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ON THE LOCALITY OF THE PSEUDOVARIETY DG

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Abstract The pseudovariety **DG** of all finite monoids all of whose regular $\mathcal{D}$-classes are subgroups is shown to be local, that is, it is verified that the pseudovariety g**DG** of finite categories generated by **DG** coincides with the pseudovariety ℓ**DG** of all finite categories whose local monoids all belong to **DG**. Yet more general statements of this kind are deduced, yielding results such as that, for every prime number p, the pseudovariety $\mathbf{DG}_p$ of all finite monoids all of whose regular $\mathcal{D}$-classes are p-groups is local, or that the pseudovarieties $\mathbf{DG}_{\mathrm{sol}}$ and $\mathbf{DG}_{\mathrm{nil}}$ of all finite monoids all of whose regular $\mathcal{D}$-classes are, respectively, solvable groups and nilpotent groups are local.

Keywords: pseudovarieties of finite monoids and finite categories;
locally finite varieties of monoids and categories;
finitely generated relatively free monoids and categories;
Mal′cev products of locally finite varieties of groups

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Introduction

Tilson in his seminal paper [**15**] laid the foundations of the theory of varieties of small categories and of pseudovarieties of finite categories. His work has grown up mainly from the need of these tools in the investigations of semidirect products of pseudovarieties of finite semigroups and monoids. Among other things, he introduced the notions of local varieties of monoids and local pseudovarieties of finite monoids. With every pseudovariety **V** of finite monoids, two pseudovarieties of finite categories are naturally associated. Namely, one can consider the smallest and the greatest pseudovarieties of finite categories having the property that the one-vertex categories in these pseudovarieties, which can be viewed as ordinary monoids, constitute exactly the given monoid pseudovariety **V**. The first of these two pseudovarieties of categories is the pseudovariety g**V** generated by the monoid pseudovariety **V**, which is viewed as a class of finite one-vertex categories. The second of the mentioned two pseudovarieties of categories is the pseudovariety ℓ**V** consisting of all finite categories all of whose local monoids belong to **V**. Naturally, one has $g\mathbf{V} \subseteq \ell\mathbf{V}$. Then the pseudovariety **V** of finite monoids is said to be local if $g\mathbf{V} = \ell\mathbf{V}$. It can be readily seen that this happens if and only if every finite category whose local monoids all belong to **V** has the property that it itself divides a finite monoid from **V** (which monoid is viewed here as a one-vertex category again). The concept of division of categories appearing in this context is another notion introduced in [**15**].

Many important pseudovarieties of finite monoids have consecutively been shown to be local in the sense specified above. Just to mention some of the most significant instances of such pseudovarieties, remember that Jones and Trotter have reported in [**8**] that the pseudovariety **DS** of all finite monoids all of whose regular $\mathcal{D}$-classes are subsemigroups (and therefore they are completely simple semigroups) is local, and that Almeida has proved in [**2**] that the pseudovariety **DO** of all finite monoids all of whose regular $\mathcal{D}$-classes are orthodox subsemigroups (and therefore they are rectangular groups) is local, and that also the pseudovariety **DA** of all finite monoids all of whose regular $\mathcal{D}$-classes are aperiodic subsemigroups (and therefore they are rectangular bands) is local as well. On the other hand, it has been noticed by Tilson in § 15 of [**15**] that from the earlier work of Knast [**10**] it follows, among other things, that the pseudovariety **J** of all finite $\mathcal{J}$-trivial monoids (that is, finite monoids all of whose regular $\mathcal{D}$-classes are trivial subsemigroups) is not local. In these circumstances, the still open question of whether the pseudovariety **DG** of all finite monoids all of whose regular $\mathcal{D}$-classes are subgroups is local, which has been raised by Almeida in [**2**], became of particular interest.

It is the purpose of the present paper to provide a positive answer to this question. However, the techniques used to tackle this problem in this paper allow us to obtain considerably more information than only this single fact. This has been incidentally the case also in the papers [**8**] and [**2**] quoted above. Thus, in [**8**], Jones and Trotter have encompassed, in fact, all subpseudovarieties of the monoid pseudovariety **DS** which arise by taking, for any given non-trivial extension closed pseudovariety **H** of finite groups, the collection of all finite monoids from **DS** whose maximal subgroups all belong to **H**. In [**2**], Almeida has even proved that, for whatever pseudovariety **H** of finite groups, the collection of all finite monoids from the pseudovariety **DO** whose maximal subgroups all belong to **H** forms a local subpseudovariety of **DO**. In particular, if **H** is the pseudovariety of trivial groups, the monoid pseudovariety **DA** arises in this way, which entails that it is a local pseudovariety. Turning now to the monoid pseudovariety **DG**, the requirement that the maximal subgroups of the monoids in **DG** should belong to any given pseudovariety **H** of finite groups sorts out from **DG** the subpseudovariety **DH** consisting of all finite monoids all of whose regular $\mathcal{D}$-classes are groups from **H**. Now, if **H** is the pseudovariety of trivial groups, then **DH** becomes the pseudovariety **J** of all finite $\mathcal{J}$-trivial monoids which is not local. Thus the apparently non-trivial question arises for which pseudovarieties **H** of finite groups the corresponding monoid pseudovarieties **DH** are local. In the present paper, significant positive results bearing on this question are obtained. On the other hand, it is shown that **J** is by far not the only monoid pseudovariety of the form **DH** which is not local.

The present paper differs substantially from the papers [**8**] and [**2**], as far as the methods applied to gain the mentioned results are concerned. In [**8**], double semidirect product decomposition techniques are used to treat the finite categories in ℓ**DS** with the aim to show that they are all members of g**DS**. However, the basic results on double semidirect products of pseudovarieties of finite categories that the approach adopted in [**8**] depends on have been correctly proved only subsequently by Steinberg in [**14**]. In [**2**], syntactic techniques are employed involving the concepts of implicit operations, pseudoidentities and relatively free profinite objects, which concepts have been extended from pseudovarieties of finite monoids to pseudovarieties of finite categories by Almeida and Weil in [**3**]. The key idea of the proof in [**2**] is to show that every category pseudoidentity satisfied in $g\mathbf{DO}$ (in $g\mathbf{DA}$) is also valid in $\ell\mathbf{DO}$ (in $\ell\mathbf{DA}$). In contrast to these advanced methods, the approach applied in the present paper is much more elementary, but, on the other hand, rather laborious. It takes advantage of the possibility to convert, in a sense, the given problem from pseudovarieties of finite monoids and categories to locally finite varieties of monoids and categories, which offers the opportunity to employ the classic techniques including usual identities and finitely generated free monoids and categories. From this point of view, this approach resembles rather the way how the pseudovariety of all finite completely regular monoids has been shown to be local by Jones in [**7**]. Additionally, the description of the free groups in the Mal'cev products of varieties of groups by means of Cayley graphs and suitable semidirect products will have its role to play in this paper. This device can either be drawn from the results of Smel'kin [**12**] who described the free groups in the Mal'cev products of varieties of groups by means of verbal wreath products, or it can be taken over directly from more general results of this kind involving Cayley graphs and semidirect products which were obtained later on in the theory of regular semigroups (see, for example, [**9**]).

This paper begins in § 1 with a survey of the notions related to graphs and categories that are needed in the subsequent sections. They are mostly adopted from the paper [**15**] by Tilson, to which paper we refer for a thorough exposition of this subject. Section 2 summarizes the necessary information on the pseudovariety of monoids **DG**, which is used in § 3 to introduce the appropriate locally finite varieties of monoids. Likewise, § 4 contains the needed information on the pseudovariety of categories $\ell\mathbf{DG}$, which is then used in § 5 to conceive the corresponding locally finite varieties of categories. In §§ 6 and 7, some suitable subvarieties of the monoid varieties appearing in § 3 are considered and various necessary technical results related to the word problem for free monoids in these subvarieties are deduced. Section 8 goes back to the monoid varieties from § 3 and provides some further specific information regarding the word problem for free monoids in these varieties that will come in handy later in the paper. Section 9 reviews the required knowledge about the Mal'cev products of varieties of groups and offers the description of the free groups in these Mal'cev products in terms of Cayley graphs and semidirect products of relatively free groups from the varieties figuring in these products. Section 10 contains a note on directed and undirected graphs which is applied substantially in the considerations of the next section. In § 11, the tools prepared in the preceding sections are used to prove that the monoid pseudovariety **DG** is local. In § 12, the proof provided in § 11 is checked up to reveal that, in fact, for a large family of pseudovarieties **H** of finite groups, it furnishes the arguments confirming that the corresponding monoid pseudovarieties **DH** are local as well. For instance, from one of the general results obtained in § 12 it follows that this finding concerns all non-trivial extension closed pseudovarieties **H** of finite groups. Furthermore, improving slightly some of the arguments given in § 11,

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one obtains the proof of a general result which yields, as a special case, that the same is true of pseudovarieties **H** that arise as the joins of non-empty families of non-trivial extension closed pseudovarieties of finite groups. The mentioned joins are taken in the lattice of all pseudovarieties of finite groups. In this way, it turns out, for example, that the pseudovarieties $\mathbf{DG}_{\mathrm{sol}}$ and $\mathbf{DG}_{\mathrm{nil}}$ of all finite monoids all of whose regular $\mathcal{D}$-classes are, respectively, solvable groups and nilpotent groups are instances of local pseudovarieties of finite monoids. Finally, in § 13, it is shown that, for the pseudovariety **Ab** of all finite abelian groups, the monoid pseudovariety **DAb** is not local, providing thus an example indicating that the positive results mentioned so far have their limits beyond which they cannot be further extended.

1. Graphs and categories

A graph Γ consists of a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of edges together with two mappings $\alpha, \omega : E(\Gamma) \to V(\Gamma)$ assigning to every edge $e \in E(\Gamma)$ its beginning $\alpha(e)$ and its end $\omega(e)$. We say that two edges $e, f \in E(\Gamma)$ are coterminal if $\alpha(e) = \alpha(f)$ and $\omega(e) = \omega(f)$. For any vertices $v, w \in V(\Gamma)$, we write $\Gamma(v, w)$ for the set of all edges $e \in E(\Gamma)$ such that $\alpha(e) = v$ and $\omega(e) = w$. We call $\Gamma(v, w)$ a hom-set of Γ. For any vertex $v \in V(\Gamma)$, we write shortly $\Gamma(v)$ instead of $\Gamma(v, v)$.

By a category, viewed as an algebraic structure, we mean a graph C endowed with an associative partial binary operation of multiplication which, for arbitrary vertices $u, v, w \in V(C)$, assigns to any edges $e \in C(u, v)$ and $f \in C(v, w)$ an edge $ef \in C(u, w)$, and having, for every vertex $v \in V(C)$, an identity edge $\mathtt{1}_v \in C(v)$ acting as an identity element with respect to the multiplication, whenever it is defined. For any vertex $v \in V(C)$, the set $C(v)$ together with the multiplication from C restricted to $C(v)$ forms a monoid, called the local monoid of C at v.

Let Γ, Δ be graphs. A graph mapping $\varphi : \Gamma \to \Delta$ consists of two mappings $\varphi_V : V(\Gamma) \to V(\Delta)$ and $\varphi_E : E(\Gamma) \to E(\Delta)$ such that, for every edge $e \in E(\Gamma)$, $\alpha(\varphi_E(e)) = \varphi_V(\alpha(e))$ and $\omega(\varphi_E(e)) = \varphi_V(\omega(e))$. Note that usually, we will write simply $\varphi(v)$ instead of $\varphi_V(v)$, for any $v \in V(\Gamma)$, and $\varphi(e)$ instead of $\varphi_E(e)$, for any $e \in E(\Gamma)$.

Let C, D be categories. A homomorphism $\psi : C \to D$ of categories is a graph mapping such that, for any vertices $u, v, w \in V(C)$ and any edges $e \in C(u, v)$ and $f \in C(v, w)$, the equality $\psi(ef) = \psi(e)\psi(f)$ holds, and for any vertex $v \in V(C)$, the equality $\psi(\mathtt{1}_v) = \mathtt{1}_{\psi(v)}$ also holds.

Let Γ be a graph. An equivalence relation $\sim$ on Γ consists of a family of equivalence relations $\sim_{v,w}$ on the hom-sets $\Gamma(v, w)$ for all pairs of vertices $v, w \in V(\Gamma)$. Often, we will write simply $\sim$ instead of $\sim_{v,w}$, for any $v, w \in V(\Gamma)$. In this way, the quotient graph $\Gamma/{\sim}$ arises whose vertices are those of Γ and whose edges are the equivalence classes of coterminal edges of Γ with respect to the equivalence relation $\sim$.

Let C be a category. A congruence on C is an equivalence relation $\sim$ on the underlying graph of C having the following property. For any two coterminal edges $e, f \in E(C)$ satisfying $e \sim f$ and for any edges $c, d \in E(C)$ such that $\omega(c) = \alpha(e) = \alpha(f)$ and $\alpha(d) = \omega(e) = \omega(f)$, also $ced \sim cfd$ is satisfied. Then the quotient graph $C/{\sim}$ carries the structure of a category, called the quotient category, with the multiplication of edges inherited from C in the standard way.

Let Γ be a graph. By a path in Γ we mean a finite sequence of consecutive edges from $E(\Gamma)$, that is, a sequence $p = e_1e_2 \cdots e_m$ where m is a positive integer and $e_1, e_2, \dots, e_m \in E(\Gamma)$ are edges such that, for every $i \in \{2, \dots, m\}$, $\omega(e_{i-1}) = \alpha(e_i)$. Then m is the length of the path p. If $\alpha(e_1) = v$ and $\omega(e_m) = w$, then we say that p is a path in Γ from v to w. If $\alpha(e_1) = \omega(e_m)$, that is, if $v = w$, then we say that p is a loop in Γ on v. Moreover, for every vertex $v \in V(\Gamma)$, we add an empty path 1_v from v to v of length 0. The set of all paths in Γ is endowed with the partial binary operation of concatenation of consecutive paths. The empty paths then act as identity elements. In this way, a category Γ^* arises, having $V(\Gamma)$ for its set of vertices and having the paths in Γ for its edges. That is, $V(\Gamma^*) = V(\Gamma)$ and $E(\Gamma^*)$ is the set of all paths in Γ. For any vertices $v, w \in V(\Gamma)$, $\Gamma^*(v, w)$ is the set of all paths in Γ from v to w. This category Γ^* is called the free category over the graph Γ.

Let Γ be a graph, let C be a category and let $\vartheta : \Gamma \to C$ be a graph mapping. Let Γ^* be the free category over Γ. Then there exists a unique homomorphism of categories $\theta : \Gamma^* \to C$ extending ϑ, that is, having the property that the restriction of θ to the graph Γ is ϑ.

Let Γ be a non-empty graph. By a path identity over Γ we mean an ordered pair $p \doteq q$ of coterminal paths $p, q \in E(\Gamma^*)$. We say that the path identity $p \doteq q$ is satisfied in a category C if, for every graph mapping $\vartheta : \Gamma \to C$, we have $\theta(p) = \theta(q)$ where $\theta : \Gamma^* \to C$ is the homomorphism of categories extending ϑ.

Notice, incidentally, that the notion of the free category over a graph Γ is an extension of the standard notion of the free monoid on a set X of variables. We will use the common notation X^* for the free monoid on X. Elements of X^* will be called words over X. We will denote by 1 the empty word, that is, the identity of X^*. In connection with monoids, for any non-empty set X of variables, as usual, by identities over X we will mean arbitrary ordered pairs $s \doteq t$ of words $s, t \in X^*$.

Besides identities over non-empty sets of variables in the case of monoids and path identities over non-empty graphs in the case of categories, we will need also the notions of pseudoidentities for finite monoids and path pseudoidentities for finite categories. We will not need here these notions in their full generality, as they were introduced in the monograph [**1**] by Almeida for pseudovarieties of monoids and in the paper [**3**] by Almeida and Weil for pseudovarieties of categories, so that we content ourselves here only with the following partial description of the most frequently encountered kinds of these pseudoidentities and, in particular, path pseudoidentities. We will use only the unary implicit operation $(\cdot)^\omega$ assigning, to every element a in every finite monoid S, the unique idempotent a^ω in the subsemigroup of S generated by a. Thus, having a non-empty set X of variables, as usual, alongside the ordinary words over X, we can also form more varied words over X using iteratively concatenation and the mentioned unary implicit operation $(\cdot)^\omega$. Then, by a pseudoidentity over X we mean any ordered pair $s \doteq t$ of words s, t over X formed in the way just described. It is obvious what it means that a pseudoidentity $s \doteq t$ of this kind is satisfied in a finite monoid S.

The situation with path pseudoidentities is somewhat more delicate. Let again Γ be a non-empty graph. Aside from the usual paths in Γ introduced above, we can also form more diverse paths in Γ of the following kind. Our description of these paths will proceed by induction. We start with the ordinary paths in Γ defined previously. Next we continue in the following way. Let r be any loop in Γ, that is, let r be any path such that $\alpha(r) = \omega(r)$. Then we may construct, in addition, also the path r^ω in Γ such that $\alpha(r^\omega) = \alpha(r) = \omega(r) = \omega(r^\omega)$. That is, we view r^ω again as a loop in Γ on the same vertex as r. Note that if C is a finite category, if $\vartheta : \Gamma \to C$ is a graph mapping and if $\theta : \Gamma^* \to C$ is the homomorphism of categories extending ϑ, then $\theta(r)$ belongs to a local monoid of C, and so $\theta(r^\omega)$ can be interpreted as the idempotent $\theta(r)^\omega$. In that way, we may construct many new loops of this kind in Γ. Furthermore, assume that we have several consecutive paths $p_1, p_2, \ldots, p_m$ in Γ (either ordinary paths and loops or the new loops just constructed). It means that $\omega(p_{i-1}) = \alpha(p_i)$ holds for all $i \in \{2, \ldots, m\}$. Then we can concatenate these paths to get the new path $p_1 p_2 \cdots p_m$. If, in addition, $\alpha(p_1) = \omega(p_m)$ holds, we thus actually get a new loop. This loop then can again be used to construct a further new loop $(p_1 p_2 \cdots p_m)^\omega$ in Γ, as before. The way how this last loop is interpreted in a finite category C, whenever a graph mapping $\vartheta : \Gamma \to C$ is given, is now obvious from what has been said above on this subject. As before, a large collection of further new loops in Γ may arise in that way. Then, again, we may concatenate these new paths and new loops to get further new paths in Γ. It is now clear that we may iterate these constructions. In this way, we eventually obtain the more diverse paths in Γ involving the unary implicit operation $(\cdot)^\omega$, which will appear in the path pseudoidentities needed in this paper. Thus, by a path pseudoidentity over Γ we mean any ordered pair $p \simeq p$ of coterminal paths p, q in Γ which are formed in the way described above in this paragraph. It is then again obvious what it means that a path pseudoidentity $p \simeq q$ of this kind is satisfied in a finite category C.

Let now X be any non-empty set of variables. We may view X as a set of edges of a graph Δ having only one vertex. That is, in this manner, elements of X are viewed as loops of length 1 on the single vertex of Δ. In that way, words of X^* can then be viewed as paths in Δ and identities over X can thus be treated as path identities over Δ. In the same way, pseudoidentities over X become path pseudoidentities over Δ. Whenever we will deal with identities or pseudoidentities over a set X of variables and we will wish to emphasize that we treat them as path identities and path pseudoidentities in the manner just described, we will call them loop identities and loop pseudoidentities, since, from this point of view, they are formed of edges that are loops in the above-mentioned graph Δ.

Let Γ be a graph and let C be a category. We say that the category C is generated by the graph Γ, if $V(\Gamma) = V(C)$, $E(\Gamma) \subseteq E(C)$, and there is no category D such that $V(\Gamma) = V(D) = V(C)$, the identity edges 1_v are the same in both categories D and C, for all vertices $v \in V(D) = V(C)$, and $E(\Gamma) \subseteq E(D) \subsetneqq E(C)$ holds. It means that, for every edge $f \in E(C)$, except the identity edges 1_v for vertices $v \in V(C)$, there exist a positive integer m and consecutive edges $e_1, e_2, \ldots, e_m \in E(\Gamma)$ such that $f = e_1 e_2 \cdots e_m$ holds in C.

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Let C and D be categories and let $\psi : C \to D$ be a homomorphism. We say that this homomorphism ψ is faithful if it is injective on hom-sets of C, that is, if for arbitrary vertices $v, w \in V(C)$, the restriction of ψ to $C(v, w)$ is injective. We say that the homomorphism ψ is quotient if it is bijective on the vertices of C, that is, if it determines a bijection of $V(C)$ onto $V(D)$, and if it is surjective on hom-sets of C, that is, if for arbitrary vertices $v, w \in V(C)$, the restriction of ψ to $C(v, w)$ is a surjection of $C(v, w)$ onto $D(\psi(v), \psi(w))$.

We say that a category C divides a category D if there exists a category B, a faithful homomorphism $\varphi : B \to D$ and a quotient homomorphism $\psi : B \to C$. Then we write $C \prec D$. Note that then the category B is isomorphic to a uniquely determined subcategory $\bar{B}$ of the direct product of categories $C \times D$. Really, then $V(\bar{B}) = \{(\psi(v), \varphi(v)) : v \in V(B)\}$ and $E(\bar{B}) = \{(\psi(e), \varphi(e)) : e \in E(B)\}$, where the mappings α, ω pertaining to the subcategory $\bar{B}$ are determined by those coming up from the direct product $C \times D$: for every $e \in E(B)$, we have $\alpha(\psi(e), \varphi(e)) = (\alpha(\psi(e)), \alpha(\varphi(e)))$ and $\omega(\psi(e), \varphi(e)) = (\omega(\psi(e)), \omega(\varphi(e)))$. Hence it follows, in particular, that if C and D are finite categories then, in this situation, the category B must also be finite.

A class $\mathcal{W}$ of categories in the above sense is said to be a variety of categories if it is closed under taking arbitrary direct products of categories and under division of categories. It is well known that a Birkhoff-type theorem holds for varieties of categories saying that a class $\mathcal{W}$ of categories is a variety if and only if it is determined by a collection of path identities.

In every variety $\mathcal{V}$ of monoids, for every non-empty set X of variables, there exists a free monoid on X relative to $\mathcal{V}$ which can be represented in the form $X^*/{\equiv_{\mathcal{V}}}$, where X^* is the free monoid over X and $\equiv_{\mathcal{V}}$ is the congruence on X^* consisting of all pairs (s, t) of ordinary words $s, t \in X^*$ such that the identity $s \doteq t$ over X holds in $\mathcal{V}$.

Similarly, in every variety $\mathcal{W}$ of categories, for every non-empty graph Γ, there exists a free category C on Γ relative to $\mathcal{W}$. In order to be more accurate, remember that, similarly as in the case of usual varieties of monoids, this free category C on Γ is considered together with a graph mapping $\iota : \Gamma \to C$ and it is determined, up to isomorphism, by the following property. For every category D in $\mathcal{W}$ and for every graph mapping $\vartheta : \Gamma \to D$, there exists a unique homomorphism of categories $\psi : C \to D$ such that ψ composed with ι yields ϑ. (The graph mapping $\iota : \Gamma \to C$ is then an embedding provided that the variety $\mathcal{W}$ does not consist only of locally trivial categories.) For the further considerations in this paper, the following fact is of more interest. This free category C on Γ relative to $\mathcal{W}$ can be represented in the form $\Gamma^*/{\equiv_{\mathcal{W}}}$, where Γ^* is, as before, the free category on Γ and $\equiv_{\mathcal{W}}$ is the congruence on Γ^* consisting of all pairs (p, q) of ordinary coterminal paths $p, q \in E(\Gamma^*)$ such that the path identity $p \doteq q$ over Γ holds in $\mathcal{W}$.

A class $\mathbf{W}$ of finite categories is said to be a pseudovariety of finite categories if it is closed under taking finitary direct products of finite categories and under division of finite categories. For every class U of finite categories, there exists the smallest pseudovariety $\mathbf{W}$ of categories such that $U \subseteq \mathbf{W}$. We say that $\mathbf{W}$ is the pseudovariety of categories generated by the class U. This pseudovariety $\mathbf{W}$ consists of all finite categories that divide some finitary direct product of categories from U.

Every monoid S can be viewed as a category having a single vertex in such a way that S is just the local monoid of this category at its unique vertex. Thus classes of monoids can be viewed as classes of categories.

For every pseudovariety $\mathbf{V}$ of finite monoids, we denote by $g\mathbf{V}$ the pseudovariety of finite categories generated by the class $\mathbf{V}$. Furthermore, we denote by $\ell\mathbf{V}$ the pseudovariety of all finite categories all of whose local monoids are in $\mathbf{V}$. Then, of course, $g\mathbf{V} \subseteq \ell\mathbf{V}$. We say that the monoid pseudovariety $\mathbf{V}$ is local if $g\mathbf{V} = \ell\mathbf{V}$. According to the previous notes, this happens if every finite category in $\ell\mathbf{V}$ divides a finite monoid in $\mathbf{V}$.

