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IDENTIFICATION AND ISOTROPY CHARACTERIZATION OF DEFORMED RANDOM FIELDS THROUGH EXCURSION SETS

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Abstract

A deterministic application $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ deforms bijectively and regularly the plane and allows the construction of a deformed random field $X \circ \theta\colon \mathbb{R}^2 \to \mathbb{R}$ from a regular, stationary, and isotropic random field $X\colon \mathbb{R}^2 \to \mathbb{R}$. The deformed field $X \circ \theta$ is, in general, not isotropic (and not even stationary), however, we provide an explicit characterization of the deformations θ that preserve the isotropy. Further assuming that X is Gaussian, we introduce a weak form of isotropy of the field $X \circ \theta$, defined by an invariance property of the mean Euler characteristic of some of its excursion sets. We prove that deformed fields satisfying this property are strictly isotropic. In addition, we are able to identify θ, assuming that the mean Euler characteristic of excursion sets of $X \circ \theta$ over some basic domain is known.

Keywords: Random field; space deformation; excursion set; Euler characteristic; isotropy

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1. Introduction

Deformed fields are a class of nonstationary and nonisotropic fields obtained by deforming a fixed stationary and isotropic random field using a deterministic function that transforms bijectively the index set. Deformed fields are useful to model spatial and physical phenomena that are in many cases not stationary nor isotropic. For example, they are currently widely used in cosmology to model the cosmic microwave background (CMB) deformed anisotropically by the gravitational lensing effect, with mass reconstruction as an objective; see [15].

Our framework is two-dimensional: we set $X\colon \mathbb{R}^2 \to \mathbb{R}$ the underlying stationary and isotropic field, $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ a bijective deterministic function, and $X_\theta = X \circ \theta$ the deformed field. In fact, most studies on the deformed field model deal with dimension two. The reason for this is that it is the simplest case of multi-dimensionality, the results can be illustrated easily through simulations and it still covers many possible applications, particularly in image analysis. For instance, deformed fields are involved in the 'shape from texture' issue, that is, the problem of recovering a three-dimensional textured surface from a two-dimensional projection; see [9].

The model of deformed fields was introduced in a spatial statistics framework by Sampson and Guttorp [19], with only a stationarity assumption on X. It was also studied through the covariance function; see [17] and [18]. Allard *et al.* [2] investigated the case of a linear deformation with a matrix representation as the product of a diagonal and a rotation matrix, producing what the authors called 'geometric anisotropy'. In [10], the deformed field model

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was studied as a particular case of a model of a deterministic deformation operator applied to a stationary field X. Many authors have proposed different methods to estimate θ, as we will see a little further on in this introduction when we come to our own contribution to the estimation matter.

Unless otherwise specified, the type of stationarity and isotropy that we consider consists in an invariance of the field's law under translations or, respectively, rotations. Even though the underlying field X is stationary and isotropic, many deformations transform the index space $\mathbb{R}^2$ in such a way that the stationarity and/or the isotropy are lost when it comes to the deformed field. The deformations preserving stationarity are the linear deformations. Concerning isotropy, a natural question arises as to which deformations θ preserve the isotropy for any underlying field X; we address this in Section 3. We state an explicit form for this kind of deformation and we call them spiral deformations. We note that the question of preserving the isotropy for one fixed underlying field X is different, and we address it in Section 5.

For the rest of the paper, we address the following practical problem: the covariance function of the underlying field X and the deformation θ are unknown. We attempt to study and even to identify θ through observations of some excursion sets of X_θ above fixed levels. For this, we add some assumptions on X (Gaussianity, a little more than $\mathcal{C}^2$-regularity, and a nondegeneracy assumption) and on θ ($\mathcal{C}^2$-regularity), which are precisely described and justified in Section 2, and we focus on the mean Euler characteristic of the excursion sets. The Euler characteristic is an additive topological functional that is defined on a large class of compact sets. Heuristically, the Euler characteristic of a set is determined by its topology: for a two-dimensional compact set, it is the number of connected components minus the number of holes in this set; for a one-dimensional set, it is simply the number of closed intervals that compose the set. Note that a modified version of the Euler characteristic of excursion sets will be more suitable to address our problem. The equations of the expectation of the (modified) Euler characteristic of an excursion set of X_θ can be found in Section 4.

More precisely, let T be a rectangle domain or segment in $\mathbb{R}^2$. We are interested in the Euler characteristic χ of the excursion set of X_θ restricted to T above a level $u \in \mathbb{R}$, $A_u(X_\theta, T) = \{t \in T,\ X(\theta(t)) \geq u\}$. However, we may study equivalently the stationary and isotropic field X on the transformed set $\theta(T)$ instead of the deformed field X_θ on the set T, since

$$\chi(A_u(X_\theta, T)) = \chi(A_u(X, \theta(T))).$$

In Section 5 we introduce the notion of χ-isotropic deformation: it applies to a deformation θ such that, for any level u and for any rectangle T, $\mathbb{E}[\chi(A_u(X_\theta, T))]$ does not vary under any rotation of T. This is, in particular, true if the deformed field is isotropic, hence this property can be viewed as a weak notion of isotropy. However, it turns out that this weak notion implies the strong one (isotropy in law), that is, the χ-isotropic deformations are exactly the spiral deformations.

In Section 6 we tackle the problem of identifying θ, assuming that we only have at our disposal the mean Euler characteristic of some excursion sets of the deformed field. The problem of estimating a deformation θ using the observation of the deformed random field X_θ is originally a problem in spatial statistics and it has been studied using different approaches since its first introduction. Initially, Sampson and Guttorp [19] used several observations on a sparse grid to estimate θ. Another approach is to use only one observation of the deformed field on a dense grid; see [3], [5], [10], [13], and [14] for this method with an underlying field that is stationary and/or isotropic. These papers involve convergence results on quadratic variations and quasi-conformal theory. The study in [4] applies, in particular, to deformed fields of the form $\{X(x+\nabla\eta(x)),\ x\in\mathbb{R}^2\}$ (where $\eta\colon\mathbb{R}^2\to\mathbb{R}$ is a deterministic function), which modelize the CMB of [15]. The authors proposed a method to estimate function η that corresponds to the gravitational lensing of the CMB.

Our approach differs from the previous ones, since our observations are limited to realizations of X_θ over a fixed level, and not to the whole realization. Our method is closer to the one of [8], where the inference of the deformation was based on the size and shape of the level curves of the deformed fields; however, the author restricted the deformations to linear ones given by symmetric, positive, and definite matrices. An analogous approach was adopted in [6] thanks to more general functionals of the level sets. With our sparse observations, we are able (as well as [5]) to compute the complex dilatation of θ up to a conformal map, at every point of the domain. The complex dilatation provides a characterization of the deformation.

In this paper we prove four main results. In Theorem 1 we state that the deformations preserving isotropy are exactly the spiral deformations. In Theorem 2, the class of deformations satisfying the invariance condition of the mean Euler characteristic of excursion sets is identified with the spiral deformations. A consequence of this theorem is Corollary 1. Roughly speaking, it states that three notions of preservation of isotropy coincide and correspond to the set of spiral deformations. In Section 6 we show how to almost entirely identify θ through the mean Euler characteristic of its excursion sets over basic domains. The general case is described by Method 2. We conclude in Section 6.2 by limiting ourselves to spiral deformations, and we finally propose an estimation method based on one single observation of the deformed field.

2. Notation and assumptions

For any compact set A in $\mathbb{R}^2$, we write $\dim(A)$ for its Hausdorff dimension; if $\dim(A)=1$, we write $|A|_1$ as its one-dimensional Hausdorff measure; if $\dim(A)=2$, we write $|A|_2$ as its two-dimensional Hausdorff measure; we also write ∂A as the frontier of A and $\mathring{A}$ its interior.

We work in a fixed orthonormal basis in $\mathbb{R}^2$ and we use the same notation for a linear application defined on $\mathbb{R}^2$ and taking values in $\mathbb{R}^2$ and for its matrix in this basis of $\mathbb{R}^2$. We denote by $\mathrm{O}(2)$ the group of orthogonal transformations in $\mathbb{R}^2$ and by $\mathrm{SO}(2)$ the group of rotations in $\mathbb{R}^2$. For any $\alpha\in\mathbb{R}/2\pi\mathbb{Z}$, ρ_α denotes the rotation of angle α. The Euclidian norm in $\mathbb{R}^2$ is written as $\|\cdot\|$.

For any real s, we write

$$[0,s]=\begin{cases}\{x\in\mathbb{R},\ 0\le x\le s\} & \text{if } s\ge 0,\\ \{x\in\mathbb{R},\ s\le x\le 0\} & \text{if } s<0.\end{cases}$$

We say that a set T in $\mathbb{R}^2$ is a segment if there exists $(a,b)\in(\mathbb{R}^2)^2$ with $a\neq b$ such that $T=\{a+t(b-a),\ t\in[0,1]\}$. For any $(s,t)\in\mathbb{R}^2$, we write $T(s,t)=[0,s]\times[0,t]$ and we say that a set T in $\mathbb{R}^2$ is a rectangle if there exist $(s,t)\in(\mathbb{R}\setminus\{0\})^2$, $\rho\in\mathrm{SO}(2)$, and a translation τ such that $T=\rho\circ\tau(T(s,t))$.

If $f=(f_1,f_2)\colon\mathbb{R}^2\to\mathbb{R}^2$, with $f_i\colon\mathbb{R}^2\to\mathbb{R}$ for $i\in\{1,2\}$, is a differentiable function, for any $x=(s,t)\in\mathbb{R}^2$, we use the notation $J_f^1(x)$ for the vector $\partial_s f(x)=(\partial_s f_1(x),\partial_s f_2(x))$, $J_f^2(x)$ for the vector $\partial_t f(x)$, and $J_f(x)$ for the Jacobian matrix of f at point x. More generally, if M is a 2×2 matrix, for $i\in\{1,2\}$, we write M^i as the ith column of M.

