in α_1 (the so-called undetermined part). Following [35], we define the filling of $\alpha_2 \in (\mathcal{A} \cup \{?\})^{\mathbb{Z}}$ into the holes of α_1 by

$$(\alpha_1 \triangleleft \alpha_2)(j) := \begin{cases} \alpha_1(j) & \text{for } j \notin n\mathbb{Z} + r, \\ \alpha_2\left(\dfrac{j-r}{n}\right) & \text{for } j \in n\mathbb{Z} + r. \end{cases}$$

In other words, we fill $\alpha_2(0)$ into the first hole at a non-negative position, $\alpha_2(1)$ into the second hole at a non-negative position, and so on, while we fill $\alpha_2(-1)$ into first hole at a negative position, $\alpha_2(-2)$ into second hole at a negative position, and so on. If both α_1 and α_2 are periodic words with holes, then $\alpha_1 \triangleleft \alpha_2$ is a periodic word with holes too. Thus we can fill a word $\alpha_3 \in (\mathcal{A} \cup \{?\})^{\mathbb{Z}}$ into the holes of $\alpha_1 \triangleleft \alpha_2$ and so on.

While a similar hole-filling procedure occurs in the construction of all kinds of Toeplitz words, simple Toeplitz words are distinguished by the type of periodic words α_k that are used. They are described by a sequence $(a_k)_{k \in \mathbb{N}_0}$ of letters $a_k \in \mathcal{A}$, a sequence $(n_k)_{k \in \mathbb{N}_0}$ of period lengths $n_k \in \mathbb{N}$, $n_k \geq 2$ and a sequence $(r_k)_{k \in \mathbb{N}_0}$ of non-negative integer positions $0 \leq r_k < n_k$. From these sequences, we define the two-sided infinite periodic words

$$(a_k^{n_k-1}?)^{\infty} := \ldots a_k \ldots a_k ? \underbrace{a_k \ldots a_k}_{n_k-1\text{-times}} ?a_k \ldots a_k ? \ldots$$

with period n_k and undetermined part $n_k\mathbb{Z} + r_k$. To keep track of which letters are still to be inserted, we define $\mathcal{A}_k := \{a_j : j \geq k\}$ for $k \in \mathbb{N}_0$. Following [35], we use $\widetilde{\mathcal{A}} := \bigcap_{k \geq 0} \mathcal{A}_k$ to denote the eventual alphabet, that is, the set of letters that appear infinitely often in the sequence (a_k). Since $\mathcal{A}$ is a finite set, there exists a number $\widetilde{K}$ such that $a_k \in \widetilde{\mathcal{A}}$ and $\mathcal{A}_k = \widetilde{\mathcal{A}}$ hold for all $k \geq \widetilde{K}$.

To construct a simple Toeplitz word, we insert $(a_1^{n_1-1}?)^{\infty}$ into the holes of $(a_0^{n_0-1}?)^{\infty}$, then insert $(a_2^{n_2-1}?)^{\infty}$ in the remaining holes of the obtained word, then insert $(a_3^{n_3-1}?)^{\infty}$ and so on. Thus we obtain a sequence $(\omega_k)_{k \in \mathbb{N}_0}$ of two-sided infinite words defined by

$$\omega_k := (a_0^{n_0-1}?)^{\infty} \triangleleft (a_1^{n_1-1}?)^{\infty} \triangleleft (a_2^{n_2-1}?)^{\infty} \triangleleft \ldots \triangleleft (a_k^{n_k-1}?)^{\infty}.$$

PROPOSITION 2.1. *The word ω_k is periodic with period $n_0 \cdot n_1 \cdot \ldots \cdot n_k$ and has undetermined part $U_k = n_0 \cdot \ldots \cdot n_k \mathbb{Z} + [r_0 + \sum_{j=1}^{k} r_j \cdot n_0 \cdot \ldots \cdot n_{j-1}]$. In particular, there is exactly one undetermined position per period.*

Proof. We proceed by induction. For $k = 0$, the claim is clearly true since $\omega_0 = (a_0^{n_0-1}?)^\infty$ has, by definition, period n_0 and undetermined part $n_0 \mathbb{Z} + r_0$. Now assume that the claim about the period and the undetermined part holds for ω_k. By definition, we have $\omega_{k+1} = \omega_k \vartriangleleft (a_{k+1}^{n_{k+1}-1}?)^\infty$. Since $(a_{k+1}^{n_{k+1}-1}?)^\infty$ has undetermined part $n_{k+1}\mathbb{Z} + r_{k+1}$, the undetermined part of ω_{k+1} is

$$n_0 \cdot \ldots \cdot n_k \cdot (n_{k+1}\mathbb{Z} + r_{k+1}) + \left[r_0 + \sum_{j=1}^{k} r_j \cdot n_0 \cdot \ldots \cdot n_{j-1} \right]$$

$$= n_0 \cdot \ldots \cdot n_{k+1}\mathbb{Z} + \left[r_0 + \sum_{j=1}^{k+1} r_j n_0 \cdot \ldots \cdot n_{j-1} \right].$$

In particular, the period of ω_{k+1} has to be at least $n_0 \cdot \ldots \cdot n_{k+1}$, since this is the distance between two undetermined positions. On the other hand, we obtained ω_{k+1} by inserting the word $(a_{k+1}^{n_{k+1}-1}?)^\infty$ with period n_{k+1} into the word ω_k with period $\prod_{j=0}^{k} n_j$ and exactly one hole per period. Hence the resulting word ω_{k+1} has a period of at most $n_{k+1} \cdot \prod_{j=0}^{k} n_j$. Since the undetermined part has the same period as the word, there is exactly one undetermined position per period. $\square$

For later use, we define $p^{(k)}$ to be the block of letters between two consecutive holes in ω_k. By the above proposition, $p^{(k)}$ is well defined and its length is given by $|p^{(k)}| + 1 = n_0 \cdot \ldots \cdot n_k$ for all $k \geq 0$. In addition, it is convenient to define $|p^{(-1)}| = 0$ such that the relation

$$|p^{(k)}| + 1 = n_k \cdot (|p^{(k-1)}| + 1)$$

holds for all $k \geq 0$. Moreover, it is easy to see from the definition of ω_k that the blocks $p^{(k)}$ satisfy the recursion relation

$$p^{(0)} = a_0^{n_0-1} \quad \text{and} \quad p^{(k+1)} = \underbrace{p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)} a_{k+1} p^{(k)}}_{n_{k+1}\text{-times } p^{(k)} \text{ and } (n_{k+1}-1)\text{-times } a_{k+1}}.$$

Now we proceed toward the definition of a simple Toeplitz word by taking the limit $\omega_\infty := \lim_{k\to\infty} \omega_k$ in $(\mathcal{A} \cup \{?\})^{\mathbb{Z}}$. The undetermined parts U_k of the words ω_k form a decreasing sequence of sets. The undetermined part $U_\infty := \bigcap_{k\geq 0} U_k$ of ω_∞ is either empty or a single position. If U_∞ is empty and the limit word ω_∞ is not periodic, then ω_∞ called a normal Toeplitz word. If U_∞ is not empty, then we insert an arbitrary letter $\tilde{a} \in \tilde{\mathcal{A}}$ into the single undetermined position. If the resulting word $\omega_\infty^{(\tilde{a})}$ is not periodic, then $\omega_\infty^{(\tilde{a})}$ is called an extended Toeplitz word. Following [35], we call a word a simple Toeplitz word if it is either a normal Toeplitz word or an extended Toeplitz word. Note that other definitions are used in the literature as well; see §1.1 in the introduction. The sequences $(a_k)_{k\in\mathbb{N}_0}$, $(n_k)_{k\in\mathbb{N}_0}$ and $(r_k)_{k\in\mathbb{N}_0}$ are called a coding of the obtained simple Toeplitz word. In the following, we will always assume that $a_k \neq a_{k+1}$ for all k, since subsequent occurrences of the same letter can be expressed as a single occurrence where the period length n_k

is increased accordingly. Moreover, we will always assume that $\#\widetilde{\mathcal{A}} \geq 2$, since Toeplitz words must not be periodic and $\#\widetilde{\mathcal{A}} = 1$ (that is, an eventually constant sequence of letters) implies a periodic word.

PROPOSITION 2.2. *Let* (a_k), (n_k) *and* (r_k) *be as defined above. Let* ω *denote the word* ω_∞ *if* $U_\infty = \emptyset$ *and the word* $\omega_\infty^{(\widetilde{a})}$ *with arbitrary* $\widetilde{a} \in \widetilde{\mathcal{A}}$ *if* $U_\infty \neq \emptyset$. *Then* ω *is periodic if and only if* $\#\widetilde{\mathcal{A}} = 1$ *holds.*

Proof. First assume that $\widetilde{\mathcal{A}}$ contains only a single element $\widetilde{a}$. Then $a_k = \widetilde{a}$ holds for all $k \geq \widetilde{K}$ and thus every hole in $\omega_{\widetilde{K}-1}$ is filled with the letter $\widetilde{a}$ to obtain ω. In the case of $U_\infty \neq \emptyset$, the remaining undetermined position is filled with the letter $\widetilde{a}$ as well. Since $\omega_{\widetilde{K}-1}$ is a periodic word over $\mathcal{A} \cup \{?\}$, replacing every letter ? by the letter $\widetilde{a}$ yields a periodic word ω.

To prove the converse, we follow the proof of Proposition 6.2 in [**40**]. Assume that $\#\widetilde{\mathcal{A}} \geq 2$ holds and that there exists a period p of ω. Choose two letters $\widetilde{a} \neq \widetilde{b}$ that appear infinitely often in the coding sequence and let $k_0 \in \mathbb{N}$ be large enough such that $p < |p^{(k_0)}| + 1$ holds. Let $j_{\widetilde{a}}, j_{\widetilde{b}} \in U_{k_0}$ be two positions that are undetermined in ω_{k_0} such that $\omega(j_{\widetilde{a}}) = \widetilde{a}$ and $\omega(j_{\widetilde{b}}) = \widetilde{b}$ hold. Recall that ω_{k_0} and, in particular, its undetermined part U_{k_0} are periodic with period $|p^{(k_0)}| + 1$. Thus $j_{\widetilde{a}} - j_{\widetilde{b}}$ is a multiple of $|p^{(k_0)}| + 1$ and we obtain the following contradiction:

$$
\begin{aligned}
\widetilde{a} &= \omega(j_{\widetilde{a}}) \\
&= \omega(j_{\widetilde{a}} + p) \quad \text{since } p \text{ is a period of } \omega \\
&= \omega_{k_0}(j_{\widetilde{a}} + p) \quad \text{since } j_{\widetilde{a}} \in U_{k_0} \text{ and } p < |p^{(k_0)}| + 1 \text{ imply } j_{\widetilde{a}} + p \notin U_{k_0} \\
&= \omega_{k_0}(j_{\widetilde{b}} + p) \quad \text{since } (j_{\widetilde{a}} + p) - (j_{\widetilde{b}} + p) \text{ is a multiple of } |p^{(k_0)}| + 1 \\
&= \omega(j_{\widetilde{b}} + p) \quad \text{since } j_{\widetilde{b}} \in U_{k_0} \text{ and } p < |p^{(k_0)}| + 1 \text{ imply } j_{\widetilde{b}} + p \notin U_{k_0} \\
&= \omega(j_{\widetilde{b}}) \quad \text{since } p \text{ is a period of } \omega \\
&= \widetilde{b}. \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \square
\end{aligned}
$$

2.2. *Subshifts of simple Toeplitz words.* For the remainder of this section, our focus changes from elements $\omega \in \mathcal{A}^{\mathbb{Z}}$ to subsets $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$. We will define the shift map as well as subshifts and state some properties of (elements of) subshifts associated to simple Toeplitz words.

In the previous subsection, we defined a topology on $\mathcal{A}^{\mathbb{Z}}$. With respect to this topology, the (left-)shift, defined by

$$
\mathrm{T} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \quad \text{with } (\mathrm{T}\,\omega)(j) := \omega(j+1),
$$

is a homeomorphism. A closed subset of $\mathcal{A}^{\mathbb{Z}}$ that is invariant under the shift T is called a subshift. A subshift $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ is called minimal if the T-orbit of every element $\omega \in \Omega$ is dense in Ω. It is called uniquely ergodic if there is a unique T-invariant Borel probability measure on Ω. For a (finite of infinite) word ω, we use $\mathrm{Sub}(\omega)$ to denote the set of all finite subwords that occur in ω. The empty word of length zero is considered to be a subword of every ω. For a subshift Ω, we define its language as $\mathrm{Sub}(\Omega) := \bigcup_{\omega \in \Omega} \mathrm{Sub}(\omega)$.

To a two-sided infinite word ω we associate a subshift $\Omega_\omega := \overline{\{T^k \omega : k \in \mathbb{N}\}}$ by taking the closure of the orbit of ω under the shift. For the remainder of this paper, we will always assume that Ω is the subshift associated to a simple Toeplitz word ω, which is defined by the sequences $(a_k)_{k \in \mathbb{N}_0}$, $(n_k)_{k \in \mathbb{N}_0}$ and $(r_k)_{k \in \mathbb{N}_0}$. Such a word ω is the limit of a sequence of periodic words ω_k, where ω_{k+1} is a completion of ω_k (that is, some holes are filled and non-hole positions remain unchanged). If ω is a normal Toeplitz word, then for every position $j \in \mathbb{Z}$ there exists an index k such that $\omega_k(j) \in \mathcal{A}$ holds. In addition, we have seen in Proposition 2.1 that there is precisely one hole per period in ω_k and thus

$$\frac{\text{number of undetermined positions in } \omega_k \text{ per period}}{\text{length of the period of } \omega_k} \xrightarrow{k \to \infty} 0$$

holds. Toeplitz words with these properties are called regular. Their associated subshift is always minimal and uniquely ergodic (see [**29**, corollary to Theorem 5] for Toeplitz words over two letters and see, e.g., [**16**] for Toeplitz words over arbitrary finite alphabets). In particular, the subshifts associated to normal Toeplitz words are minimal and uniquely ergodic.

When we construct a simple Toeplitz word ω with coding sequences (a_k), (n_k) and (r_k), the sequence (r_k) describes into which hole to map the origin. Therefore a change of r_k causes a shift of the resulting word. Since we construct the subshift by taking the orbit closure of ω, such a shift of the word does not change Ω. This was made precise in [**35**, Proposition 2.3].

PROPOSITION 2.3. [**35**] *If two simple Toeplitz words $\omega, \widetilde{\omega}$ have the coding sequences (a_k), (n_k), (r_k) and (a_k), (n_k), $(\widetilde{r_k})$, respectively, then the associated subshifts Ω_ω, $\Omega_{\widetilde{\omega}}$ are equal.*

Therefore, we will from now on omit (r_k) and speak about the subshift generated by the sequences (a_k) and (n_k). In addition, we can now extend minimality and unique ergodicity to the subshifts that are associated to arbitrary simple Toeplitz words. The underlying idea is that, if necessary, we can change (r_k) into a sequence $(\widetilde{r_k})$ such that $\widetilde{\omega}$ is a normal simple Toeplitz word (cf. [**35**, Corollary 2.1]).

PROPOSITION 2.4. [**35**] *For every simple Toeplitz word ω, the subshift Ω_ω is minimal and uniquely ergodic.*

In [**35**, Proposition 2.4] it was shown that, in a certain sense, the converse of Proposition 2.3 holds as well.

PROPOSITION 2.5. [**35**] *If the subshift Ω is associated to a simple Toeplitz word ω with coding sequences (a_k), (n_k) and (r_k), then every element $\widetilde{\omega} \in \Omega_\omega$ is a simple Toeplitz word with coding sequences (a_k), (n_k) and $(\widetilde{r_k})$.*

For every $\omega \in \Omega$ we can therefore find a sequence of periodic words ω_k, as described in Proposition 2.1, that converge to ω (with the possible exception of one undetermined position). Thus, for every $k \in \mathbb{N}_0$ and every $\omega \in \Omega$, we can write ω as

$$\omega = \ldots p^{(k)} \star p^{(k)} \star p^{(k)} \star p^{(k)} \ldots,$$

where $\star$ denotes elements from $\mathcal{A}_{k+1} = \{a_j : j \geq k + 1\}$. Since all $\omega \in \Omega$ have the same letters in the sequence (a_k) and the same period length (n_k), the $p^{(k)}$-blocks are the same for all $\omega \in \Omega$. Based on these blocks, we can give an alternative definition of the subshift associated to a simple Toeplitz word ω: let (a_k) and (n_k) be the coding sequences of ω and define the $p^{(k)}$-blocks as in the previous subsection by $p^{(0)} = a_0^{n_0-1}$ and $p^{(k+1)} = p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)}$ with $(n_{k+1} - 1)$-times a_{k+1} and n_{k+1}-times $p^{(k)}$. Since $p^{(k)}$ is a prefix of $p^{(k+1)}$ for all $k \geq 0$, there exits a unique one-sided infinite word ϖ such that $p^{(k)}$ is a prefix of ϖ for all $k \geq 0$. We define a subshift by

$$\widehat{\Omega}_\omega := \{\varrho \in \mathcal{A}^{\mathbb{Z}} : \mathrm{Sub}(\varrho) \subseteq \mathrm{Sub}(\varpi)\}.$$

PROPOSITION 2.6. *For every simple Toeplitz word ω, the equality $\widehat{\Omega}_\omega = \Omega_\omega$ holds.*

Proof. First, we show that $\widehat{\Omega}_\omega \subseteq \Omega_\omega$. Let $\varrho \in \widehat{\Omega}_\omega$ and $J \in \mathbb{N}$. The finite subword $\varrho|_{[-J,J]}$ of ϱ is a subword of ϖ as well. Thus it is contained in the prefix $p^{(k)}$ of ϖ for every sufficiently large k. Because of the decomposition $\omega = \ldots \star p^{(k)} \star p^{(k)} \star \ldots$, there exists a k_J such that $\mathrm{T}^{k_J} \omega|_{[-J,J]} = \varrho|_{[-J,J]}$. We obtain a sequence (k_J) with $\varrho = \lim_{J \to \infty} \mathrm{T}^{k_J} \omega|_{[-J,J]}$ and thus $\varrho \in \Omega_\omega$.