2. The monoid pseudovariety DG

Let $\mathbf{DG}$ be the class of all finite monoids all of whose regular $\mathcal{D}$-classes are groups. Then $\mathbf{DG}$ is a pseudovariety of finite monoids. It is well known that this pseudovariety is determined by the pseudoidentity

$$(xy)^\omega \simeq (yx)^\omega.$$

For completeness, we sketch an argument. Let S be any finite monoid and let n be a positive integer such that, for every element $s \in S$, s^n is an idempotent of S. Then, for arbitrary elements $a, b \in S$, $(ab)^n$ and $(ba)^n$ are idempotents lying in the same $\mathcal{D}$-class of S. On the other hand, for any two idempotents $e, f \in S$ lying in the same $\mathcal{D}$-class D of S, there exist elements $a', b' \in D$ such that $e = a'b'$ and $f = b'a'$, and hence also $e = (a'b')^n$ and $f = (b'a')^n$. Thus, if all regular $\mathcal{D}$-classes of S are groups, then $(ab)^n = (ba)^n$ holds for all $a, b \in S$, which means that S satisfies the above pseudoidentity. Conversely, if S satisfies this pseudoidentity, then, according to the previous notes, it turns out that every regular $\mathcal{D}$-class of S contains only one idempotent, and therefore it is a group.

For any positive integer n, let $\mathcal{V}_n$ be the variety of monoids determined by the identities

$$x^{2n} \simeq x^n$$

and

$$(xy)^n \simeq (yx)^n.$$

Then, for every finite monoid S in the pseudovariety $\mathbf{DG}$, there exists a positive integer n such that S belongs to the variety $\mathcal{V}_n$. For, it suffices to take n such that, for every element $s \in S$, s^n is an idempotent of S. Consequently, $\mathbf{DG}$ is contained in the union $\bigcup_{n=1}^{\infty} \mathcal{V}_n$.

Let X be any non-empty set of variables. Let X^* be the free monoid on X. For every word $w \in X^*$, we will denote by $\mathrm{c}(w)$ the set of all variables of X contained in w.

Proposition 2.1. *For any positive integer n and for any two words $u, w \in X^*$ such that $\mathrm{c}(u) = \mathrm{c}(w)$, the identity*

$$u^n \simeq w^n$$

is satisfied in the variety $\mathcal{V}_n$.

Proof. It is enough to verify this claim only for non-empty words u, w. Thus let S be any monoid in $\mathcal{V}_n$. Let $\vartheta : X \to S$ be any mapping and let $\theta : X^* \to S$ be the homomorphism of monoids extending ϑ. In order to prove the above claim, we have to show that $\theta(u)^n = \theta(w)^n$.

As a preparation, we will show that, for any word $v \in X^*$ such that $\mathrm{c}(v) \subseteq \mathrm{c}(u)$, there exist words $p, q \in X^*$ such that $\mathrm{c}(p) \subseteq \mathrm{c}(u)$, $\mathrm{c}(q) = \mathrm{c}(u)$ and $\theta(u)^n = \theta(p)\theta(q)^n\theta(v)$. We will proceed by induction on the length of v. If v is the empty word, there is nothing to prove. Thus let v be non-empty and let $v = zv'$ where $z \in X$ and $v' \in X^*$. Then, by the induction hypothesis, there exist words $p', q' \in X^*$ such that $\mathrm{c}(p') \subseteq \mathrm{c}(u)$, $\mathrm{c}(q') = \mathrm{c}(u)$ and $\theta(u)^n = \theta(p')\theta(q')^n\theta(v')$. Since $\mathrm{c}(v) \subseteq \mathrm{c}(u) = \mathrm{c}(q')$, we can write $q' = szt$ for some words $s, t \in X^*$. Since S satisfies the identity $(xy)^n \simeq (yx)^n$, we have $\theta(szt)^n = \theta(tsz)^n$. Since S also satisfies the identity $x^{2n} \simeq x^n$, we hence get that

$$\begin{aligned}\theta(u)^n &= \theta(p')\theta(q')^n\theta(v') = \theta(p')\theta(szt)^n\theta(v') = \theta(p')\theta(tsz)^n\theta(v')\\ &= \theta(p')\theta(tsz)^n\theta(tsz)^n\theta(v') = \theta(p'ts)\theta(zts)^{n-1}\theta(zts)^n\theta(zv')\\ &= \theta(p'ts)\theta(zts)^{n-1}\theta(zts)^n\theta(v).\end{aligned}$$

Thus we can put $p = p'ts(zts)^{n-1}$ and $q = zts$. This verifies the preparatory claim stated at the beginning of this paragraph.

Now, we apply this observation for $v = w^n$. Thus, there exist words $p, q \in X^*$ such that $\mathrm{c}(p) \subseteq \mathrm{c}(u)$, $\mathrm{c}(q) = \mathrm{c}(u)$ and $\theta(u)^n = \theta(p)\theta(q)^n\theta(w)^n$. Hence we obtain that

$$\begin{aligned}\theta(u)^n &= \theta(p)\theta(q)^n\theta(w)^n\\ &= \theta(p)\theta(q)^n\theta(w)^n\theta(w)^n\\ &= \theta(u)^n\theta(w)^n.\end{aligned}$$

Analogously we also obtain that $\theta(u)^n\theta(w)^n = \theta(w)^n$. These two equalities together yield that $\theta(u)^n = \theta(w)^n$, as required. □

Hence it follows straightforwardly that, for any two words $u, w \in X^*$ such that $\mathrm{c}(u) = \mathrm{c}(w)$, the pseudoidentity

$$u^\omega \simeq w^\omega$$

is satisfied in the pseudovariety **DG**.

Proposition 2.2. *For any positive integer n and for any words $u, v, w \in X^*$ such that $\mathrm{c}(v) \subseteq \mathrm{c}(u) = \mathrm{c}(w)$, the identity*

$$u^n v \simeq v w^n$$

is satisfied in the variety $\mathcal{V}_n$.

Proof. Again, it is enough to verify this claim only for non-empty words u, w. Thus, as above, let S be any monoid in $\mathcal{V}_n$, let $\vartheta : X \to S$ be any mapping and let $\theta : X^* \to S$ be the homomorphism of monoids extending ϑ. Then we have to show that $\theta(u)^n\theta(v) = \theta(v)\theta(w)^n$. However, since $\mathrm{c}(u) = \mathrm{c}(w)$, by Proposition 2.1 we have the

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equality $\theta(u)^n = \theta(w)^n$. Since $\mathrm{c}(v) \subseteq \mathrm{c}(w)$, by the same proposition we also have the equalities $\theta(w)^n = \theta(vw)^n$ and $\theta(w)^n = \theta(wv)^n$. Hence we obtain that

$$\begin{aligned}\theta(u)^n\theta(v) &= \theta(w)^n\theta(v) = \theta(vw)^n\theta(v)\\ &= \theta(v)\theta(wv)^n = \theta(v)\theta(w)^n,\end{aligned}$$

as required. □

As before, it hence follows immediately that, for any words $u, v, w \in X^*$ such that $\mathrm{c}(v) \subseteq \mathrm{c}(u) = \mathrm{c}(w)$, the pseudoidentity

$$u^\omega v \simeq v w^\omega$$

is satisfied in the pseudovariety **DG**.

The following consequences of the fact that the identities described in Propositions 2.1 and 2.2 hold in the variety $\mathcal{V}_n$ will be used subsequently.

Let n be any positive integer and let S be any monoid in $\mathcal{V}_n$. Then, for any positive integers h, k and arbitrary elements $a_1, \dots, a_h \in S$ and $b_1, \dots, b_k \in S$ such that $\{a_1, \dots, a_h\} = \{b_1, \dots, b_k\}$, we have the equality

$$(a_1 \cdots a_h)^n = (b_1 \cdots b_k)^n,$$

that is, we thus obtain each time the same idempotent of S. Furthermore, in this situation, for any positive integer ℓ and arbitrary elements $c_1, \dots, c_\ell \in \{a_1, \dots, a_h\}$, we have the equality

$$(a_1 \cdots a_h)^n c_1 \cdots c_\ell = c_1 \cdots c_\ell (a_1 \cdots a_h)^n.$$

In addition, this element lies in the maximal subgroup of S containing the idempotent $(a_1 \cdots a_h)^n$. In particular, for all $i \in \{1, \dots, \ell\}$, the elements $(a_1 \cdots a_h)^n c_i$ lie in this maximal subgroup, and we also have the equality

$$(a_1 \cdots a_h)^n c_1 \cdots c_\ell = (a_1 \cdots a_h)^n c_1 \cdots (a_1 \cdots a_h)^n c_\ell.$$

To verify the latter statements, one only has to notice that, in the same way as above, we also get the equalities

$$(a_1 \cdots a_h)^n = (c_1 \cdots c_\ell a_1 \cdots a_h)^n = (a_1 \cdots a_h c_1 \cdots c_\ell)^n,$$

which shows that the element $(a_1 \cdots a_h)^n c_1 \cdots c_\ell$ lies in the same $\mathcal{H}$-class of S as the idempotent $(a_1 \cdots a_h)^n$. Thus this element belongs to the maximal subgroup of S containing the mentioned idempotent, as stated above. The last statement given above follows straightforwardly.

Proposition 2.3. *For any finite monoid S in the pseudovariety* **DG**, *there exist positive integers n and k such that all identities of the form*

$$w_1 \cdots w_k \simeq u^n w_1 \cdots w_k,$$

for arbitrary words $u \in X^*$ and $w_1, \dots, w_k \in X^*$ having the property that $\mathrm{c}(u) = \mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k)$, are satisfied in S.

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Proof. Let n be a positive integer such that, for every element $s \in S$, s^n is an idempotent of S. Then S belongs to the variety $\mathcal{V}_n$. Let k be any positive integer which is greater than the number of elements of S. Let $\vartheta : X \to S$ be any mapping and let $\theta : X^* \to S$ be the homomorphism of monoids extending ϑ. Then it will be sufficient if we show that, for any word $u \in X^*$ and arbitrary words $w_1, \dots, w_k \in X^*$ such that $\mathrm{c}(u) = \mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k)$, we have the equality $\theta(w_1) \cdots \theta(w_k) = \theta(u)^n \theta(w_1) \cdots \theta(w_k)$.

Let us consider the elements $\theta(w_1)$, $\theta(w_1)\theta(w_2)$, $\theta(w_1)\theta(w_2)\theta(w_3)$, $\dots$, $\theta(w_1) \cdots \theta(w_k)$. Since the number of elements in this sequence is greater then the number of elements in S, there exist $i, j \in \{1, \dots, k\}$ such that $i < j$ and $\theta(w_1) \cdots \theta(w_i) = \theta(w_1) \cdots \theta(w_i) \cdots \theta(w_j)$. Hence it follows that

$$\theta(w_1) \cdots \theta(w_i) = \theta(w_1) \cdots \theta(w_i)(\theta(w_{i+1}) \cdots \theta(w_j))^{\ell}$$

for any positive integer ℓ. In particular, this holds for $\ell = n$. Furthermore, since S is in $\mathcal{V}_n$, by Proposition 2.2 we know that S satisfies the identity

$$u^n w_1 \cdots w_i \doteq w_1 \cdots w_i (w_{i+1} \cdots w_j)^n,$$

since $\mathrm{c}(u) = \mathrm{c}(w_1 \cdots w_i) = \mathrm{c}(w_{i+1} \cdots w_j)$. It means that we further have

$$\theta(w_1) \cdots \theta(w_i)(\theta(w_{i+1}) \cdots \theta(w_j))^n = \theta(u)^n \theta(w_1) \cdots \theta(w_i).$$

Hence we finally deduce that

$$\begin{aligned}
\theta(w_1) \cdots \theta(w_k) &= \theta(w_1) \cdots \theta(w_i)\theta(w_{i+1}) \cdots \theta(w_j)\theta(w_{j+1}) \cdots \theta(w_k) \\
&= \theta(w_1) \cdots \theta(w_i)(\theta(w_{i+1}) \cdots \theta(w_j))^{n+1}\theta(w_{j+1}) \cdots \theta(w_k) \\
&= \theta(u)^n \theta(w_1) \cdots \theta(w_i)\theta(w_{i+1}) \cdots \theta(w_j)\theta(w_{j+1}) \cdots \theta(w_k) \\
&= \theta(u)^n \theta(w_1) \cdots \theta(w_k),
\end{aligned}$$

as required. □

This observation prompts us to consider, for any positive integers n and k, the variety of monoids $\mathcal{V}_{n,k}$ determined by the following identities, drawn up, in the case of the last collection of identities, over arbitrary non-empty sets X of variables:

$$\begin{aligned}
x^n &\doteq x^{2n}, \\
(xy)^n &\doteq (yx)^n,
\end{aligned}$$

and

$$w_1 \cdots w_k \doteq u^n w_1 \cdots w_k \quad \text{for arbitrary words } u \in X^* \text{ and } w_1, \dots, w_k \in X^* \text{ such that } \mathrm{c}(u) = \mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k).$$

Then, of course, $\mathcal{V}_{n,k} \subseteq \mathcal{V}_n$ and, as we have seen in Proposition 2.3 and its proof, for every finite monoid S in the pseudovariety **DG**, there exist positive integers n and k such that S belongs to the variety $\mathcal{V}_{n,k}$. Consequently, **DG** is contained in the union $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{V}_{n,k}$.

Let $\mathcal{K}$ be any locally finite variety of groups. Then $\mathcal{K}$ is a variety of groups of finite exponent, which means that $\mathcal{K}$ can be viewed as a variety of monoids. Assume next that the set X of variables is infinite. Then the variety $\mathcal{K}$ is determined by the set of all identities of the form $v \doteq 1$ which hold in $\mathcal{K}$, where $v \in X^*$.

Proposition 2.4. *For any finite monoid S in the pseudovariety* **DG**, *there exist a positive integer n and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq 1$ such that, for every word $v \in X^*$ having the property that the identity*

$$v \doteq 1$$

holds in $\mathcal{K}$, the identity

$$v^{n+1} \doteq v^n$$

is satisfied in S.

Proof. Let $\mathcal{K}$ be the variety of groups generated by all maximal subgroups of the monoid S. Let again n be a positive integer such that, for every element $s \in S$, s^n is an idempotent of S. Then S belongs to the variety $\mathcal{V}_n$ and, of course, $\mathcal{K}$ satisfies the identity $x^n \doteq 1$. Let $v \in X^*$ be any word such that the identity $v \doteq 1$ holds in $\mathcal{K}$. Let $\vartheta : X \to S$ be any mapping and let $\theta : X^* \to S$ be the homomorphism of monoids extending ϑ. Then we have to show that the equality $\theta(v)^{n+1} = \theta(v)^n$ holds.

Note that $\theta(v)^n$ is an idempotent of S. Remember also that, by virtue of the consequences of Propositions 2.1 and 2.2 discussed in the text preceding Proposition 2.3, for any variable $z \in X$ which is contained in v, we have $\theta(v)^n\theta(z) = \theta(z)\theta(v)^n$, and this element lies in the maximal subgroup of S containing $\theta(v)^n$.

Now let $v = z_1 z_2 \cdots z_m$ where $z_1, z_2, \ldots, z_m \in X$ and m is the length of v. Then, applying the previous notes repeatedly, we consecutively obtain that

$$\begin{aligned}
\theta(v)^{n+1} &= \theta(v)^n\theta(v) = \theta(v)^n\theta(z_1)\theta(z_2)\cdots\theta(z_m) \\
&= \theta(v)^n\theta(v)^n\theta(z_1)\theta(z_2)\cdots\theta(z_m) \\
&= \theta(v)^n\theta(z_1)\theta(v)^n\theta(z_2)\cdots\theta(z_m) \\
&\ \ \vdots \\
&= \theta(v)^n\theta(z_1)\theta(v)^n\theta(z_2)\cdots\theta(v)^n\theta(z_m).
\end{aligned}$$

It means that the element $\theta(v)^{n+1}$ is the product of the elements $\theta(v)^n\theta(z_1)$, $\theta(v)^n\theta(z_2)$, $\ldots$, $\theta(v)^n\theta(z_m)$, which all lie in the maximal subgroup of S containing the idempotent $\theta(v)^n$. Since the identity $v \doteq 1$, that is, $z_1 z_2 \cdots z_m \doteq 1$ holds in this maximal subgroup, we hence get that $\theta(v)^{n+1}$ is the idempotent of this subgroup. This entails that $\theta(v)^{n+1} = \theta(v)^n$, as required. □

3. The monoid varieties $\mathcal{V}_{n,k}(\mathcal{K})$

Let again X be an infinite set of variables. Let $x, y \in X$ be distinct variables. Let $\mathcal{K}$ be any locally finite variety of groups, let n be any positive integer such that the identity

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$x^n \doteq 1$ is satisfied in $\mathcal{K}$ and let k be an arbitrary positive integer. Consider the variety of monoids $\mathcal{V}_{n,k}(\mathcal{K})$ determined by the following identities:

$$x^n \doteq x^{2n},$$
$$(xy)^n \doteq (yx)^n,$$
$$v^{n+1} \doteq v^n \qquad \text{for all words } v \in X^* \text{ such that the identity } v \doteq 1 \text{ is satisfied in } \mathcal{K},$$

and

$$w_1 \cdots w_k \doteq u^n w_1 \cdots w_k \qquad \text{for arbitrary words } u \in X^* \text{ and } w_1, \ldots, w_k \in X^* \text{ such that } \mathrm{c}(u) = \mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k).$$

Proposition 3.1. *For every finite monoid S in the pseudovariety* **DG**, *there exist positive integers n, k and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq 1$ such that S lies in the variety $\mathcal{V}_{n,k}(\mathcal{K})$.*

Proof. Let n be a positive integer such that, for every element $s \in S$, s^n is an idempotent of S. Then S satisfies the identity $x^n \doteq x^{2n}$. Since S lies in **DG**, it hence satisfies also the identity $(xy)^n \doteq (yx)^n$. Further, let $\mathcal{K}$ be the variety of groups generated by all maximal subgroups of S. Then, of course, $\mathcal{K}$ satisfies the identity $x^n \doteq 1$. In Proposition 2.4 and its proof, we have seen that then S satisfies the identities $v^{n+1} \doteq v^n$ for all words $v \in X^*$ such that the identity $v \doteq 1$ holds in $\mathcal{K}$. Moreover, we have shown in Proposition 2.3 and its proof that if k is an integer which is greater than the number of elements of S, then S satisfies also all identities of the form $w_1 \cdots w_k \doteq u^n w_1 \cdots w_k$, for arbitrary words $u \in X^*$ and $w_1, \ldots, w_k \in X^*$ such that $\mathrm{c}(u) = \mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k)$. □

Note that, for the given positive integers n, k and the given locally finite variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq 1$, we have $\mathcal{V}_{n,k}(\mathcal{K}) \subseteq \mathcal{V}_n$. Thus the monoid variety $\mathcal{V}_{n,k}(\mathcal{K})$ satisfies also the identities given in Propositions 2.1 and 2.2 for this positive integer n.

Proposition 3.2. *Let n, k be arbitrary positive integers and let $\mathcal{K}$ be any locally finite variety of groups satisfying the identity $x^n \doteq 1$. Then the monoid variety $\mathcal{V}_{n,k}(\mathcal{K})$ is also locally finite, that is, every finitely generated monoid in $\mathcal{V}_{n,k}(\mathcal{K})$ is finite.*

Proof. We have to verify that, for every monoid S in $\mathcal{V}_{n,k}(\mathcal{K})$ containing a finite subset $A \subseteq S$ such that A generates S, the monoid S is itself finite. We will proceed by induction on the number m of elements of the set A of generators. We will show that for every positive integer m there exists a positive integer $\ell = \ell(m)$ having the following property. In every monoid S of $\mathcal{V}_{n,k}(\mathcal{K})$ generated by a finite subset $A \subseteq S$ containing m generators, every product $q_1 \cdots q_\varkappa$ of arbitrary elements $q_1, \ldots, q_\varkappa \in A$, where $\varkappa > \ell$, is equal to some product $r_1 \cdots r_\lambda$ of some elements $r_1, \ldots, r_\lambda \in A$, where $\lambda < \varkappa$. This property will ensure that the monoid S is finite.

For $m = 1$ this assertion is obvious since every monoid in $\mathcal{V}_{n,k}(\mathcal{K})$ satisfies the identity $x^n \doteq x^{2n}$, so that one can take $\ell(1) = 2n - 1$. Thus suppose next that $m > 1$.

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Remember that the group variety $\mathcal{K}$ is locally finite, which means that every finitely generated group in $\mathcal{K}$ is finite. In particular, this concerns the free groups in $\mathcal{K}$ on finite sets of generators. Moreover, $\mathcal{K}$ can be viewed as a monoid variety. This entails that for every positive integer h there exists some positive integer $\wp = \wp(h)$ having the following property. In every group G of $\mathcal{K}$ generated by a finite subset $B \subseteq G$ containing h generators, every product $c_1 \cdots c_\mu$ of arbitrary elements $c_1, \dots, c_\mu \in B$ is equal to some product $d_1 \cdots d_\nu$ of some elements $d_1, \dots, d_\nu \in B$ such that $\nu \leqslant \wp$.

Now let S be any monoid of $\mathcal{V}_{n,k}(\mathcal{K})$ generated by a finite subset $A \subseteq S$ containing m generators. Let $A = \{a_1, \dots, a_m\}$ where $a_1, \dots, a_m$ are pairwise distinct elements. Consider any product $q_1 \cdots q_\varkappa$ of arbitrary elements $q_1, \dots, q_\varkappa \in A$ where

$$\varkappa > k\ell(m-1) + k - 1 \quad \text{and} \quad \varkappa > \wp(m) + mn.$$

Let further $x_1, \dots, x_m \in X$ be pairwise distinct variables. Let $\vartheta : X \to S$ be any mapping such that $\vartheta(x_1) = a_1, \dots, \vartheta(x_m) = a_m$ and let $\theta : X^* \to S$ be the homomorphism of monoids extending ϑ. Let $v \in X^*$ be the word of the form $v = y_1 \cdots y_\varkappa$, where $y_1, \dots, y_\varkappa \in \{x_1, \dots, x_m\}$ and, for each $i \in \{1, \dots, \varkappa\}$, y_i is the variable for which $\vartheta(y_i) = q_i$. Then, of course, $\theta(v) = q_1 \cdots q_\varkappa$. Since $\varkappa > k\ell(m-1) + k - 1$, one can write $v = w_1 \cdots w_k$ where $w_1, \dots, w_k \in X^*$ and each of the words $w_1, \dots, w_k$ has length greater than $\ell(m-1)$. Put $t_1 = \theta(w_1), \dots, t_k = \theta(w_k)$. Then we have

$$q_1 \cdots q_\varkappa = t_1 \cdots t_k.$$

Assume first that, for some $\varepsilon \in \{1, \dots, k\}$, the word w_ε does not contain all variables $x_1, \dots, x_m$. Then, since the length of w_ε is greater than $\ell(m-1)$, by the induction hypothesis, the element $t_\varepsilon = \theta(w_\varepsilon)$ can be obtained as a product of a shorter sequence of generators from A, that is, it can be expressed in the form $t_\varepsilon = \theta(w'_\varepsilon)$ for some word $w'_\varepsilon \in X^*$ whose length is less than that of w_ε and which is composed only of those variables which are contained in w_ε. Consequently, also the product $q_1 \cdots q_\varkappa$ itself can be obtained as a product of a shorter sequence of factors from A, as needed above.