Let X be a Gaussian, stationary, and isotropic random field, defined on $\mathbb{R}^2$ and taking real values, and we write $C\colon\mathbb{R}^2\to\mathbb{R}$ as its covariance function. Since X is stationary, we may assume it is centred. We will also assume that $C(0)=1$ since if not, we consider the field $(1/\sqrt{C(0)})X$ instead of X. As for the regularity of X, we make the assumption that almost every realization of X is of class $\mathcal{C}^2$ on $\mathbb{R}^2$. As a consequence, C is of class $\mathcal{C}^4$. We denote by $X'(t)$ and $X''(t)$ the gradient vector and the Hessian matrix of X at point t, respectively, and by $C''(t)$ the Hessian matrix of C at point t. In order to be able to apply the mean Euler characteristic of excursion sets equation, we make a boundedness assumption on the covariance functions of the $X''_{i,j}$ for $(i, j) \in \{1, 2\}^2$. We also assume that for any $t \in \mathbb{R}^2$, the joint distribution of $(X'_i(t), X''_{i,j}(t))_{(i,j)\in\{1,2\}^2,\, i\le j}$ is not degenerate. Therefore, the covariance matrix of $X'(0)$ is not degenerate; since X is isotropic, there exists $\lambda > 0$ such that $\operatorname{cov}(X'(0)) = \lambda I_2$. If $\lambda \neq 1$, X_θ is nevertheless equal to $\tilde{X}_{\tilde{\theta}}$ with $\tilde{\theta} = \sqrt{\lambda}\theta$ and with $\tilde{X}(\cdot) = X(\sqrt{\lambda}^{-1}\cdot)$ satisfying $\operatorname{cov}(\tilde{X}'(0)) = I_2$. Consequently, without loss of generality, we assume that $C''(0) = -I_2$.

We now gather all the assumptions on X that will be in force in Sections 4–6.

Assumption 1. *We assume that:*

- *X is Gaussian;*
- *X is stationary and isotropic;*
- *X is almost surely of class $\mathcal{C}^2$;*
- *there exist $\varepsilon > 0$, $\alpha > 0$, $K > 0$, such that for all $t \in \mathbb{R}^2$,*

$$\|t\| \le \varepsilon \quad \Longrightarrow \quad \left| \frac{\partial^4 C}{\partial t_i^2 \partial t_j^2}(t) - \frac{\partial^4 C}{\partial t_i^2 \partial t_j^2}(0) \right| \le K |\log(\|t\|)|^{-(1+\alpha)};$$

- *for all $t \in \mathbb{R}^2$, the joint distribution of $(X'_i(t), X''_{i,j}(t))_{(i,j)\in\{1,2\}^2,\, i\le j}$ is not degenerate;*
- *X is centred, $C(0) = 1$ and $C''(0) = -I_2$.*

Our aim in Section 6 is to identify the deformation θ assuming that we only have at our disposal the expectation of $\chi(A_u(X_\theta, T))$ for different sets T and for a fixed level u. However, it is not possible to distinguish between θ and another deformation $\tilde{\theta}$ such that the random fields X_θ and $X_{\tilde{\theta}}$ have the same law. Due to the stationarity and the isotropy of X, it occurs if $\tilde{\theta} = \rho \circ \theta + u$, where $\rho \in \mathrm{SO}(2)$ and $u \in \mathbb{R}^2$. Therefore, we can only hope to determine a deformation θ up to a left-composition with a rotation and a translation.

Consequently, without loss of generality, we make the assumption that $\theta(0) = 0$. If θ is differentiable, we will also assume that for any $x \in \mathbb{R}^2$, $\det(J_\theta(x))$ is positive or, in other words, that θ preserves orientation. Indeed, the function $x \mapsto \det(J_\theta(x))$ is continuous on $\mathbb{R}^2$ and does not take zero value; hence, it takes either only positive or only negative values. If, for all $x \in \mathbb{R}^2$, $\det(J_\theta(x)) < 0$, we can replace θ by $\sigma \circ \theta$, where $\sigma \in \mathrm{O}(2)$ is the symmetry with respect to the axis of abscissa; then for any $x \in \mathbb{R}^2$, $J_{\sigma\circ\theta}(x) = \sigma \circ J_\theta(x)$ and so the Jacobian determinant of $\sigma \circ \theta$ is positive on $\mathbb{R}^2$. These two transformations on θ (translation along vector $-\theta(0)$ and left-composition with σ) do not change the law of X_θ. Note that the class of linear as well as tensorial deformations considered as examples in Section 6 are stable under those transformations made in order to simplify our study.

We denote by $\mathcal{D}^0(\mathbb{R}^2)$ the set of continuous and bijective functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with a continuous inverse, taking value 0 at 0. For $i \in \{1, 2\}$, we denote by $\mathcal{D}^i(\mathbb{R}^2)$ the set of $\mathcal{C}^i$-diffeomorphisms from $\mathbb{R}^2$ to $\mathbb{R}^2$ taking value 0 at 0. We call such functions deformations (in $\mathcal{D}^i(\mathbb{R}^2)$ for $i \in \{0, 1, 2\}$, according to the section of this paper).

Note that the assumptions on X and θ just listed are not all in force in Section 3, where we soften the regularity assumptions and we replace the Gaussian hypothesis on X by the assumption of the existence of a second moment.

3. For which θ is X_θ isotropic?

In this section, Assumption 1 is *not* in force. We only assume that X is stationary, isotropic, and that it admits a second moment. Considering θ in $\mathcal{D}^1(\mathbb{R}^2)$, we denote by C_θ the covariance function of the deformed field X_θ. Since the field X is stationary, for any $(x, y) \in (\mathbb{R}^2)^2$,

$$C_\theta(x, y) = \operatorname{cov}(X_\theta(x), X_\theta(y)) = C(\theta(x) - \theta(y)). \tag{1}$$

In the following, we present the deformations θ in $\mathcal{D}^1(\mathbb{R}^2)$ that leave the field X_θ isotropic for any stationary and isotropic field X. Note that the underlying field X is not fixed. Our approach is analogous to the one of [18], where the objective was, starting with a random field Y with a known covariance function, to find a deformation θ such that $Y = X \circ \theta$ with $X : \mathbb{R}^2 \to \mathbb{R}$ a stationary, or stationary and isotropic random field.

We begin with a short introduction of the notation relative to the polar representation. We denote by S the transformation of polar coordinates to Cartesian coordinates in the plane deprived of the origin:

$$S : (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2 \setminus \{0\}, \qquad (r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi).$$

We define $\mathcal{D}^0((0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z})$ as the set of continuous and bijective functions $\hat{\theta} : (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \to (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ with continuous inverses. For any deformation $\theta \in \mathcal{D}^0(\mathbb{R}^2)$, we write $\theta_0 = \theta_{|\mathbb{R}^2\setminus\{0\}}$, define the deformation $\hat{\theta} \in \mathcal{D}^0((0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z})$ by $\hat{\theta} = S^{-1} \circ \theta_0 \circ S$, and we denote by $\hat{\theta}^1$ and $\hat{\theta}^2$ its coordinate functions.

Proposition 1. *The mapping $\mathcal{D}^0(\mathbb{R}^2) \to \mathcal{D}^0((0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z})$, $\theta \mapsto \hat{\theta}$, is injective and it is a group morphism, that is, if η and θ belong to $\mathcal{D}^0(\mathbb{R}^2)$ then $\widehat{\eta \circ \theta} = \hat{\eta} \circ \hat{\theta}$. Moreover, the coordinate functions of the composition $\widehat{\eta \circ \theta}$ are*

$$\widehat{\eta \circ \theta}^1 = \hat{\eta}^1 \circ \hat{\theta} \quad \textit{and} \quad \widehat{\eta \circ \theta}^2 = \hat{\eta}^2 \circ \hat{\theta}.$$

Proof. The above application is obviously injective and if η and θ belong to $\mathcal{D}^0(\mathbb{R}^2)$ then

$$(\eta \circ \theta)_0 = \eta_0 \circ \theta_0 = (S \circ \hat{\eta} \circ S^{-1}) \circ (S \circ \hat{\theta} \circ S^{-1}) = S \circ \hat{\eta} \circ \hat{\theta} \circ S^{-1};$$

hence, we have the homomorphism property. Consequently, for $i \in \{1, 2\}$, the coordinate function $\widehat{\eta \circ \theta}^i$ satisfies

$$(\widehat{\eta \circ \theta}^1, \widehat{\eta \circ \theta}^2) = \widehat{\eta \circ \theta} = \hat{\eta} \circ \hat{\theta} = (\hat{\eta}^1 \circ \hat{\theta}, \hat{\eta}^2 \circ \hat{\theta}). \qquad \square$$

Definition 1. Let $\theta \in \mathcal{D}^0(\mathbb{R}^2)$. The deformation θ is a spiral deformation if there exist $f : (0, +\infty) \to (0, +\infty)$ strictly increasing and surjective, $g : (0, +\infty) \to \mathbb{R}/2\pi\mathbb{Z}$, and $\varepsilon \in \{\pm 1\}$ such that θ satisfies

$$\hat{\theta}(r, \varphi) = (f(r), g(r) + \varepsilon\varphi) \quad \text{for all } (r, \varphi) \in (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}. \tag{2}$$

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Remark 1. Note that the set of spiral deformations forms a group for the operation of composition. The choice of f strictly increasing is due to the conditions of continuity and inversibility on θ and to the fact that $\theta(0) = 0$. The 2π-periodicity of $\hat{\theta}^2$ entails that the coefficient ε in the angular part of (2) is an integer and the inversibility of θ implies that ε belongs to $\{\pm 1\}$. If we consider only deformations with positive Jacobian determinants, in accordance with our explanations in Section 2, then we can set $\varepsilon = 1$. Indeed, the positivity of the Jacobian determinant of θ is equivalent to the positivity of the one of $\hat{\theta}$; see (17) by way of justification.