For the converse, first assume that ω is a normal simple Toeplitz word. Thus, for every $J \in \mathbb{N}$, there exists a number k_J such that all positions in $\omega|_{[-J,J]}$ are determined in the word $\omega_{k_J} = \ldots p^{(k_J)}?p^{(k_J)}?p^{(k_J)} \ldots$. Hence $\omega|_{[-J,J]}$ is contained in $p^{(k_J)}$. Now assume that ω is an extended simple Toeplitz word. Then, for every J, there is a number k such that $\omega|_{[-J,J]}$ is contained in $p^{(k)}?p^{(k)}$, where $?$ denotes the position of U_∞. Let $\widetilde{a} \in \widetilde{\mathcal{A}}$ denote the letter that is filled into U_∞. Since $\widetilde{a}$ appears infinitely often in the coding sequence, there exists a number $k_J > k$ such that $a_{k_J} = \widetilde{a}$. Now $\omega|_{[-J,J]}$ is contained in $p^{(k)}\widetilde{a}p^{(k)}$, which is contained in $p^{(k_J-1)}\widetilde{a}p^{(k_J-1)}$, which is contained in $p^{(k_J)}$. Thus, independent of which kind of simple Toeplitz word ω is, there is a number k_J such that $\omega|_{[-J,J]}$ is contained in a $p^{(k_J)}$-block. Hence every subword of ω is a subword of ϖ and we obtain $\omega \in \widehat{\Omega}_\omega$. $\square$

We conclude the section by defining a class of simple Toeplitz subshifts that will serve as our main example throughout the whole paper. As mentioned in the introduction, one motivation for the study of simple Toeplitz subshifts is their connection to self-similar groups: for the family of Grigorchuk's groups, the orbital Schreier graphs have a similar structure to Toeplitz words (see [26]). Essentially, in each step, every second 'hole' is filled. In the first step, the filling always corresponds to the generator a, and in the kth step $(k \geq 2)$, it corresponds to the value of $\omega_k \in \{\pi_b, \pi_c, \pi_d\}$. Thus we consider the following simple Toeplitz subshift.

Example. Following convention, we consider the four letter alphabet $\mathcal{A} = \{a, x, y, z\}$. We are interested in the subshifts defined by the constant sequence $(n_k)_{k \in \mathbb{N}_0} = (2, 2, 2, \ldots)$ and a sequence $(b_k) \in \mathcal{A}^{\mathbb{N}_0}$ with $b_0 = a$ and $b_k \in \{x, y, z\}$ for $k \geq 1$. Here $b_k = b_{k+1}$ is allowed, but, otherwise, the same construction is used as in the simple Toeplitz case. In particular, we assume that (b_k) is not eventually constant. Via $x \leftrightarrow \pi_d$, $y \leftrightarrow \pi_c$, $z \leftrightarrow \pi_b$, the sequence b_k corresponds to a sequence ω_k, which defines an element in the family of Grigorchuk's groups. Within the scope of this text, we will therefore call these subshifts

generalized Grigorchuk subshifts, in contrast to the (standard) Grigorchuk subshift that is defined in the example below. Note, however, that these notions are not standard terminology. To describe the subshifts in accordance with our definition of simple Toeplitz subshifts, we have to express subsequent occurrences of the same letter (b_k) as a single letter a_m with associated period length $n_m \geq 2$. Assume that $b_k \neq b := b_{k+1} = \ldots = b_{k+j} \neq b_{k+j+1}$ and note that

$$\underbrace{(b?)^\infty \triangleleft \ldots \triangleleft (b?)^\infty}_{j \text{ times}} = (\underbrace{b \ldots b}_{2^j - 1 \text{ times}} ?)^\infty = (b^{2^j - 1}?)^\infty$$

holds. Hence the generalized Grigorchuk subshifts are precisely those simple Toeplitz subshifts with $\mathcal{A} = \{a, x, y, z\}$, $a_0 = a$ and $a_k \in \{x, y, z\}$ for $k \geq 1$, where $n_0 = 2$ and, for every $k \in \mathbb{N}$, there exists a number $j_k \geq 1$ such that $n_k = 2^{j_k}$ holds. Their block length is given by

$$|p^{(k)}| + 1 = 2^{j_k} \cdot (|p^{(k-1)}| + 1) = 2^{1+j_1+\ldots+j_k}.$$

As a special case, we will sometimes consider the subshift that corresponds to Grigorchuk's group. This group is obtained when $(\omega_k)_k$ is the periodic sequence $\omega = (\pi_d, \pi_c, \pi_b, \ldots)$.

Example. As above, consider $\mathcal{A} = \{a, x, y, z\}$. Let $(a_k)_{k \in \mathbb{N}_0} = (a, x, y, z, x, y, z, \ldots)$ be 3-periodic from a_1 on and let $(n_k)_{k \in \mathbb{N}_0} = (2, 2, 2, \ldots)$ be the constant sequence with value two. The associated subshift $\widehat{\Omega}$ is precisely the subshift that is linked to Grigorchuk's group (cf. [26]) and we will refer to it as *Grigorchuk subshift*. The length of the $p^{(k)}$-blocks is given by $|p^{(k)}| + 1 = 2^{k+1}$. Moreover $\widetilde{\mathcal{A}} = \{x, y, z\}$ and $\widetilde{K} = 1$ hold.

Note that our notion of a generalized Grigorchuk subshift includes, as a special instance, what is called an *l-Grigorchuk subshift* in [17]. These are the subshifts that are obtained from $\mathcal{A} = \{a, x, y, z\}$, $(a_k) = (a, x, y, z, x, y, z, \ldots)$ and $(n_k) = (2, 2^{l_1}, 2^{l_2}, 2^{l_3}, \ldots)$ with $l_1, l_2, l_3, \ldots \in \mathbb{N}$.

3. Subword complexity

The aim of this section is to give an explicit formula for the complexity function of a simple Toeplitz subshift. For a different subclass of Toeplitz words, this problem has for example been studied in [10]. There, one-sided infinite words are studied that are obtained by a hole-filling procedure from a single word with holes. This word is repeatedly inserted into itself and it is shown that the complexity of the obtained words grows like a polynomial. In addition, it is shown in [34] with a more general construction procedure that, for every rational number $r \in \mathbb{Q}$, a Toeplitz word can be obtained such that $\mathcal{C}(L)$ grows like L^r. While both works cover cases that are much more general, they also provide bounds for the special case of a single hole per period: it follows from [10, Theorem 5] that the complexity is dominated by a linear function if one word with holes is repeatedly inserted into itself. Theorem 9 in [34] yields that this is also the case for a sequence of periodic words with holes instead of a single word, provided that the length of the periods is bounded. In the following, we provide an explicit formula for the complexity of simple

Toeplitz words, which implies that the complexity is still dominated by a linear function (cf. Proposition 3.11).

Our main strategy is similar to the one that was employed in [**24**] for the special case of the Grigorchuk subshift (see arxiv version of [**25**] as well). First, we prove an upper bound for the complexity at certain points. Then we prove a lower bound for the growth rate of the complexity function. Together, these inequalities determine the complexity. The same technique was also employed in [**17**] to obtain the complexity of l-Grigorchuk subshifts.

3.1. *Inequalities for the complexity and its growth.* In the following, we will denote the cardinality of a set A by $\#A$, its characteristic function by $\mathbb{1}_A$ and its complement by A^c. Recall that $\mathrm{Sub}(\Omega)$ denotes the language of the subshift Ω, that is, the set of all finite words that occur in elements of Ω. The complexity function counts how many words of a given finite length there are and thus measures (one aspect of) how ordered the subshift is. It is defined as

$$\mathcal{C} : \mathbb{N}_0 \to \mathbb{N}, \quad L \mapsto \#\{u \in \mathrm{Sub}(\Omega) : |u| = L\}.$$

Since the empty word is always considered to be an element of the language, we have $\mathcal{C}(0) = 1$. Moreover, we define the growth rate of the complexity as $\mathcal{G}(n) := \mathcal{C}(n+1) - \mathcal{C}(n)$ for $n \in \mathbb{N}_0$. First, we establish an upper bound for the complexity at the lengths $|p^{(k)}| + 1$ with $k \geq 0$.

PROPOSITION 3.1. *For every word $u \in \mathrm{Sub}(\Omega)$ of length $|u| \leq |p^{(k)}| + 1$, there is a letter $a \in \mathcal{A}_{k+1}$ such that u is a subword in $p^{(k)}ap^{(k)}$.*

Proof. This follows immediately from the decomposition $\omega = \ldots p^{(k)} \star p^{(k)} \star p^{(k)} \star p^{(k)} \ldots$ with letters $\star \in \mathcal{A}_{k+1}$, which exists for all $\omega \in \Omega$ and all $k \geq 0$. $\square$

PROPOSITION 3.2. *For all $k \geq 0$, the following inequality holds:*

$$\mathcal{C}(|p^{(k)}| + 1) \leq (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1).$$

Proof. First consider the case when $k \geq 1$. By Proposition 3.1, it is sufficient to study subwords of length $|p^{(k)}| + 1$ of $p^{(k)}ap^{(k)}$ for $a \in \mathcal{A}_{k+1}$. For a letter $a \neq a_k$, there are at most $|p^{(k)}| + 1$ many subwords. If $a = a_k$ is possible at all (that is, if $a_k \in \mathcal{A}_{k+1}$ holds), then this case will result in at most $|p^{(k-1)}| + 1$ subwords. The reason is that the subwords in

$$p^{(k)}a_kp^{(k)} = [p^{(k-1)}a_kp^{(k-1)} \ldots p^{(k-1)}]a_k[p^{(k-1)}a_kp^{(k-1)} \ldots p^{(k-1)}]$$

repeat already when reaching the second $p^{(k-1)}$. This yields

$$\mathcal{C}(|p^{(k)}| + 1) \leq \#(\mathcal{A}_{k+1}\backslash\{a_k\}) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1)$$

for $k \geq 1$. The case when $k = 0$ is similar: for every $a \in \mathcal{A}_1\backslash\{a_0\}$, there are at most $|p^{(0)}| + 1$ subwords in $p^{(0)}ap^{(0)}$ and, because of $p^{(0)} = a_0^{n_0-1}$, there is only one factor of length $|p^{(0)}| + 1$ in $p^{(0)}a_0p^{(0)}$, namely, $a_0^{n_0}$. Our definition of $|p^{(-1)}| = 0$ allows us to write

$$\mathcal{C}(|p^{(0)}| + 1) \leq \#(\mathcal{A}_1\backslash\{a_0\}) \cdot (|p^{(0)}| + 1) + \mathbb{1}_{\mathcal{A}_1}(a_0) \cdot (|p^{(-1)}| + 1).$$

Finally, we note that $\#(\mathcal{A}_{k+1}\backslash\{a_k\}) = \#\mathcal{A}_k - 1$ holds for all $k \geq 0$: if $a_k \in \mathcal{A}_{k+1}$ holds, then this implies that $\mathcal{A}_{k+1} = \mathcal{A}_k$ and $\#(\mathcal{A}_{k+1}\backslash\{a_k\}) = \#\mathcal{A}_{k+1} - 1 = \#\mathcal{A}_k - 1$. If, on the

other hand, $a_k \notin \mathcal{A}_{k+1}$ holds, then this implies that $\#\mathcal{A}_{k+1} = \#\mathcal{A}_k - 1$ and $\#(\mathcal{A}_{k+1}\backslash\{a_k\}) = \#\mathcal{A}_{k+1} = \#\mathcal{A}_k - 1$. $\qquad\square$

Remark. In Proposition 3.5, we will show that the converse inequality holds as well, which thus proves equality of the terms.

PROPOSITION 3.3. *For all $k \geq 0$ and $0 \leq L \leq |p^{(k)}| - |p^{(k-1)}| - 1$, the inequality $\mathcal{G}(L) := \mathcal{C}(L+1) - \mathcal{C}(L) \geq \#\mathcal{A}_k - 1$ holds.*

Proof. Consider the suffix v_1 of $p^{(k)}$ that consists of the last L letters. We will show that v_1 is a right special word, that is, a word with more than one extension to the right. More precisely, we will show that there are $\#\mathcal{A}_k$ different extensions, which implies that the complexity increases by at least $\#\mathcal{A}_k - 1$ when we increase the word length by one.

First, note that v_1 can be extended by all letters in $\mathcal{A}_{k+1}$. If $a_k \in \mathcal{A}_{k+1}$ holds, then the equality $\#\mathcal{A}_{k+1} = \#\mathcal{A}_k$ follows and the proof is complete. For $a_k \notin \mathcal{A}_{k+1}$, we will show that v_1 can, nevertheless, be extended to the right by a_k. Thus there are $\#\mathcal{A}_{k+1} + 1 = \#\mathcal{A}_k$ different extensions, which proves the statement.

For $k \geq 1$, it follows from the decomposition $p^{(k)} = p^{(k-1)} \ldots p^{(k-1)} a_k p^{(k-1)}$ and $L \leq |p^{(k)}| - (|p^{(k-1)}| + 1)$ that v_1 is a suffix of the first $(n_k - 1) \cdot |p^{(k-1)}| + n_k - 2$ letters in $p^{(k)}$ and therefore can be followed by a_k. For $k = 0$ and $n_0 > 2$, the suffix is given by $v_1 = a_0^L$ with $L \leq n_0 - 2$ and this can be extended to the right by a_0, as can be seen in $p^{(0)} = a_0^{n_0 - 1}$. For $k = 0$ and $n_0 = 2$, the proposition follows immediately from $\mathcal{C}(0) = 1$ and $\mathcal{C}(1) = \#\mathcal{A} = \#\mathcal{A}_0$. $\qquad\square$

PROPOSITION 3.4. *Let $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq 2|p^{(k-1)}| - |p^{(k-2)}|$. If $a_{k-1} \in \mathcal{A}_k$ holds, then $\mathcal{G}(L)$ is at least one greater than stated in Proposition 3.3.*

Proof. We will show that there is a word v_2 of length L that is right special and different from the suffix v_1 of $p^{(k)}$ that was considered in Proposition 3.3. First, assume that $k \geq 2$ and note that the word $p^{(k-1)} a_{k-1} p^{(k-1)}$ occurs in the subshift, since $a_{k-1} \in \mathcal{A}_k$ holds by assumption. From the decomposition into $p^{(k-1)}$-blocks and single letters, it is clear that $p^{(k-1)} a_{k-1} p^{(k-1)}$ has to be followed by a_k. Since $k \geq 2$, we can decompose $p^{(k-1)}$ in $p^{(k-2)}$-blocks and single letters a_{k-1} (see Figure 2). Let v_2 be the suffix of length L of $p^{(k-1)} a_{k-1} p^{(k-1)}$. The decomposition shows that v_2 can be extended to the right with both a_k and a_{k-1}. Moreover, it follows from $L \geq |p^{(k-1)}| + 1$ that the word v_2 ends with $a_{k-1} p^{(k-1)}$. It is therefore different from the suffix v_1 of $p^{(k)} = p^{(k-1)} a_k \ldots a_k p^{(k-1)}$ of length L, which ends with $a_k p^{(k-1)}$.

For $k = 1$, we define $v_2 = a_0^L$. Because $a_0 \in \mathcal{A}_1$, this word occurs in $p^{(0)} a_0 p^{(0)} = a_0^{n_0 - 1} a_0 a_0^{n_0 - 1}$. From the decomposition into $p^{(0)}$-blocks and single letters, it is clear that $p^{(0)} a_0 p^{(0)}$ has to be followed by a_1. For every length $n_0 \leq L \leq 2 \cdot (n_0 - 1)$, v_2 can therefore be extended to the right by both a_0 and a_1. Moreover, it ends with $a_0^{n_0}$, while the suffix v_1 of length L of $p^{(1)}$ ends with $a_1 a_0^{n_0 - 1}$. $\qquad\square$

Remark. Note that $a_{k-1} \in \mathcal{A}_{k+1}$ holds if and only if $a_{k-1} \in \mathcal{A}_k$ holds: clearly, $a_{k-1} \in \mathcal{A}_{k+1}$ implies that $a_{k-1} \in \mathcal{A}_k$, as $\mathcal{A}_{k+1} \subseteq \mathcal{A}_k$ holds for all k. Conversely, $a_{k-1} \in \mathcal{A}_k = \{a_j : j \geq k\}$ implies that $a_{k-1} \in \mathcal{A}_{k+1} = \{a_j : j \geq k + 1\}$ since we assumed that $a_{k-1} \neq a_k$.

PROPOSITION 3.5. *For all $k \geq 0$, the following inequality holds:*

$$\mathcal{C}(|p^{(k)}| + 1) \geq (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1).$$

Proof. We proceed by induction and express the complexity as a telescoping sum of the growth of the complexity. For $k = 0$, the growth is bounded from below by Proposition 3.3 and we obtain

$$
\begin{aligned}
\mathcal{C}(|p^{(0)}| + 1) &= \mathcal{G}(|p^{(0)}|) + \sum_{L=0}^{|p^{(0)}|-1} \mathcal{G}(L) + \mathcal{C}(0) \\
&\geq \#\mathcal{A}_1 - 1 + (\#\mathcal{A}_0 - 1) \cdot |p^{(0)}| + 1 \\
&= (\#\mathcal{A}_0 - 1) \cdot (|p^{(0)}| + 1) + \mathbb{1}_{\mathcal{A}_1}(a_0) \cdot (|p^{(-1)}| + 1),
\end{aligned}
$$

where we used $1 - \#\mathcal{A}_0 + \#\mathcal{A}_1 = \mathbb{1}_{\mathcal{A}_1}(a_0)$ and $|p^{(-1)}| + 1 = 1$ in the last line. Now assume that the claim is true for $k - 1$. For $k \geq 1$, we employ the bounds from the Propositions 3.3 and 3.4. This yields

$$
\begin{aligned}
&\mathcal{C}(|p^{(k)}| + 1) \\
&= \sum_{L=|p^{(k-1)}|+1}^{|p^{(k)}|-|p^{(k-1)}|-1} \mathcal{G}(L) + \sum_{L=|p^{(k)}|-|p^{(k-1)}|}^{|p^{(k)}|} \mathcal{G}(L) + \mathcal{C}(|p^{(k-1)}| + 1) \\
&\geq (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| - 2|p^{(k-1)}| - 1) + (\#\mathcal{A}_{k+1} - 1) \cdot (|p^{(k-1)}| + 1) \\
&\quad + \mathbb{1}_{\mathcal{A}_{k+1}}(a_{k-1}) \cdot (|p^{(k-1)}| - |p^{(k-2)}|) + \mathcal{C}(|p^{(k-1)}| + 1) \\
&= (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1 - 2|p^{(k-1)}| - 2) + (\#\mathcal{A}_k - 1 + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) - 1) \\
&\quad \cdot (|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_{k-1}) \cdot (|p^{(k-1)}| - |p^{(k-2)}|) + \mathcal{C}(|p^{(k-1)}| + 1) \\
&= (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1) - (\#\mathcal{A}_k - \mathbb{1}_{\mathcal{A}_k}(a_{k-1})) \\
&\quad \cdot (|p^{(k-1)}| + 1) - \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) \cdot (|p^{(k-2)}| + 1) + \mathcal{C}(|p^{(k-1)}| + 1) \\
&= (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1) \\
&\quad - [(\#\mathcal{A}_{k-1} - 1) \cdot (|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) \cdot (|p^{(k-2)}| + 1)] + \mathcal{C}(|p^{(k-1)}| + 1) \\
&\geq (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1),
\end{aligned}
$$

where we used $\#\mathcal{A}_k + 1 - \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) = \#\mathcal{A}_{k-1}$ in the next to last line and the induction hypothesis in the last line. $\qquad\square$

Propositions 3.2 and 3.5 yield the exact value of the complexity function at all points of the form $|p^{(k)}| + 1$. In particular, the subwords of the words $p(k)ap(k)$ with $a \in \mathcal{A}_{k+1}$, that were counted in the proof of Proposition 3.2, are pairwise different. Moreover, the complexity grows exactly by the amount given as lower bounds in Propositions 3.3 and 3.4. If the growth was faster, then the value at $|p^{(k)}| + 1$ could not be obtained.