Thus assume further that we have $\mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k) = \{x_1, \dots, x_m\}$. Since the monoid S is from $\mathcal{V}_{n,k}(\mathcal{K})$, and therefore it satisfies also the identity $w_1 \cdots w_k \simeq (x_1 \cdots x_m)^n w_1 \cdots w_k$ for the above words $w_1, \dots, w_k$, applying the homomorphism θ, we hence obtain the equality

$$t_1 \cdots t_k = (a_1 \cdots a_m)^n t_1 \cdots t_k.$$

From this and the previous equality we get the equality

$$q_1 \cdots q_\varkappa = (a_1 \cdots a_m)^n q_1 \cdots q_\varkappa.$$

Now, since $S \in \mathcal{V}_n$, according to the consequences of Propositions 2.1 and 2.2 explained in the text preceding Proposition 2.3, we hence obtain the equality

$$q_1 \cdots q_\varkappa = (a_1 \cdots a_m)^n q_1 \cdots (a_1 \cdots a_m)^n q_\varkappa.$$

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In this way, the product $q_1 \cdots q_\varkappa$ becomes expressed as the product of the elements $(a_1 \cdots a_m)^n q_1, \ldots, (a_1 \cdots a_m)^n q_\varkappa$. In addition, by the consequences of Propositions 2.1 and 2.2 just mentioned, we know that all these elements, that is, the elements $(a_1 \cdots a_m)^n a_1, \ldots, (a_1 \cdots a_m)^n a_m$ belong to the maximal subgroup G of S containing the idempotent $(a_1 \cdots a_m)^n$. Since S is from $\mathcal{V}_{n,k}(\mathcal{K})$ and therefore it satisfies also the identities $v^{n+1} \doteq v^n$ for all words $v \in X^*$ such that the identity $v \doteq \mathtt{1}$ holds in $\mathcal{K}$, we see that G belongs to the variety $\mathcal{K}$. According to the above remarks on the finitely generated groups in $\mathcal{K}$, we know that the product $(a_1 \cdots a_m)^n q_1 \cdots (a_1 \cdots a_m)^n q_\varkappa$ can be expressed as a product of no more than $\wp(m)$ factors taken from among the elements $(a_1 \cdots a_m)^n a_1, \ldots, (a_1 \cdots a_m)^n a_m$. That is, we get that

$$(a_1 \cdots a_m)^n q_1 \cdots (a_1 \cdots a_m)^n q_\varkappa = (a_1 \cdots a_m)^n s_1 \cdots (a_1 \cdots a_m)^n s_\eta$$

for some $\eta \leqslant \wp(m)$ and $s_1, \ldots, s_\eta \in A$. Referring to the consequences of Propositions 2.1 and 2.2 once more, we also have the equality

$$(a_1 \cdots a_m)^n s_1 \cdots s_\eta = (a_1 \cdots a_m)^n s_1 \cdots (a_1 \cdots a_m)^n s_\eta.$$

From this and the previous equalities we eventually deduce that

$$q_1 \cdots q_\varkappa = (a_1 \cdots a_m)^n s_1 \cdots s_\eta,$$

where the product on the right is composed of $\eta + mn \leqslant \wp(m) + mn$ factors from the set A. Since $\varkappa > \wp(m) + mn$, there are fewer factors in that product than in $q_1 \cdots q_\varkappa$, as required above. Altogether this shows that one can take

$$\ell(m) = \max\{k\ell(m-1) + k - 1, \wp(m) + mn\}.$$

□

4. The category pseudovariety ℓDG

Remember that by ℓ**DG** we denote the class of all finite categories all of whose local monoids belong to **DG**. Then ℓ**DG** is a pseudovariety of finite categories. In § 2, we have seen how **DG** is determined by means of a single pseudoidentity. With this fact in mind, we can state that ℓ**DG** is determined by the loop pseudoidentity

$$(xy)^\omega \doteq (yx)^\omega.$$

For any positive integer n, let $\mathcal{W}_n$ be the variety of categories determined by the loop identities

$$x^{2n} \doteq x^n$$

and

$$(xy)^n \doteq (yx)^n.$$

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Then, for every finite category C in the pseudovariety $\ell\mathbf{DG}$, there exists a positive integer n such that C belongs to the variety $\mathcal{W}_n$. For, it suffices to take n such that, for every vertex $v \in V(C)$ and for every element a in the local monoid $C(v)$, a^n is an idempotent of $C(v)$. Consequently, $\ell\mathbf{DG}$ is contained in the union $\bigcup_{n=1}^{\infty} \mathcal{W}_n$.

Let Γ be any non-empty graph. Let Γ^* be the free category over Γ. For any path p in Γ, that is, for any element p of $E(\Gamma^*)$, we will denote by $\mathrm{c}(p)$ the set of all edges of Γ contained in p.

Proposition 4.1. *For any positive integer n, for any non-empty graph Γ and for any two coterminal loops p, q in Γ such that $\mathrm{c}(p) = \mathrm{c}(q)$, the path identity*

$$p^n \doteq q^n$$

is satisfied in the variety $\mathcal{W}_n$.

Proof. It is enough to consider only non-empty loops p, q. Thus let C be any category in $\mathcal{W}_n$. Let $\vartheta : \Gamma \to C$ be any graph mapping and let $\theta : \Gamma^* \to C$ be the homomorphism of categories extending ϑ. In order to prove the above claim, we have to show that $\theta(p)^n = \theta(q)^n$.

As a preparation, we will show that, for any path r in Γ such that $\mathrm{c}(r) \subseteq \mathrm{c}(p)$ and $\omega(r) = \alpha(p) = \omega(p)$, there exist a path s and a loop t in Γ such that $\mathrm{c}(s) \subseteq \mathrm{c}(p)$, $\mathrm{c}(t) = \mathrm{c}(p)$, $\alpha(s) = \alpha(p)$, $\omega(s) = \alpha(t) = \omega(t) = \alpha(r)$ and $\theta(p)^n = \theta(s)\theta(t)^n\theta(r)$. We proceed by induction on the length of r. If r is the empty path, there is nothing to prove. Thus let r be non-empty and let $r = zr'$ where $z \in E(\Gamma)$ and r' is a path in Γ. Then, by the induction hypothesis, there exist a path s' and a loop t' in Γ such that $\mathrm{c}(s') \subseteq \mathrm{c}(p)$, $\mathrm{c}(t') = \mathrm{c}(p)$, $\alpha(s') = \alpha(p)$, $\omega(s') = \alpha(t') = \omega(t') = \alpha(r')$ and $\theta(p)^n = \theta(s')\theta(t')^n\theta(r')$. Since $\mathrm{c}(r) \subseteq \mathrm{c}(p) = \mathrm{c}(t')$, we can write $t' = uzw$ for some paths u, w in Γ. Note that then $\alpha(w) = \omega(z) = \alpha(r') = \omega(w) = \alpha(u)$, so that uz and w are coterminal loops in Γ. Since C satisfies the above-mentioned loop identity $(xy)^n \doteq (yx)^n$, we thus obtain that $\theta(uzw)^n = \theta(wuz)^n$. Since C satisfies also the loop identity $x^{2n} \doteq x^n$, we hence get that

$$\begin{aligned}\theta(p)^n &= \theta(s')\theta(t')^n\theta(r') = \theta(s')\theta(uzw)^n\theta(r') = \theta(s')\theta(wuz)^n\theta(r') \\ &= \theta(s')\theta(wuz)^n\theta(wuz)^n\theta(r') = \theta(s'wu)\theta(zwu)^{n-1}\theta(zwu)^n\theta(zr') \\ &= \theta(s'wu)\theta(zwu)^{n-1}\theta(zwu)^n\theta(r).\end{aligned}$$

Thus we can take $s = s'wu(zwu)^{n-1}$ and $t = zwu$. Since $\alpha(z) = \omega(u)$, we see that t is a loop in Γ. This verifies the preparatory claim stated at the beginning of this paragraph.

Now, we apply this observation for $r = q^n$. Thus, there exist a path s and a loop t in Γ such that $\mathrm{c}(s) \subseteq \mathrm{c}(p)$, $\mathrm{c}(t) = \mathrm{c}(p)$, $\alpha(s) = \alpha(p)$, $\omega(s) = \alpha(t) = \omega(t) = \alpha(q) = \alpha(p)$ and $\theta(p)^n = \theta(s)\theta(t)^n\theta(q)^n$. Hence we obtain that

$$\begin{aligned}\theta(p)^n &= \theta(s)\theta(t)^n\theta(q)^n \\ &= \theta(s)\theta(t)^n\theta(q)^n\theta(q)^n \\ &= \theta(p)^n\theta(q)^n.\end{aligned}$$

Analogously we also obtain that $\theta(p)^n\theta(q)^n = \theta(q)^n$. These two equalities together yield that $\theta(p)^n = \theta(q)^n$, as required. □

Hence it follows that, for any non-empty graph Γ and for any two coterminal loops p, q in Γ such that $\mathrm{c}(p) = \mathrm{c}(q)$, the path pseudoidentity

$$p^\omega \doteq q^\omega$$

is satisfied in the category pseudovariety $\ell\mathbf{DG}$.

Proposition 4.2. *For any positive integer n, for any non-empty graph Γ and for arbitrary paths p, q, r in Γ such that p, q are loops, $\alpha(p) = \omega(p) = \alpha(r)$, $\omega(r) = \alpha(q) = \omega(q)$ and $\mathrm{c}(r) \subseteq \mathrm{c}(p) = \mathrm{c}(q)$, the path identity*

$$p^n r \doteq r q^n$$

is satisfied in the variety $\mathcal{W}_n$.

Proof. It is enough to verify this assertion only for non-empty loops p, q. Thus, as before, let C be any category in $\mathcal{W}_n$, let $\vartheta : \Gamma \to C$ be any graph mapping and let $\theta : \Gamma^* \to C$ be the homomorphism of categories extending ϑ. Then we have to show that $\theta(p)^n\theta(r) = \theta(r)\theta(q)^n$.

Since $\mathrm{c}(r) \subseteq \mathrm{c}(q)$, there exist paths s, t in Γ such that $q = st$ and $\omega(s) = \alpha(t) = \alpha(r)$. Note that then also $\alpha(s) = \omega(t) = \omega(r)$. It means that rs is a loop in Γ which is coterminal with p and sr is a loop in Γ which is coterminal with q. Then rsp is also a loop in Γ coterminal with p, and as $\mathrm{c}(p) = \mathrm{c}(q)$, it follows that $\mathrm{c}(rsp) = \mathrm{c}(p)$. Thus, by Proposition 4.1, we have the equality $\theta(p)^n = \theta(rsp)^n$. Moreover, spr is also a loop in Γ, this time coterminal with q, and we likewise get that $\mathrm{c}(spr) = \mathrm{c}(q)$. Therefore, by Proposition 4.1 again, we have the equality $\theta(spr)^n = \theta(q)^n$. Hence, altogether we obtain that

$$\theta(p)^n\theta(r) = \theta(rsp)^n\theta(r) = \theta(r)\theta(spr)^n = \theta(r)\theta(q)^n,$$

as required. □

As before, it hence follows again that, for any non-empty graph Γ and for any paths p, q, r in Γ such that p, q are loops, $\alpha(p) = \omega(p) = \alpha(r)$, $\omega(r) = \alpha(q) = \omega(q)$ and $\mathrm{c}(r) \subseteq \mathrm{c}(p) = \mathrm{c}(q)$, the path pseudoidentity

$$p^\omega r \doteq r q^\omega$$

is satisfied in the category pseudovariety $\ell\mathbf{DG}$.

Corollary 4.3. *Let n be any positive integer, let Γ be any non-empty graph and let p, q, r be arbitrary paths in Γ such that p, q are loops, $\alpha(p) = \omega(p) = \alpha(r)$ and $\omega(r) = \alpha(q) = \omega(q)$. If $\mathrm{c}(r) \subseteq \mathrm{c}(p)$ and $\mathrm{c}(q) \subseteq \mathrm{c}(p)$, then the path identity*

$$p^n r \doteq p^n r q^n$$

is satisfied in the variety $\mathcal{W}_n$, and if $\mathrm{c}(r) \subseteq \mathrm{c}(q)$ and $\mathrm{c}(p) \subseteq \mathrm{c}(q)$, then the path identity

$$r q^n \doteq p^n r q^n$$

is satisfied in the variety $\mathcal{W}_n$.

Proof. We verify the first of these two assertions; the second one is dual. Clearly, it is enough to do so only for non-empty loops p, q. Thus, once again, let C be any category in $\mathcal{W}_n$, let $\vartheta : \Gamma \to C$ be any graph mapping and let $\theta : \Gamma^* \to C$ be the homomorphism of categories extending ϑ. Then we have to show that $\theta(p)^n\theta(r) = \theta(p)^n\theta(r)\theta(q)^n$.

Since p, q are loops and $\mathrm{c}(q) \subseteq \mathrm{c}(p)$, there exist paths s, t in Γ such that $p = st$ and $\omega(s) = \alpha(t) = \alpha(q) = \omega(q) = \omega(r)$. Then ts is a loop in Γ which is coterminal with q. Hence also tsq^n is a loop in Γ which is coterminal with q. Moreover, $\mathrm{c}(tsq^n) = \mathrm{c}(p)$. Therefore, by Proposition 4.2, we have $\theta(p)^n\theta(r) = \theta(r)\theta(tsq^n)^n$. Thus, altogether we obtain that

$$\begin{aligned}\theta(p)^n\theta(r) &= \theta(r)\theta(tsq^n)^n \\ &= \theta(r)\theta(tsq^n)^n\theta(q^n) \\ &= \theta(p)^n\theta(r)\theta(q)^n,\end{aligned}$$

as desired. □

Once again, it hence follows that, for any non-empty graph Γ and for any paths p, q, r in Γ such that p, q are loops, $\alpha(p) = \omega(p) = \alpha(r)$ and $\omega(r) = \alpha(q) = \omega(q)$, the following holds. If $\mathrm{c}(r) \subseteq \mathrm{c}(p)$ and $\mathrm{c}(q) \subseteq \mathrm{c}(p)$, then the path pseudoidentity

$$p^\omega r \triangleq p^\omega r q^\omega$$

is satisfied in the category pseudovariety $\ell\mathbf{DG}$. If $\mathrm{c}(r) \subseteq \mathrm{c}(q)$ and $\mathrm{c}(p) \subseteq \mathrm{c}(q)$, then the path pseudoidentity

$$r q^\omega \triangleq p^\omega r q^\omega$$

is satisfied in the category pseudovariety $\ell\mathbf{DG}$.

The following consequences of the fact that the path identities described in Propositions 4.1 and 4.2 hold in $\mathcal{W}_n$ will also be used subsequently.

Let n be any positive integer and let C be any category in $\mathcal{W}_n$. Consider any sequences of arbitrary consecutive edges $e_1, \ldots, e_h \in E(C)$ and $f_1, \ldots, f_k \in E(C)$, where h, k are any positive integers. It means that these edges of C satisfy the conditions $\omega(e_{i-1}) = \alpha(e_i)$ and $\omega(f_{j-1}) = \alpha(f_j)$ for all $i \in \{2, \ldots, h\}$ and $j \in \{2, \ldots, k\}$. Assume, in addition, that the mentioned edges satisfy also the condition $\alpha(e_1) = \omega(e_h) = \alpha(f_1) = \omega(f_k)$. Then $e_1 \cdots e_h$ and $f_1 \cdots f_k$ are elements of the same local monoid $C(v)$ of C where $v = \alpha(e_1) = \omega(e_h) = \alpha(f_1) = \omega(f_k)$. If, moreover, we have $\{e_1, \ldots, e_h\} = \{f_1, \ldots, f_k\}$, then, by Proposition 4.1, we have the equality

$$(e_1 \cdots e_h)^n = (f_1 \cdots f_k)^n,$$

that is, we thus obtain each time the same idempotent of $C(v)$. Furthermore, in this situation, take any positive integer ℓ and consider arbitrary consecutive edges $g_1, \ldots, g_\ell \in \{e_1, \ldots, e_h\}$, that is, edges satisfying the conditions $\omega(g_{i-1}) = \alpha(g_i)$ for all $i \in \{2, \ldots, \ell\}$. Assume, in addition, that also the condition $\alpha(g_1) = \omega(g_\ell) = \alpha(e_1) = \omega(e_h)$ holds. Then

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$g_1 \cdots g_\ell$ is also an element of the local monoid $C(v)$ and, by Proposition 4.2, we have the equality

$$(e_1 \cdots e_h)^n g_1 \cdots g_\ell = g_1 \cdots g_\ell (e_1 \cdots e_h)^n.$$

In addition, this last element lies in the maximal subgroup of the local monoid $C(v)$ containing the idempotent $(e_1 \cdots e_h)^n$. In order to verify this statement, we have only to notice that, using Proposition 4.1 once again, we also get the equalities

$$(e_1 \cdots e_h)^n = (g_1 \cdots g_\ell e_1 \cdots e_h)^n = (e_1 \cdots e_h g_1 \cdots g_\ell)^n,$$

which shows that the element $(e_1 \cdots e_h)^n g_1 \cdots g_\ell$ lies in the same $\mathcal{H}$-class of $C(v)$ as the idempotent $(e_1 \cdots e_h)^n$. Thus this element belongs to the maximal subgroup of $C(v)$ containing the mentioned idempotent, as stated above.

Proposition 4.4. *For any finite category C in the pseudovariety ℓ**DG**, there exist positive integers n and k such that, for any non-empty graph Γ, all path identities of the form*

$$q_1 \cdots q_k \doteq p^n q_1 \cdots q_k$$

over Γ are satisfied in C, where p is any loop in Γ and $q_1, \ldots, q_k$ are any paths in Γ satisfying the conditions $\alpha(p) = \omega(p) = \alpha(q_1)$ and $\omega(q_{i-1}) = \alpha(q_i)$, for all $i \in \{2, \ldots, k\}$, and having the property that $\mathrm{c}(p) = \mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k)$.

Proof. Let n be a positive integer such that, for every vertex $v \in V(C)$ and for every element a in the local monoid $C(v)$, a^n is an idempotent of $C(v)$. Then C belongs to the variety of categories $\mathcal{W}_n$. Let h be any positive integer which is greater than the number of elements of each local monoid of C and let k be any positive integer which is greater then $2h$. Let Γ be any non-empty graph, let Γ^* be the free category over Γ, let $\vartheta : \Gamma \to C$ be any graph mapping, and let $\theta : \Gamma^* \to C$ be the homomorphism of categories extending ϑ. Then it will be sufficient if we show that, for any loop p in Γ and any paths $q_1, \ldots, q_k$ in Γ such that $\alpha(p) = \omega(p) = \alpha(q_1)$, $\omega(q_1) = \alpha(q_2)$, $\ldots$, $\omega(q_{k-1}) = \alpha(q_k)$, and the condition $\mathrm{c}(p) = \mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k)$ is satisfied, we have the equality $\theta(q_1) \cdots \theta(q_k) = \theta(p)^n \theta(q_1) \cdots \theta(q_k)$.

Since $\mathrm{c}(p) = \mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k)$, for every $i \in \{1, \ldots, k\}$, there exist consecutive paths s_i, t_i in Γ such that $q_i = s_i t_i$ and $\omega(s_i) = \alpha(t_i) = \alpha(p) = \omega(p)$. Put $r_1 = q_1 s_2$ and $r_j = t_{2j-2} q_{2j-1} s_{2j}$ for all $j \in \{2, \ldots, h\}$. Then $q_1 \cdots q_k = r_1 \cdots r_h t_{2h} q_{2h+1} \cdots q_k$. Moreover, we have $\mathrm{c}(p) = \mathrm{c}(r_1) = \cdots = \mathrm{c}(r_h)$ and $r_1, \ldots, r_h$ are loops in Γ on the same vertex $\alpha(p) = \omega(p)$ of Γ. It means that the elements $\theta(r_1), \ldots, \theta(r_h)$ all occur in the same local monoid of C. Consider now the elements $\theta(r_1)$, $\theta(r_1)\theta(r_2)$, $\theta(r_1)\theta(r_2)\theta(r_3)$, $\ldots$, $\theta(r_1) \cdots \theta(r_h)$ of this local monoid. Since the number of elements in this sequence is greater then the number of elements in every local monoid of C, there exist $i, j \in \{1, \ldots, h\}$ such that $i < j$ and $\theta(r_1) \cdots \theta(r_i) = \theta(r_1) \cdots \theta(r_i) \cdots \theta(r_j)$. Hence it follows that

$$\theta(r_1) \cdots \theta(r_i) = \theta(r_1) \cdots \theta(r_i)(\theta(r_{i+1}) \cdots \theta(r_j))^\ell$$

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holds for every positive integer ℓ. In particular, this holds for $\ell = n$. Furthermore, since C is in $\mathcal{W}_n$, by Proposition 4.2 we know that C satisfies the path identity

$$p^n r_1 \cdots r_i \doteq r_1 \cdots r_i (r_{i+1} \cdots r_j)^n,$$

since $\mathrm{c}(p) = \mathrm{c}(r_1 \cdots r_i) = \mathrm{c}(r_{i+1} \cdots r_j)$. It means that we further have

$$\theta(r_1) \cdots \theta(r_i)(\theta(r_{i+1}) \cdots \theta(r_j))^n = \theta(p)^n \theta(r_1) \cdots \theta(r_i).$$

Hence we finally deduce that

$$\begin{aligned}
&\theta(q_1) \cdots \theta(q_k) \\
&\quad = \theta(r_1) \cdots \theta(r_h)\theta(t_{2h})\theta(q_{2h+1}) \cdots \theta(q_k) \\
&\quad = \theta(r_1) \cdots \theta(r_i)\theta(r_{i+1}) \cdots \theta(r_j)\theta(r_{j+1}) \cdots \theta(r_h)\theta(t_{2h})\theta(q_{2h+1}) \cdots \theta(q_k) \\
&\quad = \theta(r_1) \cdots \theta(r_i)(\theta(r_{i+1}) \cdots \theta(r_j))^{n+1}\theta(r_{j+1}) \cdots \theta(r_h)\theta(t_{2h})\theta(q_{2h+1}) \cdots \theta(q_k) \\
&\quad = \theta(p)^n\theta(r_1) \cdots \theta(r_i)\theta(r_{i+1}) \cdots \theta(r_j)\theta(r_{j+1}) \cdots \theta(r_h)\theta(t_{2h})\theta(q_{2h+1}) \cdots \theta(q_k) \\
&\quad = \theta(p)^n\theta(q_1) \cdots \theta(q_k),
\end{aligned}$$

as required. □

This fact prompts us to consider, for arbitrary positive integers n and k, the variety of categories $\mathcal{W}_{n,k}$ determined by the following loop identities:

$$x^n \doteq x^{2n},$$
$$(xy)^n \doteq (yx)^n,$$

and by the following path identities:

$q_1 \cdots q_k \doteq p^n q_1 \cdots q_k$ for arbitrary non-empty graphs Γ, for arbitrary loops p in Γ and for arbitrary paths $q_1, \dots, q_k$ in Γ such that $\alpha(p) = \omega(p) = \alpha(q_1)$, $\omega(q_1) = \alpha(q_2)$, ..., $\omega(q_{k-1}) = \alpha(q_k)$, and $\mathrm{c}(p) = \mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k)$.

Then, clearly, $\mathcal{W}_{n,k} \subseteq \mathcal{W}_n$ and, as we have seen in Proposition 4.4 and its proof, for every finite category C in the pseudovariety $\ell\mathbf{DG}$, there exist positive integers n and k such that C belongs to the variety $\mathcal{W}_{n,k}$. Consequently, $\ell\mathbf{DG}$ is contained in the union $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{W}_{n,k}$.

Let again $\mathcal{K}$ be any locally finite variety of groups. Remember once more that then $\mathcal{K}$ is a variety of groups of finite exponent, which means that $\mathcal{K}$ can be viewed as a variety of monoids. Assume next that X is an infinite set of variables. Then, as it has already been mentioned in § 2, the variety $\mathcal{K}$ is determined by the set of all identities of the form $v \doteq 1$ which hold in $\mathcal{K}$, where $v \in X^*$.

Proposition 4.5. *For any finite category C in the pseudovariety $\ell\mathbf{DG}$, there exist a positive integer n and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq 1$ such that, for every word $v \in X^*$ having the property that the identity*

$$v \doteq 1$$

holds in $\mathcal{K}$, the loop identity

$$v^{n+1} \simeq v^n$$

is satisfied in C.

Proof. Let $\mathcal{K}$ be the variety of groups generated by all maximal subgroups of the local monoids of C. Let again n be a positive integer such that, for every element a of any local monoid of C, a^n is an idempotent of C. Then C belongs to the variety of categories $\mathcal{W}_n$ and all local monoids of C belong to the monoid variety $\mathcal{V}_n$. Of course, the group variety $\mathcal{K}$ then satisfies the identity $x^n \simeq \mathtt{1}$. Let $v \in X^*$ be any word such that the identity $v \simeq \mathtt{1}$ holds in $\mathcal{K}$. Then we have to show that the loop identity $v^{n+1} \simeq v^n$ holds in C. However, this amounts to showing that the identity $v^{n+1} \simeq v^n$ holds in every local monoid of C. But local monoids of C all belong to the monoid variety $\mathcal{V}_n$ and their subgroups all lie in the group variety $\mathcal{K}$. Therefore, this last statement can be verified arguing in the same way as in the proof of Proposition 2.4. □

5. The category varieties $\mathcal{W}_{n,k}(\mathcal{K})$

Let again $\mathcal{K}$ be any locally finite variety of groups, let n be any positive integer such that the identity $x^n \simeq \mathtt{1}$ is satisfied in $\mathcal{K}$ and let k be an arbitrary positive integer. Consider the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ determined by the following loop identities:

$$x^n \simeq x^{2n},$$

$$(xy)^n \simeq (yx)^n,$$

$v^{n+1} \simeq v^n$ for arbitrary non-empty sets X of variables and for all words $v \in X^*$ such that the identity $v \simeq \mathtt{1}$ is satisfied in $\mathcal{K}$,

and by the following path identities:

$q_1 \cdots q_k \simeq p^n q_1 \cdots q_k$ for arbitrary non-empty graphs Γ, for arbitrary loops p in Γ and for arbitrary paths $q_1, \ldots, q_k$ in Γ such that $\alpha(p) = \omega(p) = \alpha(q_1)$, $\omega(q_1) = \alpha(q_2)$, $\ldots$, $\omega(q_{k-1}) = \alpha(q_k)$, and $\mathrm{c}(p) = \mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k)$.