Example 1. (*Linear spiral deformations.*) A linear spiral deformation is a deformation with polar representation either $(r, \varphi) \mapsto (\lambda r, \varphi + \alpha)$ or $(r, \varphi) \mapsto (\lambda r, -\varphi + \alpha)$ with $\lambda > 0$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, that is, it is of the form $\lambda\rho$ with $\rho \in \mathrm{O}(2)$.

In [6] and [8], the deformations were restricted to the ones given by symmetric, positive, and definite matrices. In that case, the field X_θ is isotropic if and only if the two positive eigenvalues of θ are equal. In the following theorem, we also determine the deformations preserving isotropy but in the general case.

Theorem 1. *The set of spiral deformations in $\mathcal{D}^1(\mathbb{R}^2)$ coincides with the set of deformations in $\mathcal{D}^1(\mathbb{R}^2)$ such that for any stationary and isotropic field X, X_θ is isotropic.*

Proof. To prove the direct implication, let us assume that a deformation θ is a spiral deformation with polar representation (2) and let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. We recall that ρ_α denotes the rotation of angle α in $\mathbb{R}^2$. Then

$$\begin{aligned}\hat{\theta} \circ \hat{\rho}_\alpha(r, \varphi) &= (f(r), g(r) + \varepsilon(\varphi + \alpha)) \\ &= (f(r), g(r) + \varepsilon\varphi + \varepsilon\alpha) \\ &= \hat{\rho}_{\varepsilon\alpha} \circ \hat{\theta}(r, \varphi) \quad \text{for all } (r, \varphi) \in (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}.\end{aligned}$$

Therefore, θ satisfies the following property:

$$\text{for all } \rho \in \mathrm{SO}(2), \quad \text{there exists } \rho' \in \mathrm{SO}(2)) \text{ such that } \theta \circ \rho = \rho' \circ \theta.$$

This entails $X_\theta \circ \rho = X \circ \rho' \circ \theta$. The isotropy of X implies that $X \circ \rho'$ has the same law as X. Consequently, $X_\theta \circ \rho$ has the same law as X_θ. Thus, the isotropy of X_θ is proved.

We now turn to the converse implication. Let θ in $\mathcal{D}^1(\mathbb{R}^2)$ be such that for any stationary and isotropic field X, the field X_θ is isotropic. Hence, its covariance function, given by (1), is invariant under the action of any rotation, that is,

$$C_\theta(\rho(x), \rho(y)) = C_\theta(x, y) \quad \text{for all } \rho \in \mathrm{SO}(2),\ (x, y) \in (\mathbb{R}^2)^2.$$

In particular, if we use the Gaussian covariance function $C(x) = \exp(-\|x\|^2)$, we obtain

$$\|\theta(\rho(x)) - \theta(\rho(y))\| = \|\theta(x) - \theta(y)\| \quad \text{for all } \rho \in \mathrm{SO}(2),\ (x, y) \in (\mathbb{R}^2)^2. \tag{3}$$

Taking $y = 0$, from (3) we deduce that $\hat{\theta}^1$ is radial. We set, for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and for any $r > 0$, $\hat{\theta}^1(r, \varphi) = f(r)$. Since θ is bijective, continuous, and $\theta(0) = 0$, f is necessarily strictly increasing with $\lim_{r\to 0} f(r) = 0$ and $\lim_{r\to+\infty} f(r) = +\infty$.

To infer the form of $\hat{\theta}^2$, we fix $r > 0$ and for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, we use the complex representation to write (3) using $x = r\mathrm{e}^{\mathrm{i}\varphi}$, $y = r$, and for any angle α of the rotation ρ. Dividing the equality by $f(r)$, we obtain

$$|\mathrm{e}^{\mathrm{i}\hat{\theta}^2(r,\varphi+\alpha)} - \mathrm{e}^{\mathrm{i}\hat{\theta}^2(r,\alpha)}| = |\mathrm{e}^{\mathrm{i}\hat{\theta}^2(r,\varphi)} - \mathrm{e}^{\mathrm{i}\hat{\theta}^2(r,0)}|;$$

hence,

$$|1 - \mathrm{e}^{\mathrm{i}(\hat{\theta}^2(r,\varphi+\alpha)-\hat{\theta}^2(r,\alpha))}| = |1 - \mathrm{e}^{\mathrm{i}(\hat{\theta}^2(r,\varphi)-\hat{\theta}^2(r,0))}|.$$

Since each term belongs to $\{z \in \mathbb{C}\colon |z| = 1\}$, a geometric interpretation of the above equality requires that for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, there exists $\varepsilon(r, \varphi, \alpha) \in \{\pm 1\}$ such that

$$\hat{\theta}^2(r, \varphi + \alpha) - \hat{\theta}^2(r, \alpha) = \varepsilon(r, \varphi, \alpha)(\hat{\theta}^2(r, \varphi) - \hat{\theta}^2(r, 0)). \tag{4}$$

Assuming that there exists $\varphi \neq 0$ such that $\hat{\theta}^2(r, \varphi) - \hat{\theta}^2(r, 0) = 0$, from (4) we deduce that $\hat{\theta}^2(r, \cdot)$ is constant on $\mathbb{R}/2\pi\mathbb{Z}$, which contradicts the bijectivity of θ. Consequently, for any $\varphi \neq 0$,

$$\varepsilon(r, \varphi, \alpha) = \frac{\hat{\theta}^2(r, \varphi + \alpha) - \hat{\theta}^2(r, \alpha)}{\hat{\theta}^2(r, \varphi) - \hat{\theta}^2(r, 0)}.$$

This implies that ε is continuous from $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}\backslash\{0\} \times \mathbb{R}/2\pi\mathbb{Z}$ onto $\{\pm 1\}$. A connexity argument applies and implies that ε is constant. We write $\varepsilon(r, \varphi, \alpha) = \varepsilon \in \{\pm 1\}$.

We fix $r > 0$. For any $(\varphi, \alpha) \in (\mathbb{R}/2\pi\mathbb{Z})^2$, we can write (4) as

$$\hat{\theta}^2(r, \varphi + \alpha) = \hat{\theta}^2(r, \alpha) + \varepsilon(\hat{\theta}^2(r, \varphi) - \hat{\theta}^2(r, 0)).$$

By differentiating the above equality with respect to α, for a fixed $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, we deduce that $\partial_\varphi \hat{\theta}^2(r, \cdot)$ is constant on $\mathbb{R}/2\pi\mathbb{Z}$. Therefore, there exist $k(r) \in \{\pm 1\}$ and $g(r) \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\hat{\theta}^2(r, \varphi) = k(r)\varphi + g(r) \quad \text{for all } r > 0,\ \varphi \in \mathbb{R}/2\pi\mathbb{Z}.$$

Note that the reason why $k(r)$ must belong to $\{\pm 1\}$ has already been explained in Remark 1. Finally, since $\hat{\theta}^2$ is continuous, $k(r)$ is necessarily constant, which concludes the proof. □

Remark 2. Considering the proof of Theorem 1, we could state an equivalent version of it, requiring only one fixed stationary and isotropic random field X such that its covariance function is injective: the deformations θ in $\mathcal{D}^1(\mathbb{R}^2)$ such that X_θ is isotropic are the spiral deformations.

4. Expectation equations

4.1. Euler characteristic of an excursion set

The Euler characteristic is defined on a large subset of compact sets. There are several ways to define this additive topological functional; see, for example, [1] and [16]. However, we are actually only interested in the Euler characteristic of excursion sets, which can be computed as a result of specific equations. From now on, X is a random field assumed to satisfy Assumption 1 and θ is a deformation in $\mathcal{D}^2(\mathbb{R}^2)$. Consequently, even though X_θ is, in general, not stationary nor isotropic, it is Gaussian and its realizations are almost surely of class $\mathcal{C}^2$. Moreover, if T is a rectangle or a segment in $\mathbb{R}^2$ then the set $\theta(T)$ and its frontier $\delta\theta(T) = \theta(\delta T)$ are regular stratified manifolds; see [1, Definition 9.2.2].

We start by introducing the general equation for the expectation of the Euler characteristic of an excursion set of X_θ, above a d-dimensional rectangle T and then we show how it adapts to dimensions $d = 1$ and $d = 2$. We first explain why we may study equivalently the stationary and isotropic field X on the transformed set $\theta(T)$ or the nonstationary and anisotropic field X_θ on the set T. The deformation θ is an homeomorphism and it satisfies $A_u(X_\theta, T) = \theta^{-1}(A_u(X, \theta(T)))$; therefore, the sets $A_u(X_\theta, T)$ and $A_u(X, \theta(T))$ are homotopic. Since the

Euler characteristic is a homotopy invariant (see [16, Theorem 13.36]), the above relation leads to

$$\chi(A_u(X_\theta, T)) = \chi(A_u(X, \theta(T))).$$

Consequently, we can focus on $\mathbb{E}[\chi(A_u(X, \theta(T)))]$, which can be computed thanks to [1, Theorem 12.4.2]. This theorem is based on the explicit expression of $\chi(A_u(X, \theta(T)))$ given by the Morse formula (see [1, Equation (12.4.9)]) and it is applicable here due to Assumption 1. We write $(H_i)_{i\in\mathbb{N}}$ for the Hermite polynomials and, for any real x, $H_{-1}(x) = \sqrt{2\pi}\,\Psi(x)\exp(x^2/2)$, where Ψ is the tail probability of a standard Gaussian variable. Then

$$\mathbb{E}[\chi(A_u(X_\theta, T))] = \mathbb{E}[\chi(A_u(X, \theta(T)))] = \sum_{0\le i\le d} \mathcal{L}_i(\theta(T))\rho_i(u) \tag{5}$$

with $\rho_i(u) = (2\pi)^{-(i+1)/2} H_{i-1}(u)\mathrm{e}^{-u^2/2}$ for all $0 \le i \le d$, and with $\mathcal{L}_i(\theta(T))$ the ith Lipschitz–Killing curvature of $\theta(T)$. Thanks to the isotropy assumption on X and to the hypothesis $C''(0) = -I_2$, the Lipschitz–Killing curvatures have a very simple expression (see [1, Section 12.5]), that is,

$$\mathcal{L}_1(\theta(T)) = |\theta(T)|_1, \qquad \mathcal{L}_0(\theta(T)) = \chi(\theta(T)) = 1 \quad \text{if } d = 1,$$
$$\mathcal{L}_2(\theta(T)) = |\theta(T)|_2, \quad \mathcal{L}_1(\theta(T)) = \tfrac{1}{2}|\partial\theta(T)|_1, \quad \mathcal{L}_0(\theta(T)) = \chi(\theta(T)) = 1 \quad \text{if } d = 2.$$