COROLLARY 3.6. *For all $k \geq 0$, the complexity of $|p^{(k)}| + 1$ is given by*

$$\mathcal{C}(|p^{(k)}| + 1) = (\#\mathcal{A}_k - 1) \cdot (|p^{(k)}| + 1) + \mathbb{1}_{\mathcal{A}_{k+1}}(a_k) \cdot (|p^{(k-1)}| + 1).$$
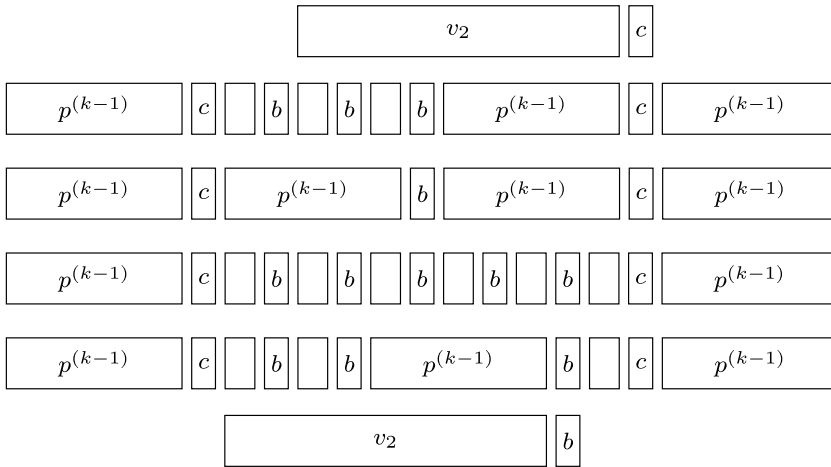
FIGURE 2. Two different extensions of v_2 in the decomposition of $p^{(k-1)}bp^{(k-1)}$. For the sake of readability, we use the abbreviations $b := a_{k-1}$ and $c := a_k$.

The growth rate of the complexity function is given by

$$\mathcal{G}(L) = \#\mathcal{A}_0 - 1 \quad \text{for } 0 \leq L \leq |p^{(0)}| - 1,$$

$$\text{by} \quad \mathcal{G}(L) = \#\mathcal{A}_1 - 1 \quad \text{for } L = |p^{(0)}|$$

$$\text{and by} \quad \mathcal{G}(L) = \#\mathcal{A}_k - 1$$

$$- \begin{cases} 0 & \text{if } |p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| - |p^{(k-1)}| - 1, \\ \#\mathcal{A}_k - \#\mathcal{A}_{k+1} & \text{if } |p^{(k)}| - |p^{(k-1)}| \leq L \leq |p^{(k)}|, \end{cases}$$

$$+ \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) & \text{if } |p^{(k-1)}| + 1 \leq L \leq 2|p^{(k-1)}| - |p^{(k-2)}|, \\ 0 & \text{if } 2|p^{(k-1)}| - |p^{(k-2)}| + 1 \leq L \leq |p^{(k)}|, \end{cases}$$

for $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}|$. Note that $\#\mathcal{A}_k - \#\mathcal{A}_{k+1} = \mathbb{1}_{\mathcal{A}_{k+1}^c}(a_k)$ holds for all k.

3.2. *Computing the complexity.* With the growth $\mathcal{G}(L) = \mathcal{C}(L + 1) - \mathcal{C}(L)$ known for all $L \geq 0$, we can now prove an explicit formula for the complexity function. For the computation, we have to distinguish three cases and we will split up the result into three statements accordingly.

Since the additional increase observed in Proposition 3.4 occurs only for $L \geq |p^{(0)}| + 1$, we treat the complexity in the case of $L \leq |p^{(0)}| + 1$ separately.

PROPOSITION 3.7. (Complexity function I) *The first values of the complexity function are*

$$\mathcal{C}(L) = (\#\mathcal{A}_0 - 1)L + 1 \quad \text{for } 0 \leq L \leq |p^{(0)}|$$

and

$$\mathcal{C}(L) = (\#\mathcal{A}_0 - 1)L + \mathbb{1}_{\mathcal{A}_1}(a_0) \quad \text{for } L = |p^{(0)}| + 1.$$

Proof. The complexity of the length $L = |p^{(0)}| + 1$ is already known from Corollary 3.6. The formula for $0 \leq L \leq |p^{(0)}|$ follows from $\mathcal{C}(0) = 1$ and $\mathcal{G}(L) = \mathcal{A}_0 - 1$ for $0 \leq L \leq |p^{(0)}| - 1$. $\square$

The following two theorems will deal with the case $L \geq |p^{(0)}| + 2$. It turns out that it is important to distinguish between $n_k = 2$ and $n_k > 2$, since the order in which the cases in the formula for $\mathcal{G}(L)$ in Corollary 3.6 change is different for $n_k = 2$ and $n_k > 2$.

THEOREM 3.8. (Complexity function II) *For $k \geq 1$ with $n_k = 2$, the complexity function in the range $|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| + 1$ is given by*

$$\mathcal{C}(L) = (\#\mathcal{A}_{k+1} - 1)L + (\#\mathcal{A}_{k-1} - \#\mathcal{A}_{k+1})(|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1})$$
$$\cdot \begin{cases} -|p^{(k-1)}| + |p^{(k-2)}| + L & \text{if } |p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| - |p^{(k-2)}|, \\ |p^{(k-1)}| + 1 & \text{if } |p^{(k)}| - |p^{(k-2)}| + 1 \leq L \leq |p^{(k)}| + 1. \end{cases}$$

Proof. Because of $n_k = 2$, we have $|p^{(k)}| = 2|p^{(k-1)}| + 1$ and the growth given in Corollary 3.6 simplifies to

$$\mathcal{G}(L) = \#\mathcal{A}_{k+1} - 1 + \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) & \text{if } |p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| - |p^{(k-2)}| - 1, \\ 0 & \text{if } |p^{(k)}| - |p^{(k-2)}| \leq L \leq |p^{(k)}|. \end{cases}$$

We can now compute the complexity from the value of $\mathcal{C}(|p^{(k-1)}| + 1)$ and the growth. First, we consider the case of $|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| - |p^{(k-2)}|$:

$$\begin{aligned} \mathcal{C}(L) &= \mathcal{C}(|p^{(k-1)}| + 1) + \sum_{j=|p^{(k-1)}|+1}^{L-1} \mathcal{G}(j) \\ &= (\#\mathcal{A}_{k-1} - 1) \cdot (|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) \cdot (|p^{(k-2)}| + 1) \\ &\quad + (L - |p^{(k-1)}| - 1) \cdot (\#\mathcal{A}_{k+1} - 1 + \mathbb{1}_{\mathcal{A}_k}(a_{k-1})) \\ &= (\#\mathcal{A}_{k-1} - \#\mathcal{A}_{k+1})(|p^{(k-1)}| + 1) - \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) \cdot (|p^{(k-1)}| - |p^{(k-2)}|) \\ &\quad + (\#\mathcal{A}_{k+1} - 1 + \mathbb{1}_{\mathcal{A}_k}(a_{k-1}))L. \end{aligned}$$

Similarly, we obtain the complexity for $|p^{(k)}| - |p^{(k-2)}| + 1 \leq L \leq |p^{(k)}| + 1$:

$$\begin{aligned} \mathcal{C}(L) &= \mathcal{C}(|p^{(k-1)}| + 1) + \sum_{j=|p^{(k-1)}|+1}^{L-1} \mathcal{G}(j) \\ &= (\#\mathcal{A}_{k-1} - 1) \cdot (|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) \cdot (|p^{(k-2)}| + 1) \\ &\quad + (\#\mathcal{A}_{k+1} - 1)(L - 1 - |p^{(k-1)}|) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(|p^{(k-1)}| - |p^{(k-2)}|) \\ &= (\#\mathcal{A}_{k-1} - \#\mathcal{A}_{k+1})(|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) \cdot (|p^{(k-1)}| + 1) \\ &\quad + (\#\mathcal{A}_{k+1} - 1)L. \end{aligned}$$ $\square$

THEOREM 3.9. (Complexity function III) *For $k \geq 1$ with $n_k > 2$, the complexity function in the range $|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| + 1$ is given by*

$$
\mathcal{C}(L) = (|p^{(k-1)}| + 1) + (\#\mathcal{A}_k - 1)L + \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(L - 2|p^{(k-1)}| + |p^{(k-2)}| - 1) \\ 0 \\ -\mathbb{1}_{\mathcal{A}_{k+1}^c}(a_k)(L - |p^{(k)}| + |p^{(k-1)}|) \end{cases}
$$

$$
\begin{cases} \text{if } |p^{(k-1)}| + 2 \leq L \leq 2|p^{(k-1)}| - |p^{(k-2)}| + 1, \\ \text{if } 2|p^{(k-1)}| - |p^{(k-2)}| + 2 \leq L \leq |p^{(k)}| - |p^{(k-1)}|, \\ \text{if } |p^{(k)}| - |p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| + 1. \end{cases}
$$

Proof. To shorten the notation, we introduce the abbreviations

$$
\begin{aligned}
I_1 &:= \mathbb{Z} \cap [|p^{(k-1)}| + 2, 2|p^{(k-1)}| - |p^{(k-2)}| + 1], \\
I_2 &:= \mathbb{Z} \cap [2|p^{(k-1)}| - |p^{(k-2)}| + 2, |p^{(k)}| - |p^{(k-1)}|], \\
I_3 &:= \mathbb{Z} \cap [|p^{(k)}| - |p^{(k-1)}| + 1, |p^{(k)}| + 1].
\end{aligned}
$$

We proceed similarly to the proof of Theorem 3.8. Because of $n_k > 2$, we have $|p^{(k)}| - |p^{(k-1)}| = (n_k - 1)(|p^{(k-1)}| + 1) > 2|p^{(k-1)}| - |p^{(k-2)}|$ and the growth given in Corollary 3.6 simplifies to

$$
\mathcal{G}(L) = \#\mathcal{A}_k - 1 + \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1}) & \text{if } L + 1 \in I_1, \\ 0 & \text{if } L + 1 \in I_2, \\ -(\#\mathcal{A}_k - \#\mathcal{A}_{k+1}) & \text{if } L + 1 \in I_3. \end{cases}
$$

We compute the complexity from the value of $\mathcal{C}(|p^{(k-1)}| + 1)$ and the growth:

$$
\begin{aligned}
\mathcal{C}(L) &= \mathcal{C}(|p^{(k-1)}| + 1) + \sum_{j=|p^{(k-1)}|+1}^{L-1} \mathcal{G}(j) \\
&= (\#\mathcal{A}_{k-1} - 1)(|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(|p^{(k-2)}| + 1) \\
&\quad + (\#\mathcal{A}_k - 1)(L - |p^{(k-1)}| - 1) \\
&\quad + \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(L - |p^{(k-1)}| - 1) \\ \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(|p^{(k-1)}| - |p^{(k-2)}|) \\ \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(|p^{(k-1)}| - |p^{(k-2)}|) - (\#\mathcal{A}_k - \#\mathcal{A}_{k+1})(L - |p^{(k)}| + |p^{(k-1)}|) \end{cases} \\
&\qquad \begin{cases} \text{if } L \in I_1, \\ \text{if } L \in I_2, \\ \text{if } L \in I_3, \end{cases} \\
&= (\#\mathcal{A}_{k-1} - \#\mathcal{A}_k)(|p^{(k-1)}| + 1) + \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(|p^{(k-1)}| + 1) + (\#\mathcal{A}_k - 1)L \\
&\quad + \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(L - 2|p^{(k-1)}| + |p^{(k-2)}| - 1) & \text{if } L \in I_1, \\ 0 & \text{if } L \in I_2, \\ -(\#\mathcal{A}_k - \#\mathcal{A}_{k+1})(L - |p^{(k)}| + |p^{(k-1)}|) & \text{if } L \in I_3, \end{cases}
\end{aligned}
$$

$$= (|p^{(k-1)}| + 1) + (\#\mathcal{A}_k - 1)L$$

$$+ \begin{cases} \mathbb{1}_{\mathcal{A}_k}(a_{k-1})(L - 2|p^{(k-1)}| + |p^{(k-2)}| - 1) & \text{if } L \in I_1, \\ 0 & \text{if } L \in I_2, \\ -(\#\mathcal{A}_k - \#\mathcal{A}_{k+1})(L - |p^{(k)}| + |p^{(k-1)}|) & \text{if } L \in I_3. \end{cases}$$

Now the relation $\#\mathcal{A}_k - \#\mathcal{A}_{k+1} = \mathbb{1}_{\mathcal{A}_{k+1}^c}(a_k)$ yields the claim. □

Recall from §2.1 that $\widetilde{K}$ denotes the number from which $\mathcal{A}_k$ is equal to $\widetilde{\mathcal{A}}$. For $k \geq \widetilde{K} + 1$, the above formulas for the complexity simplify and the difference between $n_k = 2$ and $n_k > 2$ vanishes.

COROLLARY 3.10. *For $k \geq \widetilde{K} + 1$, the complexity function in the range $|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| + 1$ is given by*

$$\mathcal{C}(L) = \begin{cases} \#\widetilde{\mathcal{A}} \cdot L - |p^{(k-1)}| + |p^{(k-2)}| & \text{if } |p^{(k-1)}| + 2 \leq L \leq 2|p^{(k-1)}| - |p^{(k-2)}| + 1, \\ (\#\widetilde{\mathcal{A}} - 1) \cdot L + |p^{(k-1)}| + 1 & \text{if } 2|p^{(k-1)}| - |p^{(k-2)}| + 2 \leq L \leq |p^{(k)}| + 1. \end{cases}$$

Example. For the Grigorchuk subshift, we have $|p^{(k)}| = 2^{k+1} - 1$, $\#\mathcal{A}_0 = 4$, $\#\widetilde{\mathcal{A}} = 3$ and $\widetilde{K} = 1$. In particular, $\mathbb{1}_{\mathcal{A}_1}(a_0) = 0$ holds. From Proposition 3.7 and Theorem 3.8 for $k = 1$ and from Corollary 3.10 for $k \geq 2$, we recover the complexity function that was given for the Grigorchuk subshift in [**24**, Theorem 1]:

$$\begin{aligned} \mathcal{C}(L) &= 3L + 1 & \text{for } 0 \leq L \leq 1, \\ \mathcal{C}(L) &= 3L & \text{for } L = 2, \\ \mathcal{C}(L) &= 2L + 2 & \text{for } 3 \leq L \leq 4 \end{aligned}$$

and

$$\mathcal{C}(L) = \begin{cases} 3 \cdot L - 2^k + 2^{k-1} & \text{if } 2^k + 1 \leq L \leq 2^{k+1} - 2^{k-1}, \\ 2 \cdot L + 2^k & \text{if } 2^{k+1} - 2^{k-1} + 1 \leq L \leq 2^{k+1} \end{cases} \quad \text{for } k \geq 2.$$

PROPOSITION 3.11. *For $k \geq \widetilde{K} + 1$ and $|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| + 1$, the quotient $\mathcal{C}(L)/L$ lies between*

$$\#\widetilde{\mathcal{A}} - 1 < \min\left\{\#\widetilde{\mathcal{A}} - \frac{n_k - 1}{n_k}, \#\widetilde{\mathcal{A}} - \frac{n_{k-1} - 1}{n_{k-1}}\right\} \leq \min_{|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| + 1} \frac{\mathcal{C}(L)}{L}$$

and

$$\max_{|p^{(k-1)}| + 2 \leq L \leq |p^{(k)}| + 1} \frac{\mathcal{C}(L)}{L} = \#\widetilde{\mathcal{A}} - \frac{n_{k-1} - 1}{2n_{k-1} - 1} \leq \#\widetilde{\mathcal{A}} - \frac{1}{3},$$

where the maximum is taken at $L = 2|p^{(k-1)}| - |p^{(k-2)}| + 1$.