Proposition 5.1. *For every finite category C in the pseudovariety ℓ**DG**, there exist positive integers n, k and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \simeq \mathtt{1}$ such that C lies in the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$.*

Proof. Let n be a positive integer such that, for every element a of any local monoid of C, a^n is an idempotent of C. Then C satisfies the loop identity $x^n \simeq x^{2n}$. Since C lies in ℓ**DG**, it satisfies also the loop identity $(xy)^n \simeq (yx)^n$. Furthermore, let $\mathcal{K}$ be the variety of groups generated by all maximal subgroups of all local monoids of C. Then, of course, $\mathcal{K}$ satisfies the identity $x^n \simeq \mathtt{1}$, and we have seen in Proposition 4.5 and its proof that then C satisfies the loop identities $v^{n+1} \simeq v^n$ for all words $v \in X^*$ such that the identity $v \simeq \mathtt{1}$ holds in $\mathcal{K}$. Moreover, we have also shown in Proposition 4.4 and its proof that then there exists some (sufficiently large) positive integer k such that C satisfies all path identities of the form $q_1 \cdots q_k \doteq p^n q_1 \cdots q_k$ for arbitrary loops p and arbitrary paths $q_1, \ldots, q_k$ in any non-empty graph Γ such that $\alpha(p) = \omega(p) = \alpha(q_1)$, $\omega(q_1) = \alpha(q_2)$, …, $\omega(q_{k-1}) = \alpha(q_k)$, and $\mathrm{c}(p) = \mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k)$. □

Note that, for the given positive integers n, k and the given locally finite variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq \mathtt{1}$, we have $\mathcal{W}_{n,k}(\mathcal{K}) \subseteq \mathcal{W}_n$. Thus the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ satisfies also the path identities given in Propositions 4.1 and 4.2 and in Corollary 4.3, for the mentioned positive integer n.

Proposition 5.2. *Let n, k be arbitrary positive integers and let $\mathcal{K}$ be any locally finite variety of groups satisfying the identity $x^n \doteq \mathtt{1}$. Then the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ is also locally finite, that is, every category in $\mathcal{W}_{n,k}(\mathcal{K})$ generated by a finite graph is itself finite.*

Proof. We need to verify that the following holds. If C is a category in $\mathcal{W}_{n,k}(\mathcal{K})$ such that $V(C)$ is a finite set and there exists a finite subset $F \subseteq E(C)$ such that the graph Δ obtained from C by omitting those edges which do not belong to F generates the category C, then the category C itself is finite. Besides, it is clearly enough to do so assuming that $k > 1$. We will proceed by induction on the number m of elements of the set F of edges of the graph Δ. We will show that for every positive integer m there exists a positive integer $\ell = \ell(m)$ having the following property. In every category C of $\mathcal{W}_{n,k}(\mathcal{K})$ having only finitely many vertices and possessing a subset F of edges containing m edges such the graph Δ obtained from C by deleting all edges beyond F generates this category C, every product $a_1 \cdots a_\varkappa$ of arbitrary consecutive edges $a_1, \ldots, a_\varkappa \in F$, where $\varkappa > \ell$, is equal to some product $b_1 \cdots b_\lambda$ of some consecutive edges $b_1, \ldots, b_\lambda \in F$, where $\lambda < \varkappa$. This property will ensure that the category C is finite.

For $m = 1$ this assertion is obvious. One only has to consider the situation when the unique edge in F is a loop. Then this edge belongs to a local monoid of C, and as C is in $\mathcal{W}_{n,k}(\mathcal{K})$, this local monoid satisfies the identity $x^n \doteq x^{2n}$. Therefore, one can take $\ell(1) = 2n - 1$. Thus suppose further that $m > 1$.

Remember once again that the group variety $\mathcal{K}$ is locally finite. This means that every finitely generated group in $\mathcal{K}$ is finite. This concerns, of course, also the free groups in $\mathcal{K}$ on finite sets of generators. In addition, $\mathcal{K}$ can be viewed as a monoid variety. Altogether this again ensures that for every positive integer h there exists some positive integer $\wp = \wp(h)$ having the following property. In every group G of $\mathcal{K}$ generated by a finite subset $Z \subseteq G$ containing h generators, every product $\eta_1 \cdots \eta_\mu$ of arbitrary elements $\eta_1, \ldots, \eta_\mu \in Z$ is equal to some product $\rho_1 \cdots \rho_\nu$ of some elements $\rho_1, \ldots, \rho_\nu \in Z$ such that $\nu \leqslant \wp$.

Now let C be any category in $\mathcal{W}_{n,k}(\mathcal{K})$ having only finitely many vertices and possessing a subset F of edges containing m edges such the graph Δ obtained from C by deleting all edges beyond F generates this category C. Let $F = \{f_1, \ldots, f_m\}$ where $f_1, \ldots, f_m$ are pairwise distinct edges. Let further Γ be an isomorphic copy of the graph Δ. Let $E(\Gamma) = \{e_1, \ldots, e_m\}$ and let $\xi : \Gamma \to \Delta$ be the pertinent isomorphism such that $\xi(e_1) = f_1$, …, $\xi(e_m) = f_m$. Let $\zeta : \Delta \to C$ be the inclusion mapping of the graph Δ into the category C and let $\vartheta : \Gamma \to C$ be the composition of the mappings ξ and ζ. Let Γ^* be the free category over the graph Γ and let $\theta : \Gamma^* \to C$ be the homomorphism of categories extending ϑ. Consider now any product $a_1 \cdots a_\varkappa$ of arbitrary consecutive edges $a_1, \ldots, a_\varkappa \in F$ in C where

$$\varkappa > k\ell(m-1) + k - 1 \quad \text{and} \quad \varkappa > \wp(m)(2n(m-1)+1) + nm^2 + 2.$$

For every $i \in \{1, \ldots, \varkappa\}$, let g_i be the edge of $E(\Gamma)$ for which $\xi(g_i) = a_i$. Note that then the sequence $p = g_1 \cdots g_\varkappa$ consists of consecutive edges of $E(\Gamma)$ and so it is a path in Γ, and hence an edge of the free category Γ^*. Moreover, $\theta(p) = a_1 \cdots a_\varkappa$. Since $\varkappa > k\ell(m-1) + k - 1$, one can write $p = q_1 \cdots q_k$ where $q_1, \ldots, q_k$ are consecutive paths in Γ and each of the paths $q_1, \ldots, q_k$ has length greater than $\ell(m-1)$. Put further $r_1 = \theta(q_1)$, …, $r_k = \theta(q_k)$. Then, in the category C, we have

$$a_1 \cdots a_\varkappa = r_1 \cdots r_k.$$

Assume first that, for some $\varepsilon \in \{1, \ldots, k\}$, the path q_ε does not contain some of the edges $e_1, \ldots, e_m$. Then, since the length of q_ε is greater than $\ell(m-1)$, by the induction hypothesis, the element $r_\varepsilon = \theta(q_\varepsilon)$ can be obtained as a product of a shorter sequence of consecutive edges from F, that is, it can be expressed in the form $r_\varepsilon = \theta(q'_\varepsilon)$ for some path q'_ε in Γ which is coterminal with q_ε, whose length is less than that of q_ε and which is composed only of those edges of $E(\Gamma)$ which are contained in q_ε. Consequently, also the product $a_1 \cdots a_\varkappa$ in C itself can be obtained as a product of a shorter sequence of consecutive edges from F, as desired above.

Thus assume further that we have $\mathrm{c}(q_1) = \cdots = \mathrm{c}(q_k) = \{e_1, \ldots, e_m\}$. Since, by our assumption, $k > 1$, we may consider the path $q_1 q_2$. As $\mathrm{c}(q_1) = \mathrm{c}(q_2) = \{e_1, \ldots, e_m\}$, there exists a loop u in Γ which is an initial segment of the path $q_1 q_2$ such that $\mathrm{c}(u) = \{e_1, \ldots, e_m\}$ and $\alpha(u) = \omega(u) = \alpha(q_1)$. Furthermore, it is not difficult to realize that one can select at most $(m-1)^2 + 1$ consecutive edges from the loop u which again form a loop w in Γ such that $\mathrm{c}(w) = \{e_1, \ldots, e_m\}$ and $\alpha(w) = \omega(w) = \alpha(q_1)$. Put $\varpi = \theta(w)$. Since the category C is from $\mathcal{W}_{n,k}(\mathcal{K})$, it therefore satisfies also the path identity $q_1 \cdots q_k \simeq w^n q_1 \cdots q_k$ for the above paths $q_1, \ldots, q_k$ and the loop w, that is, it satisfies the path identity $g_1 \cdots g_\varkappa \simeq w^n g_1 \cdots g_\varkappa$ over the graph Γ. Consequently, in the category C, we have the equality

$$a_1 \cdots a_\varkappa = \varpi^n a_1 \cdots a_\varkappa.$$

Once again, since there is the path $q_1 q_2$ in Γ and $\mathrm{c}(q_1) = \mathrm{c}(q_2) = \{e_1, \ldots, e_m\}$, for every $i \in \{1, \ldots, m\}$ there exist a path c_i in Γ from $\alpha(w) = \omega(w)$ to $\alpha(e_i)$ and a path d_i in Γ from $\omega(e_i)$ to $\alpha(w) = \omega(w)$. In addition, we may assume that whenever $i, j \in \{1, \ldots, m\}$ are such that $\alpha(e_i) = \alpha(e_j)$, then $c_i = c_j$, and whenever these indices i, j are such that $\omega(e_i) = \omega(e_j)$, then $d_i = d_j$. Moreover, all these paths c_i, d_i can be chosen so that their lengths are not greater than $m-1$. Now, since $g_1, \ldots, g_\varkappa \in \{e_1, \ldots, e_m\}$, for every $\pi \in \{1, \ldots, \varkappa\}$, there exists some $i \in \{1, \ldots, m\}$ such that $g_\pi = e_i$. In this situation, put $s_\pi = c_i$ and $t_\pi = d_i$. Put also $\sigma_\pi = \theta(s_\pi)$ and $\tau_\pi = \theta(t_\pi)$. Notice that, for every $\pi \in \{2, \ldots, \varkappa\}$, $t_{\pi-1} s_\pi$ is a loop in Γ on the vertex $\omega(g_{\pi-1}) = \alpha(g_\pi)$. Then, from the

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last path identity valid in $\mathcal{W}_{n,k}(\mathcal{K})$ which is mentioned above, and from the first path identity given in Corollary 4.3 which is valid in $\mathcal{W}_n$, and hence also in $\mathcal{W}_{n,k}(\mathcal{K})$, one can consecutively deduce the path identity

$$g_1 g_2 \dots g_\varkappa \doteq w^n g_1 (t_1 s_2)^n g_2 (t_2 s_3)^n \cdots (t_{\varkappa-1} s_\varkappa)^n g_\varkappa$$

over the graph Γ, so that this path identity also holds in $\mathcal{W}_{n,k}(\mathcal{K})$. It means that, in the category C, we have the equality

$$a_1 a_2 \cdots a_\varkappa = \varpi^n a_1 (\tau_1 \sigma_2)^n a_2 (\tau_2 \sigma_3)^n \cdots (\tau_{\varkappa-1} \sigma_\varkappa)^n a_\varkappa,$$

where the elements $a_1 \tau_1$ and $\sigma_\pi (\tau_{\pi-1} \sigma_\pi)^{n-1} a_\pi \tau_\pi$ for all $\pi \in \{2, \dots, \varkappa - 1\}$ belong to the local monoid of C at the vertex $\alpha(\varpi) = \omega(\varpi) = \alpha(a_1)$. Moreover, all these elements are obtained by multiplying appropriately selected edges from F. Hence, using the first path identity given in Corollary 4.3 once again, from the previous equality valid in the category C, we successively get that, in C, we also have the equality

$$a_1 a_2 \cdots a_\varkappa = \varpi^n a_1 \tau_1 \varpi^n \sigma_2 (\tau_1 \sigma_2)^{n-1} a_2 \tau_2 \cdots \\ \cdots \varpi^n \sigma_{\varkappa-1} (\tau_{\varkappa-2} \sigma_{\varkappa-1})^{n-1} a_{\varkappa-1} \tau_{\varkappa-1} \sigma_\varkappa (\tau_{\varkappa-1} \sigma_\varkappa)^{n-1} a_\varkappa.$$

Thus the product $a_1 a_2 \cdots a_\varkappa$ becomes expressed as the product of the elements $\varpi^n a_1 \tau_1$, $\varpi^n \sigma_\pi (\tau_{\pi-1} \sigma_\pi)^{n-1} a_\pi \tau_\pi$ for all $\pi \in \{2, \dots, \varkappa - 1\}$, and the element $\sigma_\varkappa (\tau_{\varkappa-1} \sigma_\varkappa)^{n-1} a_\varkappa$. From the consequences of Propositions 4.1 and 4.2 stated in the text preceding Proposition 4.4 we know that all elements in this list except the last one belong to the maximal subgroup G of the local monoid of C at $\alpha(a_1)$ containing the idempotent ϖ^n. Consider now in detail the product

$$\varpi^n \sigma_2 (\tau_1 \sigma_2)^{n-1} a_2 \tau_2 \cdots \varpi^n \sigma_{\varkappa-1} (\tau_{\varkappa-2} \sigma_{\varkappa-1})^{n-1} a_{\varkappa-1} \tau_{\varkappa-1}$$

which forms a substantial part of the product displayed above. In order to handle it further, put $\gamma_i = \theta(c_i)$ and $\delta_i = \theta(d_i)$ for all $i \in \{1, \dots, m\}$. Using once more the fact that there is the path $q_1 q_2$ in Γ, for every $i \in \{1, \dots, m\}$, we can find some $j_i \in \{1, \dots, m\}$ such that $\omega(e_{j_i}) = \alpha(e_i)$. Then we may consider the elements $\varpi^n \gamma_i (\delta_{j_i} \gamma_i)^{n-1} f_i \delta_i$ for all $i \in \{1, \dots, m\}$. There are altogether m such elements, all of them lie in the maximal subgroup G of the local monoid of C at $\alpha(a_1)$ containing the idempotent ϖ^n, and, according to our previous considerations, all factors $\varpi^n \sigma_\pi (\tau_{\pi-1} \sigma_\pi)^{n-1} a_\pi \tau_\pi$ of the above product for $\pi \in \{2, \dots, \varkappa-1\}$ appear among these m elements. Now, since the category C is from $\mathcal{W}_{n,k}(\mathcal{K})$, and therefore the local monoids of C satisfy also the identities $v^{n+1} \doteq v^n$ for all words v over arbitrary sets X of variables such that the identity $v \doteq 1$ holds in $\mathcal{K}$, we see that the group G belongs to the variety $\mathcal{K}$. According to the earlier remarks on the finitely generated groups in $\mathcal{K}$, we thus know that the last product displayed above can be expressed as a product of at most $\wp(m)$ factors taken from among the elements $\varpi^n \gamma_i (\delta_{j_i} \gamma_i)^{n-1} f_i \delta_i$ for all $i \in \{1, \dots, m\}$. Therefore, having the last equality displayed above in mind, and using the first path identity given in Corollary 4.3 once more, in the end, we see that the product $a_1 a_2 \cdots a_\varkappa$ can eventually be expressed as

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the product of the element $\varpi^n a_1\tau_1$, several (at most $\wp(m)$) factors taken from among the elements $\gamma_i(\delta_{j_i}\gamma_i)^{n-1}f_i\delta_i$ for $i \in \{1,\dots,m\}$, and the element $\sigma_\varkappa(\tau_{\varkappa-1}\sigma_\varkappa)^{n-1}a_\varkappa$. By our previous estimates, however, this product is obtained by multiplying at most $n((m-1)^2+1)+m+\wp(m)(2n(m-1)+1)+2(n-1)(m-1)+m$ edges, that is, at most $\wp(m)(2n(m-1)+1)+nm^2+2$ edges from the set F. As $\varkappa > \wp(m)(2n(m-1)+1)+nm^2+2$, there are fewer edges from F in that product than in $a_1\cdots a_\varkappa$, as required above. This finally shows that one can take

$$\ell(m) = \max\{k\ell(m-1)+k-1, \wp(m)(2n(m-1)+1)+nm^2+2\}.$$

□

6. The monoid varieties $\mathcal{U}_k$

Let again X be an infinite set of variables. Let $k > 2$ be an arbitrary integer. Consider the variety of monoids $\mathcal{U}_k$ determined by the following identities:

$$u_1\cdots u_k \doteq w_1\cdots w_k \quad \text{for arbitrary words } u_1,\dots,u_k,w_1,\dots,w_k \in X^* \text{ such that } \mathrm{c}(u_1)=\cdots=\mathrm{c}(u_k)=\mathrm{c}(w_1)=\cdots=\mathrm{c}(w_k).$$

Note that if n is any integer such that $k \leqslant n$ then the variety $\mathcal{U}_k$ is obviously a subvariety of the previously defined variety of monoids $\mathcal{V}_{n,k}(\mathcal{K})$ for any locally finite variety of groups $\mathcal{K}$ satisfying the identity $x^n \doteq 1$.

In this and the next sections, we intend to gather some partial information regarding mainly the word problem for the free monoids in the variety $\mathcal{U}_k$. The free monoid in $\mathcal{U}_k$ on the set X can be represented in the form $X^*/{\equiv_{\mathcal{U}_k}}$ where $\equiv_{\mathcal{U}_k}$ is the congruence on X^* consisting of all pairs (u,w) of words $u,w \in X^*$ such that the identity $u \doteq w$ holds in $\mathcal{U}_k$.

Let $u \in X^*$ be any non-empty word. We say that a word $v \in X^*$ is a *subword of* u if there exist some words $s,t \in X^*$ such that $u = svt$. Notice, however, that we will always view subwords of the word u as its marked segments, that is, we will consider them together with their positions in u. It means that whenever we deal with a subword v of u, we always have in mind its concrete occurrence in the word u.

We will have to deal in this and the subsequent sections with non-empty words $v \in X^*$ having the property that there exist words $v_1,\dots,v_k \in X^*$ such that $v = v_1\cdots v_k$ and $\mathrm{c}(v_1)=\cdots=\mathrm{c}(v_k)=\mathrm{c}(v)$. We then say that the word v has the *k-factorization property.*

Let $u \in X^*$ be any non-empty word. Note that every non-empty subword v of u can be expanded in its given position in u to a maximal subword v' of u such that $\mathrm{c}(v')=\mathrm{c}(v)$. That is, v in its position in u becomes contained as a subword in v', and the maximality of the subword v' in u means that whenever v'' is a subword of u containing v' in its position in u as a subword and having the property that $\mathrm{c}(v'')=\mathrm{c}(v)$, then v'' coincides with v' in u. Obviously, the *maximal subword v' of u* is uniquely determined by the position of the initial subword v in u and by its set $\mathrm{c}(v)$ of variables, where $\mathrm{c}(v) \subseteq \mathrm{c}(u)$.

We will next be concerned with non-empty subwords of the word $u \in X^*$ having the k-factorization property. Clearly, if some non-empty subword v of u has the k-factorization property, then the maximal subword v' extending v in u in the way

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specified in the previous paragraph again has the k-factorization property. Thus, to be more specific than before, we will have to deal with *maximal subwords of u having the k-factorization property.*

The following fact is now obvious.

Lemma 6.1. *Let $u, w \in X^*$ be any non-empty words such that $u \equiv_{\mathcal{U}_k} w$. Then there exist a non-negative integer m, words $v_0, v_1, \ldots, v_m \in X^*$ such that $u = v_0$, $w = v_m$, and for every $i \in \{1, \ldots, m\}$, there exist words $a_i, b_i \in X^*$ and non-empty words $c_i, d_i \in X^*$ such that $v_{i-1} = a_i c_i b_i$, $v_i = a_i d_i b_i$, and the words c_i, d_i satisfy $\mathrm{c}(c_i) = \mathrm{c}(d_i)$ and they have both the k-factorization property. In addition, for every $i \in \{1, \ldots, m\}$, the words a_i, b_i, c_i, d_i can be picked so that the words c_i, d_i are maximal subwords of $a_i c_i b_i$, $a_i d_i b_i$, respectively, having the k-factorization property.* □

Assume that, in the given word $u \in X^*$, there are two maximal subwords v and w of u having the k-factorization property such that both $\mathrm{c}(v) \nsubseteq \mathrm{c}(w)$ and $\mathrm{c}(w) \nsubseteq \mathrm{c}(v)$ hold. Assume, in addition, that the subword v precedes the subword w in u and that the subwords v and w overlap in u. That is, in the given positions of the subwords v and w in u, there is a subword r of u which is at the same time a final segment of v and an initial segment of w, and it is this segment r in which the subwords v and w overlap in u. We allow that the overlap r may also be empty, so that, in this case, the subwords v and w only touch each other. In any case, we necessarily have $\mathrm{c}(r) \subsetneqq \mathrm{c}(v)$ and $\mathrm{c}(r) \subsetneqq \mathrm{c}(w)$, by the above assumption about $\mathrm{c}(v)$ and $\mathrm{c}(w)$. Hence it follows that, if $v_1, \ldots, v_k \in X^*$ and $w_1, \ldots, w_k \in X^*$ are the factors such that $v = v_1 \cdots v_k$, $w = w_1 \cdots w_k$ and $\mathrm{c}(v_1) = \cdots = \mathrm{c}(v_k) = \mathrm{c}(v)$, $\mathrm{c}(w_1) = \cdots = \mathrm{c}(w_k) = \mathrm{c}(w)$, then r is a proper final segment of v_k and a proper initial segment of w_1. Note that then $v_k = \bar{v}_k r$ and $w_1 = r\bar{w}_1$ for some non-empty words $\bar{v}_k, \bar{w}_1 \in X^*$ and that we have $\mathrm{c}(\bar{v}_k r \bar{w}_1) = \mathrm{c}(v) \cup \mathrm{c}(w)$.

We proceed to generalize slightly the notions introduced so far in this section. We will next be interested in non-empty words $v \in X^*$ having the property that there exist non-empty words $v_1, \ldots, v_k \in X^*$ such that $v = v_1 \cdots v_k$, $\mathrm{c}(v_2) = \cdots = \mathrm{c}(v_{k-1}) = \mathrm{c}(v)$ and $\mathrm{c}(v_1), \mathrm{c}(v_k) \subseteq \mathrm{c}(v)$. We then say that the word v has the *weak k-factorization property.* If, in addition, $\mathrm{c}(v_1) = \mathrm{c}(v)$ also holds, then we say that the word v has the *right weak k-factorization property.* Likewise, if $\mathrm{c}(v_k) = \mathrm{c}(v)$ holds in addition, then we say that the word v has the *left weak k-factorization property.* Remember once again that every non-empty subword v of u can be expanded in its given position in u to a maximal subword v' of u such that $\mathrm{c}(v') = \mathrm{c}(v)$. Clearly, if the subword v has the weak (right weak, left weak) k-factorization property, then the maximal subword v' has again the weak (right weak, left weak) k-factorization property.