Thus, if T is a segment in $\mathbb{R}^2$ then $\theta(T)$ is a one-dimensional manifold, and applying (5) with $d = 1$ yields

$$\mathbb{E}[\chi(A_u(X, \theta(T)))] = \mathrm{e}^{-u^2/2}\frac{|\theta(T)|_1}{2\pi} + \Psi(u). \tag{6}$$

If T is a two-dimensional rectangle in $\mathbb{R}^2$, we obtain

$$\mathbb{E}[\chi(A_u(X, \theta(T)))] = \mathrm{e}^{-u^2/2}\bigg(u\frac{|\theta(T)|_2}{(2\pi)^{3/2}} + \frac{|\partial\theta(T)|_1}{4\pi}\bigg) + \Psi(u). \tag{7}$$

We now state a continuity result on the mean Euler characteristic of excursion sets. From the proposition we see that if T is a segment in $\mathbb{R}^2$, the mean Euler characteristic of the excursion set of X_θ above T may be seen as the limit of the mean Euler characteristic of excursion sets of X_θ over a sequence of two-dimensional sets, decreasing in the sense of set inclusion and approaching T.

Proposition 2. *Let T be a segment in $\mathbb{R}^2$. Let v be a unit vector orthogonal to T and for any $\rho > 0$, let T_ρ be the rectangle $\{t + \delta v, t \in T, -\rho \le \delta \le \rho\}$. Then, for any random field X satisfying Assumption 1, as ρ decreases towards* 0,

$$\mathbb{E}[\chi(A_u(X_\theta, T_\rho))] \to \mathbb{E}[\chi(A_u(X_\theta, T))], \qquad \rho \to 0.$$

Proof. We write $T = \{a + \lambda(b - a), \lambda \in [0, 1]\}$, where a and b belong to $\mathbb{R}^2$. The set $\theta(T)$ is one-dimensional while for any $\rho > 0$, $\theta(T_\rho)$ is two-dimensional. Therefore, according to (6) and (7),

$$\mathbb{E}[\chi(A_u(X, \theta(T))] = \mathrm{e}^{-u^2/2}\frac{|\theta(T)|_1}{2\pi} + \Psi(u),$$
$$\mathbb{E}[\chi(A_u(X, \theta(T_\rho)))] = \mathrm{e}^{-u^2/2}\bigg(u\frac{|\theta(T_\rho)|_2}{(2\pi)^{3/2}} + \frac{|\partial\theta(T_\rho)|_1}{4\pi}\bigg) + \Psi(u) \quad \text{for all } \rho > 0.$$

For any sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive terms decreasing towards 0, the sequence of sets $(\theta(T_{\rho_n}))_{n\in\mathbb{N}}$ decreases to $\bigcap_{n\in\mathbb{N}}\theta(T_{\rho_n})=\theta(T)$; thus, the limit of $|\theta(T_{\rho_n})|_2$ as $n\to\infty$ is 0.

For any $\rho>0$, the frontier of $\theta(T_\rho)$ is

$$\begin{aligned}\partial\theta(T_\rho)&=\theta(\partial T_\rho)\\&=\{\theta(t+\rho v),\ t\in T\}\cup\{\theta(t-\rho v),\ t\in T\}\\&\quad\cup\{\theta(a+\delta v),\ \delta\in(-\rho,\rho)\}\cup\{\theta(b+\delta v),\ \delta\in(-\rho,\rho)\}.\end{aligned}$$

As ρ tends to 0, the one-dimensional measure of each of the first two sets of this disjoint union tends to $|\theta(T)|_1$, while the one of the last two tends to 0; therefore, $|\partial\theta(T_\rho)|_1\to 2|\theta(T)|_1$, completing the proof. □

Remark 3. Proposition 2 could be adapted in various ways. First, we could generalize it to a one-dimensional compact set T satisfying certain regularity assumptions. Besides, the sequence of sets $\{T_\rho,\ \rho>0\}$ approaching T could be defined differently, for instance as the sequence of ρ-tubes around T, that is,

$$T_\rho=\{z\in\mathbb{R}^2\colon \operatorname{dist}(T,z)\le\rho\}\quad\text{for all }\rho>0,$$

where $\operatorname{dist}(T,z)=\min_{x\in T}\{\|x-z\|\}$.

4.2. Modified Euler characteristic of an excursion set

For our approach in Section 6, where we want to identify θ by considering some well-chosen excursion sets of X_θ, it will be easier to limit ourselves to the term of highest index in (5). This is possible if we study the so-called modified Euler characteristic instead of the Euler characteristic of the excursion sets. Note that the modified Euler characteristic (denoted by ϕ) is only defined for excursion sets. It was introduced by Estrade and León [12] (where the authors acknowledge [1, Lemma 11.7.1] as a source of inspiration) in order to simplify the study of the Euler characteristic of excursion sets; see also [11].

Let us define it properly. Let d be a positive integer and let M be a d-dimensional compact set in $\mathbb{R}^d$. Let $f\colon\mathbb{R}^d\to\mathbb{R}$ be a function of class $\mathcal{C}^2$. We define, for $k\in\{0,\dots,d\}$,

$$\mu_k(f,u,M)=\#\{t\in\mathring{M}\colon f(t)\ge u,\ f'(t)=0,\ \operatorname{index}(f''(t))=d-k\},$$

where '#' denotes the cardinality and 'index$(f''(t))$' is the number of negative eigenvalues of the Hessian matrix of f at point t. The modified Euler characteristic of the excursion set of f above level u is then defined as

$$\phi(A_u(f,M))=\sum_{0\le k\le d}(-1)^k\mu_k(f,u,M).$$

Then

$$\phi(A_u(f,M))=\begin{cases}\#\{\text{local maxima of } f \text{ above } u \text{ in } \mathring{M}\}\\\quad-\#\{\text{local minima of } f \text{ above } u \text{ in } \mathring{M}\} & \text{if d=1,} \quad (8)\\ 2\#\{\text{local extrema of } f \text{ above } u \text{ in } \mathring{M}\}\\\quad-\#\{\text{stationary points of } f \text{ above } u \text{ in } \mathring{M}\} & \text{if d=2.}\quad (9)\end{cases}$$

We extend the definition in order to apply when $f\colon\mathbb{R}^2\to\mathbb{R}$ and M is a one-dimensional parametric curve of class $\mathcal{C}^2$ in $\mathbb{R}^2$. In this case, we consider $f_{|M}$ as a function defined on $\mathbb{R}$ and we define $\phi(A_u(f,M))$ as $\phi(A_u(f_{|M},M))$, which is given by (8).

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As with the Euler characteristic, in our setting when T is a rectangle or a segment in $\mathbb{R}^2$, we may write

$$\phi(A_u(X_\theta, T)) = \phi(A_u(X, \theta(T))).$$

Let us explain this, considering the case where T is a rectangle. For any $t \in \mathring{T}$, $(X_\theta)'(t) = J_\theta(t)^T X'(\theta(t))$. Since J_θ is invertible, there exists a one-to-one correspondence between the stationary points of X restricted to $\theta(\mathring{T})$ and the stationary points of X_θ restricted to $\mathring{T}$. Moreover, it is obvious that a point t is a local maximum of X_θ above u in $\mathring{T}$ if and only if $\theta(t)$ is a local maximum of X above u in $\mathring{\theta}(T)$. The same holds for the minima. Therefore, (9) implies that $\phi(A_u(X_\theta, T)) = \phi(A_u(X, \theta(T)))$.

Remark 4. (*Additivity property.*) The modified Euler characteristic of excursion sets, like the Euler characteristic, satisfies an additivity property. Let T and T' be two rectangles in $\mathbb{R}^2$ such that $T \cap T' = \varnothing$. Then

$$\phi(A_u(X_\theta, T \cup T')) = \phi(A_u(X_\theta, T)) + \phi(A_u(X_\theta, T')).$$

This property is a consequence of (9) by defining the modified Euler characteristic of an excursion set of X_θ as the alternate sum of numbers of stationary points of different types of X_θ in the considered domain. Even if the rectangles T and T' have a nonempty but one-dimensional intersection, the additivity property is still satisfied. Indeed, in this case, according to Bulinskaya lemma (see [1, Lemma 11.2.10]), almost surely X_θ admits no stationary points in $T \cap T'$; consequently, the modified Euler characteristic of the excursion set of X_θ over $T \cap T'$ is almost surely 0.