Proof. From Corollary 3.10, we obtain

$$\frac{\mathcal{C}(L)}{L} = \begin{cases} \#\widetilde{\mathcal{A}} - \dfrac{|p^{(k-1)}| - |p^{(k-2)}|}{L} & \text{if } |p^{(k-1)}| + 2 \leq L \leq 2|p^{(k-1)}| - |p^{(k-2)}| + 1, \\ \#\widetilde{\mathcal{A}} - 1 + \dfrac{|p^{(k-1)}| + 1}{L} & \text{if } 2|p^{(k-1)}| - |p^{(k-2)}| + 2 \leq L \leq |p^{(k)}| + 1, \end{cases}$$

for $k \geq \widetilde{K} + 1$. Therefore, the maximum in the interval $[|p^{(k-1)}| + 2, |p^{(k)}| + 1]$ is taken at either $L_1 := 2|p^{(k-1)}| - |p^{(k-2)}| + 1$ or $L_2 := 2|p^{(k-1)}| - |p^{(k-2)}| + 2$. Direct computation shows that the value at L_1 is greater.

$$
\begin{aligned}
\frac{\mathcal{C}(L_1)}{L_1} &= \#\widetilde{\mathcal{A}} - \frac{|p^{(k-1)}| - |p^{(k-2)}|}{2|p^{(k-1)}| - |p^{(k-2)}| + 1} = \#\widetilde{\mathcal{A}} - 1 + \frac{|p^{(k-1)}| + 1}{2|p^{(k-1)}| - |p^{(k-2)}| + 1} \\
&> \frac{\mathcal{C}(L_2)}{L_2}.
\end{aligned}
$$

Thus the maximum is taken at L_1 and by dividing both numerator and denominator of $\mathcal{C}(L_1)/L_1$ by $|p^{(k-2)}| + 1$ we obtain the claimed expression. Similarly, the minimum in the interval is taken at either $L_3 = |p^{(k-1)}| + 2$ or at $L_4 := |p^{(k)}| + 1$. While the value of the minimum depends on the values of n_{k-1} and n_k, a short calculation yields the claim since

$$
\begin{aligned}
\frac{\mathcal{C}(L_3)}{L_3} &= \#\widetilde{\mathcal{A}} - \frac{|p^{(k-1)}| - |p^{(k-2)}|}{|p^{(k-1)}| + 2} > \#\widetilde{\mathcal{A}} - \frac{(|p^{(k-1)}| + 1)(1 - 1/n_{k-1})}{|p^{(k-1)}| + 1} \\
&= \#\widetilde{\mathcal{A}} - \frac{n_{k-1} - 1}{n_{k-1}}
\end{aligned}
$$

and

$$
\frac{\mathcal{C}(L_4)}{L_4} = \#\widetilde{\mathcal{A}} - 1 + \frac{|p^{(k-1)}| + 1}{|p^{(k)}| + 1} = \#\widetilde{\mathcal{A}} - 1 + \frac{1}{n_k} = \#\widetilde{\mathcal{A}} - \frac{n_k - 1}{n_k}. \qquad \square
$$

Remark. In particular, Proposition 3.11 yields

$$
\#\widetilde{\mathcal{A}} - 1 \leq \liminf_{L \to \infty} \frac{\mathcal{C}(L)}{L} \leq \limsup_{L \to \infty} \frac{\mathcal{C}(L)}{L} \leq \#\widetilde{\mathcal{A}} - \frac{1}{3} < \#\widetilde{\mathcal{A}}.
$$

Besides the regularity of the Toeplitz word, this is an alternative way to prove unique ergodicity for simple Toeplitz words with $\#\widetilde{\mathcal{A}} \leq 3$: by [8, Theorem 1.5], every minimal subshift with $\limsup_{L \to \infty} \mathcal{C}(L)/L < 3$ is uniquely ergodic. Such a proof was, for example, used in [17].

4. *De Bruijn graphs and palindrome complexity*
In this section, we investigate a sequence of graphs which are called de Bruijn graphs or Rauzy graphs. The Lth graph in this sequence encodes in its vertices which words of length L occur in the subshift. In its edges, it shows by which letter(s) a word can be extended to the right or to the left. To construct the graphs, we rely heavily on the results about the complexity and their proofs from the previous §3. The construction is carried out in detail in the first subsection. In the second subsection, palindromes are discussed. We prove that reflection symmetry in the graphs corresponds to reflection symmetry of the words. This yields an explicit formula for the palindrome complexity.

4.1. *Description of the de Bruijn graphs.*
Let Ω be a subshift. The Lth de Bruijn graph $\mathcal{G}_L = (\mathcal{V}_L, \mathcal{E}_L)$ associated to Ω is a directed graph with vertices

$$
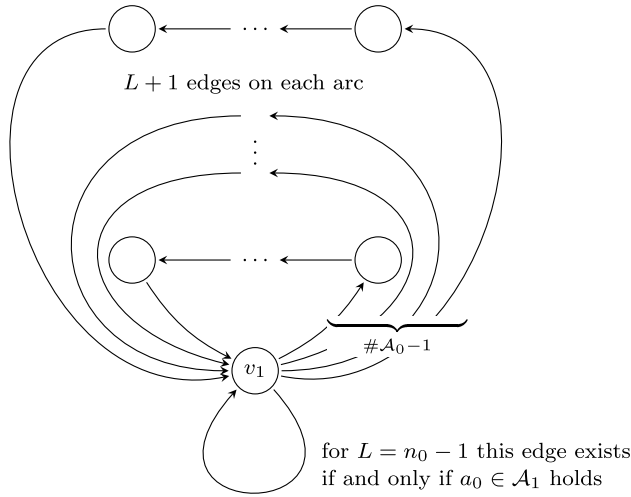\mathcal{V}_L := \{ u \in \mathrm{Sub}(\Omega) : |u| = L \}
$$

FIGURE 3. The de Bruijn graph for $1 \leq L \leq |p^{(0)}|$.

and an edge $(u, v) \in \mathcal{E}_L \subseteq \mathcal{V}_L \times \mathcal{V}_L$ from $u = u(1) \ldots u(L) \in \mathcal{V}_L$ to $v = v(1) \ldots v(L) \in \mathcal{V}_L$ if

$$u(2) \ldots u(L) = v(1) \ldots v(L-1) \quad \text{and} \quad uv(L) = u(1)v \in \text{Sub}(\Omega)$$

hold. The edges can be interpreted as words of length $L + 1$, and thus encode the possible extensions of words of length L. Hence, the results from the previous §3 about right special words and their extensions describe the branching points of the graph and their branching behaviour. We know from the proof of Proposition 3.3 that the suffix v_1, which consists of the last L letters of $p^{(k)}$, is right special and can be extended by all letters in $\mathcal{A}_k$ if $k \geq 0$ and $L \leq |p^{(k)}| - |p^{(k-1)}| - 1$. Another right special word is given in the proof of Proposition 3.4 for $L \geq |p^{(0)}| + 1$. We will discuss its implications later, but first we consider $1 \leq L \leq |p^{(0)}| = n_0 - 1$, where Proposition 3.4 does not apply.

The suffix of length $L \leq |p^{(0)}|$ of $p^{(0)}$ is given by $v_1 = a_0^L$. For $L \leq |p^{(0)}| - |p^{(-1)}| - 1 = n_0 - 2$, the extension with all letters in $\mathcal{A}_0$ is possible. Extension by a_0 yields an edge back to the vertex v_1. Extension by a letter $b \in \mathcal{A}_0 \backslash \{a_0\}$ yields an edge from v_1 to the word $a_0^{L-1}b$. Since v_1 is the prefix of $p^{(0)}$ as well, the word v_1bv_1 exists and we get a loop with $L + 1$ edges from v_1 via $a_0^{L-1}b$ back to v_1. The words encountered in the loop are clearly pairwise different and also different from all words encountered when we extend v_1 with a letter different from b. We obtain the graph shown in Figure 3. For $L = n_0 - 1$, the extensions with letters $b \neq a_0$ remain the same, but the edge from v_1 directly back to itself exists if and only if $a_0 \in \mathcal{A}_1$ holds.

Next we turn to the case $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}|$. To keep the notation short, we define $r := L \bmod (|p^{(k-1)}| + 1)$. By u_1 (respectively, v_1) we denote the prefix (respectively, the suffix) of length L of $p^{(k)}$. It follows from the decomposition $p^{(k)} = p^{(k-1)}a_k p^{(k-1)}a_k \ldots a_k p^{(k-1)}$ that v_1 is obtained when u_1 is shifted $|p^{(k-1)}| - r$ positions to the right.
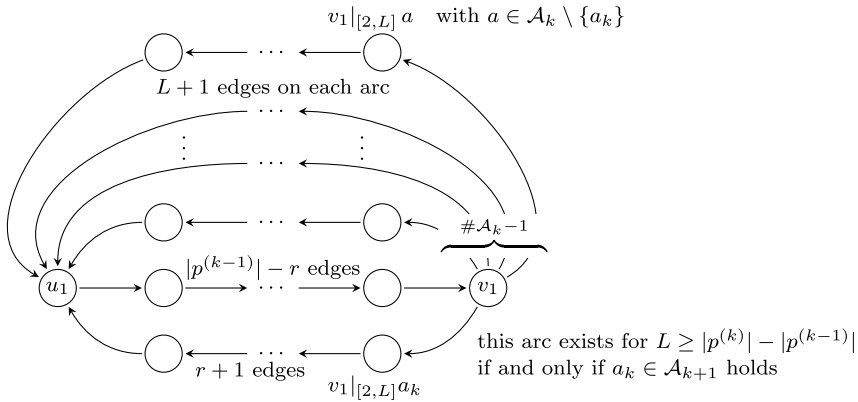
FIGURE 4. The de Bruijn graph $\mathcal{G}_L$ for $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}|$ when either $a_{k-1} \notin \mathcal{A}_k$ or $L > 2|p^{(k-1)}| - |p^{(k-2)}|$ holds. The number of edges of an arc refers to the distance between u_1 and v_1.
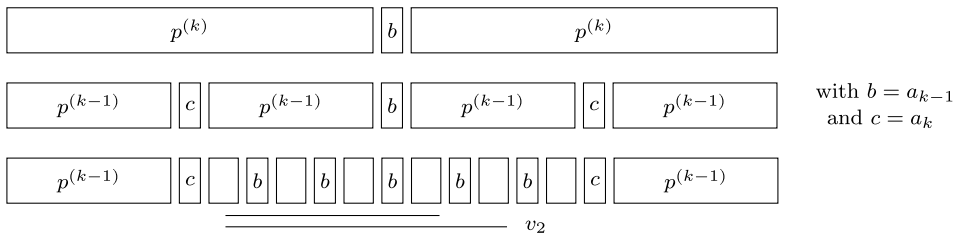

FIGURE 5. Location of the first instance of v_2 that is reached by shifting from v_1: two examples for different word lengths L.

As before, we know that v_1 can be extended to the right by all letters in $\mathcal{A}_k$ if $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| - |p^{(k-1)}| - 1$ holds (see Proposition 3.3). For $|p^{(k)}| - |p^{(k-1)}| \leq L \leq |p^{(k)}|$, the extension of v_1 by a_k exists if and only if $a_k \in \mathcal{A}_{k+1}$ holds. When v_1 is extended with a letter in $\mathcal{A}_k \backslash \{a_k\}$, shifting v_1 by $L + 1$ positions along this extension will result in u_1. When v_1 is extended with a_k, then already a shift by $r + 1$ positions will result in u_1. This yields the graph in Figure 4. If Proposition 3.4 does not apply because either $a_{k-1} \notin \mathcal{A}_k$ or $2|p^{(k-1)}| - |p^{(k-2)}| < L$ holds, then this is the de Bruijn graph for L.

If Proposition 3.4 applies, then the de Bruijn graph has to be adjusted, since there is another right special word $v_2 \neq v_1$. It is the suffix of length L of $p^{(k-1)} a_{k-1} p^{(k-1)}$ and can be extended with both, a_{k-1} and a_k. Note that every suffix of $p^{(k-1)} a_{k-1} p^{(k-1)}$ is contained in $p^{(k)} a_{k-1} p^{(k)}$. To see how many positions we have to shift from the suffix v_1 of the first $p^{(k)}$ to reach v_2, we observe that, after the shift, the right end of the word has to align with the right end of a $p^{(k-2)}$-block. When we reach v_2 for the first time, the leftmost letter of the word has to be in the leftmost $p^{(k-2)} a_{k-1}$ of the rightmost $p^{(k-1)}$-block of the left $p^{(k)}$-block (see Figure 5).

Let $\widetilde{r} := L \bmod (|p^{(k-2)}| + 1)$. Starting in v_1, we reach the branching point v_2 for the first time after $r + 1 + |p^{(k-2)}| - \widetilde{r}$ shifts. The vertex v_2 lies on the arc that represents the extension of v_1 by $a_{k-1} \in \mathcal{A}_k \backslash \{a_k\}$. When we follow the arc from v_1 to v_2, we pass through the prefix of length L of $p^{(k-1)} a_{k-1} p^{(k-1)}$. Denote this prefix by u_2. We reach it
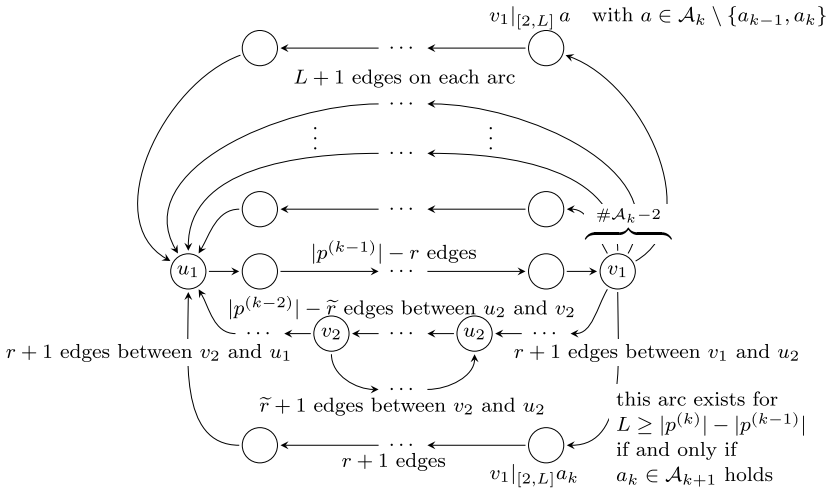
FIGURE 6. The de Bruijn graph $\mathcal{G}_L$ for $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq 2|p^{(k-1)}| - |p^{(k-2)}|$ when $a_{k-1} \in \mathcal{A}_k$ holds. Unless stated otherwise, the number of edges refers to the distance between u_1 and v_1.

after $r + 1$ shifts from v_1. Continuing from u_2 along the arc, we reach v_2, where the arc splits in two paths. One path is the extension of v_2 with a_{k-1}. On this path, we reach the prefix u_2 after $\widetilde{r} + 1$ shifts and upon further shifting proceed once more to v_2. The other path is the extension of v_2 with a_k. This path leads us from v_2 to u_1, which is reached after $L - |p^{(k-1)}| = r + 1$ shifts. When we take these facts into account, we obtain the de Bruijn graph as shown in Figure 6.

From what we have seen in §3, it follows that this describes the de Bruijn graph completely. More precisely, we know from Corollary 3.6 that there are no other branching points besides the ones described in Propositions 3.3 and 3.4 and that these cannot branch into more paths than discussed. Moreover, we know that every word of length $L \leq |p^{(k)}|$ is contained in a word $p^{(k)} a p^{(k)}$ with $a \in \mathcal{A}_{k+1}$ (see Proposition 3.1). For all $a \in \mathcal{A}_{k+1}$, the extension of v_1 by a is included in the graph, together with all other words that follow when we shift further. Thus, neither edges nor vertices are missing.

Example. For the Grigorchuk subshift, $|p^{(0)}| = 1$ and $a = a_0 \notin \mathcal{A}_1$ hold. Thus, we obtain the graph $\mathcal{G}_1$ as shown in Figure 7(a) as a special case of Figure 3. Because of $a_0 \notin \mathcal{A}_1$, it is Figure 4 that describes the de Bruijn graph for $k = 1$. The result is shown in the Figures 7(b) and 7(c). For $k \geq 2$, we always have $a_{k-1} \in \mathcal{A}_k$. Therefore the graph $\mathcal{G}_L$ is given by Figure 6 for $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| - |p^{(k-2)}| - 1$ and by Figure 4 for $|p^{(k)}| - |p^{(k-2)}| \leq L \leq |p^{(k)}|$. The existence condition for the bottommost arc in each figure is always satisfied, since $a_k \in \mathcal{A}_{k+1}$ holds for all $k \geq 1$. Using $\mathcal{A}_k = \{a_{k-1}, a_k, a_{k+1}\}$ and $|p^{(k)}| = 2^{k+1} - 1$, we obtain the graphs shown in Figures 8 and 9.

4.2. *Application: palindrome complexity.*   A word is called a palindrome if it remains the same when read backwards, that is, if

$$u = u(1) \ldots u(L) = u(L) \ldots u(1) =: u^{\mathsf{R}}$$

(a) $\mathcal{G}_1$ with $v_1 = a$

(b) $\mathcal{G}_2$ with $u_1 = ax$, $v_1 = xa$ and $r = 0$

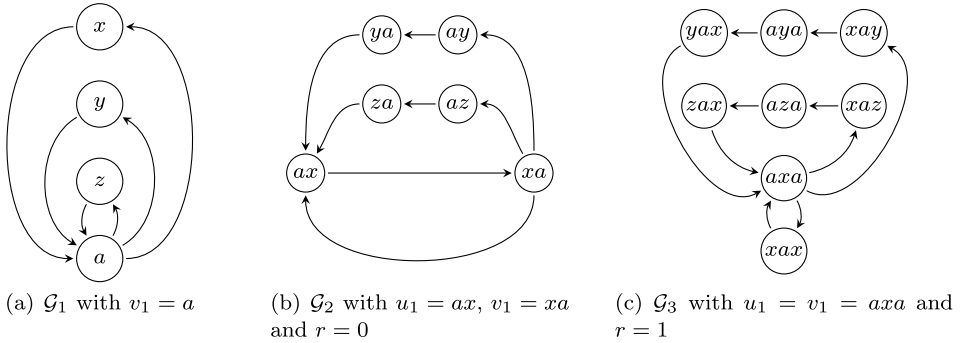(c) $\mathcal{G}_3$ with $u_1 = v_1 = axa$ and $r = 1$

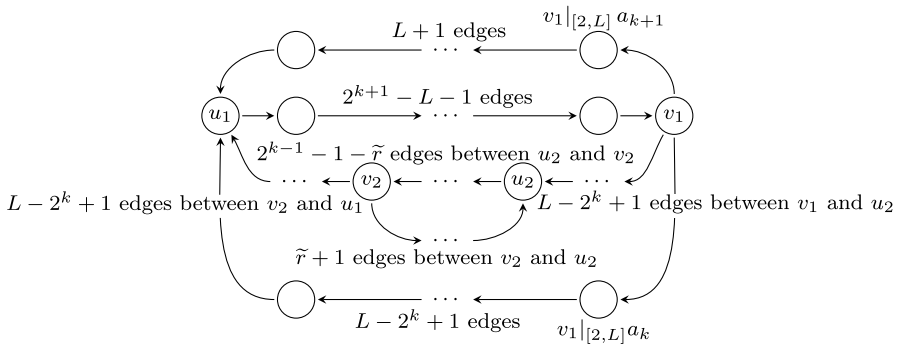FIGURE 7. The first de Bruijn graphs for the Grigorchuk subshift.

FIGURE 8. The de Bruijn graph for the Grigorchuk subshift for $k \geq 2$ and $2^k \leq L \leq 2^{k+1} - 2^{k-1} - 1$. We have $r = L - 2^k$. Unless stated otherwise, the number of edges refers to the distance between u_1 and v_1.
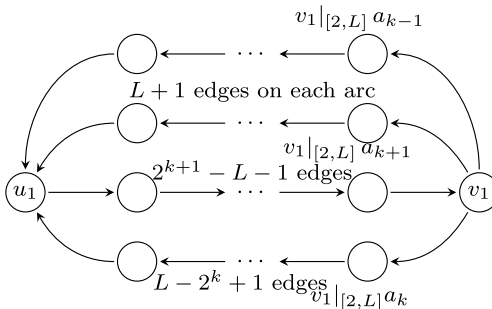

FIGURE 9. The de Bruijn graph for the Grigorchuk subshift for $k \geq 2$ and $2^{k+1} - 2^{k-1} \leq L \leq 2^{k+1} - 1$. We have $r = L - 2^k$. The number of edges of an arc refers to the distance between u_1 and v_1.

holds, where u^{R} denotes the reflection of the word u at its midpoint. Note that $\bullet^{\mathsf{R}}$ defines an involution on the set of subwords of length L of the subshift.