Motivated by the considerations in the text following Lemma 6.1, we continue by introducing another somewhat complicated notion. Thus let $u \in X^*$ be any non-empty word and let ℓ be any positive integer. If $\ell = 1$ then let $v^{(1)} = v^{(\ell)}$ be any maximal subword of u having the k-factorization property. On the other hand, if $\ell > 1$ then let $v^{(1)}, \ldots, v^{(\ell)}$ be any consecutive subwords of u such that $v^{(1)}$ is a maximal subword of u having the right weak k-factorization property, $v^{(2)}, \ldots, v^{(\ell-1)}$ are maximal subwords of u having the weak k-factorization property and $v^{(\ell)}$ is a maximal subword of u having the left weak k-factorization property. In this case, suppose, in addition, that $\mathrm{c}(v^{(i-1)}) \nsubseteq \mathrm{c}(v^{(i)})$

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and $\mathrm{c}(v^{(i)}) \nsubseteq \mathrm{c}(v^{(i-1)})$ hold for all $i \in \{2, \ldots, \ell\}$, and that, for all $i \in \{2, \ldots, \ell\}$ again, the maximal subwords $v^{(i-1)}$ and $v^{(i)}$ overlap in u. That is, for every $i \in \{2, \ldots, \ell\}$, there is a subword $r^{(i)}$ of u which is at the same time a final segment of $v^{(i-1)}$ and an initial segment of $v^{(i)}$, and it is this segment $r^{(i)}$ in which the subwords $v^{(i-1)}$ and $v^{(i)}$ overlap. We allow that the overlap $r^{(i)}$ may also be empty, so that, in such a case, the subwords $v^{(i-1)}$ and $v^{(i)}$ touch each other. In any case, for all $i \in \{2, \ldots, \ell\}$, we necessarily have $\mathrm{c}(r^{(i)}) \subsetneqq \mathrm{c}(v^{(i-1)})$ and $\mathrm{c}(r^{(i)}) \subsetneqq \mathrm{c}(v^{(i)})$, by the above assumption about $\mathrm{c}(v^{(i-1)})$ and $\mathrm{c}(v^{(i)})$. Moreover, assume that the factors of the maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ assuring their (right, left) weak k-factorization property can be chosen so that the following holds. For every $i \in \{1, \ldots, \ell\}$, let $v_1^{(i)}, \ldots, v_k^{(i)} \in X^*$ be non-empty words such that $v^{(i)} = v_1^{(i)} \cdots v_k^{(i)}$ and such that, for $i = 1$, we have

$$\mathrm{c}(v_1^{(1)}) = \mathrm{c}(v_2^{(1)}) = \cdots = \mathrm{c}(v_{k-1}^{(1)}) = \mathrm{c}(v^{(1)}) \quad \text{and} \quad \mathrm{c}(v_k^{(1)}) \subseteq \mathrm{c}(v^{(1)}),$$

for all $i \in \{2, \ldots, \ell - 1\}$, we have

$$\mathrm{c}(v_2^{(i)}) = \cdots = \mathrm{c}(v_{k-1}^{(i)}) = \mathrm{c}(v^{(i)}) \quad \text{and} \quad \mathrm{c}(v_1^{(i)}), \mathrm{c}(v_k^{(i)}) \subseteq \mathrm{c}(v^{(i)}),$$

and, at last, for $i = \ell$, we have

$$\mathrm{c}(v_2^{(\ell)}) = \cdots = \mathrm{c}(v_{k-1}^{(\ell)}) = \mathrm{c}(v_k^{(\ell)}) = \mathrm{c}(v^{(\ell)}) \quad \text{and} \quad \mathrm{c}(v_1^{(\ell)}) \subseteq \mathrm{c}(v^{(\ell)}).$$

Furthermore, assume also that then, for every $i \in \{2, \ldots, \ell\}$, the subword $r^{(i)}$ is actually a proper final segment of $v_k^{(i-1)}$ and a proper initial segment of $v_1^{(i)}$, and, in addition, if $\bar{v}_k^{(i-1)}, \bar{v}_1^{(i)} \in X^*$ are the non-empty words for which $v_k^{(i-1)} = \bar{v}_k^{(i-1)} r^{(i)}$ and $v_1^{(i)} = r^{(i)} \bar{v}_1^{(i)}$ hold, then

$$\mathrm{c}(\bar{v}_k^{(i-1)} r^{(i)} \bar{v}_1^{(i)}) = \mathrm{c}(v^{(i-1)}) \cup \mathrm{c}(v^{(i)})$$

is fulfilled. If all these conditions are satisfied then we say that $v^{(1)}, \ldots, v^{(\ell)}$ form a *properly intersecting sequence of maximal subwords of u relative to the k-factorization property*, or shortly, we say that $v^{(1)}, \ldots, v^{(\ell)}$ form a *properly intersecting k-sequence in u*. The segment $\tilde{v}$ of the word u composed of the maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ then has the form

$$\tilde{v} = v_1^{(1)} v_2^{(1)} \cdots v_{k-1}^{(1)} \bar{v}_k^{(1)} r^{(2)} \bar{v}_1^{(2)} v_2^{(2)} \cdots v_{k-1}^{(2)} \bar{v}_k^{(2)} r^{(3)} \cdots \\ \cdots r^{(\ell-1)} \bar{v}_1^{(\ell-1)} v_2^{(\ell-1)} \cdots v_{k-1}^{(\ell-1)} \bar{v}_k^{(\ell-1)} r^{(\ell)} \bar{v}_1^{(\ell)} v_2^{(\ell)} \cdots v_{k-1}^{(\ell)} v_k^{(\ell)}$$

and it will be called a *raw k-nest in u*. This terminology extends to the case when $\ell = 1$ and the segment $\tilde{v}$ of u consists only of the single maximal subword $v^{(1)} = v^{(\ell)}$ having the k-factorization property.

We will need later yet the following enhancement of the notion we have just introduced. Suppose that everything is given as in the previous paragraph, with the following amendments. For some $\varepsilon \in \{1, \ldots, \ell\}$, we require, in addition, that the following conditions are satisfied. If $\ell = 1$ then, of course, $\varepsilon = 1$, and we want that the subword $v^{(1)} = v^{(\ell)}$ of u has even the $2k$-factorization property. If $\ell > 1$ and $\varepsilon = 1$, then

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we demand that the subword $v^{(1)} = v^{(\varepsilon)}$ has the right weak $2k$-factorization property. If $\ell > 1$ and $\varepsilon = \ell$, then we demand that the subword $v^{(\ell)} = v^{(\varepsilon)}$ has the left weak $2k$-factorization property. And if $\ell > 2$ and $1 < \varepsilon < \ell$, then we require that the subword $v^{(\varepsilon)}$ has the weak $2k$-factorization property. In addition, we assume that the $2k$ factors $v_1^{(\varepsilon)}, \dots, v_k^{(\varepsilon)}, v_{k+1}^{(\varepsilon)}, \dots, v_{2k}^{(\varepsilon)}$ of the subword $v^{(\varepsilon)}$ assuring its $2k$-factorization property in case $\ell = 1$, or its (right, left) weak $2k$-factorization property in case $\ell > 1$ can be chosen so that the following requests are satisfied. Besides the standard requirements including the equality $v^{(\varepsilon)} = v_1^{(\varepsilon)} \cdots v_k^{(\varepsilon)} v_{k+1}^{(\varepsilon)} \cdots v_{2k}^{(\varepsilon)}$ and the appropriate equalities or inclusions involving the sets of variables $\mathrm{c}(v^{(\varepsilon)})$ and $\mathrm{c}(v_1^{(\varepsilon)}), \dots, \mathrm{c}(v_k^{(\varepsilon)}), \mathrm{c}(v_{k+1}^{(\varepsilon)}), \dots, \mathrm{c}(v_{2k}^{(\varepsilon)})$, we impose also the following requirements which adjust accordingly some of the subsequent conditions requested in the previous paragraph. If $\ell > 1$ and $1 < \varepsilon$ then we want that the overlap $r^{(\varepsilon)}$ is a proper final segment of $v_k^{(\varepsilon-1)}$ and a proper initial segment of $v_1^{(\varepsilon)}$, and if $\bar{v}_k^{(\varepsilon-1)}, \bar{v}_1^{(\varepsilon)} \in X^*$ are the non-empty words for which $v_k^{(\varepsilon-1)} = \bar{v}_k^{(\varepsilon-1)} r^{(\varepsilon)}$ and $v_1^{(\varepsilon)} = r^{(\varepsilon)} \bar{v}_1^{(\varepsilon)}$ hold, then we demand that

$$\mathrm{c}(\bar{v}_k^{(\varepsilon-1)} r^{(\varepsilon)} \bar{v}_1^{(\varepsilon)}) = \mathrm{c}(v^{(\varepsilon-1)}) \cup \mathrm{c}(v^{(\varepsilon)}).$$

Further, if $\ell > 1$ and $\varepsilon < \ell$ then we want that the overlap $r^{(\varepsilon+1)}$ is a proper final segment of $v_{2k}^{(\varepsilon)}$ and a proper initial segment of $v_1^{(\varepsilon+1)}$, and if $\bar{v}_{2k}^{(\varepsilon)}, \bar{v}_1^{(\varepsilon+1)} \in X^*$ are the non-empty words for which $v_{2k}^{(\varepsilon)} = \bar{v}_{2k}^{(\varepsilon)} r^{(\varepsilon+1)}$ and $v_1^{(\varepsilon+1)} = r^{(\varepsilon+1)} \bar{v}_1^{(\varepsilon+1)}$ hold, then we demand that

$$\mathrm{c}(\bar{v}_{2k}^{(\varepsilon)} r^{(\varepsilon+1)} \bar{v}_1^{(\varepsilon+1)}) = \mathrm{c}(v^{(\varepsilon)}) \cup \mathrm{c}(v^{(\varepsilon+1)}).$$

If these additional conditions are satisfied for the given index $\varepsilon \in \{1, \dots, \ell\}$, then we say that the subwords $v^{(1)}, \dots, v^{(\ell)}$ form a *properly intersecting k-sequence in u enhanced at the subword $v^{(\varepsilon)}$*.

We will need yet the following device. Let $p \in X^*$ be any non-empty word. Let $\varkappa, \lambda$ be any non-negative integers and let $u^{(1)}, \dots, u^{(\varkappa)}, v, w^{(1)}, \dots, w^{(\lambda)}$ form a properly intersecting sequence of maximal subwords of p relative to the k-factorization property in the sense just explained. Let s be any maximal subword of p having the k-factorization property such that either this word s in its position in p is a subword of some of the words $u^{(1)}, \dots, u^{(\varkappa)}, v, w^{(1)}, \dots, w^{(\lambda)}$ or else the word s and the segment of the word p composed of the subwords $u^{(1)}, \dots, u^{(\varkappa)}, v, w^{(1)}, \dots, w^{(\lambda)}$ have no intersection in p and they even do not touch each other in p. Furthermore, let $t \in X^*$ be any word having the k-factorization property such that $\mathrm{c}(s) = \mathrm{c}(t)$. Let the word $q \in X^*$ arise from p by replacing the subword s in its given position in p with the word t. Then t is again a maximal subword of q having the k-factorization property and $p \equiv_{\mathcal{U}_k} q$. Now we define what is the *trace $v^{\#}$ of the subword v of p in the word q*. If s is a subword of v then $v^{\#}$ arises by replacing s with t in v. If the words v and s do not intersect at all and if they do not touch each other, then $v^{\#}$ remains identical with v. Note that, in this case, s may still be a subword of some of the words $u^{(1)}, \dots, u^{(\varkappa)}, w^{(1)}, \dots, w^{(\lambda)}$. Otherwise, either $\varkappa > 0$ and s is a subword of $u^{(\varkappa)}$ intersecting or touching the word v from the

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left or else $\lambda > 0$ and s is a subword of $w^{(1)}$ intersecting or touching the word v from the right. Remember yet that now s is not a subword of v, so that the subwords s and v actually overlap in p or possibly they touch each other. Hence, in this case, since the subwords $u^{(1)}, \ldots, u^{(\varkappa)}, v, w^{(1)}, \ldots, w^{(\lambda)}$ form a properly intersecting k-sequence in p and $k > 2$, omitting from v its initial or final segment which formed the overlap of v with s, we obtain a subword $\bar{v}$ of both words p and q (with exactly determined positions in these words) satisfying $\mathrm{c}(v) = \mathrm{c}(\bar{v})$. Then $v^{\#}$ is obtained as the maximal subword of q containing $\bar{v}$ such that $\mathrm{c}(v^{\#}) = \mathrm{c}(v)$. Furthermore, we will see below that, under the given circumstances, after replacing s with t in p, the segment of p composed of the maximal subwords $u^{(1)}, \ldots, u^{(\varkappa)}, v, w^{(1)}, \ldots, w^{(\lambda)}$ will transform to a segment of q composed of the same number of maximal subwords, say, $\hat{u}^{(1)}, \ldots, \hat{u}^{(\varkappa)}, v^{\#}, \hat{w}^{(1)}, \ldots, \hat{w}^{(\lambda)}$ which again constitute a properly intersecting sequence of maximal subwords of q relative to the k-factorization property. And it is, indeed, the trace $v^{\#}$ of the subword v which occurs amidst this sequence of subwords. Returning to the notation used in the preceding parts of this section, we get that, more generally, the following then holds.

Lemma 6.2. *Let $u \in X^*$ be any non-empty word and let s be any maximal subword of u having the k-factorization property. Let $t \in X^*$ be another word having the k-factorization property such that $\mathrm{c}(s) = \mathrm{c}(t)$ holds. Replacing the subword s in u with the word t, we obtain a word z from u such that t in its position in z is again a maximal subword of z having the k-factorization property. Besides, $u \equiv_{\mathcal{U}_k} z$ then holds. Let further ℓ be any positive integer and let $v^{(1)}, \ldots, v^{(\ell)}$ be any properly intersecting sequence of maximal subwords of u relative to the k-factorization property. Assume, in addition, that the above subword s in its position in u is a subword of one of the maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$. Then, after replacing s with t in u, the segment of u composed of the maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ will transform to a segment of z composed of ℓ consecutive subwords $w^{(1)}, \ldots, w^{(\ell)}$ of z which again constitute a properly intersecting sequence of maximal subwords of z relative to the k-factorization property. Moreover, for every $i \in \{1, \ldots, \ell\}$, the subword $w^{(i)}$ then arises as the trace of the subword $v^{(i)}$ of u in the word z.*

Proof. In order to verify the statements concerning the mentioned properly intersecting k-sequences in u and z, we have to distinguish two possibilities.

Either the maximal subword s is equal to $v^{(\vartheta)}$ for some $\vartheta \in \{1, \ldots, \ell\}$. Then, of course, this maximal subword $v^{(\vartheta)}$ has the k-factorization property. After the replacement of s with t in u this maximal subword will change to a maximal subword of z which we may denote by $w^{(\vartheta)}$ and which, this time, is equal to t and, consequently, it again has the k-factorization property. If $\vartheta > 1$ then also the maximal subword $v^{(\vartheta-1)}$ of u may change to yield a maximal subword of z, which we may accordingly denote by $w^{(\vartheta-1)}$, because the overlap $r^{(\vartheta)}$ of $v^{(\vartheta-1)}$ and $v^{(\vartheta)}$ in u may then change. Namely, the subword $w^{(\vartheta-1)}$ is obtained by cutting off this overlap $r^{(\vartheta)}$ from $v^{(\vartheta-1)}$ and then by extending the rest of this subword to the right so as to make of it the maximal subword $w^{(\vartheta-1)}$ of z such that $\mathrm{c}(w^{(\vartheta-1)}) = \mathrm{c}(v^{(\vartheta-1)})$. However, anyway, if $v^{(\vartheta-1)} = v_1^{(\vartheta-1)} \cdots v_k^{(\vartheta-1)}$ is the factorization of $v^{(\vartheta-1)}$ existing in connection with the appearance of $v^{(\vartheta-1)}$ in the properly

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intersecting k-sequence $v^{(1)}, \dots, v^{(\ell)}$ in u, then the replacement of s with t in u may affect only its last factor $v_k^{(\vartheta-1)}$, so that the subword $w^{(\vartheta-1)}$ of z then can be written in the form $w^{(\vartheta-1)} = v_1^{(\vartheta-1)} \cdots v_{k-1}^{(\vartheta-1)} w_k^{(\vartheta-1)}$. The overlap of $w^{(\vartheta-1)}$ and $w^{(\vartheta)}$ in z thus remains a proper final segment of $w_k^{(\vartheta-1)}$ and, at the same time, a proper initial segment of the first factor of $w^{(\vartheta)}$ in its factorization into k segments, each of which contains all variables of $w^{(\vartheta)}$ (remember that this subword has the k-factorization property). Similarly, if $\vartheta < \ell$ then also the maximal subword $v^{(\vartheta+1)}$ of u may change to yield a maximal subword of z denoted by $w^{(\vartheta+1)}$ in an analogous manner, so that the corresponding notes on the overlap of $w^{(\vartheta)}$ and $w^{(\vartheta+1)}$ in z hold, as well. The other maximal subwords in the sequence $v^{(1)}, \dots, v^{(\ell)}$ remain unchanged, so that, merely renaming them, we thus obtain the sequence $w^{(1)}, \dots, w^{(\ell)}$ of maximal subwords of z which clearly satisfies all properties making of it again a properly intersecting sequence of maximal subwords of z relative to the k-factorization property. Moreover, it is then also obvious that, for every $i \in \{1, \dots, \ell\}$, the subword $w^{(i)}$ is the trace of the subword $v^{(i)}$ of u in z, as stated above.

Or else the maximal subword s is a proper subword of $v^{(\vartheta)}$ for some $\vartheta \in \{1, \dots, \ell\}$. This entails that $\mathrm{c}(s) \subsetneqq \mathrm{c}(v^{(\vartheta)})$. Therefore, if $v^{(\vartheta)} = v_1^{(\vartheta)} \cdots v_k^{(\vartheta)}$ is the factorization of $v^{(\vartheta)}$ stemming from the circumstance that $v^{(\vartheta)}$ occurs in the properly intersecting k-sequence $v^{(1)}, \dots, v^{(\ell)}$ in u, then s must appear as a proper subword of $v_{\nu-1}^{(\vartheta)} v_\nu^{(\vartheta)}$ for some $\nu \in \{2, \dots, k\}$. Since both words s and t have the k-factorization property and $k > 2$, we thus come to the following conclusion. When the maximal subword $v^{(\vartheta)}$ of u gets changed to the maximal subword $w^{(\vartheta)}$ of z by replacing the subword s with t in it, it is always possible to choose the required factorization $w^{(\vartheta)} = w_1^{(\vartheta)} \cdots w_k^{(\vartheta)}$ of $w^{(\vartheta)}$ so that $\mathrm{c}(v_1^{(\vartheta)}) \subseteq \mathrm{c}(w_1^{(\vartheta)})$ and $\mathrm{c}(v_k^{(\vartheta)}) \subseteq \mathrm{c}(w_k^{(\vartheta)})$. The subsequent discussion is then nearly as simple as before. The last-mentioned fact makes it possible to conclude that the sequence of maximal subwords $w^{(1)}, \dots, w^{(\ell)}$ of z that we eventually obtain satisfies all properties needed to become again a properly intersecting sequence of maximal subwords of z relative to the k-factorization property. In order to complete the arguments, it is enough to note the following. As before, if $\vartheta > 1$ then also the maximal subword $v^{(\vartheta-1)}$ of u may transform to another maximal subword $w^{(\vartheta-1)}$ of z. In addition to the considerations parallel to those in the previous paragraph, one has to take here into account yet the possibility that s may at the same time be a subword of $v^{(\vartheta-1)}$, so that s is then a subword of the overlap of $v^{(\vartheta-1)}$ and $v^{(\vartheta)}$ in u. But this situation is fairly easy to handle. Similarly, if $\vartheta < \ell$ then analogous deliberations apply also to the maximal subword $v^{(\vartheta+1)}$ of u and the maximal subword $w^{(\vartheta+1)}$ of z corresponding to it. The other maximal subwords in the sequence $v^{(1)}, \dots, v^{(\ell)}$ again remain unchanged. As before, it is likewise easy to see that, for every $i \in \{1, \dots, \ell\}$, the subword $w^{(i)}$ is the trace of the subword $v^{(i)}$ of u in z, as claimed above. □

We will need later yet the following modification of the previous lemma.

Lemma 6.3. *Let $u \in X^*$ be any non-empty word. Let s be any maximal subword of u having the k-factorization property. Let $t \in X^*$ be another word having the k-factorization property such that $\mathrm{c}(s) = \mathrm{c}(t)$ holds. Then, as before, replacing the subword s in u with the word t, we obtain a word z from u such that t appears in z as*

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a maximal subword having the k-factorization property. Let further ℓ be any positive integer, let $\varepsilon \in \{1, \ldots, \ell\}$ be any index and let $v^{(1)}, \ldots, v^{(\ell)}$ be any properly intersecting k-sequence in u enhanced at the subword $v^{(\varepsilon)}$. Furthermore, assume that the above subword s in its position in u is a subword of one of the maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$, and, moreover, if it happens that s is a subword of $v^{(\varepsilon)}$, then assume, in addition, that s is a proper subword of $v^{(\varepsilon)}$, so that $\mathrm{c}(s) \subsetneqq \mathrm{c}(v^{(\varepsilon)})$. Then, after replacing s with t in u, the segment of u composed of the maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ will transform to a segment of z composed of ℓ consecutive subwords $w^{(1)}, \ldots, w^{(\ell)}$ of z which are again maximal subwords of z and constitute a properly intersecting k-sequence in z, this time enhanced at the subword $w^{(\varepsilon)}$. Moreover, for every $i \in \{1, \ldots, \ell\}$, the subword $w^{(i)}$ then arises as the trace of the subword $v^{(i)}$ of u in the word z.

Proof. The modified statements in this lemma follow by inspecting the arguments used in the proof of the previous lemma. □

7. Identities in the varieties $\mathcal{U}_{2k}$

Let again, as before, X be an infinite set of variables, and let $k > 2$ be an arbitrary integer.

Let $u \in X^*$ be any non-empty word. Consider now the collection of all maximal subwords of u having the k-factorization property, each considered together with its position in u. It may happen, however, that there are some maximal subwords v and w of u having the k-factorization property such that, in the positions of these words in u, the word v is a subword of the word w, but distinct from w. Then, of course, we necessarily have $\mathrm{c}(v) \subsetneqq \mathrm{c}(w)$. Such maximal subwords v of u which are covered in the way just described with some other maximal subword w of u possessing the k-factorization property will next be deleted from our initial collection of all maximal subwords of u having the k-factorization property. The maximal subwords of u having the k-factorization property that remain in our collection after this deletion will then be called the *truly maximal subwords of u having the k-factorization property.*

In the given word $u \in X^*$, two distinct truly maximal subwords v and w of u having the k-factorization property may still overlap. However, then we have both $\mathrm{c}(v) \nsubseteq \mathrm{c}(w)$ and $\mathrm{c}(w) \nsubseteq \mathrm{c}(v)$, since otherwise we would have a contradiction with the maximality of either the subword w or the subword v. Therefore, everything that has already been said in the previous section about two overlapping maximal subwords of u having the k-factorization property remains true. Thus recall the notes appearing in the text following Lemma 6.1 once more in this place.

Consider again our collection of all truly maximal subwords of u with the k-factorization property. Some of these subwords may overlap in the way described in the text from the previous section that has just been remembered. However, since $k > 2$, each of these overlaps may affect only two consecutive subwords. Now, having this in mind, we may consider the maximal chains of consecutively overlapping truly maximal subwords of u having the k-factorization property. We will call them briefly the *maximal k-chains in u*. That is, every two adjacent subwords in any such maximal k-chain in u overlap in the way described in the previous section (including the possibility that the overlap is empty and that the two adjacent subwords touch each other). Notice that, according to the notes in the previous section, the subwords of every such maximal k-chain in u form a properly intersecting k-sequence in u, which notion has also been introduced in that section. The segment of the word u composed of the subwords of u which form any such maximal k-chain in u will then be called a *maximal genuine k-nest in u*. There may be several such maximal genuine k-nests in u, but these nests already do not overlap and they even do not touch each other, but they are separated from each other with some other non-empty segments of u. It means that the structure of the word u from this point of view can, in general, be described as follows:

$$u = \varsigma_0 \tilde{v}^{[1]} \varsigma_1 \tilde{v}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{v}^{[m]} \varsigma_m,$$

where m is a non-negative integer, $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ are all maximal genuine k-nests in u and $\varsigma_0, \varsigma_1, \dots, \varsigma_m \in X^*$ are segments of u and the segments $\varsigma_1, \varsigma_2, \dots, \varsigma_{m-1}$ are non-empty. Let $\tilde{v}$ be any of the maximal genuine k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$. Then, for some positive integer ℓ, which may vary for different k-nests, $\tilde{v}$ is composed of ℓ consecutively overlapping truly maximal subwords $v^{(1)}, \dots, v^{(\ell)}$ of u having the k-factorization property. It means that, for every $i \in \{1, \dots, \ell\}$, there exist words $v_1^{(i)}, \dots, v_k^{(i)} \in X^*$ such that $v^{(i)} = v_1^{(i)} \cdots v_k^{(i)}$ and $\mathrm{c}(v_1^{(i)}) = \cdots = \mathrm{c}(v_k^{(i)}) = \mathrm{c}(v^{(i)})$. Furthermore, there exist some words $r^{(2)}, \dots, r^{(\ell)} \in X^*$ such that, for every $i \in \{2, \dots, \ell\}$, $r^{(i)}$ is a proper final segment of $v_k^{(i-1)}$ and a proper initial segment of $v_1^{(i)}$. Thus, for every $i \in \{2, \dots, \ell\}$ again, there exist non-empty words $\bar{v}_k^{(i-1)}, \bar{v}_1^{(i)} \in X^*$ for which $v_k^{(i-1)} = \bar{v}_k^{(i-1)} r^{(i)}$ and $v_1^{(i)} = r^{(i)} \bar{v}_1^{(i)}$. Then we have

$$\tilde{v} = v_1^{(1)} v_2^{(1)} \cdots v_{k-1}^{(1)} \bar{v}_k^{(1)} r^{(2)} \bar{v}_1^{(2)} v_2^{(2)} \cdots v_{k-1}^{(2)} \bar{v}_k^{(2)} r^{(3)} \cdots \\ \cdots r^{(\ell-1)} \bar{v}_1^{(\ell-1)} v_2^{(\ell-1)} \cdots v_{k-1}^{(\ell-1)} \bar{v}_k^{(\ell-1)} r^{(\ell)} \bar{v}_1^{(\ell)} v_2^{(\ell)} \cdots v_{k-1}^{(\ell)} v_k^{(\ell)}.$$

We will again have to consider a somewhat more general situation than is the one just described. Namely, we will weaken some of the assumptions made above. In particular, assume further that $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ are not necessarily all maximal genuine k-nests in u, but suppose that they are only some raw k-nests in u, which notion has also been introduced in the previous section. That is, $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ are now only some segments of u composed of the subwords of some properly intersecting k-sequences in u. Suppose again that these segments are separated from each other with some other non-empty segments of u, as above. Then, if we remember the definition of the properly intersecting k-sequences in u and the corresponding raw k-nests in u given in the previous section, we see that the structure of the word u can now be described in the same way as above, weakening accordingly the conditions imposed on the above words $v_1^{(i)}, v_2^{(i)}, \dots, v_k^{(i)}$, for all $i \in \{1, \dots, \ell\}$, just as in the description of a raw k-nest $\tilde{v}$ in u given in the previous section.