Under our assumptions on X, $\phi(A_u(X, \theta(T)))$ is one of the additive terms that appear in the expression of $\chi(A_u(X, \theta(T)))$ given by the Morse formula, the other terms depending only on the behaviour of X on the border of the set $\theta(T)$; see [12, Section 2.3] for more details. Thus, as in the proof of [1, Theorem 12.4.2], the term of highest index in the sum (5) corresponds to the expectation of $\phi(A_u(X, \theta(T)))$, that is,

$$\mathbb{E}[\phi(A_u(X_\theta, T))] = \mathbb{E}[\phi(A_u(X, \theta(T)))] = \mathcal{L}_d(\theta(T))\rho_d(u).$$

It follows that

$$\mathbb{E}[\phi(A_u(X_\theta, T))] = \mathbb{E}[\phi(A_u(X, \theta(T)))] = \begin{cases} \mathrm{e}^{-u^2/2}\dfrac{|\theta(T)|_1}{2\pi} & \text{if } \dim(T) = 1, \quad (10) \\ \mathrm{e}^{-u^2/2} u \dfrac{|\theta(T)|_2}{(2\pi)^{3/2}} & \text{if } \dim(T) = 2. \quad (11) \end{cases}$$

The expectation of the modified Euler characteristic of excursion sets does not satisfy the same continuity result as the one stated in Proposition 2 concerning the Euler characteristic. Indeed, consider, for example, $T_N = [a, b] \times [-N^{-1}, N^{-1}]$ for $N \in \mathbb{N} \setminus \{0\}$. Then (11) yields

$$\mathbb{E}[\phi(A_u(X, \theta(T_N)))] \to 0, \qquad N \to +\infty,$$

whereas, according to Proposition 2,

$$\mathbb{E}[\phi(A_u(X, \theta(T)))] = \mathrm{e}^{-u^2/2}\frac{|\theta([a, b] \times \{0\})|_1}{2\pi} \neq 0.$$

To conclude this section we provide an integral expression of the variance of the modified Euler characteristic of an excursion set of X over $\theta(T)$, where T is a rectangle. This will be

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useful in Section 6.2 when we address estimation matters. The following equation was proved in [11, Proposition 1] and it adapts easily if the domain is of the form $\theta(T)$ and if the random field X satisfies Assumption 1, as well as the following additional assumption: for any $t \in \mathbb{R}^2$, the random vector $(X(0), X(t))$ is not degenerate. Then

$$\begin{aligned}\operatorname{var}[\phi(A_u(X_\theta, T)] &= \operatorname{var}[\phi(A_u(X, \theta(T)))] \\ &= \int_{\mathbb{R}^2} |\theta(T) \cap (\theta(T) - t)|_2 (G(u, t) D(t)^{-1/2} - h(u)^2)\,\mathrm{d}t \\ &\quad + |\theta(T)|_2 (2\pi)^{-1} g(u),\end{aligned} \tag{12}$$

where

$$G(u, t) = \mathbb{E}[\mathbf{1}_{[u,+\infty)}(X(0))\mathbf{1}_{[u,+\infty)}(X(t)) \det(X''(0)) \det(X''(t)) \mid X'(0) = X'(t) = 0],$$
$$D(t) = (2\pi)^4 \det(I_2 - C''(t)^2), \qquad g(u) = \mathbb{E}[\mathbf{1}_{[u,+\infty)}(X(0)) |\det(X''(0))|)],$$
$$h(u) = (2\pi)^{-3/2} u \mathrm{e}^{-u^2/2}.$$

5. Notion of χ-isotropic deformation

In this section, the underlying field X is fixed and it satisfies Assumption 1. The deformations belong to $\mathcal{D}^2(\mathbb{R}^2)$. We define χ-isotropic deformations, characterized by an invariance condition of the mean Euler characteristic of some excursion sets of the associated deformed field. We show that the only deformations that satisfy this invariance property are the spiral deformations, that is, the ones that were proved to preserve isotropy in Section 3.

Definition 2. (χ*-isotropic deformation.*) A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is χ-isotropic if for any rectangle T in $\mathbb{R}^2$, for any $u \in \mathbb{R}$, and for any $\rho \in \mathrm{SO}(2)$,

$$\mathbb{E}[\chi(A_u(X_\theta, \rho(T))] = \mathbb{E}[\chi(A_u(X_\theta, T)]. \tag{13}$$

Remark 5. Note that the notion of χ-isotropy seems to be dependant on the underlying random field X involved in (13). However, after the statement and the proof of Theorem 2, it will be clear that it is in fact not the case. It will also be clear that an equivalent definition of χ-isotropic deformations could be given by replacing 'for any $u \in \mathbb{R}$' by 'for a fixed $u \neq 0$'. Besides, an equivalent version of Definition 2, using the modified Euler characteristic instead of the Euler characteristic of excursion sets, will appear to hold: a deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is χ-isotropic if for any rectangle *or segment* T in $\mathbb{R}^2$, for any $u \in \mathbb{R}$, and for any $\rho \in \mathrm{SO}(2)$,

$$\mathbb{E}[\phi(A_u(X_\theta, \rho(T))] = \mathbb{E}[\phi(A_u(X_\theta, T)].$$

Example 2. Spiral deformations (defined in Section 3) are χ-isotropic deformations. Indeed, if a deformation θ is such that X_θ is isotropic then it satisfies the above definition, since for any $\rho \in \mathrm{SO}(2)$, $X_{\theta\circ\rho}$ has the same law as X_θ. But according to Theorem 1, the deformations that preserve isotropy are exactly the spiral deformations.

The following theorem is the main result of Section 5.

Theorem 2. *The χ-isotropic deformations are exactly the spiral deformations in $\mathcal{D}^2(\mathbb{R}^2)$.*

Proof. Spiral deformations are χ-isotropic deformations according to Example 2; we prove that they are the only χ-isotropic deformations thanks to two lemmas and one result from [7].

The first lemma yields a characterization of χ-isotropic deformations involving invariance properties of the Jacobian matrix under rotation. To formulate it, we need to introduce an equivalence relation, denoted by '$\overset{\mathrm{SO}(2)}{\sim}$', on the space of invertible matrices of size 2×2: if M and N are two square matrices of size 2×2, $M \overset{\mathrm{SO}(2)}{\sim} N$ if there exists $\rho \in \mathrm{SO}(2)$ such that $M = \rho N$. Note that $M \overset{\mathrm{SO}(2)}{\sim} N$ is equivalent to the conditions $\|M^i\| = \|N^i\|$ for $i \in \{1, 2\}$ and $\det(M) = \det(N)$.

Lemma 1. *A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is χ-isotropic if and only if*

$$J_{\theta\circ\rho}(x) \overset{\mathrm{SO}(2)}{\sim} J_\theta(x) \quad \textit{for all } \rho \in \mathrm{SO}(2),\ x \in \mathbb{R}^2. \tag{14}$$

Proof. Let $\theta \in \mathcal{D}^2(\mathbb{R}^2)$. As explained above, condition (14) is equivalent to the following condition:

$$\|J^i_{\theta\circ\rho}(x)\| = \|J^i_\theta(x)\| \quad \text{for all } i \in \{1, 2\},\ \rho \in \mathrm{SO}(2),\ x \in \mathbb{R}^2, \tag{15a}$$

$$\det(J_{\theta\circ\rho}(x)) = \det(J_\theta(x)) \quad \text{for all } \rho \in \mathrm{SO}(2),\ x \in \mathbb{R}^2, \tag{15b}$$

First, we assume that θ is a χ-isotropic deformation. We fix $\rho \in \mathrm{SO}(2)$, $(s, t) \in (\mathbb{R} \setminus \{0\})^2$, and $u \in \mathbb{R} \setminus \{0\}$. Equation (13) is satisfied for rectangle $T = T(s, t)$; thus, (7) applied at two different levels u and u' implies that $|\theta \circ \rho(T(s, t))|_2 = |\theta(T(s, t))|_2$, whence

$$\int_{[0,s]} \int_{[0,t]} |\det(J_{\theta\circ\rho}(x, y))| \,\mathrm{d}x \,\mathrm{d}y = \int_{[0,s]} \int_{[0,t]} |\det(J_\theta(x, y)| \,\mathrm{d}x \,\mathrm{d}y.$$

Differentiating the above equality twice with respect to s and t yields, for any $(s, t) \in (\mathbb{R} \setminus \{0\})^2$, $|\det(J_{\theta\circ\rho}(s, t))| = |\det(J_\theta(s, t))|$, however, $|\det(J_{\theta\circ\rho}(s, t))| = |\det(J_\theta(\rho(s, t)))|$ and the Jacobian determinant of θ has a fixed sign on $\mathbb{R}^2$; hence, (15b) is satisfied. We now prove (15a) for $i = 1$, for instance. For any $n \in \mathbb{N} \setminus \{0\}$, according to the definition of χ-isotropy,

$$\begin{aligned}&\mathbb{E}[\chi(A_u(X_\theta, [0, s] \times [t - n^{-1}, t + n^{-1}]))]\\&\qquad = \mathbb{E}[\chi(A_u(X_\theta, \rho([0, s] \times [t - n^{-1}, t + n^{-1}])))].\end{aligned}$$

Then we apply Proposition 2 to the set $[0, s] \times \{t\}$ and the intersection of the sets $\{[0, s] \times [t - n^{-1}, t + n^{-1}],\ n \in \mathbb{N} \setminus \{0\}\}$ (respectively, to the set $\rho([0, s] \times \{t\})$ and the intersection of the sets $\{\rho([0, s] \times [t - n^{-1}, t + n^{-1}]),\ n \in \mathbb{N} \setminus \{0\}\}$). This yields

$$\mathbb{E}[\chi(A_u(X_\theta, [0, s] \times \{t\}))] = \mathbb{E}[\chi(A_u(X_\theta, \rho([0, s] \times \{t\})))]$$

and, using (6),

$$|\theta \circ \rho([0, s] \times \{t\})|_1 = |\theta([0, s] \times \{t\})|_1,$$

which can be written as

$$\int_{[0,s]} \|J^1_{\theta\circ\rho}(x, t)\| \,\mathrm{d}x = \int_{[0,s]} \|J^1_\theta(x, t)\| \,\mathrm{d}x.$$

Differentiating this integral equality with respect to s, we obtain $\|J^1_{\theta\circ\rho}(s, t)\| = \|J^1_\theta(s, t)\|$. Similarly, we obtain $\|J^2_{\theta\circ\rho}(s, t)\| = \|J^2_\theta(s, t)\|$. Hence, we have proved the direct implication of Lemma 1 and we now turn to the converse implication.