Example. For every $k \in \mathbb{N}_0$, the word $p^{(k)}$ is a palindrome: for $p^{(0)} = a_0^{n_0 - 1}$, this is obviously true and for $k \geq 1$ it follows by induction from the decomposition

$$p^{(k+1)\mathsf{R}} = p^{(k)\mathsf{R}} a_{k+1} p^{(k)\mathsf{R}} \ldots p^{(k)\mathsf{R}} = p^{(k)} a_{k+1} p^{(k)} \ldots p^{(k)} = p^{(k+1)}.$$
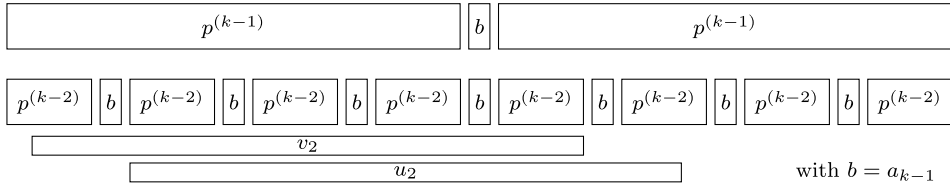
FIGURE 10. Location of v_2 and u_2 in $p^{(k-1)}a_{k-1}p^{(k-1)}$.

Similarly to the subword complexity, we define the palindrome complexity as

$$\mathcal{P} : \mathbb{N} \to \mathbb{N} \quad L \mapsto \#\{u \in \mathrm{Sub}(\Omega) : |u| = L \text{ and } u^{\mathsf{R}} = u\}.$$

To investigate the palindrome complexity of simple Toeplitz subshifts, we will show that reflection symmetry of words corresponds to reflection symmetry in the de Bruijn graphs. Thus, palindromes are precisely the vertices that lie on the reflection axis. Recall that u_1 (respectively, v_1) in $\mathcal{V}_L$ denote the prefix (respectively, suffix) of length L of $p^{(k)}$. Note that $u_1 = v_1$ holds for $1 \le L \le |p^{(0)}|$.

PROPOSITION 4.1. *Reflection at the midpoint of a word corresponds in the Figures 3, 4 and 6 to reflection of the graph at a vertical axis through the middle of the graph.*

Proof. First, we consider the arcs from v_1 to u_1, except for the arc from v_2 to u_2 in Figure 6, which will be considered later. The vertices on every arc from v_1 to u_1 are the subwords that occur in a word of the type $p^{(k)}ap^{(k)}$, $a \in \mathcal{A}_k$, between the suffix of the first $p^{(k)}$ and the prefix of the second $p^{(k)}$. The word $p^{(k)}|_{[|p^{(k)}|-L+1+j,|p^{(k)}|]}ap^{(k)}|_{[1,j-1]}$ is the jth vertex on such an arc, counted from v_1. Here, an empty interval as index means that no letter of this $p^{(k)}$ occurs. The reversed word is

$$p^{(k)}|_{[1,j-1]}^{\mathsf{R}}ap^{(k)}|_{[|p^{(k)}|-L+1+j,|p^{(k)}|]}^{\mathsf{R}} = p^{(k)}|_{[|p^{(k)}|-j+2,|p^{(k)}|]}ap^{(k)}|_{[1,L-j]}.$$

This is the jth vertex on the same arc when counted from u_1. Thus reversing the words corresponds to reflection of the path in the middle.

Now we deal with the path from u_1 to v_1. Let $u := p^{(k)}|_{[j,j+L-1]}$ denote a subword of length L that is contained in $p^{(k)}$ and starts at the jth letter from the left. Then the reflected word is $u^{\mathsf{R}} = p^{(k)}|_{[|p^{(k)}|-j-L+2,|p^{(k)}|-j+1]}$, that is, u^{R} is the subword of $p^{(k)}$ that ends at the jth letter from the right. The jth vertex on the path from u_1 to v_1 is precisely the subword that starts at the jth letter from the left. Hence, reversing the words corresponds to reflecting the path in the middle.

Finally, we consider the arc from v_2 to u_2. Because $L \le 2|p^{(k-1)}| - |p^{(k-2)}|$, there is a copy of u_2 that begins at the start of the second $p^{(k-2)}$-block in $p^{(k-1)}a_{k-1}p^{(k-1)}$ (see Figure 10). Moreover, we can find a copy of v_2 that begins in the first $p^{(k-2)}$-block in $p^{(k-1)}a_{k-1}p^{(k-1)}$. The path from v_2 to u_2 corresponds to the words between them.

Recall the notation $\tilde{r} := L \bmod (|p^{(k-2)}| + 1)$. Moreover, let x_j denote the jth vertex after v_2 on the path to u_2, where $j = 0$ is v_2 and $j = \tilde{r} + 1$ is u_2. The already established symmetry of the arcs between v_1 and u_1 yields the equality $x_0^{\mathsf{R}} = v_2^{\mathsf{R}} = u_2 = x_{\tilde{r}+1}$. For

$j = 1, \ldots, \widetilde{r}$, the jth vertex after v_2 on the path to u_2 is the word

$$
\begin{aligned}
x_j &= p^{(k-1)}|_{[|p^{(k-2)}|+1-\widetilde{r}+j, |p^{(k-1)}|]} a_{k-1} p^{(k-1)}|_{[1, L-|p^{(k-1)}|+|p^{(k-2)}|-\widetilde{r}+j-1]} \\
&= [p^{(k-2)}|_{[|p^{(k-2)}|+1-\widetilde{r}+j, |p^{(k-2)}|]} a_{k-1} p^{(k-2)} \ldots p^{(k-2)}] a_{k-1} \\
&\quad [p^{(k-2)} \ldots p^{(k-2)} a_{k-1} p^{(k-2)}|_{[1, j-1]}].
\end{aligned}
$$

Now we obtain the reflection symmetry of the arc from v_2 to u_2 from the fact that the $p^{(k-2)}$-blocks are palindromes:

$$
\begin{aligned}
x_j{}^{\mathsf{R}} &= p^{(k-2)}|_{[1, j-1]}{}^{\mathsf{R}} a_{k-1} p^{(k-2)} \ldots p^{(k-2)} a_{k-1} p^{(k-2)}|_{[|p^{(k-2)}|+1-\widetilde{r}+j, |p^{(k-2)}|]}{}^{\mathsf{R}} \\
&= p^{(k-2)}|_{[|p^{(k-2)}|-j+2, |p^{(k-2)}|]} a_{k-1} p^{(k-2)} \ldots p^{(k-2)} a_{k-1} p^{(k-2)}|_{[1, \widetilde{r}-j]} \\
&= x_{\widetilde{r}+1-j}. \hspace{6cm} \square
\end{aligned}
$$

The symmetry of the graphs with respect to $\bullet^{\mathsf{R}}$ implies that the number of palindromes of length L is the number of arcs in $\mathcal{G}_L$ with an even number of edges. The formula for the palindrome complexity follows now from the description of the graphs in the previous subsection (see also Figures 3, 4 and 6). As before, we denote $r := L \bmod (|p^{(k-1)}| + 1)$ and $\widetilde{r} := L \bmod (|p^{(k-2)}| + 1)$.

COROLLARY 4.2. *For $1 \leq L \leq |p^{(0)}|$, the palindrome complexity is given by*

$$
\mathcal{P}(L) = (\#\mathcal{A}_0 - 1) \cdot (L \bmod 2) + 1.
$$

For $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}|$, the palindrome complexity is given by

$$
\begin{aligned}
\mathcal{P}(L) = {}& (\#\mathcal{A}_k - 1) \cdot (L \bmod 2) + (|p^{(k-1)}| + 1 - r \bmod 2) \\
&+ (r \bmod 2) \cdot
\begin{cases}
1 & \text{if } |p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| - |p^{(k-1)}| - 1, \\
\mathbb{1}_{\mathcal{A}_{k+1}}(a_k) & \text{if } |p^{(k)}| - |p^{(k-1)}| \leq L \leq |p^{(k)}|,
\end{cases} \\
&+
\begin{cases}
0 \\
(\widetilde{r} \bmod 2) + (|p^{(k-2)}| + 1 - \widetilde{r} \bmod 2) - (L \bmod 2)
\end{cases} \\
&\quad
\begin{cases}
\text{if } & a_{k-1} \notin \mathcal{A}_k \text{ or } L > 2|p^{(k-1)}| - |p^{(k-2)}|, \\
\text{if } & a_{k-1} \in \mathcal{A}_k \text{ and } L \leq 2|p^{(k-1)}| - |p^{(k-2)}|.
\end{cases}
\end{aligned}
$$

Example. For a generalized Grigorchuk subshift and every $k \geq 0$, the length $|p^{(k)}| + 1$ is a power of two and is thus even. Therefore $(r \bmod 2) = (L \bmod 2) = (\widetilde{r} \bmod 2)$ holds and the palindrome complexity simplifies for a generalized Grigorchuk subshift to

$$
\mathcal{P}(L) = (\#\mathcal{A}_0 - 1) \cdot (L \bmod 2) + 1 \quad \text{for } 1 \leq L \leq |p^{(0)}|
$$

and

$$
\begin{aligned}
\mathcal{P}(L) = \Bigg(\#\mathcal{A}_k + {}&
\begin{cases}
1 & \text{if } |p^{(k-1)}| + 1 \leq L \leq |p^{(k)}| - |p^{(k-1)}| - 1, \\
\mathbb{1}_{\mathcal{A}_{k+1}}(a_k) & \text{if } |p^{(k)}| - |p^{(k-1)}| \leq L \leq |p^{(k)}|,
\end{cases} \\
+{}&
\begin{cases}
0 & \text{if } a_{k-1} \notin \mathcal{A}_k \text{ or } L > 2|p^{(k-1)}| - |p^{(k-2)}|, \\
1 & \text{if } a_{k-1} \in \mathcal{A}_k \text{ and } L \leq 2|p^{(k-1)}| - |p^{(k-2)}|,
\end{cases} \Bigg) \cdot (L \bmod 2)
\end{aligned}
$$

for $k \geq 1$ and $|p^{(k-1)}| + 1 \leq L \leq |p^{(k)}|$.

Example. For the (standard) Grigorchuk subshift, $\#\mathcal{A}_0 = 4$ and $a_0 \notin \mathcal{A}_1$ hold. Moreover, we have $|p^{(k)}| + 1 = 2^{k+1}$ for all $k \geq 0$, as well as $\#\mathcal{A}_k = 3$ and $a_k \in \mathcal{A}_{k+1}$ for all $k \geq 1$. This yields

$$\mathcal{P}(L) = \begin{cases} 4 \cdot (L \bmod 2) & \text{if } 1 \leq L \leq 3, \\ 5 \cdot (L \bmod 2) & \text{if } 2^k \leq L \leq 2^{k+1} - 2^{k-1} - 1 \text{ for } k \geq 2, \\ 4 \cdot (L \bmod 2) & \text{if } 2^{k+1} - 2^{k-1} \leq L \leq 2^{k+1} - 1 \text{ for } k \geq 2. \end{cases}$$

5. *Repetitivity*

In this section, we continue our study of combinatorial properties of simple Toeplitz subshifts by investigating the repetitivity function. In the first part of the section, we introduce notation and two tools that describe combinatorial properties of the coding sequence (a_k). In the second part, we give an explicit formula for the repetitivity function of a simple Toeplitz subshift for all $L \geq |p^{(m_1)}| - |p^{(m_1-1)}| + 1$. In the third part, we characterize α-repetitivity of the subshift as well as the special case of linear repetitivity (i.e., $\alpha = 1$).

5.1. *General notions.* Recall that $\mathrm{Sub}(\Omega)$ denotes the set of finite words that appear in elements of the subshift Ω. The repetitivity function is defined as $\mathcal{R} : \mathbb{N} \to \mathbb{N}$, $L \mapsto \min\{\widetilde{L} : \text{every word of length } L \text{ in } \mathrm{Sub}(\Omega) \text{ is contained in every word of length } \widetilde{L} \text{ in } \mathrm{Sub}(\Omega)\}$. It is easy to see that the repetitivity function is strictly monotonically increasing. In order to compute the repetitivity of Ω, we need certain information about the structure of the coding sequence (a_k). The first one is expressed by the values of the function

$$F : \mathbb{N}_0 \to \mathbb{N}, \quad k \mapsto \min\{j > k : \{a_{k+1}, \ldots, a_j\} = \mathcal{A}_{k+1}\}.$$

This function describes at which point all letters that occur after a_k have occurred at least once. Note that $F(k) \geq k + \#\mathcal{A}_{k+1} \geq k + \#\widetilde{\mathcal{A}}$ holds for all $k \in \mathbb{N}_0$.

PROPOSITION 5.1. *The function F is monotonically increasing.*

Proof. We show that $F(k+1) \geq F(k)$ holds for every $k \in \mathbb{N}_0$. In the case of $\mathcal{A}_{k+2} = \mathcal{A}_{k+1}$, this follows from the definition of F, since

$$\begin{aligned} F(k+1) &= \min\{j : \{a_{k+2}, \ldots, a_j\} = \mathcal{A}_{k+1}\} \\ &\geq \min\{j : \{a_{k+1}, \ldots, a_j\} = \mathcal{A}_{k+1}\} \\ &= F(k) \end{aligned}$$

holds. In the case of $\mathcal{A}_{k+2} \neq \mathcal{A}_{k+1}$, we have $\mathcal{A}_{k+2} = \mathcal{A}_{k+1} \backslash \{a_{k+1}\}$. Now the claim follows from the relations

$$\begin{aligned} F(k+1) &= \min\{j : \{a_{k+2}, \ldots, a_j\} = \mathcal{A}_{k+1} \backslash \{a_{k+1}\}\} \\ &= \min\{j : \{a_{k+1}, \ldots, a_j\} = \mathcal{A}_{k+1}\} \\ &= F(k). \end{aligned} \qquad \square$$

As can be seen in the above proof, F is, in general, monotonically, but not strictly monotonically, increasing. The second piece information we need about (a_k) to compute

the repetitivity is at which positions the value of F actually changes. Therefore, we define the sequence $(m_i)_{i\in\mathbb{N}_0}$ by $m_0 = 0$ and $F(m_i) = F(m_i + 1) = \ldots = F(m_{i+1} - 1) < F(m_{i+1})$. It denotes those positions where F is growing. The following two propositions give alternative descriptions of (m_i) that will be helpful when computing the repetitivity.

PROPOSITION 5.2. *The sequence (m_i) can be characterized by $m_0 = 0$ and the recursion relation $m_{i+1} := \max\{j \le F(m_i) : \{a_j, a_{j+1}, \ldots, a_{F(m_i)}\} = \mathcal{A}_{m_i+1}\}$.*

Proof. Let (m_i) denote the previously defined sequence of points at which F is increasing and let $(\widetilde{m}_i)$ denote the sequence defined by the recursion relation in this proposition. We have $m_0 = 0 = \widetilde{m}_0$, by definition, and we proceed by induction, so assume that $m_i = \widetilde{m}_i$ holds.

By definition of F, the equality $\{a_{\widetilde{m}_i+1}, \ldots, a_{F(\widetilde{m}_i)}\} = \mathcal{A}_{\widetilde{m}_i+1}$ holds, which implies that $\widetilde{m}_{i+1} \ge \widetilde{m}_i + 1$. From $\{a_{\widetilde{m}_{i+1}}, \ldots, a_{F(\widetilde{m}_i)}\} = \mathcal{A}_{\widetilde{m}_i+1} \supseteq \mathcal{A}_{\widetilde{m}_{i+1}}$, the inequality

$$F(\widetilde{m}_{i+1} - 1) = \min\{j > \widetilde{m}_{i+1} - 1 : \{a_{\widetilde{m}_{i+1}}, \ldots, a_j\} = \mathcal{A}_{\widetilde{m}_{i+1}}\} \le F(\widetilde{m}_i) = F(m_i)$$

follows, which yields $\widetilde{m}_{i+1} - 1 \le m_i \le m_{i+1} - 1$. On the other hand, we know from the definition of m_{i+1} that $F(m_{i+1} - 1) = F(m_i)$ holds, which implies that $\{a_{m_{i+1}}, \ldots, a_{F(m_i)}\} = \mathcal{A}_{m_{i+1}} \subseteq \mathcal{A}_{m_i+1}$ and thus $\widetilde{m}_{i+1} \ge m_{i+1}$. $\square$

PROPOSITION 5.3. *For $k \ge 1$, the equality $a_k = a_{F(k)}$ holds if and only if $k = m_i$ for some $i \ge 1$.*

Proof. First, assume that $k = m_i$. The definition of $F(m_i - 1)$ yields $\{a_{m_i}, \ldots, a_{F(m_i-1)}\} = \mathcal{A}_{m_i}$. Since F is increasing at m_i, we obtain $F(m_i - 1) \le F(m_i) - 1$ and thus $\{a_{m_i}, \ldots, a_{F(m_i)-1}\} = \mathcal{A}_{m_i}$. Moreover, it follows from the definition of $F(m_i)$ that

$$\{a_{m_i+1}, \ldots, a_{F(m_i)-1}\} = \mathcal{A}_{m_i+1}\backslash\{a_{F(m_i)}\} \subseteq \mathcal{A}_{m_i}\backslash\{a_{F(m_i)}\}$$

holds, which implies that $a_{m_i} = a_{F(m_i)}$.

Now assume that $a_k = a_{F(k)}$ holds for a certain $k \ge 1$. Since a_k appears again at position $F(k) \ge k + 1$, we have $\mathcal{A}_k = \mathcal{A}_{k+1}$. Using $a_k = a_{F(k)}$ and the definition of $F(k)$ we obtain

$$\{a_k, \ldots, a_{F(k)-1}\} = \{a_{k+1}, \ldots, a_{F(k)}\} = \mathcal{A}_{k+1} = \mathcal{A}_k$$

and thus $F(k - 1) \le F(k) - 1$. Therefore F is increasing at k and there exists an $i \ge 1$ such that $k = m_i$ holds. $\square$

Example. Consider a simple Toeplitz subshift with $\#\widetilde{\mathcal{A}} = 2$. Let $k \ge \widetilde{K}$, that is, let k be large enough such that $a_k \in \widetilde{\mathcal{A}}$ and $\mathcal{A}_k = \widetilde{\mathcal{A}}$ hold. For these k, the sequence (a_k) will alternate between the two letters in $\widetilde{\mathcal{A}}$. This implies that $F(k) = k + 2$ for all $k \ge \widetilde{K} - 1$ and hence $m_{i+1} = m_i + 1$.