Assume next that the word $u \in X^*$ has the structure described above with the mentioned weaker assumptions on the segments $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$. That is, these segments are only some raw k-nests in u. In this situation, let us replace in the word u one of its maximal subwords s having the k-factorization property with another word t having the k-factorization property such that $\mathrm{c}(s) = \mathrm{c}(t)$. We thus obtain from u a word z in which t is a maximal subword having the k-factorization property. Suppose, in addition, that the word s in its position in u was a subword of one of the maximal subwords constituting some of the raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ of u. Then, according to Lemma 6.2, after replacing s with t in u, the raw k-nest of u comprising the maximal subword containing s as a subword will transform to a raw k-nest of z comprising a maximal subword containing t as a subword and having a very similar structure. More precisely, if we denote, for the sake of simplicity, by $\tilde{v}$ the raw k-nest of u containing the maximal subword having s for its subword, and if we denote by $\tilde{\omega}$ the new raw k-nest of z arising from $\tilde{v}$ by replacing s with t in it, and if $\tilde{v}$ is composed of the subwords of the properly intersecting k-sequence $v^{(1)}, \dots, v^{(\ell)}$, then $\tilde{\omega}$ will be composed of the subwords of the properly intersecting k-sequence $\omega^{(1)}, \dots, \omega^{(\ell)}$ where, for every $i \in \{1, \dots, \ell\}$, the subword $\omega^{(i)}$ is the trace of the subword $v^{(i)}$ of u in z. In particular, if $\vartheta \in \{1, \dots, \ell\}$ is the index for which s is a subword of $v^{(\vartheta)}$, then t is a subword of $\omega^{(\vartheta)}$. Note also that, beyond the raw k-nests $\tilde{v}$ and $\tilde{\omega}$, the word z does not differ from the word u. Thus the raw k-nests of u specified above give rise to the respective raw k-nests of z.

We may now repeat this procedure with the word z, with another maximal subword σ of z having the k-factorization property and contained in one of the maximal subwords of some of the raw k-nests of z originating from the given raw k-nests of u in the way just specified, and with another word τ having the k-factorization property such that $\mathrm{c}(\sigma) = \mathrm{c}(\tau)$.

In this manner, we may repeat this procedure several times. As the outcome of this process, we will get a sequence of words $f_0, f_1, \dots, f_h \in X^*$ such that $f_0 = u$ and, for every $i \in \{1, \dots, h\}$, the word f_i has arisen from the word f_{i-1} using some variant of the procedure described above. Notice, in this connection, that the original raw k-nests of u consecutively give rise to the respective raw k-nests of the words $f_1, \dots, f_h$, using repeatedly the traces of the maximal subwords constituting these k-nests, as discussed above. Let us now denote by η the last word f_h in this sequence. Then, of course, $u \equiv_{\mathcal{U}_k} \eta$. Furthermore, it follows that if the word u had the structure as given above, then the word η has the form

$$\eta = \varsigma_0 \tilde{w}^{[1]} \varsigma_1 \tilde{w}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{w}^{[m]} \varsigma_m,$$

where $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\dots$, $\tilde{w}^{[m]}$ are some raw k-nests in η and $\varsigma_0, \varsigma_1, \dots, \varsigma_m$ are the same words as above. Moreover, we then have $\tilde{v}^{[1]} \equiv_{\mathcal{U}_k} \tilde{w}^{[1]}$, $\tilde{v}^{[2]} \equiv_{\mathcal{U}_k} \tilde{w}^{[2]}$, $\dots$, $\tilde{v}^{[m]} \equiv_{\mathcal{U}_k} \tilde{w}^{[m]}$. We then say that *the word η has been deduced k-tamely from the word u with respect to the given collection of raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ in u*. In addition, if $\tilde{v}$ is any of the raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$, if $\tilde{w}$ is the respective raw k-nest among $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\dots$, $\tilde{w}^{[m]}$, and if $\tilde{v}$ is composed of ℓ subwords $v^{(1)}, \dots, v^{(\ell)}$ of some properly intersecting k-sequence in u, then $\tilde{w}$ is composed of ℓ subwords $w^{(1)}, \dots, w^{(\ell)}$ of a properly intersecting k-sequence in η such that $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, $\dots$, $\mathrm{c}(v^{(\ell)}) = \mathrm{c}(w^{(\ell)})$. Furthermore, for every $i \in \{1, \dots, \ell\}$, there exist non-empty words $w_1^{(i)}, \dots, w_k^{(i)} \in X^*$ such that $w^{(i)} = w_1^{(i)} \cdots w_k^{(i)}$ and such that these words satisfy all other conditions imposed formerly in order to make hereby of $w^{(1)}, \dots, w^{(\ell)}$ a properly intersecting k-sequence

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in η. Among other things, this means that, for every $i \in \{2, \ldots, \ell\}$, there exists a word $\rho^{(i)} \in X^*$ which is a proper final segment of $w_k^{(i-1)}$ and a proper initial segment of $w_1^{(i)}$. Thus, for every $i \in \{2, \ldots, \ell\}$ again, there exist non-empty words $\bar{w}_k^{(i-1)}, \bar{w}_1^{(i)} \in X^*$ such that $w_k^{(i-1)} = \bar{w}_k^{(i-1)} \rho^{(i)}$ and $w_1^{(i)} = \rho^{(i)} \bar{w}_1^{(i)}$. Then we have

$$\tilde{w} = w_1^{(1)} w_2^{(1)} \cdots w_{k-1}^{(1)} \bar{w}_k^{(1)} \rho^{(2)} \bar{w}_1^{(2)} w_2^{(2)} \cdots w_{k-1}^{(2)} \bar{w}_k^{(2)} \rho^{(3)} \cdots \\ \cdots \rho^{(\ell-1)} \bar{w}_1^{(\ell-1)} w_2^{(\ell-1)} \cdots w_{k-1}^{(\ell-1)} \bar{w}_k^{(\ell-1)} \rho^{(\ell)} \bar{w}_1^{(\ell)} w_2^{(\ell)} \cdots w_{k-1}^{(\ell)} w_k^{(\ell)}.$$

Our next objective is to verify the following fact.

Lemma 7.1. *Let $u \in X^*$ be any non-empty word. Let $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ be all maximal genuine k-nests in u, that is, let $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ be all consecutive segments of the word u composed of the subwords of u which form the maximal k-chains in u. By definition, these maximal k-chains consist of consecutively overlapping truly maximal subwords of u having the k-factorization property. Let further $\eta \in X^*$ be any word which has been deduced k-tamely from the word u with respect to the mentioned collection of all maximal genuine k-nests in u. Let $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ be the raw k-nests in η arising from the maximal genuine k-nests in u during the process of the k-tame derivation of η from u described in the previous paragraph. Then there exists no non-empty subword of the word η having the $2k$-factorization property which is not contained as a subword in one of the maximal subwords forming some of the properly intersecting k-sequences in η that constitute the mentioned raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ of η.*

Proof. By contradiction, assume that there is a non-empty maximal subword p in η having the $2k$-factorization property such that p in its position in η is not contained as a subword in any of the maximal subwords constituting the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ of η. Note that then, of course, p has also the k-factorization property and it is a maximal subword of η with this property. Consider now the collection of all maximal subwords of η that occur in the properly intersecting k-sequences in η constituting the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ of η. Delete now from this collection all subwords that, in their positions in η, appear as subwords of the subword p. Note that, by our assumption, all these deleted subwords are proper subwords of p. We next show that the subwords that remained in our collection after the mentioned deletion together with the subword p can again be naturally aligned to give rise to a new set of raw k-nests of the word η which are again separated from each other with some other non-empty segments of η. This can be realized in the following manner.

If there are some raw k-nests among $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ that have no intersection with the subword p and even do not touch this subword, then the maximal subwords of η constituting these raw k-nests are retained and these raw k-nests themselves remain unaltered. To be more concrete, let $\varkappa \geqslant 1$ be the largest integer such that the raw k-nests $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}$ have no intersection with p, do not touch p and occur to the left of p in η, and let $\lambda \leqslant m$ be the smallest integer such that the raw k-nests $\tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ have no intersection with p, do not touch p and occur to the right of p in η. Then they are the raw k-nests $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ which are retained unaltered. Note that we have $\varkappa = \lambda + 1$ if and only if there are no raw k-nests among $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\dots$, $\tilde{w}^{[m]}$ intersecting or touching the subword p. In the opposite case, we have $\varkappa \leqslant \lambda$ and hence also $\varkappa \leqslant m$ and $\lambda \geqslant 1$, Then all raw k-nests among $\tilde{w}^{[\varkappa]}$, $\dots$, $\tilde{w}^{[\lambda]}$ intersect the subword p or touch it. If there are raw k-nests among $\tilde{w}^{[\varkappa]}$, $\dots$, $\tilde{w}^{[\lambda]}$ that appear as subwords of the subword p in its position in η, then all maximal subwords of η constituting these raw k-nests are deleted, so that these raw k-nests disappear completely. In particular, this concerns the raw k-nests $\tilde{w}^{[\varkappa+1]}$, $\dots$, $\tilde{w}^{[\lambda-1]}$, provided that this last list of raw k-nests is not empty.

Then we have still to deal with the possibility that some of the raw k-nests $\tilde{w}^{[\varkappa]}$ and $\tilde{w}^{[\lambda]}$ intersects the subword p or touches this subword, but it is not a subword of p. If $\varkappa < \lambda$ and $\tilde{w}^{[\varkappa]}$ intersects p in this manner, then $\tilde{w}^{[\varkappa]}$ begins in η before p (not at the same place as p) and it ends in η within p. Likewise, if $\varkappa < \lambda$ and $\tilde{w}^{[\lambda]}$ intersects p in that manner, then $\tilde{w}^{[\lambda]}$ ends in η after p (not at the same place as p) and it begins in η within p. If $\varkappa = \lambda$ then any of these possibilities may occur. However, in this case, it may also happen that the whole word p appears as a subword of the raw k-nest $\tilde{w}^{[\varkappa]} = \tilde{w}^{[\lambda]}$ and that this raw k-nest overreaches p both at its beginning and at its end.

Thus assume, for instance, that $\tilde{w}^{[\varkappa]}$ begins in η earlier than p and that it intersects or touches p. Write, for the sake of simplicity, only $\tilde{w}$ instead of $\tilde{w}^{[\varkappa]}$. Then let $w^{(1)}, \dots, w^{(\ell)}$ be the maximal subwords of the properly intersecting k-sequence in η which gives rise to the raw k-nest $\tilde{w}$. Let $\vartheta \in \{1, \dots, \ell\}$ be the smallest index for which the subword $w^{(\vartheta)}$ intersects or touches the subword p. Then $w^{(\vartheta)}$ is not a subword of p, so that $w^{(\vartheta)}$ either overlaps or touches p. Moreover, we then have $\mathrm{c}(w^{(\vartheta)}) \nsubseteq \mathrm{c}(p)$. We now show that either $\vartheta = \ell$ or the subword $w^{(\vartheta+1)}$ is already a subword of p. Thus assume that $\vartheta < \ell$. If p has at its beginning some common non-empty segment with $w^{(\vartheta)}$ lying before the overlap of $w^{(\vartheta)}$ with $w^{(\vartheta+1)}$, then $\mathrm{c}(p) \nsubseteq \mathrm{c}(w^{(\vartheta+1)})$. Since p is not a subword of $w^{(\vartheta)}$, it must then go through the whole overlap of $w^{(\vartheta)}$ with $w^{(\vartheta+1)}$ and then continue at least a bit further in $w^{(\vartheta+1)}$ beyond $w^{(\vartheta)}$. Hence we get that also $\mathrm{c}(p) \nsubseteq \mathrm{c}(w^{(\vartheta)})$. In this situation, since the word p has the k-factorization property and $k > 2$, it follows that p must contain the whole word $w^{(\vartheta+1)}$ as its subword and that, in fact, it must then continue yet somewhat farther to the right. On the other hand, if p begins within the overlap of $w^{(\vartheta)}$ with $w^{(\vartheta+1)}$, then p must contain the whole final segment of $w^{(\vartheta+1)}$ starting at the same place as p, and then p must again continue yet somewhat farther to the right, since it is not a subword of $w^{(\vartheta+1)}$. Hence we again get that $\mathrm{c}(p) \nsubseteq \mathrm{c}(w^{(\vartheta)})$ and $\mathrm{c}(p) \nsubseteq \mathrm{c}(w^{(\vartheta+1)})$. However, since the word $w^{(\vartheta+1)}$ has the weak k-factorization property, it occurs as a member of a properly intersecting k-sequence in η and $k > 2$, it hence follows that $\mathrm{c}(w^{(\vartheta+1)}) \subseteq \mathrm{c}(p)$. Thus, by its maximality, the word p must, in fact, contain the whole word $w^{(\vartheta+1)}$ as its subword, as before. Moreover, since the word p has the k-factorization property, $\mathrm{c}(p) \nsubseteq \mathrm{c}(w^{(\vartheta)})$, $\mathrm{c}(w^{(\vartheta)}) \nsubseteq \mathrm{c}(p)$ and $\mathrm{c}(w^{(\vartheta+1)}) \subseteq \mathrm{c}(p)$, it turns out that the subwords $w^{(1)}, \dots, w^{(\vartheta)}, p$ form a properly intersecting k-sequence in η, having in view that the subwords $w^{(1)}, \dots, w^{(\vartheta)}, w^{(\vartheta+1)}, \dots, w^{(\ell)}$ formed a properly intersecting k-sequence in η. Remember that this conclusion follows in this way provided that $\vartheta < \ell$. If $\vartheta = \ell$ then, instead of the last inclusion involving $\mathrm{c}(w^{(\vartheta+1)})$ which has been quoted above, the fact that the word $w^{(\vartheta)} = w^{(\ell)}$ now has the left weak k-factorization property

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has to be invoked here. In addition, the k-sequence $w^{(1)}, \ldots, w^{(\vartheta)}, p$ is enhanced at the subword p, in the sense introduced in the previous section.

On the other side, if $\tilde{w}^{[\lambda]}$ ends in η farther than p and if it intersects or touches p, we may carry out analogous considerations also with this raw k-nest. Again, for the sake of simplicity, we write $\tilde{\tilde{w}}$ instead of $\tilde{w}^{[\lambda]}$ and we let $\hat{w}^{(1)}, \ldots, \hat{w}^{(\ell')}$ be the maximal subwords of the properly intersecting k-sequence in η which gives rise to the raw k-nest $\tilde{\tilde{w}}$. Let $\imath \in \{1, \ldots, \ell'\}$ be the largest index for which the subword $\hat{w}^{(\imath)}$ intersects or touches the subword p. Then $\hat{w}^{(\imath)}$ is not a subword of p and $\mathrm{c}(\hat{w}^{(\imath)}) \nsubseteq \mathrm{c}(p)$. However, analogously as above it can be shown that then either $\imath = 1$ or the subword $\hat{w}^{(\imath-1)}$ is already a subword of p. Anyway, as before, it again turns out that, this time, the subwords $p, \hat{w}^{(\imath)}, \ldots, \hat{w}^{(\ell')}$ form a properly intersecting k-sequence in η. In addition, this k-sequence is again enhanced at the subword p, in the sense introduced in the previous section.

Now we are in a position to draw appropriate conclusions from the considerations carried out in the previous three paragraphs. If $\varkappa < \lambda$, if $\tilde{w}^{[\varkappa]}$ begins in η earlier than p and if it intersects or touches p, and if, coincidentally, $\tilde{w}^{[\lambda]}$ ends in η farther than p and if it intersects or touches p, then, returning to our previous notation for the maximal subwords of η constituting the raw k-nests $\tilde{w}^{[\varkappa]}$ and $\tilde{w}^{[\lambda]}$, we see that the subwords $w^{(1)}, \ldots, w^{(\vartheta)}, p, \hat{w}^{(\imath)}, \ldots, \hat{w}^{(\ell')}$ altogether form a properly intersecting k-sequence in η, which is enhanced at the subword p. And this whole k-sequence then gives rise to a new raw k-nest in η, which we will denote shortly by $\tilde{p}$. If we add this new raw k-nest to the previous raw k-nests that we have retained above, we obtain the new set $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}, \tilde{p}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ of raw k-nests in η, which we have promised to provide in the first paragraph of this proof. If $\varkappa \leqslant \lambda$, if $\tilde{w}^{[\varkappa]}$ begins in η earlier than p and if it intersects or touches p, but if $\tilde{w}^{[\lambda]}$ does not end farther than p in η, then the new raw k-nest $\tilde{p}$ arises merely from the k-sequence $w^{(1)}, \ldots, w^{(\vartheta)}, p$. Similarly, if $\varkappa \leqslant \lambda$, if $\tilde{w}^{[\lambda]}$ ends in η farther than p and if it intersects or touches p, but if $\tilde{w}^{[\varkappa]}$ does not begin earlier than p in η, then the new raw k-nest $\tilde{p}$ arises merely from the k-sequence $p, \hat{w}^{(\imath)}, \ldots, \hat{w}^{(\ell')}$. If $\varkappa \leqslant \lambda$ and if neither $\tilde{w}^{[\varkappa]}$ begins earlier than p in η, nor $\tilde{w}^{[\lambda]}$ ends in η farther than p, that is, if both raw k-nests $\tilde{w}^{[\varkappa]}$ and $\tilde{w}^{[\lambda]}$ are subwords of p, then the new raw k-nest $\tilde{p}$ consists merely of the sole word p. Similarly, if $\varkappa = \lambda + 1$ then, as mentioned above, there is no raw k-nest among $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$ intersecting or touching p, so that the new raw k-nest $\tilde{p}$ consists again only of the sole word p. In any of the cases just mentioned, however, the new raw k-nest $\tilde{p}$ is again enhanced at the subword p, and as before, the new set $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}, \tilde{p}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ of raw k-nests in η arises, which we have promised to provide at the beginning of this proof. And, at last, if $\varkappa = \lambda$ and if the word p appears as a subword of the raw k-nest $\tilde{w}^{[\varkappa]} = \tilde{w}^{[\lambda]}$ and if this raw k-nest overreaches p in η both on the left and on the right, then the new raw k-nest $\tilde{p}$ arises from the whole properly intersecting k-sequence in η given at the beginning of this paragraph which, in this case, can be written in the simplified form $w^{(1)}, \ldots, w^{(\vartheta)}, p, w^{(\imath)}, \ldots, w^{(\ell)}$, since $\tilde{w}^{[\varkappa]} = \tilde{w}^{[\lambda]}$, and so both these raw k-nests are composed of the subwords $w^{(1)}, \ldots, w^{(\ell)}$ of the same properly intersecting k-sequence in η. Note also that now $\vartheta + 1 < \imath$, or, equivalently, $\vartheta < \imath - 1$, since we have seen above that,

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in this situation, the maximal subwords $w^{(\vartheta+1)}$ and $w^{(\imath-1)}$ are subwords of p. Notice also that, in this case, as well, the mentioned properly intersecting k-sequence in η producing the new raw k-nest $\tilde{p}$ is enhanced at the subword p in the sense explained previously. The raw k-nests $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}, \tilde{p}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ then again form the new set of raw k-nests in η whose existence has been advised in the first paragraph of this proof.

Thus, altogether, we have to deal with the two collections of raw k-nests

$$\tilde{w}^{[1]}, \tilde{w}^{[2]}, \ldots, \tilde{w}^{[m]} \quad \text{and} \quad \tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}, \tilde{p}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$$

in η. In the second collection, the raw k-nest $\tilde{p}$ comes from a properly intersecting k-sequence in η containing the subword p and being enhanced at this subword. Moreover, if $\varkappa \leqslant \lambda$ then the just mentioned k-sequence in η may contain also several subwords from the beginning of the k-sequence giving rise to the raw k-nest $\tilde{w}^{[\varkappa]}$ and, possibly, also several subwords from the end of the k-sequence giving rise to the raw k-nest $\tilde{w}^{[\lambda]}$. In this situation, we may carry out a procedure with the word η which is similar to the one we have performed formerly with the initial word u. Thus choose in η one of its maximal subwords s' having the k-factorization property such that s' in its position in η occurs as a subword in one of the maximal subwords constituting some of the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}, \ldots, \tilde{w}^{[m]}$. Note that if this maximal subword of η containing s' as its own subword appears itself coincidentally as a subword of the word p, then, by our assumption on p, this maximal subword must be a proper subword of p, and so the more the subword s' in its position in η is a proper subword of p. Then let us replace in the word η the subword s' with another word t' having the k-factorization property such that $\mathrm{c}(s') = \mathrm{c}(t')$. We thus obtain from η a word μ in which t' in its given position is a maximal subword having the k-factorization property. We have seen in Lemma 6.2 that, after replacing s' with t' in η, the raw k-nest $\tilde{w}$ of η occurring among the k-nests $\tilde{w}^{[1]}, \tilde{w}^{[2]}, \ldots, \tilde{w}^{[m]}$ which comprises the maximal subword containing s' as a subword will transform to a raw k-nest $\tilde{\varpi}$ of μ comprising a maximal subword containing t' as a subword and consisting of the traces in μ of the maximal subwords of η which constitute the raw k-nest $\tilde{w}$. The other raw k-nests among $\tilde{w}^{[1]}, \tilde{w}^{[2]}, \ldots, \tilde{w}^{[m]}$ remain unchanged. At the same time, however, in the same way, after replacing s' with t' in η, the raw k-nest $\tilde{r}$ of η occurring among the k-nests $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}, \tilde{p}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ which comprises the maximal subword containing s' as a subword will transform to a raw k-nest $\tilde{\rho}$ of μ comprising a maximal subword containing t' as a subword and consisting of the traces in μ of the maximal subwords of η which constitute the raw k-nest $\tilde{r}$. The other raw k-nests in the list $\tilde{w}^{[1]}, \ldots, \tilde{w}^{[\varkappa-1]}$, $\tilde{p}, \tilde{w}^{[\lambda+1]}, \ldots, \tilde{w}^{[m]}$ again remain unchanged. In addition, however, we have also seen in Lemma 6.3 that, if $\tilde{r}$ is, by accident, the raw k-nest $\tilde{p}$, which is composed of the subwords of a properly intersecting k-sequence in η containing the subword p and being enhanced at this subword, then $\tilde{\rho}$ is composed of the traces in μ of the subwords of η occurring in this k-sequence, and these traces form a properly intersecting k-sequence in μ containing the trace q of p and being again enhanced at q. To sum it up, in this way, for every admissible choice of the words s' and t', as described above, the two above-mentioned collections of raw k-nests in η give rise to two collections of raw k-nests in μ. Obviously, these new collections of raw k-nests in μ are of the form $\tilde{\varpi}^{[1]}, \tilde{\varpi}^{[2]}, \ldots, \tilde{\varpi}^{[m]}$ and

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$\tilde{\varpi}^{[1]}, \ldots, \tilde{\varpi}^{[\varkappa-1]}, \tilde{q}, \tilde{\varpi}^{[\lambda+1]}, \ldots, \tilde{\varpi}^{[m]}$, where, for every $i \in \{1, 2, \ldots, m\}$, the raw k-nest $\tilde{\varpi}^{[i]}$ is composed of the traces in μ of the maximal subwords of η which constitute the raw k-nest $\tilde{w}^{[i]}$, and the raw k-nest $\tilde{q}$ is composed of the traces in μ of the subwords of η which constitute the raw k-nest $\tilde{p}$. Yet more precisely, the raw k-nest $\tilde{p}$ is composed of the subwords of a properly intersecting k-sequence in η containing the subword p and being enhanced at this subword, and the raw k-nest $\tilde{q}$ is composed of the traces of these subwords, which form a properly intersecting k-sequence in μ containing the trace q of p and being enhanced at q.