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Let T be a rectangle in $\mathbb{R}^2$. First, there exist $(s,t) \in (\mathbb{R}\setminus\{0\})^2$, $\rho_0 \in \mathrm{SO}(2)$, and a translation by vector $(a,b) \in \mathbb{R}^2$, denoted by $\tau_{a,b}$, such that $T = \rho_0 \circ \tau_{a,b}(T(s,t))$. Let $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ satisfying (15a) and (15b) for any $\rho \in \mathrm{SO}(2)$ and for any $x \in \mathbb{R}^2$. Therefore,

$$\begin{aligned}
|\theta \circ \rho(T)|_2 &= |\theta \circ \rho \circ \rho_0(\tau_{a,b}(T(s,t)))|_2 \\
&= \int_{[0,s]} \int_{[0,t]} |\det(J_{\theta\circ\rho\circ\rho_0}(a+x, b+y)|\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{[0,s]} \int_{[0,t]} |\det(J_{\theta\circ\rho_0}(a+x, b+y)|\,\mathrm{d}x\,\mathrm{d}y \\
&= |\theta(T)|_2.
\end{aligned}$$

The third equality results from (15b). Now, we express the perimeter length of $\theta \circ \rho(T)$ as

$$\begin{aligned}
|\partial\theta \circ \rho(T)|_1 &= |\partial\theta \circ \rho \circ \rho_0(\tau_{a,b}(T(s,t)))|_1 \\
&= \int_{[0,s]} \|J^1_{\theta\circ\rho\circ\rho_0}(a+x, b)\|\,\mathrm{d}x + \int_{[0,s]} \|J^1_{\theta\circ\rho\circ\rho_0}(a+x, b+t)\|\,\mathrm{d}x \\
&\quad + \int_{[0,t]} \|J^2_{\theta\circ\rho\circ\rho_0}(a, b+y)\|\,\mathrm{d}y + \int_{[0,t]} \|J^2_{\theta\circ\rho\circ\rho_0}(a+s, b+y)\|\,\mathrm{d}y \\
&= \int_{[0,s]} \|J^1_{\theta\circ\rho_0}(a+x, b)\|\,\mathrm{d}x + \int_{[0,s]} \|J^1_{\theta\circ\rho_0}(a+x, b+t)\|\,\mathrm{d}x \\
&\quad + \int_{[0,t]} \|J^2_{\theta\circ\rho_0}(a, b+y)\|\,\mathrm{d}y + \int_{[0,t]} \|J^2_{\theta\circ\rho_0}(a+s, b+y)\|\,\mathrm{d}y \\
&= |\partial\theta(T)|_1.
\end{aligned}$$

The third equality results from (15a). An application of (7) proves that $\mathbb{E}[\chi(A_u(X, \theta\circ\rho(T)))] = \mathbb{E}[\chi(A_u(X, \theta(T)))]$. Hence, θ is a χ-isotropic deformation and the proof is complete. □

Returning to the proof of Theorem 2, we are now able to state a second lemma that provides another characterization of χ-isotropic deformations involving the polar representation.

Lemma 2. *A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is a χ-isotropic deformation if and only if the functions*

$$\begin{gathered}
(r,\varphi) \mapsto (\partial_r\hat{\theta}^1(r,\varphi))^2 + (\hat{\theta}^1(r,\varphi)\partial_r\hat{\theta}^2(r,\varphi))^2, \\
(r,\varphi) \mapsto (\partial_\varphi\hat{\theta}^1(r,\varphi))^2 + (\hat{\theta}^1(r,\varphi)\partial_\varphi\hat{\theta}^2(r,\varphi))^2, \qquad (r,\varphi) \mapsto \hat{\theta}^1(r,\varphi)\det(J_{\hat{\theta}}(r,\varphi))
\end{gathered} \tag{16}$$

are radial, that is, if they do not depend on φ.

Proof. We use the notation introduced at the beginning of Section 3. The Jacobian matrix of S at point $(r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}$ is

$$J_S(r,\varphi) = \rho_\varphi \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}.$$

Consequently,

$$J_{S^{-1}}(S(r,\varphi)) = (J_S(r,\varphi))^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \rho_{-\varphi}.$$

Now for any rotation $\rho \in SO(2)$ and for any $(r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}$, we want to express $J_{(\theta\circ\rho)_0}(S(r,\varphi)) = J_{\theta_0\circ\rho_0}(S(r,\varphi))$ in terms of $J_{\widehat{\theta\circ\rho}}(r,\varphi)$.

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Since $\theta_0 = S \circ \hat{\theta} \circ S^{-1}$, we have

$$J_{\theta_0}(S(r,\varphi)) = \rho_{\hat{\theta}^2(r,\varphi)} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\theta}^1(r,\varphi) \end{pmatrix} J_{\hat{\theta}}(r,\varphi) \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \rho_{-\varphi}. \tag{17}$$

We use the characterization of χ-isotropy from Lemma 1. A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is a χ-isotropic deformation if and only if for any $(r,\varphi,\alpha) \in (0,+\infty) \times (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$J_{\theta_0 \circ \rho_\alpha}(S(r,\varphi)) = J_{\theta_0}(S(r,\varphi+\alpha))\rho_\alpha$$

is equivalent to $J_{\theta_0}(S(r,\varphi))$. Equivalently, for any $(r,\varphi,\alpha) \in (0,+\infty) \times (\mathbb{R}/2\pi\mathbb{Z})^2$, the equivalence relation

$$\begin{pmatrix} 1 & 0 \\ 0 & \hat{\theta}^1(r,\varphi+\alpha) \end{pmatrix} J_{\hat{\theta}}(r,\varphi+\alpha) \overset{\mathrm{SO}(2)}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\theta}^1(r,\varphi) \end{pmatrix} J_{\hat{\theta}}(r,\varphi)$$

holds and the above matrices have the same determinant in absolute value and the same norm of columns, which means that the functions defined by (16) do not depend on their second variable. □

To conclude the proof of Theorem 2, we refer the reader to [7] where the use of partial differential equations techniques was employed to prove that deformations satisfying (16) are spiral deformations. □

Denote by $\mathcal{S}$ the set of spiral deformations in $\mathcal{D}^2(\mathbb{R}^2)$, $\mathcal{X}$ the set of χ-isotropic deformations, $\mathcal{I}$ the set of deformations θ in $\mathcal{D}^2(\mathbb{R}^2)$ such that for any isotropic and stationary field X satisfying Assumption 1, X_θ is isotropic, and, finally, for a fixed stationary and isotropic field X satisfying Assumption 1, $\mathcal{I}(X)$ the set of deformations θ in $\mathcal{D}^2(\mathbb{R}^2)$ such that X_θ is isotropic. These sets satisfy the following chain of inclusions or equalities:

$$\mathcal{S} = \mathcal{I} \subset \mathcal{I}(X) \subset \mathcal{X} = \mathcal{S}.$$

The first and the last equalities come from Theorem 1 and Theorem 2, respectively; the first inclusion is obvious and the second one is a consequence of Example 2. As a result, the following corollary holds.

Corollary 1. *Let X be a stationary and isotropic random field satisfying Assumption 1. Then $\mathcal{S} = \mathcal{I}(X) = \mathcal{I} = \mathcal{X}$.*

To conclude, it is evident that the different notions that we have introduced so far to describe the isotropic behaviour of a deterministic deformation are in fact one and correspond to the spiral case.

6. Identification of θ through excursion sets

As explained in the introduction of this paper, we consider the case of an unknown deformation θ. We want to identify it using sparse data, that is, the observations of excursion sets of X_θ over well-chosen domains. More precisely, we assume that the mean modified Euler characteristic of some excursion sets of X_θ has been computed and we explain how we can almost uniquely characterize θ. The modified Euler characteristic is more adapted to our method than the Euler characteristic itself. This is due to the dependence of the mean Euler

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characteristic of an excursion set over a two-dimensional domain on both the perimeter length and the area of the domain, whereas its mean modified version only depends on the area; see (7) and (11). Furthermore, we limit ourselves to spiral deformations and we show that in this case, we can easily estimate θ using only one realization of the deformed field X_θ.

The underlying field X is unknown but it is still assumed to satisfy Assumption 1. The unknown deformation θ belongs to $\mathcal{D}^2(\mathbb{R}^2)$ and at each point in $\mathbb{R}^2$, its Jacobian determinant is positive.

6.1. Identification of θ

6.1.1. *The case of a linear deformation.* We state the simple case of a linear deformation that we use as a first step towards the general case. We assume that θ is a linear function and write it matricially in a fixed orthonormal basis of $\mathbb{R}^2$, that is,

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.$$

In this case, we only have to consider the excursion sets over one horizontal segment, one vertical segment, and one rectangle (product of two segments). Thus, we fix $(s, t) \in (\mathbb{R} \setminus \{0\})^2$, $u \neq 0$, and we assume that we know $\mathbb{E}[\phi(A_u(X_\theta, [0, s] \times \{0\}))]$, $\mathbb{E}[\phi(A_u(X_\theta, \{0\} \times [0, t]))]$, and $\mathbb{E}[\phi(A_u(X_\theta, T(s, t))]$. The three real numbers

$$a = \sqrt{\theta_{11}^2 + \theta_{21}^2}, \qquad b = \sqrt{\theta_{12}^2 + \theta_{22}^2}, \qquad c = \theta_{11}\theta_{22} - \theta_{21}\theta_{12} \tag{18}$$

satisfy

$$|\theta([0, s] \times \{0\})|_1 = |s|a, \qquad |\theta(\{0\} \times [0, t])|_1 = |t|b, \qquad |\theta(T(s, t))|_2 = |st|c.$$

Therefore, they are solutions to (10) and (11) and they can be used to write another expression of matrix θ: there exists $(\alpha, \beta) \in (\mathbb{R}/2\pi\mathbb{Z})^2$ such that

$$\theta = \begin{pmatrix} a\cos(\alpha) & b\cos(\beta) \\ a\sin(\alpha) & b\sin(\beta) \end{pmatrix}.$$

Let $\delta = \beta - \alpha$ be the angle between the two column vectors. It satisfies $c = ab\sin(\delta)$, whence

$$\delta \in \{\delta_0, \delta_1\},$$

where $\delta_0 = \arcsin(c/ab) \in (0, \pi/2]$ and $\delta_1 = \pi - \arcsin(c/ab) \in [\pi/2, \pi)$. Consequently, we are able to determine matrix θ up to an unknown rotation, with two possibilities concerning the angle between its two column vectors: θ belongs to the set $\mathcal{M}(a, b, c)$ defined by

$$\mathcal{M}(a, b, c) = \left\{\rho_\alpha \begin{pmatrix} a & \sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix}, \rho_\alpha \begin{pmatrix} a & -\sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix}, \alpha \in \mathbb{R}/2\pi\mathbb{Z}\right\}. \tag{19}$$

If the determinant of θ was not assumed to be positive, there would be two other possibilities, up to a rotation, because δ could take four possible values. Note that according to Example 1, X_θ is isotropic in the case where $a = b = \sqrt{c}$, which implies $\delta = \pi/2$.