Example. For a generalized Grigorchuk subshift, $\#\widetilde{\mathcal{A}}$ is either equal to 2 or 3. The case $\#\widetilde{\mathcal{A}} = 2$ was discussed in the previous example. For $\#\widetilde{\mathcal{A}} = 3$, we obtain $\widetilde{\mathcal{A}} = \{x, y, z\}$ and $\widetilde{K} = 1$. Then (m_i) is given by $m_0 = 0$ and $m_{i+1} = F(m_i) - 2$ for $i \ge 0$, which can be seen as follows. For all $i \ge 0$, the number $F(m_i)$ is minimal with $\{a_{m_i+1}, \ldots, a_{F(m_i)}\} = \mathcal{A}_{m_i+1} = \widetilde{\mathcal{A}}$. This yields $a_{F(m_i)} \notin \{a_{m_i+1}, \ldots, a_{F(m_i)-1}\}$. Thus

$a_{F(m_i)-2}, a_{F(m_i)-1}$ and $a_{F(m_i)}$ are pairwise different and $\{a_{F(m_i)-2}, a_{F(m_i)-1}, a_{F(m_i)}\} = \widetilde{\mathcal{A}}$ holds. By Proposition 5.2, we obtain $m_{i+1} = F(m_i) - 2$. Note that we have $F(k) = k + 3$ for all $k \geq 0$ for the special case of the standard Grigorchuk subshift, and thus $m_{i+1} = m_i + 1$. There is, however, no explicit formula for $F(k)$ for the generalized Grigorchuk subshift with $\#\widetilde{\mathcal{A}} = 3$, since arbitrarily long blocks in the sequence (a_k) are possible in which one letter of $\widetilde{\mathcal{A}}$ is missing.

5.2. *Computing the repetitivity function.* In this subsection, we give an explicit formula for the repetitivity function of a simple Toeplitz subshift. We start by proving lower and upper bounds for the repetitivity function. Before we do so, recall from Proposition 3.2 and Corollary 3.6 that the words that have length less than or equal to $|p^{(k)}| + 1$ are precisely the subwords of $p^{(k)} a p^{(k)}$ for $a \in \mathcal{A}_{k+1}$. Moreover, all subwords of length $|p^{(k)}| + 1$ that start in $p^{(k)} a$ for $a \in \mathcal{A}_{k+1} \backslash \{a_k\}$ and all subwords that start in $p^{(k-1)} a_k$ (provided that $a_k \in \mathcal{A}_{k+1}$ holds) are pairwise different.

PROPOSITION 5.4. *The inequality*

$$\mathcal{R}(|p^{(m_i)}| - |p^{(m_i-1)}| + 1) \geq 2 \cdot (|p^{(F(m_i)-1)}| + 1)$$

holds for all $i \geq 1$.

Proof. By definition of F, the letter $a_{F(m_i)}$ is not in $\{a_{m_i+1}, \ldots, a_{F(m_i)-1}\}$. Because $a_{F(m_i)} \in \mathcal{A}_{m_i+1}$, the word $p^{(m_i)} a_{F(m_i)} p^{(m_i)}$ occurs in the subshift. Let v denote the suffix of length $|p^{(m_i)}| + 1 - |p^{(m_i-1)}|$ of $p^{(m_i)} a_{F(m_i)}$. Now decompose $p^{(m_i)}$ into $p^{(m_i-1)}$-blocks,

$$p^{(m_i)} a_{F(m_i)} = [p^{(m_i-1)} a_{m_i} p^{(m_i-1)} \ldots p^{(m_i-1)} a_{m_i} p^{(m_i-1)}] a_{F(m_i)}$$
$$= p^{(m_i-1)} \underbrace{a_{F(m_i)} p^{(m_i-1)} \ldots p^{(m_i-1)} a_{F(m_i)} p^{(m_i-1)} a_{F(m_i)}}_{\text{the suffix } v},$$

where we used that $a_{F(m_i)} = a_{m_i}$ holds by Proposition 5.3. We conclude that n_{m_i} consecutive single letters in the $p^{(m_i-1)}$-block decomposition of v are the letter $a_{F(m_i)}$.

Now let $b \in \mathcal{A}_{F(m_i)} \backslash \{a_{F(m_i)}\}$ be another letter. The word $u := p^{(F(m_i)-1)} b p^{(F(m_i)-1)}$ occurs in the subshift since $b \in \mathcal{A}_{F(m_i)+1}$ holds. We can split $p^{(F(m_i)-1)}$ into $p^{(m_i)}$-blocks and single letters from $\{a_{m_i+1}, \ldots, a_{F(m_i)-1}\}$. Since $a_{F(m_i)}$ is not among these letters, at most $n_{m_i} - 1$ consecutive single letters in the $p^{(m_i-1)}$-block decomposition of u are the letter $a_{F(m_i)}$. Thus v is not contained in u, which yields

$$\mathcal{R}(|p^{(m_i)}| + 1 - |p^{(m_i-1)}|) = \mathcal{R}(|v|) > |u| = 2 \cdot |p^{(F(m_i)-1)}| + 1. \qquad \square$$

PROPOSITION 5.5. *The inequality*

$$\mathcal{R}(|p^{(m_i)}| + 2) \geq 2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i)}| + 1$$

holds for all $i \geq 1$.

Proof. Similarly to the previous proof, we construct a word v of length $|p^{(m_i)}| + 2$ and a word u of length $2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i)}|$ that does not contain v. Let

$b \in \mathcal{A}_{F(m_i)+1} \setminus \{a_{F(m_i)}\}$. First, note that $u := p^{(F(m_i)-1)} b p^{(F(m_i)-1)} a_{F(m_i)} p^{(m_i)}$ occurs in $p^{(F(m_i))} b p^{(F(m_i))}$ and that $v := a_{F(m_i)} p^{(m_i)} a_{m_i+1}$ occurs in $p^{(F(m_i))}$.

We have seen in the previous proof that $a_{F(m_i)}$ does not appear as a single letter in the $p^{(m_i)}$-block decomposition of $p^{(F(m_i)-1)}$. Any two $p^{(m_i)}$-blocks in $p^{(F(m_i)-1)} b p^{(F(m_i)-1)} a_{F(m_i)} p^{(m_i)}$ are therefore separated by a letter that is not $a_{F(m_i)}$, except for the next to last block and the last block. Hence, the word $v = a_{F(m_i)} p^{(m_i)} a_{m_i+1}$ does not occur in $p^{(F(m_i)-1)} b p^{(F(m_i)-1)}$. Because $a_{F(m_i)} = a_{m_i} \neq a_{m_i+1}$, it does not occur in the $a_{m_i+1} p^{(m_i)} a_{F(m_i)} p^{(m_i)}$-suffix of u either, since subwords of length $|p^{(m_i)}| + 1$ of $p^{(m_i)} a p^{(m_i)}$ are pairwise different for different letters a. Thus, v does not occur in u, which proves the claim. □

Remark. Note that, in the case of $m_{i+1} = m_i + 1$ and $n_{m_{i+1}} = 2$, the Propositions 5.4 and 5.5 give a lower bound for the repetitivity function at the same length, since

$$|p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| + 1 = n_{m_{i+1}}(|p^{(m_i)}| + 1) - |p^{(m_i)}| = |p^{(m_i)}| + 2$$

holds. The lower bound given by Proposition 5.4 is stronger (that is, higher), as can be seen from the following direct computation:

$$2(|p^{(F(m_{i+1})-1)}| + 1) = 2 \cdot n_{F(m_{i+1})-1} \cdot (|p^{(F(m_{i+1})-2)}| + 1)$$
$$\geq 2 \cdot n_{F(m_{i+1})-1} \cdot (|p^{(F(m_i)-1)}| + 1)$$
$$= 2(|p^{(F(m_i)-1)}| + 1) + 2(n_{F(m_{i+1})-1} - 1)(|p^{(F(m_i)-1)}| + 1)$$
$$> 2(|p^{(F(m_i)-1)}| + 1) + |p^{(m_i)}| + 1.$$

The reason why the case $n_{m_{i+1}} = 2$ is special, is that the word $a_{m_i+1} p^{(m_i)} a_{m_i+1}$ is not a subword of $p^{(m_i+1)}$ any more. Moreover, $m_{i+1} = m_i + 1$ implies that $a_{m_i+1} = a_{m_{i+1}} = a_{F(m_{i+1})}$ and this letter is lacking in $\{a_{m_i+2}, \ldots, a_{F(m_{i+1})-1}\}$. Thus we now need to look at a much longer word to see $a_{m_i+1} p^{(m_i)} a_{m_i+1}$ than when it appears as a subword in every $p^{(m_i+1)}$ block.

PROPOSITION 5.6. *The inequality*

$$\mathcal{R}(|p^{(m_i)}| - |p^{(m_i-1)}|) \leq 2 \cdot (|p^{(F(m_i-1)-1)}| + 1) + |p^{(m_i)}| - |p^{(m_i-1)}| - 1$$

holds for all $i \geq 1$.

Proof. To shorten the notation, we introduce $\widetilde{m} := F(m_{i-1}) - 1 = F(m_i - 1) - 1$. We have to prove that every word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i)}| - |p^{(m_i-1)}| - 1$ contains all subwords of length $|p^{(m_i)}| - |p^{(m_i-1)}|$ of all words $p^{(m_i)} a p^{(m_i)}$ with $a \in \mathcal{A}_{m_i+1}$.

Clearly, every word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i)}| - |p^{(m_i-1)}| - 1$ contains the block $p^{(\widetilde{m})}$ at least once. For $a = a_{m_i}$, it is sufficient to consider subwords of $p^{(m_i)} a p^{(m_i)}$ that start in $p^{(m_i-1)} a$. Consequently, all subwords of $p^{(m_i)} a_{m_i} p^{(m_i)}$ of length $|p^{(m_i)}| - |p^{(m_i-1)}|$ are contained in $p^{(m_i)}$, which is contained in $p^{(\widetilde{m})}$. For all $a \in \{a_{m_i+1}, \ldots, a_{\widetilde{m}}\}$, the decomposition

$$p^{(\widetilde{m})} = p^{(m_i)} a_{m_i+1} \ldots a_{m_i+1} p^{(m_i)} a_{m_i+2} p^{(m_i)} a_{m_i+1} \ldots a_{m_i+1} p^{(m_i)} \ldots a_{\widetilde{m}} \ldots$$

yields that the word $p^{(m_i)} a p^{(m_i)}$ is contained in $p^{(\widetilde{m})}$ as well.

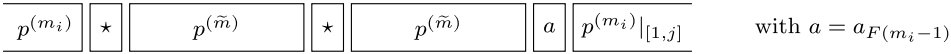| $p^{(m_i)}$ | $\star$ | $p^{(\widetilde{m})}$ | $\star$ | $p^{(\widetilde{m})}$ | a | $p^{(m_i)}|_{[1,j]}$ | with $a = a_{F(m_i-1)}$ |

FIGURE 11. Decomposition of a word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i)}| - |p^{(m_i-1)}| - 1$.

Because of $\mathcal{A}_{m_i+1} \subseteq \mathcal{A}_{m_i} = \{a_{m_i}, \ldots, a_{F(m_i-1)}\}$, the only remaining case is $a = a_{F(m_i-1)}$. For this case, we need to refine the above arguments. First, we note that a word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i)}| - |p^{(m_i-1)}| - 1$ contains a $p^{(\widetilde{m})}$-block together with both neighbouring single letters. From the $p^{(\widetilde{m})}$-block decomposition, it is clear that at least one of the neighbouring single letters has to be $a_{F(m_i-1)}$. Let us assume it is the right letter (the case where it is the left one can be treated similarly). To the right of $a_{F(m_i-1)}$, the next $p^{(m_i)}$ begins. We distinguish two cases.

First, assume that there are at least $|p^{(m_i)}| - |p^{(m_i-1)}| - 1$ letters to the right of $a_{F(m_i-1)}$. In this case, all subwords of length $|p^{(m_i)}| - |p^{(m_i-1)}|$ of $p^{(m_i)}a_{F(m_i-1)}p^{(m_i)}$, which start in $p^{(m_i)}a_{F(m_i-1)}$, are contained in our word.

Second, we have the case where there are $0 \leq j < |p^{(m_i)}| - |p^{(m_i-1)}| - 1$ letters to the right of $a_{F(m_i-1)}$ (see Figure 11). Then there are $|p^{(\widetilde{m})}| + 1 + |p^{(\widetilde{m})}| + 1 + |p^{(m_i)}| - |p^{(m_i-1)}| - j - 2$ letters to the left of $a_{F(m_i-1)}$, that is, a $p^{(\widetilde{m})}$-block, a single letter, another $p^{(\widetilde{m})}$-block, another single letter and the rightmost $|p^{(m_i)}| - |p^{(m_i-1)}| - 2 - j$ letters of $p^{(m_i)}$. Again, at least one of the single letters is $a_{F(m_i-1)}$. If it is the one in the middle, then our word contains $p^{(m_i)}a_{F(m_i-1)}p^{(m_i)}$ and the proof is complete. Otherwise, note that the right end of our word is $p^{(m_i)}a_{F(m_i-1)}p^{(m_i)}|_{[1,j]}$. This contains all subwords of length $|p^{(m_i)}| - |p^{(m_i-1)}|$ of $p^{(m_i)}a_{F(m_i-1)}p^{(m_i)}$ that start in $p^{(m_i)}|_{[1,|p^{(m_i-1)}|+j+2]}$. In addition, the left end of our word is $p^{(m_i)}|_{[|p^{(m_i-1)}|+j+3,|p^{(m_i)}|]}a_{F(m_i-1)}p^{(m_i)}$. This contains the remaining subwords of length $|p^{(m_i)}| - |p^{(m_i-1)}|$ of $p^{(m_i)}a_{F(m_i-1)}p^{(m_i)}$. $\square$

PROPOSITION 5.7. *The inequality*

$$\mathcal{R}(|p^{(m_i)}| + 1) \leq 2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i-1)}|$$

holds for all $i \geq 1$.

Proof. The proof is similar to the proof of Proposition 5.6. We write $\widetilde{m} = F(m_i) - 1$ and show that every word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i-1)}|$ contains all subwords of length $|p^{(m_i)}| + 1$ of all words of the form $p^{(m_i)}ap^{(m_i)}$ with $a \in \mathcal{A}_{m_i+1}$. First, note that every word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i-1)}|$ contains at least once the word $p^{(\widetilde{m})}$. Because of the decomposition

$$p^{(\widetilde{m})} = p^{(m_i)}a_{m_i+1} \ldots a_{m_i+1}p^{(m_i)}a_{m_i+2}p^{(m_i)}a_{m_i+1} \ldots a_{m_i+1}p^{(m_i)} \ldots a_{\widetilde{m}} \ldots,$$

all words $p^{(m_i)}ap^{(m_i)}$ with $a \in \{a_{m_i+1}, \ldots, a_{F(m_i)-1}\} = \mathcal{A}_{m_i+1} \setminus \{a_{F(m_i)}\}$ are contained in $p^{(\widetilde{m})}$. Hence only the case $a = a_{F(m_i)} = a_{m_i}$ remains. To deal with this case, note that a word of length $2 \cdot (|p^{(\widetilde{m})}| + 1) + |p^{(m_i-1)}|$ actually contains a complete $p^{(\widetilde{m})}$-block together with both neighbouring letters. At least one of these letters is $a_{\widetilde{m}+1} = a_{m_i}$ and we will assume that it is the right one (the other case can be treated similarly).

Recall that it suffices to consider those subwords of length $|p^{(m_i)}| + 1$ of $p^{(m_i)}a_{m_i}p^{(m_i)}$ that start in $p^{(m_i-1)}a_{m_i}$. If there are at least $|p^{(m_i-1)}|$ letters right of a_{m_i}, then these

subwords are all contained in our word. Now assume that there are only $0 \leq j < |p^{(m_i-1)}|$ letters right of a_{m_i}. Then there are $|p^{(m_i-1)}| - 1 - j + 1 + |p^{(\tilde{m})}| + 1 + |p^{(\tilde{m})}|$ letters left of a_{m_i}, that is, a $p^{(\tilde{m})}$-block, a single letter, another $p^{(\tilde{m})}$-block, another single letter and the rightmost $|p^{(m_i-1)}| - 1 - j$ letters of $p^{(m_i-1)}$. Again, at least one of the single letters is a_{m_i}. If it is the one in the middle, then our word contains $p^{(m_i)} a_{m_i} p^{(m_i)}$ and the proof is complete. Otherwise, note that the right end of our word is $p^{(m_i)} a_{m_i} p^{(m_i-1)}|_{[1,j]}$. This contains all subwords of length $|p^{(m_i)}| + 1$ of $p^{(m_i)} a_{m_i} p^{(m_i)}$ that start in $p^{(m_i-1)}|_{[1,1+j]}$. The left end of our word is $p^{(m_i-1)}|_{[2+j,|p^{(m_i-1)}|]} a_{m_i} p^{(m_i)}$. This contains the remaining subwords of length $|p^{(m_i)}| + 1$ of $p^{(m_i)} a_{m_i} p^{(m_i)}$. $\qquad\square$

Remark. Similarly to what we observed for the lower bounds, Propositions 5.6 and 5.7 refer to the same length in the case of $m_{i+1} = m_i + 1$ and $n_{m_{i+1}} = 2$ since, in that case,

$$|p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| = |p^{(m_{i+1}-1)}| + 1 = |p^{(m_i)}| + 1$$

holds. The upper bound given by Proposition 5.7 is stronger (that is, lower), as a direct computation shows:

$$2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i-1)}|$$
$$< 2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i)}|$$
$$= 2 \cdot (|p^{(F(m_{i+1}-1)-1)}| + 1) + |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| - 1.$$

THEOREM 5.8. *For $i \geq 1$ and $|p^{(m_i)}| - |p^{(m_i-1)}| + 1 \leq L \leq |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|$, the repetitivity function is given by*

$$\mathcal{R}(L) = \begin{cases} 2|p^{(F(m_i)-1)}| + 1 - |p^{(m_i)}| + |p^{(m_i-1)}| + L, \\ 2|p^{(F(m_i)-1)}| + 1 + L, \end{cases}$$
$$\begin{cases} \textit{for} \quad |p^{(m_i)}| - |p^{(m_i-1)}| + 1 \leq L \leq |p^{(m_i)}| + 1, \\ \textit{for} \quad |p^{(m_i)}| + 2 \leq L \leq |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|. \end{cases}$$