In order to complete these considerations, it is still necessary to observe that, for every maximal subword of η that occurs in any of the properly intersecting k-sequences in η constituting the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$, the following holds. If such a maximal subword of η is a subword of the subword p in its position in η, then the trace of this maximal subword in μ is a subword of the trace q of p in its position in μ. Note that then the maximal subword of η in question must be a proper subword of p, and so, if this statement holds, its trace in μ must be a proper subword of the trace q of p in μ. In order to prove the mentioned statement, remember that if some maximal subword of η constituting one of the raw k-nests of η mentioned above and intersecting the subword p exists at all, then $\varkappa \leqslant \lambda$ and this maximal subword must occur as a constituent of one of the raw k-nests among $\tilde{w}^{[\varkappa]}$, $\ldots$, $\tilde{w}^{[\lambda]}$. Now, in addition, we have to deal with such a maximal subword of η which appears as a subword of p. Then it is fairly easy to realize that, in fact, in order to verify the statement formulated above, only the following two cases require some attention. Note that we will use here the same notations for the maximal subwords of η constituting the raw k-nests $\tilde{w}^{[\varkappa]}$ and $\tilde{w}^{[\lambda]}$ as before. Thus we have to examine the case when the raw k-nest $\tilde{w}^{[\varkappa]}$ begins in η earlier than p, $\vartheta < \ell$ where ϑ is defined in the same way as before, the maximal subword s' of η with the k-factorization property which is then replaced with the word t' is a subword of $w^{(\vartheta)}$ in its position in η, and the maximal subword of η whose trace in μ is considered is $w^{(\vartheta+1)}$. And, analogously, we have also to examine the case when the raw k-nest $\tilde{w}^{[\lambda]}$ ends in η farther than p, $\imath > 1$ where $\imath$ is defined as before, the maximal subword s' of η with the k-factorization property which is then replaced with the word t' is a subword of $\hat{w}^{(\imath)}$ in its position in η, and the maximal subword of η whose trace in μ is considered is $\hat{w}^{(\imath-1)}$. However, by the definition of the traces in μ of the maximal subwords of η constituting the given raw k-nests in η (this time it concerns the k-nests $\tilde{w}^{[\varkappa]}$ or $\tilde{w}^{[\lambda]}$ and the k-nest $\tilde{p}$), in both cases that we have pointed out the statement formulated at the beginning of this paragraph clearly holds.

At this stage, we are ready to repeat the procedure described in the previous two paragraphs with the word μ in place of η and with another maximal subword σ' of μ having the k-factorization property and contained in one of the maximal subwords constituting some of the raw k-nests $\tilde{\varpi}^{[1]}$, $\tilde{\varpi}^{[2]}$, $\ldots$, $\tilde{\varpi}^{[m]}$ of μ mentioned above, and with the raw k-nest $\tilde{q}$ of μ coming from the properly intersecting k-sequence in μ specified above which contains the trace q of p in μ and is enhanced at q, and with another word τ' having the k-factorization property such that $\mathrm{c}(\sigma') = \mathrm{c}(\tau')$. This is possible since, in the previous two paragraphs, we have used only the property that $\tilde{p}$ is a raw k-nest com-

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ing from a properly intersecting k-sequence in η which contains the subword p and is enhanced at this subword. We have also used the property that all maximal subwords of η that occur in the properly intersecting k-sequences in η constituting the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\dots$, $\tilde{w}^{[m]}$ of η and which appear as subwords of p in their positions in η are actually proper subwords of p. But we have not used the fact that the word p itself has the $2k$-factorization property in the preceding two paragraphs.

In this way, we may repeat this procedure several times. Notice again that the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\dots$, $\tilde{w}^{[m]}$ and $\tilde{p}$ of η thus consecutively give rise to the respective raw k-nests in the words arising subsequently in this process, using repeatedly the traces of the maximal subwords constituting these k-nests, as explained above in the first step of this process in the case of the words η and μ. Moreover, the k-sequences constituting the raw k-nests stemming from the raw k-nest $\tilde{p}$ of η in the mentioned subsequent words are of the same length as the k-sequence constituting the raw k-nest $\tilde{p}$ and they are enhanced at the maximal subwords occurring at the positions corresponding to the position occupied by the maximal subword p in the k-sequence constituting the raw k-nest $\tilde{p}$ of η.

Now remember that the word η has itself appeared as the last word in the process described previously that we have accomplished originally with the initial word u. (See the text preceding the lemma we are just proving.) There we have produced a sequence of words $f_0, f_1, \dots, f_h \in X^*$ such that $f_0 = u$, $f_h = \eta$, and, for every $i \in \{1, \dots, h\}$, the word f_i has arisen from the word f_{i-1} using a variant of the same procedure as is the one specified above (a maximal subword of f_{i-1} having the k-factorization property has been replaced with another word having the k-factorization property and containing the same variables, where the selected maximal subword of f_{i-1} has been contained as a subword in a member of a properly intersecting k-sequence in f_{i-1} constituting a raw k-nest of f_{i-1} which has arisen from one of the maximal genuine k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ of u in the way outlined before). But this process can be reversed, starting with the word $f_h = \eta$ and ending with the word $f_0 = u$. And it is this particular process that we will now examine. If we first take into account only the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\dots$, $\tilde{w}^{[m]}$ of η, then we can say, similarly as before, that this time, the word u with its all maximal genuine k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\dots$, $\tilde{v}^{[m]}$ has been deduced k-tamely from the word η with respect to the just mentioned collection $\tilde{w}^{[1]}, \tilde{w}^{[2]}, \dots, \tilde{w}^{[m]}$ of raw k-nests in η. As mentioned in the previous paragraph, these raw k-nests of η consecutively give rise to the respective raw k-nests in the subsequent words arising in the process we have now in mind. The last word obtained in this process is just u and the raw k-nests in u originating thus from the given raw k-nests of η are precisely all maximal genuine k-nests of u remembered above. But now, we are also concerned with the raw k-nest $\tilde{p}$ of η. This raw k-nest likewise consecutively gives rise to its respective raw k-nests in the subsequent words arising in our process. This process ends with the word u, and so let us denote by $\tilde{\pi}$ the raw k-nest in u coming thus from $\tilde{p}$. Then it follows from our foregoing considerations that this raw k-nest $\tilde{\pi}$ is composed of a properly intersecting k-sequence in u consisting of a number of maximal subwords of u which is the same as the number of maximal subwords of η which form the properly intersecting k-sequence in η constituting the raw k-nest $\tilde{p}$. This k-sequence contains the subword p and is enhanced at p. Hence, by our earlier remarks, it follows that the k-sequence constituting the k-nest $\tilde{\pi}$ is also enhanced, and this happens at the subword which has the same sequence number in that k-sequence that the subword p had in the k-sequence constituting the raw k-nest $\tilde{p}$. Denote accordingly by π this subword in the k-sequence in u constituting $\tilde{\pi}$ that corresponds in the way just described to the subword p in the k-sequence in η constituting $\tilde{p}$.

Finally, we are in a position to deduce the desired contradiction from our preceding deliberations. Again, we have to distinguish several cases. If $\varkappa = \lambda + 1$ then the situation is transparent. In this case the raw k-nest $\tilde{p}$ in η consists only of the sole word p having the $2k$-factorization property and it is inserted in a certain place among the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ of η, but it is separated from them. Observe that, in this case, the modifications of the word η in the above process of its transformation to the word u do not affect the raw k-nest $\tilde{p}$ at all, so that the corresponding raw k-nest $\tilde{\pi}$ in u is, in fact, equal to $\tilde{p}$ and hence it consists of the sole word p having the $2k$-factorization property. It is inserted in the corresponding place among the maximal genuine k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ of u and it is separated from them, just as it has been separated from the raw k-nests of η mentioned above. The word p constituting the raw k-nest $\tilde{\pi} = \tilde{p}$ then has, of course, also the k-factorization property and it is a maximal subword of u with this property which is not contained in any of the truly maximal subwords of u having the k-factorization property, since all these subwords constitute just the maximal genuine k-nests of u mentioned above. But this contradicts the way how the collection of all truly maximal subwords of u with the k-factorization property has been obtained. Thus assume further that $\varkappa \leqslant \lambda$. In this case, let $v^{(1)}, \ldots, v^{(\ell)}$ be the sequence of truly maximal subwords of u having the k-factorization property which constitute the maximal genuine k-nest $\tilde{v}^{[\varkappa]}$ of u and let $\hat{v}^{(1)}, \ldots, \hat{v}^{(\ell')}$ be the sequence of truly maximal subwords of u having the k-factorization property which constitute the maximal genuine k-nest $\tilde{v}^{[\lambda]}$ of u. (The first of these sequences consists of ℓ subwords and the second sequence consists of ℓ' subwords, which numbers correspond to the numbers of maximal subwords of η constituting the respective raw k-nests of η.) Next assume, for instance, that the k-nest $\tilde{w}^{[\varkappa]}$ begins in η earlier than p and that the k-nest $\tilde{w}^{[\lambda]}$ ends in η farther than p (each of these k-nests, however, must intersect or touch the subword p). We will see that then the same is true of the maximal genuine k-nests $\tilde{v}^{[\varkappa]}$ and $\tilde{v}^{[\lambda]}$ of u and its subword π. Let, in the present case, the indices ϑ and $\imath$ be defined as before. Then, again by our previous considerations, we may conclude that the properly intersecting k-sequence in u constituting the raw k-nest $\tilde{\pi}$ in u consists of the subwords $v^{(1)}, \ldots, v^{(\vartheta)}, \pi, \hat{v}^{(\imath)}, \ldots, \hat{v}^{(\ell')}$ and it is enhanced at π. Remember that it means, among other things, that the subword π has the weak $2k$-factorization property. Furthermore, the maximal genuine k-nests $\tilde{v}^{[1]}, \ldots, \tilde{v}^{[\varkappa-1]}$ precede $\tilde{\pi}$ in u and they are separated from it. Likewise the maximal genuine k-nests $\tilde{v}^{[\lambda+1]}, \ldots, \tilde{v}^{[\ell']}$ follow after $\tilde{\pi}$ and they are separated from it. Moreover, if $\varkappa < \lambda$ then the truly maximal subwords $v^{(\vartheta+1)}, \ldots, v^{(\ell)}$ and $\hat{v}^{(1)}, \ldots, \hat{v}^{(\imath-1)}$ of u are proper subwords of π, since the respective maximal subwords of η were proper subwords of p. By the same argument, the whole maximal genuine k-nests $\tilde{v}^{[\varkappa+1]}, \ldots, \tilde{v}^{[\lambda-1]}$ of u are subwords of π provided that this list of k-nests is not empty, and the truly maximal subwords of u forming these k-nests are proper subwords of π. Hence it turns out that

π is not contained in any of the truly maximal subwords of u having the k-factorization property. But the word π itself has the weak $2k$-factorization property, and as $k > 2$, it therefore has also the k-factorization property and it is a maximal subword of u with this property. Clearly, this again contradicts the way how the collection of all truly maximal subwords of u with the k-factorization property has been obtained. On the other hand, if $\varkappa = \lambda$ then $\tilde{v}^{[\varkappa]} = \tilde{v}^{[\lambda]}$ and so these identical maximal genuine k-nests of u are composed of the same truly maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ of u having the k-factorization property. Assuming now, in the continuity with the preceding considerations, that the word p appears as a subword of the raw k-nest $\tilde{w}^{[\varkappa]} = \tilde{w}^{[\lambda]}$ and that this raw k-nest overreaches p in η both on the left and on the right, by our previous considerations again, we may conclude that the properly intersecting k-sequence in u constituting the raw k-nest $\tilde{\pi}$ in u can now be written in the simpler adjusted form $v^{(1)}, \ldots, v^{(\vartheta)}, \pi, v^{(\imath)}, \ldots, v^{(\ell)}$ where $\vartheta + 1 < \imath$ (this k-sequence is again enhanced at π) and that the truly maximal subwords $v^{(\vartheta+1)}, \ldots, v^{(\imath-1)}$ of u are proper subwords of π. Hence it again turns out that π is not contained in any of the truly maximal subwords of u having the k-factorization property. Once again, since the word π itself has the weak $2k$-factorization property, and as $k > 2$, it therefore has also the k-factorization property, and as π is a maximal subword of u with this property, as before, this contradicts the way how the collection of all truly maximal subwords of u with the k-factorization property has been obtained. The remaining cases, when $\varkappa \leqslant \lambda$ and either the raw k-nest $\tilde{w}^{[\lambda]}$ does not end farther then p in η or the raw k-nest $\tilde{w}^{[\varkappa]}$ does not begin earlier than p in η, and yet the case when $\varkappa \leqslant \lambda$ and both raw k-nests $\tilde{w}^{[\varkappa]}$ and $\tilde{w}^{[\lambda]}$ are subwords of p, are treated in a similar manner, coming thus to a contradiction in the same way as above. This contradiction confirms that there exists no non-empty subword of η having the $2k$-factorization property which is not contained as a subword in one of the maximal subwords constituting some of the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ of η, as claimed initially. □

We have defined previously in this section what it means that some word η has been deduced k-tamely from a word u with respect to the given collection of raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}, \ldots, \tilde{v}^{[m]}$ in u. We will next need a slight modification of this notion which we are now about introducing. Remember that, in the situation we had in mind then, the structure of the word u could be described as follows:

$$u = \varsigma_0 \tilde{v}^{[1]} \varsigma_1 \tilde{v}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{v}^{[m]} \varsigma_m,$$

where m is a non-negative integer, $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ are the selected raw k-nests in u and $\varsigma_0, \varsigma_1, \ldots, \varsigma_m \in X^*$ are segments of u and the segments $\varsigma_1, \varsigma_2, \ldots, \varsigma_{m-1}$ are non-empty. As distinct from our previous approach, in this situation, let us now replace in the word u one of its maximal subwords s having the $2k$-factorization property with another word t having the $2k$-factorization property such that $\mathrm{c}(s) = \mathrm{c}(t)$. We thus obtain from u a word z in which t is a maximal subword having the $2k$-factorization property. As before, suppose, in addition, that the word s in its position in u was a subword of one of the maximal subwords in a k-sequence giving rise to some of the raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ of u. Then, as before, on the basis of Lemma 6.2 again, after replacing s

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with t in u, the raw k-nest of u comprising the mentioned maximal subword containing s as a subword will transform to a raw k-nest of z stemming from a k-sequence comprising a maximal subword containing t as a subword. This new raw k-nest of z will be composed of the traces in z of the maximal subwords of u forming the k-sequence constituting the corresponding original raw k-nest of u. The other raw k-nests of u mentioned above will remain unchanged.

Again, in the same way as before, we may repeat this procedure several times. As the outcome of this process, we will get a sequence of words $f_0, f_1, \ldots, f_h \in X^*$ such that $f_0 = u$ and, for every $i \in \{1, \ldots, h\}$, the word f_i has arisen from the word f_{i-1} using some variant of the procedure just described. Notice again, in this connection, that the original raw k-nests of u consecutively give rise to the respective raw k-nests of the words $f_1, \ldots, f_h$ in the same manner as before. Let us now denote by η the last word f_h in this sequence. Then, this once, we have $u \equiv_{\mathcal{U}_{2k}} \eta$. Furthermore, it follows that if the word u had the structure as given above, then the word η has the form

$$\eta = \varsigma_0 \tilde{w}^{[1]} \varsigma_1 \tilde{w}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{w}^{[m]} \varsigma_m,$$

where $\tilde{w}^{[1]}, \tilde{w}^{[2]}, \ldots, \tilde{w}^{[m]}$ are certain raw k-nests in η and $\varsigma_0, \varsigma_1, \ldots, \varsigma_m$ are the same words as above. Moreover, this time, we have $\tilde{v}^{[1]} \equiv_{\mathcal{U}_{2k}} \tilde{w}^{[1]}$, $\tilde{v}^{[2]} \equiv_{\mathcal{U}_{2k}} \tilde{w}^{[2]}$, …, $\tilde{v}^{[m]} \equiv_{\mathcal{U}_{2k}} \tilde{w}^{[m]}$. Under these circumstances, we say that *the word η has been deduced $2k$-tamely from the word u with respect to the given collection of raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, …, $\tilde{v}^{[m]}$ in u.* Notice yet that the other properties quoted in the text preceding Lemma 7.1 remain valid without any modification.

Lemma 7.2. *Let $u \in X^*$ be any non-empty word, let $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, …, $\tilde{v}^{[m]}$ be all maximal genuine k-nests in u and let $\eta \in X^*$ be any word such that $u \equiv_{\mathcal{U}_{2k}} \eta$ holds. Then the word η can be deduced $2k$-tamely from the word u with respect to the collection $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, …, $\tilde{v}^{[m]}$ of all maximal genuine k-nests in u. More precisely, every derivation of the word η from the word u emerging from the fact that $u \equiv_{\mathcal{U}_{2k}} \eta$ is then $2k$-tame with respect to the mentioned collection of all maximal genuine k-nests in u.*

Proof. Since $u \equiv_{\mathcal{U}_{2k}} \eta$, according to Lemma 6.1 we know that there exist a non-negative integer h, words $g_0, g_1, \ldots, g_h \in X^*$ such that $u = g_0$, $\eta = g_h$, and for every $i \in \{1, \ldots, h\}$, there exist words $a_i, b_i \in X^*$ and non-empty words $c_i, d_i \in X^*$ such that $g_{i-1} = a_i c_i b_i$, $g_i = a_i d_i b_i$, and the words c_i and d_i satisfy $\mathrm{c}(c_i) = \mathrm{c}(d_i)$ and they are, respectively, maximal subwords of g_{i-1} and g_i having the $2k$-factorization property. Let us consider further any such derivation of the word η from the word u. We will next proceed by induction on the number h. If $h = 0$ then there is nothing to prove. Thus suppose that $h > 0$. We then have $u \equiv_{\mathcal{U}_{2k}} g_{h-1}$, so that, by the induction hypothesis, we may assume that the word g_{h-1} has been deduced $2k$-tamely during the above derivation from the word u with respect to the above-mentioned collection of all maximal genuine k-nests in u. Note that this means that the word g_{h-1} has thus also been deduced k-tamely from the word u with respect to this collection of all maximal genuine k-nests in u. Thus we may consider the raw k-nests $\tilde{\chi}^{[1]}$, $\tilde{\chi}^{[2]}$, …, $\tilde{\chi}^{[m]}$ in g_{h-1} which originated from the maximal genuine k-nests of u during the above derivation of the word g_{h-1} from the word $u = g_0$,

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as explained before. Recall that these raw k-nests are separated from each other in g_{h-1} just as the maximal genuine k-nests were separated in u. Now we have $g_{h-1} = a_h c_h b_h$ and $g_h = a_h d_h b_h$, where $\mathrm{c}(c_h) = \mathrm{c}(d_h)$ and the words c_h and d_h are, respectively, maximal subwords of the words g_{h-1} and $\eta = g_h$ having the $2k$-factorization property. However, by Lemma 7.1, there is no non-empty subword of g_{h-1} having the $2k$-factorization property which is not contained as a subword in one of the maximal subwords forming a properly intersecting k-sequence constituting some of the above-mentioned raw k-nests $\tilde{\chi}^{[1]}$, $\tilde{\chi}^{[2]}$, $\dots$, $\tilde{\chi}^{[m]}$ of the word g_{h-1}. This entails that the word c_h itself must be contained as a subword in one of the maximal subwords constituting one of these raw k-nests in g_{h-1}. But this clearly means that also the word η has been deduced $2k$-tamely during the above derivation from the word u with respect to the collection of all maximal genuine k-nests in u. □

The following terminological note will come in handy subsequently. It may happen that in the non-empty word $u \in X^*$ exactly one raw k-nest $\tilde{v}$ is marked out, and that, in addition, the word u is actually equal to this raw k-nest $\tilde{v}$. Now if $\eta \in X^*$ is any word which has been deduced k-tamely from the word u with respect to this sole raw k-nest $\tilde{v}$, then from our previous notes we know that the raw k-nest $\tilde{v}$ has thus been transformed into a raw k-nest $\tilde{w}$ of the word η, and it is clear that then this raw k-nest $\tilde{w}$ is, in fact, equal to the whole word η. In this situation, we will simply say that *the raw k-nest $\tilde{w}$ has been deduced k-tamely from the raw k-nest $\tilde{v}$*, that is, that the word η has been deduced k-tamely from the word u. Analogous simplified terminology will be used if $2k$-tame deductions will appear in this context instead of the k-tame ones.

Yet the following note will be needed.

Lemma 7.3. *Let $u \in X^*$ be any non-empty word and let $\tilde{v}$ be any of its maximal genuine k-nests. Let $v^{(1)}, \dots, v^{(\ell)}$ be the truly maximal subwords of u having the k-factorization property and forming the properly intersecting k-sequence in u that the maximal genuine k-nest $\tilde{v}$ comes from. Assume that the set $\mathrm{c}(\tilde{v})$ contains exactly ϵ distinct variables. Then $\ell < k^\epsilon$.*

Proof. We will verify this inequality by induction on ϵ. If $\epsilon = 1$ then, clearly, $\ell = 1$, so that $\ell < k^\epsilon$ holds since $k > 2$. Thus let $\epsilon > 1$. Suppose, by contradiction, that $\ell \geqslant k^\epsilon$. Then, for every $j \in \{1, \dots, k\}$, we may consider the truly maximal subwords $v^{((j-1)k^{\epsilon-1}+1)}, \dots, v^{(jk^{\epsilon-1})}$ of u. These subwords again form a properly intersecting k-sequence in u, and hence also in $\tilde{v}$, and we may consider the segment of $\tilde{v}$ composed of these subwords, which we will denote by $\tilde{v}^{(j)}$. Clearly, this segment, viewed separately from u, has itself the form of a maximal genuine k-nest composed of $k^{\epsilon-1}$ truly maximal subwords. Thus, if for some $j \in \{1, \dots, k\}$, we had $\mathrm{c}(\tilde{v}^{(j)}) \subsetneqq \mathrm{c}(\tilde{v})$, then we would obviously come to a contradiction with the induction hypothesis. Therefore, for every $j \in \{1, \dots, k\}$, we necessarily have $\mathrm{c}(\tilde{v}^{(j)}) = \mathrm{c}(\tilde{v})$. Note that, for $j \in \{2, \dots, k\}$, the segments $\tilde{v}^{(j-1)}$ and $\tilde{v}^{(j)}$ may overlap in the maximal genuine k-nest $\tilde{v}$, but $\tilde{v}$ is composed of properly intersecting truly maximal subwords of u having the k-factorization property and $k > 2$. Thus, from the previous conclusion it becomes obvious that then the

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whole maximal genuine k-nest $\tilde{v}$ has itself the k-factorization property. But this contradicts the assumption that $v^{(1)}, \ldots, v^{(\ell)}$ were the truly maximal subwords of u having the k-factorization property. This contradiction confirms that $\ell < k^{\epsilon}$, as claimed above. $\square$

8. Identities in the varieties $\mathcal{V}_{n,2k}(\mathcal{K})$

Let again $\mathcal{K}$ be any locally finite variety of groups, let n be any positive integer such that the identity $x^n \doteq 1$ is satisfied in $\mathcal{K}$ and let k be an arbitrary positive integer. Assume, in addition, that $k > 2$ and that $n \geqslant 2k$. Consider now the variety of monoids $\mathcal{V}_{n,2k}(\mathcal{K})$. Varieties of monoids of this kind were introduced in § 3. Recall, at this stage, that under the given assumptions, the variety of monoids $\mathcal{U}_{2k}$ introduced in § 6 is a subvariety of the variety $\mathcal{V}_{n,2k}(\mathcal{K})$. Our next considerations here will be motivated primarily by the desire to gain some pieces of information on the word problem for the free monoids in the variety $\mathcal{V}_{n,2k}(\mathcal{K})$.

Let X be any (finite or infinite) set of variables and let X^* be the free monoid on X. Remember that then the free monoid on X relative to the variety $\mathcal{V}_{n,2k}(\mathcal{K})$ can be represented in the form $X^*/{\equiv_{\mathcal{V}_{n,2k}(\mathcal{K})}}$ where $\equiv_{\mathcal{V}_{n,2k}(\mathcal{K})}$ is the congruence on X^* consisting of all pairs (u, w) of words $u, w \in X^*$ such that the identity $u \doteq w$ holds in $\mathcal{V}_{n,2k}(\mathcal{K})$. Likewise, the free monoid on X relative to the variety $\mathcal{U}_{2k}$ can be represented in the form $X^*/{\equiv_{\mathcal{U}_{2k}}}$ where $\equiv_{\mathcal{U}_{2k}}$ is the congruence on X^* consisting of all pairs of words in X^* giving rise to identities valid in $\mathcal{U}_{2k}$. It follows, under the above assumptions on k and n, that the congruence $\equiv_{\mathcal{V}_{n,2k}(\mathcal{K})}$ is a subset of the congruence $\equiv_{\mathcal{U}_{2k}}$ on X^*.

Let $u \in X^*$ be any non-empty word. Consider, as in the previous section, the collection of all truly maximal subwords of u having the k-factorization property. Consider also the maximal chains of consecutively overlapping truly maximal subwords of u with the k-factorization property, which we have called briefly the maximal k-chains in u, and the segments of the word u composed of the subwords of u forming these maximal k-chains in u, which we have called the maximal genuine k-nest in u. Remember from the previous section that then the structure of the word u has the form

$$u = \varsigma_0 \tilde{v}^{[1]} \varsigma_1 \tilde{v}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{v}^{[m]} \varsigma_m,$$

where m is a non-negative integer, $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\ldots$, $\tilde{v}^{[m]}$ are all maximal genuine k-nests in u and $\varsigma_0, \varsigma_1, \ldots, \varsigma_m \in X^*$ are segments of u and the segments $\varsigma_1, \varsigma_2, \ldots, \varsigma_{m-1}$ are non-empty.