Of course, due to the isotropy of X, we obtain θ up to post-composition with an unknown rotation. Our method is based on the mean Euler characteristic of excursion sets of X_θ over some sets, which only depends on θ through the perimeter and area of the set's image by θ. Consequently, we cannot differentiate between two deformations that transform any set into sets with the same perimeter and the same area.

We summarize our approach in the following method.

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Method 1. *Let*

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

be an unknown linear deformation with positive determinant.

For a fixed $(s, t) \in (\mathbb{R} \setminus \{0\})^2$ *and for fixed* $u \in \mathbb{R} \setminus \{0\}$*, we assume that* $\mathbb{E}[\phi(A_u(X_\theta, T))]$ *is known for* T *of the form* $[0, s] \times \{0\}$*,* $\{0\} \times [0, t]$ *and* $[0, s] \times [0, t]$*. Then* a*,* b*, and* c *from (18) are computable as a result of (6) and (7), and* θ *belongs to the set* $\mathcal{M}(a, b, c)$ *defined by (19).*

6.1.2. *General method.* We refer the reader to [5, Appendix] for a precise definition of the complex dilatation and for the statement of the mapping theorem that formulates a characterization of a deformation up to a conformal mapping through its complex dilatation. To be able to apply it, we add a hypothesis on θ: from now on, we assume that θ has uniformly bounded distortion, that is, the ratio of

$$\limsup_{x \to x_0} \frac{|\theta(x) - \theta(x_0)|}{|x - x_0|} \quad \text{to} \quad \liminf_{x \to x_0} \frac{|\theta(x) - \theta(x_0)|}{|x - x_0|}$$

is uniformly bounded for $x_0 \in \mathbb{R}^2$.

We fix $u \neq 0$, $S > 0$, and assume that $\mathbb{E}[\phi(A_u(X_\theta, [0, s] \times \{t\}))]$, $\mathbb{E}[\phi(A_u(X_\theta, \{s\} \times [0, t]))]$, and $\mathbb{E}[\phi(A_u(X_\theta, T(s, t))]$ are known for any $(s, t) \in [-S, S]^2$. Then for any $(s, t) \in [-S, S]^2$, we can deduce $|\theta([0, s] \times \{t\})|_1$ and $|\theta(\{s\} \times [0, t])|_1$ from (10) by simply solving a linear system. Furthermore,

$$|\theta([0, s] \times \{t\})|_1 = \int_{[0,s]} \|J_\theta^1(x, t)\| \, \mathrm{d}x = \int_{[0,s]} \sqrt{\partial_x \theta_1(x, t)^2 + \partial_x \theta_2(x, t)^2} \, \mathrm{d}x,$$
$$|\theta(\{s\} \times [0, t])|_1 = \int_{[0,t]} \|J_\theta^2(s, y)\| \, \mathrm{d}y = \int_{[0,t]} \sqrt{\partial_y \theta_1(s, y)^2 + \partial_y \theta_2(s, y)^2} \, \mathrm{d}y.$$

The first-order partial derivatives of θ are continuous. By differentiating the functions $s \mapsto |\theta([0, s] \times \{t\})|_1$ and $t \mapsto |\theta(\{s\} \times [0, t])|_1$, we obtain functions $s \mapsto \|J_\theta^1(s, t)\|$ and $t \mapsto \|J_\theta^2(s, t)\|$ on the segment $[-S, S]$.

Now considering the rectangle domains $\{T(s, t), (s, t) \in ([-S, S] \setminus \{0\})^2\}$, we assume that $\mathbb{E}[\phi(A_u(X_\theta, T(s, t)))]$ is known. Since $u \neq 0$, we can compute $|T(s, t)|_2$ using (11). Then, by differentiating the function

$$(s, t) \mapsto |\theta(T(s, t))|_2 = \int_{[0,s]} \int_{[0,t]} |\det(J_\theta(x, y))| \, \mathrm{d}x \, \mathrm{d}y$$

twice with respect to s and to t on the square $[-S, S]^2$, we obtain the function $(s, t) \mapsto |\det(J_\theta(s, t))|$ on the same square.

Now, we fix $x \in ([-S, S] \setminus \{0\})^2$ and write

$$J_\theta(x) = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix},$$

using the same notation a, b, and c defined in (18) as in the linear case, although they now depend on x. The explanations given in Section 6.1.1 apply here and, consequently, $J_\theta(x)$ belongs to $\mathcal{M}(a, b, c)$. Moreover, we express the complex dilatation μ as

$$\mu = \frac{\partial_{\bar{z}} \theta}{\partial_z \theta},$$

where $\partial_z\theta = \frac{1}{2}(\partial_s\theta_1 + \partial_t\theta_2) + (\mathrm{i}/2)(\partial_s\theta_2 - \partial_t\theta_1)$ and $\partial_{\bar{z}}\theta = \frac{1}{2}(\partial_s\theta_1 - \partial_t\theta_2) + (\mathrm{i}/2)(\partial_s\theta_2 + \partial_t\theta_1)$. At point x, a short computation yields that $\mu(x)$ takes two possible values in the set

$$\mathcal{C}(a, b, c) = \left\{\frac{1}{a^2 + b^2 + 2c}(a^2 - b^2 \pm 2\mathrm{i}\sqrt{a^2b^2 - c^2})\right\}. \tag{20}$$

The general method is summarized below.

Method 2. *Let $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ be a deformation with a positive Jacobian on $\mathbb{R}^2$. Let $S > 0$ and let $u \in \mathbb{R} \setminus \{0\}$ be fixed. Assuming that for any $x = (s, t) \in [-S, S]^2$, $\mathbb{E}[\phi(A_u(X_\theta, T))]$ is known for T of the form $[0, s] \times \{t\}$, $\{s\} \times [0, t]$, and $[0, s] \times [0, t]$, we may then compute $a = \|J^1_\theta(x)\|$, $b = \|J^2_\theta(x)\|$, and $c = \det(J_\theta(x))$. Consequently, for each $x \in [-S, S]^2$, the Jacobian matrix at point x, $J_\theta(x)$ belongs to $\mathcal{M}(a, b, c)$, defined by (19), and the complex dilatation at point x, $\mu(x)$ belongs to $\mathcal{C}(a, b, c)$, defined by (20).*

Remark 6. (*Numerical approach.*) In fact, we can only have at our disposal a finite amount of data. Let σ be a partition of $[-S, S]$. If for any $(s, t) \in \sigma^2$, $\mathbb{E}[\phi(A_u(X_\theta, T))]$ is known for T of the form $[0, s] \times \{t\}$, $\{s\} \times [0, t]$, and $[0, s] \times [0, t]$, then numerical approaches such as Runge–Kutta methods allow us to compute approximate values of $\|J^1_\theta(s, t)\|$, $\|J^2_\theta(s, t)\|$, and $\det(J_\theta(s, t))$ for any $(s, t) \in \sigma^2$, and the approximate values for $J_\theta(s, t)$ and $\mu(s, t)$.

6.1.3. *The case of a tensorial deformation.* We now study the particular case of tensorial deformations, where we can completely identify θ if we make an assumption of monotonicity on its coordinate functions. Let $\theta(s, t) = (\theta_1(s), \theta_2(t))$. Our hypotheses on θ mean that for $i \in \{1, 2\}$, $\theta_i : \mathbb{R} \to \mathbb{R}$ satisfies $\theta_i(0) = 0$, θ_i is a bijective function of class $\mathcal{C}^2$, and, therefore, it is monotonic. Note that θ transforms a rectangle $[s, v] \times [t, w]$ into another rectangle $\theta_1([s, v]) \times \theta_2([t, w])$.

Let $s \in \mathbb{R} \setminus \{0\}$. From (10), we deduce that

$$\mathbb{E}[\phi(A_u(X_\theta, [0, s] \times \{0\})] = \frac{\mathrm{e}^{-u^2/2}}{2\pi}\int_0^s |\theta_1'(x)|\,\mathrm{d}x,$$
$$\mathbb{E}[\phi(A_u(X_\theta, \{0\} \times [0, s])] = \frac{\mathrm{e}^{-u^2/2}}{2\pi}\int_0^s |\theta_2'(x)|\,\mathrm{d}x$$

and, consequently, we can state the following method.