Proof. First, we show that the lower bound in Proposition 5.4 and the upper bound in Proposition 5.7 are actually equalities and that the repetitivity function increases by exactly one in between. We use the fact that the repetitivity function is strictly increasing and its growth is therefore always at least one:

$$2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i-1)}|$$
$$\geq \mathcal{R}(|p^{(m_i)}| + 1) \quad \text{by Proposition 5.7}$$
$$= \mathcal{R}(|p^{(m_i)}| - |p^{(m_i-1)}| + 1) + \sum_{L=|p^{(m_i)}|-|p^{(m_i-1)}|+1}^{|p^{(m_i)}|} [\mathcal{R}(L+1) - \mathcal{R}(L)]$$
$$\geq 2 \cdot (|p^{(F(m_i)-1)}| + 1) + 1 \cdot |p^{(m_i-1)}| \quad \text{by Proposition 5.4.}$$

This yields

$$\mathcal{R}(L) = 2 \cdot (|p^{(F(m_i)-1)}| + 1) + L - (|p^{(m_i)}| - |p^{(m_i-1)}| + 1)$$

for L from $|p^{(m_i)}| - |p^{(m_i-1)}| + 1$ to $|p^{(m_i)}| + 1$. If $m_{i+1} = m_i + 1$ and $n_{m_{i+1}} = 2$ hold, then $|p^{(m_i)}| + 1 = |p^{(m_{i+1}-1)}| + 1 = |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|$ follows and the proof

is complete. If either $m_{i+1} > m_i + 1$ or $n_{m_{i+1}} > 2$ holds, then $|p^{(m_i)}| + 1 < |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|$ follows and we have yet to consider the lengths L from $|p^{(m_i)}| + 2$ to $|p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|$. For this, we show that the inequalities in Propositions 5.5 and 5.6 are actually equalities and that the repetitivity function increases by exactly one in between:

$$
\begin{aligned}
&2 \cdot (|p^{(F(m_{i+1}-1)-1)}| + 1) + |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| - 1 \\
&\geq \mathcal{R}(|p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|) \quad \text{by Proposition 5.6} \\
&= \mathcal{R}(|p^{(m_i)}| + 2) + \sum_{L=|p^{(m_i)}|+2}^{|p^{(m_{i+1})}|-|p^{(m_{i+1}-1)}|-1} [\mathcal{R}(L+1) - \mathcal{R}(L)] \\
&\geq 2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_i)}| + 1 + |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| - |p^{(m_i)}| - 2 \\
&\quad \text{by Proposition 5.5} \\
&= 2 \cdot (|p^{(F(m_i)-1)}| + 1) + |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| - 1 \\
&= 2 \cdot (|p^{(F(m_{i+1}-1)-1)}| + 1) + |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}| - 1. \qquad \square
\end{aligned}
$$

Remark. For $m_{i+1} = m_i + 1$ and $n_{m_{i+1}} = 2$, the repetitivity function simplifies to

$$
\mathcal{R}(L) = 2 \cdot |p^{(F(m_i)-1)}| + 1 - |p^{(m_i)}| + |p^{(m_i-1)}| + L
$$

for $|p^{(m_i)}| - |p^{(m_i-1)}| + 1 \leq L \leq |p^{(m_i+1)}| - |p^{(m_i)}|$.

Remark. Roughly speaking, the jump between $\mathcal{R}(|p^{(m_i)}| + 1)$ and $\mathcal{R}(|p^{(m_i)}| + 2)$ is caused by the fact that it is sufficient to consider the subwords of $p^{(m_i)} a p^{(m_i)}$ with $a \in \mathcal{A}_{m_i+1}$ when we are interested in all words of length $|p^{(m_i)}| + 1$, but this is not true when we wish to deal with words of length $|p^{(m_i)}| + 2$. Here we have to consider words of the type $a_1 p^{(m_i)} a_2$ for $a_1, a_2 \in \mathcal{A}_{m_i+1}$ as well, and, to see all possibilities, we have to look at a much longer word. The jump between $\mathcal{R}(|p^{(m_i)}| - |p^{(m_i-1)}|)$ and $\mathcal{R}(|p^{(m_i)}| - |p^{(m_i-1)}| + 1)$ is occurs for a similar reason: all subwords of length $|p^{(m_i)}| - |p^{(m_i-1)}|$ of $p^{(m_i)} a_{m_i} p^{(m_i)}$ are contained in $p^{(m_i)}$. This is not true for subwords of length $|p^{(m_i)}| - |p^{(m_i-1)}| + 1$, where we have to look at a much longer word to see the next occurrence of a_{m_i}. In the special cases of $n_{m_{i+1}} = 2$ and $m_{i+1} = m_i + 1$, the positions for the two jumps coincide.

5.3. *Application: α-repetitivity.* In this subsection, we will use the above formula for the repetitivity function to investigate linear repetitivity and, more generally, α-repetitivity of simple Toeplitz subshifts. Since the repetitivity is strictly increasing and clearly $1 < \#\mathcal{A} \leq \mathcal{R}(1)$ holds, we obtain $1 < \mathcal{R}(L)/L$ for all $L \geq 1$. If there exists a constant C such that $\mathcal{R}(L)/L \leq C$ holds for all $L \geq 1$, then the subshift is called linear repetitive. This was generalized in [**28**, Definition 2.9], where a subshift is called α-repetitive for $\alpha \geq 1$ if $0 < \limsup_{L \to \infty} \mathcal{R}(L)/L^\alpha < \infty$ holds. In [**17**, Theorem 4.10], α-repetitivity is characterized for l-Grigorchuk subshifts. Below we give a characterization for simple Toeplitz subshifts. For $m_i = i$, $F(m_i) = m_i + 3$ and $n_j = 2^{l_j}$, the following proposition yields precisely the result for l-Grigorchuk subshifts from [**17**].

PROPOSITION 5.9. *Let $\alpha \geq 1$. A simple Toeplitz subshift is α-repetitive if and only if the inequalities*

$$0 < \limsup_{i \to \infty} \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha} = \limsup_{i \to \infty} \frac{n_0 \cdot \ldots \cdot n_{F(m_i)-1}}{n_0^\alpha \cdot \ldots \cdot n_{m_i}^\alpha} < \infty$$

hold.

Proof. We use the result from Theorem 5.8. The quotient takes the form $\mathcal{R}(L)/L^\alpha = \text{const}/L^\alpha + L^{1-\alpha}$, where the constant depends on whether $|p^{(m_i)}| - |p^{(m_i-1)}| + 1 \leq L \leq |p^{(m_i)}| + 1$ or $|p^{(m_i)}| + 2 \leq L \leq |p^{(m_{i+1})}| - |p^{(m_{i+1}-1)}|$ holds and on the value of i. For every i, the quotient is maximal either at $L_1(i) := |p^{(m_i)}| - |p^{(m_i-1)}| + 1$ or at $L_2(i) := |p^{(m_i)}| + 2$. At these points, the upper and lower bounds

$$2 \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha} < \frac{\mathcal{R}(L_1(i))}{L_1(i)^\alpha} < \frac{2 \cdot (|p^{(F(m_i)-1)}| + 1)}{(1 - 1/n_{m_i})^\alpha \cdot (|p^{(m_i)}| + 1)^\alpha} \leq 2^{1+\alpha} \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha}$$

and

$$2^{1-\alpha} \cdot \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha} = \frac{2 \cdot (|p^{(F(m_i)-1)}| + 1)}{(2 \cdot (|p^{(m_i)}| + 1))^\alpha} < \frac{\mathcal{R}(L_2(i))}{L_2(i)^\alpha} < 3 \cdot \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha}$$

hold. If we assume that $0 < \limsup_{i \to \infty} (|p^{(F(m_i)-1)}| + 1)/(|p^{(m_i)}| + 1)^\alpha < \infty$ holds, then the above bounds yield

$$0 < \limsup_{i \to \infty} 2 \cdot \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha} \leq \limsup_{L \to \infty} \frac{\mathcal{R}(L)}{L^\alpha}$$

$$\leq \limsup_{i \to \infty} 2^{1+\alpha} \cdot \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha} < \infty,$$

and thus the subshift is α-repetitive. Conversely, if we assume that the subshift is α-repetitive, that is, $0 < \limsup_{L \to \infty} \mathcal{R}(L)/L^\alpha < \infty$ holds, then the above bounds yield

$$0 < 2^{-1-\alpha} \cdot \limsup_{L \to \infty} \frac{\mathcal{R}(L)}{L^\alpha} < \limsup_{i \to \infty} \frac{|p^{(F(m_i)-1)}| + 1}{(|p^{(m_i)}| + 1)^\alpha} < 2^{-1+\alpha} \cdot \limsup_{L \to \infty} \frac{\mathcal{R}(L)}{L^\alpha} < \infty.$$

$\square$

COROLLARY 5.10. *A simple Toeplitz subshift is linear repetitive if and only if $(\prod_{j=m_i+1}^{F(m_i)-1} n_j)_{i \geq 1}$ is bounded from above.*

Proof. By Proposition 5.9, the subshift is linear repetitive ($\alpha = 1$) if and only if

$$0 < \limsup_{i \to \infty} \prod_{j=m_i+1}^{F(m_i)-1} n_j < \infty$$

holds. Because $n_j \geq 2$ for all $j \geq 0$ and $F(k) \geq k + \#\mathcal{A}_{k+1} \geq k + 2$ for all $k \geq 0$, the product is bounded from below by 2 for all $i \geq 0$. $\square$

Remark. The product $\prod_{j=m_i+1}^{F(m_i)-1} n_j$ gives the length of the period of the word $(a_{m_i+1}^{n_{m_i+1}-1}?)^\infty \lhd \ldots \lhd (a_{F(m_i)-1}^{n_{F(m_i)-1}-1}?)^\infty$. Thus a simple Toeplitz subshift is linear repetitive if and only if the sequence of the period lengths of the words $(a_{m_i+1}^{n_{m_i+1}-1}?)^\infty \lhd \ldots \lhd (a_{F(m_i)-1}^{n_{F(m_i)-1}-1}?)^\infty$ is bounded.

COROLLARY 5.11. *On the one hand, if the sequence $(n_k)_{k \geq 0}$ is bounded, then the subshift is linear repetitive if and only if the sequence $(F(m_i) - m_i)_{i \geq 0}$ is bounded. On the other hand, if the sequence $(F(m_i) - m_i)_{i \geq 0}$ is bounded, then the subshift is linear repetitive if and only if the sequence $(n_k)_{k \geq 0}$ is bounded.*

Remark. The difference $F(k) - k$ describes how many positions in the coding sequence, starting in a_{k+1}, we have to look at to see all letters that can occur from this point on. Taking the difference $F(k) - k$ at the points $k = m_i$ ensures that we get the largest possible distances, since (m_i) denotes the positions where the value of F increases.

COROLLARY 5.12. *If $(n_k)_{k \geq 0}$ is a constant sequence, then the subshift is α-repetitive if and only if $-\infty < \limsup_{i \to \infty}[F(m_i) - \alpha \cdot m_i] < \infty$ holds.*

Proof. Let n be the constant value of the sequence $(n_k)_k$. By Proposition 5.9, a subshift is α-repetitive if and only if

$$0 < \limsup_{i \to \infty} \frac{n_0 \cdot \ldots \cdot n_{F(m_i)-1}}{n_0^\alpha \cdot \ldots \cdot n_{m_i}^\alpha} < \infty$$

$$\Longleftrightarrow \quad 0 < \limsup_{i \to \infty} n^{F(m_i) - \alpha \cdot (m_i + 1)} < \infty$$

$$\Longleftrightarrow \quad -\infty < \limsup_{i \to \infty}[F(m_i) - \alpha \cdot (m_i + 1)] < \infty. \qquad \square$$

Example. For the Grigorchuk subshift, $(n_k)_k$ is the constant sequence with value two. Moreover $m_i = i$ and $F(k) = k + 3$ hold for all $i, k \geq 0$. Hence the Grigorchuk subshift is α-repetitive if and only if $-\infty < \limsup_{i \to \infty}[(1 - \alpha)i + 3] < \infty$ holds. Thus, it is 1-repetitive, that is, linear repetitive.

COROLLARY 5.13. *Let $(F(m_i) - m_i)_{i \geq 0}$ be a constant sequence with value c. If $n_{j+c-1} = n_j^\alpha$ holds for all $j \geq 0$, then the subshift is α-repetitive. In particular, the subshift is α-repetitive if $F(m_i) - m_i = c$ is constant and $n_{j+1} = n_j^{\sqrt[c-1]{\alpha}}$ holds for all $j \geq 0$.*

Proof. Since $F(k) \geq k + 2$ holds for all $k \geq 0$, we have $c \geq 2$. The product

$$\frac{n_0 \cdot \ldots \cdot n_{F(m_i)-1}}{n_0^\alpha \cdot \ldots \cdot n_{m_i}^\alpha} = \frac{\prod_{j=0}^{c-2} n_j \cdot \prod_{j=c-1}^{m_i+c-1} n_j}{\prod_{j=0}^{m_i} n_j^\alpha} = \frac{\prod_{j=0}^{c-2} n_j \cdot \prod_{j=0}^{m_i} n_{j+c-1}}{\prod_{j=0}^{m_i} n_j^\alpha} = \prod_{j=0}^{c-2} n_j$$

is positive, finite and independent of i. Now Proposition 5.9 yields the claim. $\qquad \square$

6. *The Boshernitzan condition and Jacobi cocycles*

In this section, the Boshernitzan condition is discussed. It can be thought of as a weaker analogue of linear repetitivity and was characterized for simple Toeplitz subshifts in [**35**]. Based on another result from [**35**], we give a different characterization, which describes the Boshernitzan condition in terms of the function F and the sequence of period lengths $(n_k)_k$. As corollaries, we obtain particular simple descriptions of the Boshernitzan condition for generalized Grigorchuk subshifts and, more generally, for simple Toeplitz subshifts with either $n_k = 2^{j_k}$ or $\#\widetilde{\mathcal{A}} = 3$.

As an application, a result from [**5**] shows that the Boshernitzan condition implies a Cantor spectrum of Lebesgue measure zero for Jacobi operators on this subclass of subshifts. This is briefly discussed in the second subsection. It serves mostly as a reminder about the definition of these operators and their associated cocycles. The implications of the characterization in §6.1 are made explicit. Moreover, we recall a result from [**35**] for the special case of Schrödinger operators on simple Toeplitz subshifts, where the spectrum is always a Cantor set of Lebesgue measure zero, independent of the Boshernitzan condition.

6.1. *The Boshernitzan-condition.* Recall from §2.1 that $C_j(u)$ denotes the cylinder set of all elements in which a finite word u occurs at position j. The set of all finite subwords of a subshift Ω is denoted by Sub(Ω). As we have seen in §2.2, simple Toeplitz subshifts are uniquely ergodic due to their regularity. Now let ν denote the unique T-invariant ergodic probability measure on Ω and define

$$\eta(L) := \min\{\nu(C_1(u)) : u \in \text{Sub}(\Omega), |u| = L\}.$$

A subshift is said to satisfy the Boshernitzan condition if

$$\limsup_{L \to \infty} L \cdot \eta(L) > 0 \tag{B}$$

holds. In [**35**], a number of results related to the Boshernitzan condition are proved: it is shown that every simple Toeplitz subshift satisfies (B) if $\#\widetilde{\mathcal{A}} = 2$ holds (Proposition 4.1). Moreover, for $\#\widetilde{\mathcal{A}} \geq 3$, a description of $\eta(L)$ is given (Proposition 4.2), which is used to characterize (B) for simple Toeplitz subshifts (Corollary 4.1).

Here, a different characterization of (B) in terms of the function F and the period lengths $(n_k)_k$ is provided. Since the proof is based on the mentioned description of $\eta(L)$ from [**35**], it is stated below in our notation. As in §5, let $F(k) = \min\{j > k : \{a_{k+1}, \ldots, a_j\} = \mathcal{A}_{k+1}\}$ and recall that $\widetilde{K}$ denotes a number such that $a_k \in \widetilde{\mathcal{A}}$ and $\mathcal{A}_k = \widetilde{\mathcal{A}}$ hold for all $k \geq \widetilde{K}$. Moreover, we define $s_j := n_0 \cdot \ldots \cdot n_{j-1}$.

PROPOSITION 6.1. [**35**] *For a simple Toeplitz subshift with $\#\widetilde{\mathcal{A}} \geq 3$, there exist constants $0 < c_1 \leq c_2$ such that, for every $L > s_{\widetilde{K}}$ and j defined by the property $s_{j-1} < L \leq s_j$, the following holds:*
If $s_{j-1} < L < 2s_{j-1}$, then

$$c_1 \cdot \eta(L) \leq \min\left\{\frac{\lceil(2s_{j-1} - L)/s_{j-2}\rceil}{s_{\min\{i > j-1 : a_i = a_{j-2}\}}}, \frac{1}{s_{F(j-3)}}\right\} \leq c_2 \cdot \eta(L).$$

If $2s_{j-1} \leq L \leq s_j$, then

$$c_1 \cdot \eta(L) \leq \frac{1}{s_{F(j-2)}} \leq c_2 \cdot \eta(L).$$

Using the above description of $\eta(L)$ from [**35**], we will now prove the following proposition.

PROPOSITION 6.2. *A simple Toeplitz subshift satisfies (B) if and only if there exists a sequence $(k_r)_r$ of natural numbers with $\lim_{r \to \infty} k_r = \infty$ such that $\prod_{j=k_r+1}^{F(k_r-1)-1} n_j$ is bounded.*

Proof. For $\#\widetilde{\mathcal{A}} = 2$, the claimed equivalence is true for trivial reasons: for all r such that $k_r > \widetilde{K}$ holds, we have $F(k_r - 1) = k_r + 1$. Hence $\prod_{j=k_r+1}^{F(k_r-1)-1} n_j = 1$ is the empty product and therefore bounded. In addition, (B) is always satisfied according to [**35**, Proposition 4.1].

For $\#\widetilde{\mathcal{A}} \geq 3$, we first prove that boundedness of the product $\prod_{j=k_r+1}^{F(k_r-1)-1} n_j$ implies that $\limsup_{L\to\infty} L \cdot \eta(L) > 0$ holds. For this, we consider the subsequence (L_r) that is given by $L_r := s_{k_r+1} = n_0 \cdot \ldots \cdot n_{k_r}$. The description of $\eta(L)$ in Proposition 6.1 yields $\eta(L_r) \geq (c_2 \cdot s_{F(k_r-1)})^{-1}$. Thus we obtain

$$
\begin{aligned}
\limsup_{L\to\infty} L \cdot \eta(L) &\geq \limsup_{r\to\infty} L_r \cdot \eta(L_r) \\
&\geq \limsup_{r\to\infty} \frac{s_{k_r+1}}{c_2 \cdot s_{F(k_r-1)}} \\
&= \frac{1}{c_2} \limsup_{r\to\infty} \frac{1}{\prod_{j=k_r+1}^{F(k_r-1)-1} n_j} > 0.
\end{aligned}
$$

To prove the converse, assume that no sequence (k_r) exists for which the product is bounded. Then $\lim_{k\to\infty} \prod_{j=k+1}^{F(k-1)-1} n_j = \infty$ holds and we will show that this implies that $\lim_{L\to\infty} L \cdot \eta(L) = 0$. For every L, we define j as above by $s_{j-1} < L \leq s_j$. In the case of $s_{j-1} < L < 2s_{j-1}$, Proposition 6.1 yields

$$
\begin{aligned}
L \cdot \eta(L) &\leq \frac{L}{c_1} \cdot \min\left\{ \frac{\lceil (2s_{j-1}-L)/s_{j-2} \rceil}{s_{\min\{i>j-1:a_i=a_{j-2}\}}}, \frac{1}{s_{F(j-3)}} \right\} \\
&< \frac{2s_{j-1}}{c_1} \cdot \frac{1}{s_{F(j-3)}} \\
&= \frac{2}{c_1} \cdot \frac{1}{\prod_{i=j-1}^{F(j-3)-1} n_i} \xrightarrow{j\to\infty} 0.
\end{aligned}
$$

In the case of $2s_{j-1} \leq L \leq s_j$, Proposition 6.1 yields

$$
L \cdot \eta(L) \leq \frac{L}{c_1} \cdot \frac{1}{s_{F(j-2)}} \leq \frac{s_j}{c_1} \cdot \frac{1}{s_{F(j-2)}} = \frac{1}{c_1} \cdot \frac{1}{\prod_{i=j}^{F(j-2)-1} n_i} \xrightarrow{j\to\infty} 0. \qquad \square
$$

We will now characterize the existence of such a sequence (k_r) in terms of the sequence (m_i). Recall that (m_i) was defined as those positions where F increases. Hence, for $k = m_i + 1, \ldots, m_{i+1}$, the value of $F(k-1)$ is constant. Therefore, in this range, the product $\prod_{j=k+1}^{F(k-1)-1} n_j$ is minimal at $k = m_{i+1}$.

PROPOSITION 6.3. *There exists a sequence $(k_r)_r$ with $\lim_{r\to\infty} k_r = \infty$ such that $\prod_{j=k_r+1}^{F(k_r-1)-1} n_j$ is bounded if and only if there exists a subsequence $(m_{i_r})_r$ of (m_i) such that $\prod_{j=m_{i_r}+1}^{F(m_{i_r}-1)-1} n_j$ is bounded.*

Proof. One implication is clear: if (m_{i_r}) is such a subsequence, then we define $k_r := m_{i_r}$ and the proof is complete. For the converse implication, assume that (k_r) is a sequence such that the product $\prod_{j=k_r+1}^{F(k_r-1)-1} n_j$ is bounded. For every r, there is an index i_r such that $m_{i_r-1} \leq k_r - 1 < m_{i_r}$ holds. By definition of the sequence (m_i), this implies that

$F(m_{i_r-1}) = F(k_r - 1) = F(m_{i_r} - 1) < F(m_{i_r})$. This yields

$$\prod_{j=m_{i_r}+1}^{F(m_{i_r}-1)-1} n_j \leq \prod_{j=k_r+1}^{F(k_r-1)-1} n_j,$$

which shows that $\prod_{j=m_{i_r}+1}^{F(m_{i_r}-1)-1} n_j$ is bounded. □

Remark. As shown in Proposition 5.10, a simple Toeplitz subshift is linear repetitive if and only if the sequence $(\prod_{j=m_i+1}^{F(m_i)-1} n_j)_{i\geq 1}$ is bounded. By combining Propositions 6.2 and 6.3, we obtain that a simple Toeplitz subshift satisfies (B) if and only if there exists a subsequence $(m_{i_r})_r$ of (m_i) such that $(\prod_{j=m_{i_r}+1}^{F(m_{i_r}-1)-1} n_j)_r$ is bounded. In that sense, the Boshernitzan condition is a weaker analogue of linear repetitivity.

COROLLARY 6.4. *Let Ω be the subshift associated to a simple Toeplitz word with $\#\widetilde{\mathcal{A}} = 3$. Then Ω satisfies (B) if and only if $\lim\inf_{i\to\infty} n_{m_i+1} < \infty$ holds.*

Proof. The definitions of F and (m_i) imply that $a_{F(m_i-1)} \notin \{a_{m_i}, \ldots, a_{F(m_i-1)-1}\}$. For sufficiently large i, we have $m_i \geq m_{i-1} + 1 \geq \widetilde{K}$ and obtain $\#\{a_{m_i}, \ldots, a_{F(m_i-1)-1}\} = 2$. Since m_i is maximal with this property (cf. Proposition 5.2) and consecutive letters are different, we obtain $F(m_i - 1) = m_i + 2$ for all such i that $m_{i-1} + 1 \geq \widetilde{K}$ holds. Now Propositions 6.2 and 6.3 yield the claim. □

Remark. It was shown in [35, Corollary 4.3] that simple Toeplitz subshifts with $\#\widetilde{\mathcal{A}} \geq 3$, bounded sequence n_k and

$$\lim_{k\to\infty} (F(k-2) - k) = \infty$$

do not satisfy (B). However, for $\#\widetilde{\mathcal{A}} = 3$, the assumptions of this statement cannot be satisfied, since $\#\widetilde{\mathcal{A}} = 3$ implies that $F(m_i - 1) = m_i + 2$, as we have seen in the proof of Corollary 6.4. In particular, it follows from Corollary 6.4 that every simple Toeplitz subshift with $\#\widetilde{\mathcal{A}} = 3$ and a bounded sequence n_k satisfies (B). Note that this is not true for $\#\widetilde{\mathcal{A}} > 3$. The 4-letter coding sequence

$$a_1 a_2 a_3 \ldots = (ab)c(ab)^2 d(ab)^3 c(ab)^4 d \ldots$$

was given in [35] as an example of a simple Toeplitz word with $\lim_{k\to\infty}(F(k-2) - k) = \infty$. Hence, the associated subshift does not satisfy (B), independent on the sequence (n_k).

We conclude this subsection with a discussion of generalized Grigorchuk subshifts. Recall the reason why they were defined with period lengths of the form $n_k = 2^{j_k}$: they are obtained from the constant sequence $(n_k)_{k\in\mathbb{N}_0} = (2, 2, 2, \ldots)$ and a sequence $(b_k) \in \mathcal{A}^{\mathbb{N}_0}$, where $b_k = b_{k+1}$ is allowed, by combining consecutive occurrences of the same letter. When a letter b is repeated j times, the period length of the resulting word is 2^j. Conversely, when we have period lengths of the form $n_k = 2^{j_k}$, we can interpret the condition on (n_k) in Proposition 6.2 as a condition on the sequence (b_k).

COROLLARY 6.5. *Let Ω be a simple Toeplitz subshift with period lengths of the form $n_k = 2^{j_k}$ with $j_k \in \mathbb{N}$. Then Ω satisfies (B) if and only if there exists a constant C and a sequence $(t_r)_r$ with $\lim_{r\to\infty} t_r = \infty$ such that, for every r, the equality $\{b_{t_r}, \ldots, b_{t_r+C}\} = \{b_i : i \geq t_r\}$ holds.*

Proof. From Proposition 6.2 and $n_k = 2^{j_k}$, we obtain the equivalence

$$\Omega \text{ satisfies (B)} \iff \text{there is a sequence } (k_r)_r \text{ with } \lim_{r\to\infty} k_r = \infty \text{ such that}$$

$$\prod_{i=k_r+1}^{F(k_r-1)-1} n_i \text{ is bounded}$$

$$\iff \text{there is a sequence } (k_r)_r \text{ with } \lim_{r\to\infty} k_r = \infty \text{ such that}$$

$$\sum_{i=k_r+1}^{F(k_r-1)-1} j_i \text{ is bounded}$$

Let the letter a_k correspond to the letters $b_t = \ldots = b_{t+j_k-1}$, let the letter a_{k+1} correspond to the letters $b_{t+j_k} = \ldots = b_{t+j_k+j_{k+1}-1}$ and so on, such that the letter $a_{F(k-1)}$ corresponds to the letters $b_{t+j_k+\ldots+j_{F(k-1)-1}} = \ldots = b_{j_k+\ldots+j_{F(k-1)}-1}$. The definition of $F(k-1)$ yields

$$\mathcal{A}_k = \{a_k, \ldots, a_{F(k-1)}\}$$
$$= \{b_{t+j_k-1}, b_{t+j_k}, \ldots, b_{t+j_k+\ldots+j_{F(k-1)-1}-1}, b_{t+j_k+\ldots+j_{F(k-1)-1}}\}.$$

The set on the right-hand side contains $2 + \sum_{i=k+1}^{F(k-1)-1} j_i$ elements. When there exists a sequence k_r such that this sum is bounded, then every set of the form

$$\{b_{t_r+j_{k_r}-1}, \ldots, b_{t_r+j_{k_r}+\ldots+j_{F(k_r-1)-1}}\}$$

has the claimed property. Conversely, assume that a sequence t_r exists such that, for every r, the property $\{b_{t_r}, \ldots, b_{t_r+C}\} = \{b_i : i \geq t_r\}$ holds. Let a_{k_r} denote the letter that corresponds to b_{t_r}. Then $a_{F(k_r-1)-1}$ corresponds to a letter b_t with $t \leq t_r + C$. Since j_k denotes the multiplicity of the letter b_t corresponding to a_k, we obtain

$$\sum_{i=k_r+1}^{F(k_r-1)-1} j_i \leq (t_r + C) - (t_r + 1) + 1 = C. \qquad \square$$

Remark. The criterion for the Boshernitzan condition for generalized Grigorchuk subshifts in the previous Corollary 6.5 is similar to a criterion for the Boshernitzan condition for self-similar groups by Francoeur, Nagnibeda and Pérez [18]: let G_ω be the element of the family of Grigorchuk's groups (or, more generally, a spinal group) that is defined by the sequence $\omega = (\omega_t)_t$ of epimorphisms. The action of G_ω on the boundary ∂T of the tree T defines for every ray $\xi \in \partial T$ a rooted Schreier graph with root ξ. Consider the closure of the set of all these rooted Schreier graphs, except the one defined by the rightmost ray in the tree. Then G_ω acts on this set by shifting the root. Francoeur, Nagnibeda and Pérez show that this action satisfies the Boshernitzan condition if and only if there exists a constant C and a sequence $(t_r)_r$ with $\lim_{r\to\infty} t_r = \infty$ such that, for every r, the equality $\{\omega_{t_r}, \ldots, \omega_{t_r+C}\} = \{\omega_i : i \geq t_r\}$ holds.

6.2. *Application: Jacobi operators and cocycles.* In this subsection, we briefly discuss the implications of the Boshernitzan condition for the spectrum of Jacobi operators associated to a subshift. First, we recall the standard notions of transfer matrices and cocycles. By a result from [35], which is based on cocycles, Schrödinger operators

on simple Toeplitz subshifts always have Cantor spectrum of Lebesgue measure zero. Finally, we quote a result from [**5**], which connects the Boshernitzan condition to a Cantor spectrum of Lebesgue measure zero for Jacobi operators. As a corollary, the results about the Boshernitzan condition from the previous subsection allow conclusions about the spectrum.

We begin with the definitions of our objects of interest. Let $p : \Omega \to \mathbb{R} \backslash \{0\}$ and $q : \Omega \to \mathbb{R}$ be continuous functions. For an element $\omega \in \Omega$, the operator

$$H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}),$$
$$(H_\omega \psi)(k) = p(T^k \omega)\psi(k-1) + q(T^k \omega)\psi(k) + p(T^{k+1} \omega)\psi(k+1)$$

is called the Jacobi operator associated to ω. In the following, we will always assume that p and q take only finitely many values. Moreover, we require the dynamical system that is defined by

$$\widetilde{\Omega} := \left\{ \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} : \omega \in \Omega \right\} \quad \text{and} \quad \widetilde{T} : \widetilde{\Omega} \to \widetilde{\Omega}, \quad \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} \mapsto \begin{pmatrix} p(T \omega) \\ q(T \omega) \end{pmatrix},$$

with the product topology on $\widetilde{\Omega}$, to be aperiodic. Since simple Toeplitz subshifts are minimal, the spectrum of H_ω is, as a set, independent of $\omega \in \Omega$ and we call it the spectrum of the Jacobi operator on the subshift.

When investigating the spectrum of the Jacobi operator, important tools are transfer matrices. A solution φ to the eigenvalue equation $H_\omega \varphi = E\varphi$, with $E \in \mathbb{R}$, is determined by its value at two consecutive positions. All other values can be computed from

$$\varphi(k+1) = \frac{(E - q(T^k \omega))\varphi(k) - p(T^k \omega)\varphi(k-1)}{p(T^{k+1} \omega)}$$

and

$$\varphi(k-1) = \frac{(E - q(T^k \omega))\varphi(k) - p(T^{k+1} \omega)\varphi(k+1)}{p(T^k \omega)}.$$

When we choose $\varphi(0)$ and $\varphi(1)$ as starting values and express the above dependence with the help of matrices, we obtain, for $k > 0$, the equality

$$\begin{pmatrix} \varphi(k+1) \\ \varphi(k) \end{pmatrix} = \begin{pmatrix} \dfrac{E - q(T^k \omega)}{p(T^{k+1} \omega)} & -\dfrac{p(T^k \omega)}{p(T^{k+1} \omega)} \\ 1 & 0 \end{pmatrix}$$
$$\cdots \cdot \begin{pmatrix} \dfrac{E - q(T \omega)}{p(T^2 \omega)} & -\dfrac{p(T \omega)}{p(T^2 \omega)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi(1) \\ \varphi(0) \end{pmatrix}$$

and, for $k < 0$, the equality

$$\begin{pmatrix} \varphi(k+1) \\ \varphi(k) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{p(T^{k+2} \omega)}{p(T^{k+1} \omega)} & \dfrac{E - q(T^{k+1} \omega)}{p(T^{k+1} \omega)} \end{pmatrix}$$
$$\cdots \cdot \begin{pmatrix} 0 & 1 \\ -\dfrac{p(T \omega)}{p(\omega)} & \dfrac{E - q(\omega)}{p(\omega)} \end{pmatrix} \begin{pmatrix} \varphi(1) \\ \varphi(0) \end{pmatrix}.$$

For a continuous map $M : \Omega \to \mathrm{GL}(2, \mathbb{R})$, the associated cocycle $M : \mathbb{Z} \times \Omega \to \mathrm{GL}(2, \mathbb{R})$ is defined as

$$
(n, \omega) \mapsto
\begin{cases}
M(\mathrm{T}^{n-1}\,\omega) \cdot \ldots \cdot M(\omega) & \text{for } n > 0, \\
\mathrm{Id} & \text{for } n = 0, \\
M^{-1}(\mathrm{T}^n\,\omega) \cdot \ldots \cdot M^{-1}(\mathrm{T}^{-1}\,\omega) & \text{for } n < 0.
\end{cases}
$$

The transfer matrices are the maps

$$
M^E : \Omega \to \mathrm{GL}(2, \mathbb{R}), \quad \omega \mapsto
\begin{pmatrix}
\dfrac{E - q(\mathrm{T}\,\omega)}{p(\mathrm{T}^2\,\omega)} & -\dfrac{p(\mathrm{T}\,\omega)}{p(\mathrm{T}^2\,\omega)} \\
1 & 0
\end{pmatrix},
$$

with $E \in \mathbb{R}$, and their associated cocycles are precisely the above matrix products that determine an eigenfunction φ from two consecutive positions.

The properties of the transfer matrices are connected to properties of the spectrum of the Jacobi operator in the following way: a function $M : \Omega \to \mathrm{GL}(2, \mathbb{R})$ is called uniform, if $\lim_{n \to \infty} \ln(\|M(n, \omega)\|)/n$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω (cf. [19, p. 803]). It was shown in [5, Theorem 3], that the spectrum of the Jacobi operator on a minimal, uniquely ergodic and aperiodic subshift is a Cantor set of Lebesgue measure zero if the transfer matrix is uniform for every $E \in \mathbb{R}$. Based on this, the Cantor spectrum can be deduced in the following cases.

6.3. *Schrödinger operators.* For $p = 1$, the resulting operator

$$
\mathrm{H}_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (\mathrm{H}_\omega\,\psi)(k) = \psi(k-1) + q(\mathrm{T}^k\,\omega)\psi(k) + \psi(k+1)
$$

is called the Schrödinger operator associated to ω. In [35, Theorem 1.1], it was shown for every simple Toeplitz subshift that the transfer matrix M^E of the Schrödinger operator is uniform for all $E \in \mathbb{R}$. Thus, the spectrum of a Schrödinger operator on a simple Toeplitz subshift is always a Cantor set of Lebesgue measure zero.

6.4. *Boshernitzan condition.* For the uniformity of M^E, it is sufficient that the subshift satisfies (B). More precisely, the following proposition is obtained as [5, Corollary 4], using a result from [14].

PROPOSITION 6.6. [5] *Let* (Ω, T) *be a minimal, aperiodic subshift such that the Boshernitzan condition holds. Consider the family of the corresponding Jacobi operators* $\{\mathrm{H}_\omega\}_{\omega \in \Omega}$, *where the continuous maps* p *and* q *take finitely many values and the aperiodicity of the subshifts carries over to* $(\widetilde{\Omega}, \widetilde{\mathrm{T}})$. *Then the transfer matrix* $M^E : \Omega \to \mathrm{GL}(2, \mathbb{R})$ *is uniform for each* $E \in \mathbb{R}$. *In particular, the spectrum* Σ *is a Cantor set of Lebesgue measure zero.*

By combining this statement with the results from Proposition 6.2, Corollary 6.4 and [35, Proposition 4.1], we obtain the following corollary.

COROLLARY 6.7. *Let* Ω *be a simple Toeplitz subshift with coding* (a_k) *and* (n_k). *If there exists a sequence* $(k_r)_r$ *of natural numbers with* $\lim_{r \to \infty} k_r = \infty$ *such that* $\prod_{j=k_r+1}^{F(k_r-1)-1} n_j$

is bounded, then the Jacobi operator on the subshift has a Cantor spectrum of Lebesgue measure zero. This is particularly the case if either $\#\mathcal{A} = 2$ holds or $\#\widetilde{\mathcal{A}} = 3$ and $\liminf_{i \to \infty} n_{m_i+1} < \infty$ hold.

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