Method 3. *Let $(s, t) \mapsto \theta(s, t) = (\theta_1(s), \theta_2(t)) \in \mathcal{D}^2(\mathbb{R}^2)$ be a tensorial deformation. We fix $S > 0$ and $u \in \mathbb{R}$. We assume that for any $s \in [-S, S] \setminus \{0\}$, $\mathbb{E}[\phi(A_u(X_\theta, T))]$ is known for T of the form $[0, s] \times \{0\}$ and $\{0\} \times [0, s]$. Then we determine functions $s \mapsto |\theta_1'(s)|$ and $s \mapsto |\theta_2'(s)|$ on $[-S, S]$ using (10). If the sign of each coordinate function is known then θ is completely determined on $[-S, S]^2$.*

Example 3. Let $(\alpha, \beta) \in (\mathbb{R} \setminus \{0\})^2$, θ be defined on $[0, 1]^2$ by $\theta(s, t) = (s^\alpha, t^\beta)$, and σ be a partition of $(0, 1]$. To identify θ_1, we follow the above method adapted to a numerical approach; thus, we obtain approximate values for $\{|\theta_1'(s)|,\ s \in \sigma\}$. Constant values correspond to the $\alpha = 1$ case. Otherwise, we have $|\theta_1'(s)| = |\alpha| s^{\alpha-1}$; therefore, coefficient α can be computed through a regression method: $\alpha - 1$ is the slope of the line representing $\log(|\theta_1'(s)|) = \log(|\alpha|) + (\alpha - 1)\log(s)$ as a function of $\log(s)$ on $(0, 1]$. The same method can be used to obtain coefficient β.

Remark 7. Methods 1–3 can be easily adapted if the modified Euler characteristic ϕ is replaced by the Euler characteristic χ.

6.2. Estimation in the spiral case

Up to now we have assumed that $\mathbb{E}[\phi(A_u(X_\theta, T))]$ was known for some basic domains T, but we have yet to discuss estimation matters. Without any hypothesis on θ, it is not possible to estimate this expectation from one single realization of X_θ, since the deformed field is nonstationary, except in the linear case. Yet it is possible in the spiral case due to the isotropy of the deformed field.

In this section, in order to derive results about the variance of our estimators, we furthermore assume that for any $t \in \mathbb{R}^2$, the vector $(X(0), X(t))$ is not degenerate and that the function C and its derivatives satisfy the following condition at ∞:

$$\nu(t) \to 0 \quad \text{and} \quad \nu \in L^1(\mathbb{R}^2), \qquad \|t\| \to +\infty, \tag{21}$$

where

$$\nu(t) = \max\left\{ \left| \frac{\partial^{k_1+k_2} C}{\partial x^{k_1} \partial y^{k_2}}(t) \right|, (k_1, k_2) \in \mathbb{N}^2, k_1 + k_2 \le 4 \right\}.$$

Let $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ be a spiral deformation; we show in the following how to estimate $\|J^1_\theta(x)\|$, $\|J^2_\theta(x)\|$, and $\det(J_\theta(x))$ at each point x in a chosen domain. Then Method 2 can be applied to identify θ.

Let $x \in \mathbb{R}^2 \setminus \{0\}$ and let (r_0, φ_0) be its polar coordinates. For $N \in \mathbb{N} \setminus \{0\}$, let $T^0_N = \{(r, \varphi) \in (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}\colon r_0 \le r \le r_0 + N^{-1}, \varphi_0 \le \varphi \le \varphi_0 + 2\pi N^{-1}\}$.

For any $k \in \{0, \dots, N-1\}$, we write $T^k_N = \rho_{2k\pi/N}(T^0_N)$. We fix $u \ne 0$ and define

$$Z_N = N^{-1} \sum_{k=0}^{N-1} \phi(A_u(X_\theta, T^k_N)) = N^{-1} \sum_{k=0}^{N-1} \phi(A_u(X, \theta(T^k_N))),$$

where ϕ is the modified Euler characteristic. Recall that ϕ satisfies an additivity property; see Remark 4. Thus,

$$Z_N = N^{-1} \phi\bigg(A_u\bigg(X, \bigcup_{k=0}^{N-1} \theta(T^k_N)\bigg)\bigg) = N^{-1}\phi(A_u(X, \theta(U_N))),$$

where $U_N = \bigcup_{k=0}^{N-1} T^k_N$. We derive the asymptotic behaviour of the expectation and variance of Z_N described in the following proposition from the χ-isotropy property satisfied by θ. If $(u_N)_{N\in\mathbb{N}}$ and $(v_N)_{N\in\mathbb{N}}$ are real sequences, we write $u_N \sim v_N$, $N \to +\infty$ if there exist $N_0 \in \mathbb{N}$ and a real sequence $(\varepsilon_N)_{N \ge N_0}$ converging towards 1 such that for $N \ge N_0$, $u_N = \varepsilon_N v_N$.

Proposition 3. *There exist constants $a \ne 0$ and $c > 0$ (depending only on u) and $n \in \mathbb{N} \setminus \{0\}$ such that*

$$\mathbb{E}[Z_N] \sim a|\det(J_\theta(x))||T^0_N|_2, \qquad N \to +\infty,$$

and for $N \ge n$,

$$\operatorname{var}[Z_N] \le c\frac{|\det(J_\theta(x))||T^0_N|_2}{N}.$$

Proof. Let $N \in \mathbb{N} \setminus \{0\}$. According to Theorem 2, θ is χ-isotropic, which implies that for any $k \in \{0, \dots, N-1\}$, $\mathbb{E}[\phi(A_u(X, \theta(T^k_N)))] = \mathbb{E}[\phi(A_u(X, \theta(T^0_N)))]$ according to Definition 2. Therefore, the expectation of Z_N is

$$\mathbb{E}[Z_N] = \mathbb{E}[\phi(A_u(X, \theta(T^0_N)))] = \frac{u\mathrm{e}^{-u^2/2}}{(2\pi)^{3/2}}|\theta(T^0_N)|_2.$$

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We study the asymptotic behaviour of the sequence

$$\begin{aligned}
&||\theta(T_N^0)|_2 - |\det(J_\theta(x))||T_N^0|_2| \\
&\quad \le \int_{r_0}^{r_0+N^{-1}} \int_{\varphi_0}^{\varphi_0+2\pi N^{-1}} ||\det(J_\theta(S(r,\varphi)))| - |\det(J_\theta(S(r_0,\varphi_0)))||r \, \mathrm{d}r \, \mathrm{d}\varphi \\
&\quad \le \sup_{\varphi_0 \le \varphi \le \varphi_0+2\pi N^{-1},\, r_0 \le r \le r_0+N^{-1}} ||\det(J_\theta(S(r,\varphi)))| - |\det(J_\theta(S(r_0,\varphi_0)))|||T_N^0|_2,
\end{aligned}$$

with $\sup_{\varphi_0 \le \varphi \le \varphi_0+2\pi N^{-1},\, r_0 \le r \le r_0+N^{-1}} ||\det(J_\theta(S(r,\varphi)))| - |\det(J_\theta(S(r_0,\varphi_0)))|| \to 0$, $N \to +\infty$. Consequently,

$$|\theta(T_N^0)|_2 \sim |\det(J_\theta(x))||T_N^0|_2, \qquad N \to +\infty, \tag{22}$$

and the result for the asymptotic expectation holds.

Now we use (12) (with its notation) to obtain an integral expression of the variance of $Z_N = N^{-1}\phi(A_u(X, \theta(U_N)))$. An asymptotic upper bound is obtained under condition (21), which requires that the map $t \mapsto G(u,t)D(t)^{-1/2} - h(u)^2$ has a finite integral on $\mathbb{R}^2$, according to [11, Lemma 3]. Thus,

$$\begin{aligned}
&\operatorname{var}[\phi(A_u(X, \theta(U_N)))] \\
&\quad = \int_{\mathbb{R}^2} |\theta(U_N) \cap (\theta(U_N) - t)|_2 (G(u,t)D(t)^{-1/2} - h(u)^2) \, \mathrm{d}t + |\theta(U_N)|_2 (2\pi)^{-1} g(u) \\
&\quad \le |\theta(U_N)|_2 \bigg(\int_{\mathbb{R}^2} (G(u,t)D(t)^{-1/2} - h(u)^2) \, \mathrm{d}t + (2\pi)^{-1} g(u) \bigg) \\
&\quad \le c|\theta(U_N)|_2 \\
&\quad = cN|\theta(T_N^0)|_2,
\end{aligned}$$

where $c > 0$. (Note that the integration domain of the first integral is, in fact, the compact $\{t - t', (t,t') \in \theta(U_N)^2\}$.) Consequently, using (22), we obtain

$$\operatorname{var}[Z_N] \le cN^{-1}|\theta(T_N^0)|_2 \sim cN^{-1}|\det(J_\theta(x))||T_N^0|_2, \qquad N \to +\infty.$$

This concludes the proof. □

In Proposition 3 we see that, asymptotically, the variance of Z_N is negligible with respect to its expectation. Practically, we could obtain $|\det(J_\theta(x))|$ through a regression method since, up to a constant, it is the coefficient of the linear relation linking asymptotically $|\theta(T_N^0)|_2$ and $|T_N^0|_2$. Constant a is totally explicit and constant c may be numerically computed.

We can adopt the same approach to obtain an estimation of $\|J_\theta^i(x)\|$ for $i \in \{1, 2\}$. We will only state the asymptotic result (for $i = 1$) as the proof is very similar to the one of Proposition 3. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $S_N^0 = [x_1, x_1 + N^{-1}] \times \{x_2\}$. For any $N \in \mathbb{N} \setminus \{0\}$ and for any $k \in \{0, \ldots, N-1\}$, we write $S_N^k = \rho_{2k\pi/N}(S_N^0)$ and define

$$Y_N = N^{-1} \sum_{k=0}^{N-1} \phi(A_u(X_\theta, S_N^k)).$$

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Proposition 4. *There exist constants $d \neq 0$ and $k > 0$ (depending only on u), and $n \in \mathbb{N} \setminus \{0\}$ such that*

$$\mathbb{E}[Y_N] \sim d \| J^1_\theta(x) \| |S^0_N|_1, \qquad N \to +\infty,$$

and for $N \geq n$,

$$\operatorname{var}[Y_N] \leq k \frac{\| J^1_\theta(x) \| |S^0_N|_1}{N}.$$

Estimates of $|\det(J_\theta)|$, $\|J^1_\theta(x)\|$, and $\|J^2_\theta(x)\|$ yield a near complete characterization of the Jacobian matrix of θ, as explained in Section 6.1.

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