Next let $z \in X^*$ be any non-empty word such that $u \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} z$. Then, of course, we also have $u \equiv_{\mathcal{U}_{2k}} z$. By Lemma 7.2, we know that then the word z has been deduced $2k$-tamely from the word u with respect to the collection $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\ldots$, $\tilde{v}^{[m]}$ of all maximal genuine k-nests in u. In fact, according to Lemma 7.2, the derivation of the word z from the word u stemming from the fact that $u \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} z$ holds has been of this kind. Therefore, we may consider the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$ in z which have originated from the maximal genuine k-nests in u during this derivation of the word z from u. The way how this is carried out in detail has been explained in the previous section. From our considerations in that section we thus also know that then the word z has the form

$$z = \varsigma_0 \tilde{w}^{[1]} \varsigma_1 \tilde{w}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{w}^{[m]} \varsigma_m,$$

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where m is the same integer as above, $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$ are the just mentioned raw k-nests in z and $\varsigma_0, \varsigma_1, \ldots, \varsigma_m$ are the same words as above. Furthermore, we have also seen in the previous section that then $\tilde{v}^{[1]} \equiv_{\mathcal{U}_{2k}} \tilde{w}^{[1]}$, $\tilde{v}^{[2]} \equiv_{\mathcal{U}_{2k}} \tilde{w}^{[2]}$, $\ldots$, $\tilde{v}^{[m]} \equiv_{\mathcal{U}_{2k}} \tilde{w}^{[m]}$. In addition, since, this time, $u \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} z$ holds, it hence obviously follows that, in fact, we have

$$\tilde{v}^{[1]} \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} \tilde{w}^{[1]},\ \tilde{v}^{[2]} \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} \tilde{w}^{[2]},\ \ldots,\ \tilde{v}^{[m]} \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} \tilde{w}^{[m]}.$$

Moreover, it is clear that, for every $j \in \{1, 2, \ldots, m\}$, the raw k-nest $\tilde{w}^{[j]}$ has thus been deduced $2k$-tamely from the maximal genuine k-nest $\tilde{v}^{[j]}$.

In this connection, turn back to the locally finite variety $\mathcal{K}$ of groups. Its free group on X can, of course, be represented in the form $X^*/{\equiv_{\mathcal{K}}}$ where $\equiv_{\mathcal{K}}$ is the congruence on X^* consisting of all pairs of words in X^* creating identities valid in $\mathcal{K}$. Since, clearly, this variety $\mathcal{K}$ of groups is a subvariety of the monoid variety $\mathcal{V}_{n,2k}(\mathcal{K})$, the congruence $\equiv_{\mathcal{V}_{n,2k}(\mathcal{K})}$ is a subset of the congruence $\equiv_{\mathcal{K}}$. Therefore, from the above notes we obtain that $\tilde{v}^{[1]} \equiv_{\mathcal{K}} \tilde{w}^{[1]}$, $\tilde{v}^{[2]} \equiv_{\mathcal{K}} \tilde{w}^{[2]}$, $\ldots$, $\tilde{v}^{[m]} \equiv_{\mathcal{K}} \tilde{w}^{[m]}$.

Let further $\tilde{v}$ be any of the maximal genuine k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\ldots$, $\tilde{v}^{[m]}$ of u and let $\tilde{w}$ be the respective raw k-nest among the above raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$ of z. We have seen above that then $\tilde{v} \equiv_{\mathcal{V}_{n,2k}(\mathcal{K})} \tilde{w}$ and, consequently, also $\tilde{v} \equiv_{\mathcal{K}} \tilde{w}$ holds. From our previous notes we also know that the raw k-nest $\tilde{w}$ has thus been deduced $2k$-tamely from the maximal genuine k-nest $\tilde{v}$.

Remember that, for some positive integer ℓ, the maximal genuine k-nest $\tilde{v}$ consists of ℓ consecutively overlapping truly maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ of u having the k-factorization property, which therefore form a properly intersecting k-sequence in u, and the raw k-nest $\tilde{w}$ is composed of ℓ maximal subwords $w^{(1)}, \ldots, w^{(\ell)}$ of z which form a properly intersecting k-sequence in z. We have also seen in the previous section that we have $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, $\ldots$, $\mathrm{c}(v^{(\ell)}) = \mathrm{c}(w^{(\ell)})$. We have noticed that these properties mean, among other things, that there exist some words $r^{(2)}, \ldots, r^{(\ell)} \in X^*$ such that, for every $i \in \{2, \ldots, \ell\}$, $r^{(i)}$ is the overlap of the maximal subwords $v^{(i-1)}$ and $v^{(i)}$ in u, so that $r^{(i)}$ is a proper final segment of $v^{(i-1)}$ and a proper initial segment of $v^{(i)}$ and $\mathrm{c}(r^{(i)}) \subseteq \mathrm{c}(v^{(i-1)}) \cap \mathrm{c}(v^{(i)})$. Thus we see that, for all $i \in \{1, \ldots, \ell\}$, there exist words $\bar{v}^{(i)} \in X^*$ satisfying $\mathrm{c}(\bar{v}^{(i)}) = \mathrm{c}(v^{(i)})$ such that $v^{(1)} = \bar{v}^{(1)} r^{(2)}$, $v^{(i)} = r^{(i)} \bar{v}^{(i)} r^{(i+1)}$ holds for all $i \in \{2, \ldots, \ell - 1\}$, and $v^{(\ell)} = r^{(\ell)} \bar{v}^{(\ell)}$. Then the given maximal genuine k-nest $\tilde{v}$ of u can be written in the form

$$\tilde{v} = \bar{v}^{(1)} r^{(2)} \bar{v}^{(2)} r^{(3)} \cdots r^{(\ell-1)} \bar{v}^{(\ell-1)} r^{(\ell)} \bar{v}^{(\ell)}.$$

Furthermore, the above-mentioned properties similarly mean that there exist also some words $\rho^{(2)}, \ldots, \rho^{(\ell)} \in X^*$ such that, for every $i \in \{2, \ldots, \ell\}$, $\rho^{(i)}$ is the overlap of the maximal subwords $w^{(i-1)}$ and $w^{(i)}$ in z, so that $\rho^{(i)}$ is a proper final segment of $w^{(i-1)}$ and a proper initial segment of $w^{(i)}$ and $\mathrm{c}(\rho^{(i)}) \subseteq \mathrm{c}(w^{(i-1)}) \cap \mathrm{c}(w^{(i)})$. Since the maximal subwords $w^{(1)}, \ldots, w^{(\ell)}$ of z form a properly intersecting k-sequence in z, it again follows that, for all $i \in \{1, \ldots, \ell\}$, there exist words $\bar{w}^{(i)} \in X^*$ satisfying $\mathrm{c}(\bar{w}^{(i)}) = \mathrm{c}(w^{(i)})$ such that $w^{(1)} = \bar{w}^{(1)} \rho^{(2)}$, $w^{(i)} = \rho^{(i)} \bar{w}^{(i)} \rho^{(i+1)}$ holds for all $i \in \{2, \ldots, \ell - 1\}$, and $w^{(\ell)} = \rho^{(\ell)} \bar{w}^{(\ell)}$. Then the above raw k-nest $\tilde{w}$ of z can be written in the form

$$\tilde{w} = \bar{w}^{(1)} \rho^{(2)} \bar{w}^{(2)} \rho^{(3)} \cdots \rho^{(\ell-1)} \bar{w}^{(\ell-1)} \rho^{(\ell)} \bar{w}^{(\ell)}.$$

It is this situation, which has been described in the previous several paragraphs, that the next result will apply to. However, we will have to use this result subsequently yet under somewhat weaker assumptions, which we are about describing now. Thus suppose further that $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ are not necessarily all maximal genuine k-nests in u, but that they are again, more generally, only some raw k-nests in u, that is, they are only segments of u assembled from the subwords of some properly intersecting k-sequences in u. Nevertheless, suppose that, as above, these k-nests are separated from each other with some other non-empty segments of u, so that the overall description of the structure of the word u will look the same as above in this section. Next, differently from the considerations in the previous paragraphs, consider the variety of monoids $\mathcal{V}_{n,k}(\mathcal{K})$, for the same group variety $\mathcal{K}$ as above. Let further $z \in X^*$ be any non-empty word such that $u \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} z$. Then, as before, this time we hence get $u \equiv_{\mathcal{U}_k} z$. Now suppose that the derivation of the word z from the word u assuring that $u \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} z$ holds, and hence also $u \equiv_{\mathcal{U}_k} z$ holds, has been of such a kind that the word z has thus been deduced k-tamely from the word u with respect to the given collection $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ of raw k-nests in u. Then we may again consider the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ in z originating from the mentioned raw k-nests in u during this derivation of the word z from u. Thus it follows that the overall description of the structure of the word z will again look the same as above in this section. Furthermore, since now $u \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} z$ holds, arguing as before, we may conclude that now we actually have $\tilde{v}^{[1]} \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} \tilde{w}^{[1]}$, $\tilde{v}^{[2]} \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} \tilde{w}^{[2]}$, ..., $\tilde{v}^{[m]} \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} \tilde{w}^{[m]}$. As $\mathcal{K}$ is a subvariety of $\mathcal{V}_{n,k}(\mathcal{K})$, we hence get again that $\tilde{v}^{[1]} \equiv_{\mathcal{K}} \tilde{w}^{[1]}$, $\tilde{v}^{[2]} \equiv_{\mathcal{K}} \tilde{w}^{[2]}$, ..., $\tilde{v}^{[m]} \equiv_{\mathcal{K}} \tilde{w}^{[m]}$. Let us concentrate now on any one raw k-nest $\tilde{v}$ from among the raw k-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, ..., $\tilde{v}^{[m]}$ of u and on its respective raw k-nest $\tilde{w}$ among the raw k-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, ..., $\tilde{w}^{[m]}$ of z. Then, as we have just seen we have $\tilde{v} \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} \tilde{w}$ and, consequently, also $\tilde{v} \equiv_{\mathcal{K}} \tilde{w}$. Moreover, according to the above assumption in this paragraph, the raw k-nest $\tilde{w}$ has thus been deduced k-tamely from the raw k-nest $\tilde{v}$. Besides, for some positive integer ℓ, the raw k-nest $\tilde{v}$ is composed of ℓ maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ forming a properly intersecting k-sequence in u, and the raw k-nest $\tilde{w}$ is assembled from ℓ maximal subwords $w^{(1)}, \ldots, w^{(\ell)}$ forming a properly intersecting k-sequence in z. At that, we have $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, ..., $\mathrm{c}(v^{(\ell)}) = \mathrm{c}(w^{(\ell)})$, and everything else concerning the raw k-nests $\tilde{v}$ and $\tilde{w}$, including the detailed description of their structure, looks quite the same as in the preceding paragraph.

Lemma 8.1. *Assume the situation with the non-empty words $u \in X^*$ and $z \in X^*$ described in the previous paragraph (including the quotations referring to the text preceding this paragraph). Then it is true that, for the selected raw k-nests $\tilde{v}$ of u and $\tilde{w}$ of z, and for every $i \in \{2, \ldots, \ell\}$, there exists a word $g^{(i)} \in X^*$ satisfying*

$$\mathrm{c}(g^{(i)}) \subseteq \mathrm{c}(v^{(i-1)}) \cap \mathrm{c}(v^{(i)}) = \mathrm{c}(w^{(i-1)}) \cap \mathrm{c}(w^{(i)})$$

such that

$$\bar{v}^{(1)} r^{(2)} \bar{v}^{(2)} r^{(3)} \cdots r^{(i-1)} \bar{v}^{(i-1)} r^{(i)} \equiv_{\mathcal{K}} \bar{w}^{(1)} \rho^{(2)} \bar{w}^{(2)} \rho^{(3)} \cdots \rho^{(i-1)} \bar{w}^{(i-1)} \rho^{(i)} g^{(i)}.$$

Proof. Remember once again that $\tilde{v} \equiv_{\mathcal{V}_{n,k}(\mathcal{K})} \tilde{w}$, which yields that $\tilde{v} \equiv_{\mathcal{U}_k} \tilde{w}$, and that, by the assumption made in the previous paragraph, the raw k-nest $\tilde{w}$ has thus been deduced

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k-tamely from the raw k-nest $\tilde{v}$. It means that there exist a non-negative integer h, words $\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_h \in X^*$ such that $\tilde{v} = \tilde{f}_0$, $\tilde{w} = \tilde{f}_h$, and for every $j \in \{1, \dots, h\}$, there exist words $a_j, b_j \in X^*$ and non-empty words $c_j, d_j \in X^*$ such that $\tilde{f}_{j-1} = a_j c_j b_j$, $\tilde{f}_j = a_j d_j b_j$, and the words c_j and d_j satisfy one of the following conditions:

$\{c_j, d_j\} = \{p^n, p^{2n}\}$	for some word $p \in X^*$,
$\{c_j, d_j\} = \{(pq)^n, (qp)^n\}$	for some words $p, q \in X^*$,
$\{c_j, d_j\} = \{e^n, e^{n+1}\}$	for some word $e \in X^*$ such that the identity $e \simeq 1$ is satisfied in $\mathcal{K}$,
$\{c_j, d_j\} = \{t_1 \cdots t_k, s^n t_1 \cdots t_k\}$	for some words $s \in X^*$ and $t_1, \dots, t_k \in X^*$ such that $\mathrm{c}(s) = \mathrm{c}(t_1) = \cdots = \mathrm{c}(t_k)$.

Furthermore, since, in this way, the raw k-nest $\tilde{w}$ has been deduced k-tamely from the raw k-nest $\tilde{v}$, it hence follows, as we have already observed before, that also the words $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_{h-1}$ have the form of raw k-nests composed of ℓ maximal subwords forming properly intersecting k-sequences. More precisely, for every $j \in \{1, \dots, h-1\}$, the word $\tilde{f}_j$ is composed of ℓ maximal subwords $f_j^{(1)}, \dots, f_j^{(\ell)}$ forming a properly intersecting k-sequence such that $\mathrm{c}(f_j^{(1)}) = \mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, $\dots$, $\mathrm{c}(f_j^{(\ell)}) = \mathrm{c}(v^{(\ell)}) = \mathrm{c}(w^{(\ell)})$. Thus there exist some words $\eta_j^{(2)}, \dots, \eta_j^{(\ell)} \in X^*$ such that, for every $i \in \{2, \dots, \ell\}$, $\eta_j^{(i)}$ is the overlap of the maximal subwords $f_j^{(i-1)}$ and $f_j^{(i)}$ in $\tilde{f}_j$, so that $\eta_j^{(i)}$ is a proper final segment of $f_j^{(i-1)}$ and a proper initial segment of $f_j^{(i)}$ and $\mathrm{c}(\eta_j^{(i)}) \subseteq \mathrm{c}(f_j^{(i-1)}) \cap \mathrm{c}(f_j^{(i)})$. Consequently, for all $i \in \{1, \dots, \ell\}$, there exist words $\bar{f}_j^{(i)} \in X^*$ satisfying $\mathrm{c}(\bar{f}_j^{(i)}) = \mathrm{c}(f_j^{(i)})$ such that $f_j^{(1)} = \bar{f}_j^{(1)} \eta_j^{(2)}$, $f_j^{(i)} = \eta_j^{(i)} \bar{f}_j^{(i)} \eta_j^{(i+1)}$ holds for all $i \in \{2, \dots, \ell-1\}$, and $f_j^{(\ell)} = \eta_j^{(\ell)} \bar{f}_j^{(\ell)}$. Then, for every $j \in \{1, \dots, h-1\}$, the raw k-nest $\tilde{f}_j$ can be written in the form

$$\tilde{f}_j = \bar{f}_j^{(1)} \eta_j^{(2)} \bar{f}_j^{(2)} \eta_j^{(3)} \cdots \eta_j^{(\ell-1)} \bar{f}_j^{(\ell-1)} \eta_j^{(\ell)} \bar{f}_j^{(\ell)}.$$

We can introduce this notation also for $j = 0$ and $j = h$, in which cases, however, $\tilde{f}_0 = \tilde{v}$ and $\tilde{f}_h = \tilde{w}$, so that this description of $\tilde{f}_j$ for $j = 0$ and $j = h$ becomes identical, respectively, with the description of $\tilde{v}$ and $\tilde{w}$ given above in this section. The different notations used in these descriptions translate to the present notation in the obvious way. Moreover, with this supplement, the present analysis of the derivation of the raw k-nest $\tilde{w}$ from the raw k-nest $\tilde{v}$ can be rendered complete by the following note. Since $\tilde{w}$ has been deduced k-tamely from $\tilde{v}$, for every $j \in \{1, \dots, h\}$, there exists some $i_j \in \{1, \dots, \ell\}$ such that the word c_j in $a_j c_j b_j$ appears as a subword of the maximal subword $f_{j-1}^{(i_j)}$ of $\tilde{f}_{j-1}$ and the word d_j in $a_j d_j b_j$ appears as a subword of the maximal subword $f_j^{(i_j)}$ of $\tilde{f}_j$. Yet more precisely, the above description of $\tilde{f}_j$ in terms of its maximal subwords arises from the respective description of $\tilde{f}_{j-1}$ by replacing the subword c_j in $f_{j-1}^{(i_j)}$ with the word d_j. According to Lemma 6.2, the maximal subwords appearing in the mentioned description of $\tilde{f}_j$ then arise as the traces of the respective maximal subwords occurring in the description of $\tilde{f}_{j-1}$.

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Now, for every $j \in \{0, 1, \ldots, h\}$ and for every $i \in \{2, \ldots, \ell\}$, we will show that there exists a word $g_j^{(i)} \in X^*$ satisfying

$$\mathrm{c}(g_j^{(i)}) \subseteq \mathrm{c}(f_j^{(i-1)}) \cap \mathrm{c}(f_j^{(i)}) = \mathrm{c}(v^{(i-1)}) \cap \mathrm{c}(v^{(i)}) = \mathrm{c}(w^{(i-1)}) \cap \mathrm{c}(w^{(i)})$$

such that

$$\bar{v}^{(1)} r^{(2)} \bar{v}^{(2)} r^{(3)} \cdots r^{(i-1)} \bar{v}^{(i-1)} r^{(i)} \equiv_{\mathcal{K}} \bar{f}_j^{(1)} \eta_j^{(2)} \bar{f}_j^{(2)} \eta_j^{(3)} \cdots \eta_j^{(i-1)} \bar{f}_j^{(i-1)} \eta_j^{(i)} g_j^{(i)}.$$

We will proceed by induction on j. For $j = 0$, we have $\tilde{v} = \tilde{f}_0$, so that

$$\bar{v}^{(1)} r^{(2)} \bar{v}^{(2)} r^{(3)} \cdots r^{(i-1)} \bar{v}^{(i-1)} r^{(i)} = \bar{f}_0^{(1)} \eta_0^{(2)} \bar{f}_0^{(2)} \eta_0^{(3)} \cdots \eta_0^{(i-1)} \bar{f}_0^{(i-1)} \eta_0^{(i)}$$

holds for all $i \in \{2, \ldots, \ell\}$, and so $g_0^{(i)}$ can be the empty word $\mathtt{1}$. Assume further that $j > 0$. Then, by the induction hypothesis, for every $i \in \{2, \ldots, \ell\}$, there exists a word $g_{j-1}^{(i)} \in X^*$ satisfying

$$\mathrm{c}(g_{j-1}^{(i)}) \subseteq \mathrm{c}(f_{j-1}^{(i-1)}) \cap \mathrm{c}(f_{j-1}^{(i)}) = \mathrm{c}(v^{(i-1)}) \cap \mathrm{c}(v^{(i)}) = \mathrm{c}(w^{(i-1)}) \cap \mathrm{c}(w^{(i)})$$

such that

$$\bar{v}^{(1)} r^{(2)} \bar{v}^{(2)} r^{(3)} \cdots r^{(i-1)} \bar{v}^{(i-1)} r^{(i)} \equiv_{\mathcal{K}} \bar{f}_{j-1}^{(1)} \eta_{j-1}^{(2)} \bar{f}_{j-1}^{(2)} \eta_{j-1}^{(3)} \cdots \eta_{j-1}^{(i-1)} \bar{f}_{j-1}^{(i-1)} \eta_{j-1}^{(i)} g_{j-1}^{(i)}.$$

Now, if the word c_j in $a_j c_j b_j$ appears, for the given $i \in \{2, \ldots, \ell\}$, as a subword of one of the maximal subwords $f_{j-1}^{(i+1)}, \ldots, f_{j-1}^{(\ell)}$ of $\tilde{f}_{j-1}$, then we have

$$\bar{f}_{j-1}^{(1)} \eta_{j-1}^{(2)} \bar{f}_{j-1}^{(2)} \eta_{j-1}^{(3)} \cdots \eta_{j-1}^{(i-1)} \bar{f}_{j-1}^{(i-1)} \eta_{j-1}^{(i)} = \bar{f}_j^{(1)} \eta_j^{(2)} \bar{f}_j^{(2)} \eta_j^{(3)} \cdots \eta_j^{(i-1)} \bar{f}_j^{(i-1)} \eta_j^{(i)}.$$

Thus, in this case, we can put $g_j^{(i)} = g_{j-1}^{(i)}$. If, for this $i \in \{2, \ldots, \ell\}$, the word c_j in $a_j c_j b_j$ appears as a subword of one of the maximal subwords $f_{j-1}^{(1)}, \ldots, f_{j-1}^{(i-1)}$ of $\tilde{f}_{j-1}$, then, looking over the possible forms of the pairs of words c_j, d_j whose list has been displayed above, we come to the conclusion that now we have

$$\bar{f}_{j-1}^{(1)} \eta_{j-1}^{(2)} \bar{f}_{j-1}^{(2)} \eta_{j-1}^{(3)} \cdots \eta_{j-1}^{(i-1)} \bar{f}_{j-1}^{(i-1)} \eta_{j-1}^{(i)} \equiv_{\mathcal{K}} \bar{f}_j^{(1)} \eta_j^{(2)} \bar{f}_j^{(2)} \eta_j^{(3)} \cdots \eta_j^{(i-1)} \bar{f}_j^{(i-1)} \eta_j^{(i)}.$$

So, in this case, we can again put $g_j^{(i)} = g_{j-1}^{(i)}$. Thus it remains to examine the case when, for the given $i \in \{2, \ldots, \ell\}$, the word c_j in $a_j c_j b_j$ appears as a subword of the maximal subword $f_{j-1}^{(i)}$ of $\tilde{f}_{j-1}$. Recall that then the maximal subword $f_j^{(i)}$ of $\tilde{f}_j$ arises by replacing the subword c_j in $f_{j-1}^{(i)}$ with the word d_j. We may next assume that the word c_j in its position in $\tilde{f}_{j-1}$ is not contained as a subword in the overlap $\eta_{j-1}^{(i)}$, since if this happened, then c_j would be also a subword of the maximal subword $f_{j-1}^{(i-1)}$ of $\tilde{f}_{j-1}$, which possibility has already been settled above. On the other hand, we may assume that the word c_j intersects the overlap $\eta_{j-1}^{(i)}$ or, at the least, that it touches it, since otherwise we would again have

$$\bar{f}_{j-1}^{(1)} \eta_{j-1}^{(2)} \bar{f}_{j-1}^{(2)} \eta_{j-1}^{(3)} \cdots \eta_{j-1}^{(i-1)} \bar{f}_{j-1}^{(i-1)} \eta_{j-1}^{(i)} = \bar{f}_j^{(1)} \eta_j^{(2)} \bar{f}_j^{(2)} \eta_j^{(3)} \cdots \eta_j^{(i-1)} \bar{f}_j^{(i-1)} \eta_j^{(i)},$$

which case has also been settled above. So let σ_j be the overlap of the subwords $\eta_{j-1}^{(i)}$ and c_j in $f_{j-1}^{(i)}$ (it is empty if c_j only touches $\eta_{j-1}^{(i)}$). Remember, in this connection, that the maximal subword $f_j^{(i-1)}$ arises as the trace in $\tilde{f}_j$ of the maximal subword $f_{j-1}^{(i-1)}$ of $\tilde{f}_{j-1}$. It means that $f_j^{(i-1)}$ is obtained in the following way. First one cuts off the overlap σ_j from $\eta_{j-1}^{(i)}$ and hereby from $f_{j-1}^{(i-1)}$. The rest of $f_{j-1}^{(i-1)}$ then can be viewed as a subword of $\tilde{f}_j$ neighbouring on d_j. This subword is then expanded to the right so as to become a maximal subword of $\tilde{f}_j$ whose set of variables is $\mathrm{c}(f_{j-1}^{(i-1)})$. It is in this way that the maximal subword $f_j^{(i-1)}$ of $\tilde{f}_j$ arises. However, $f_j^{(i-1)}$ must end on the right before the end of d_j, since $\mathrm{c}(c_j) \nsubseteq \mathrm{c}(f_{j-1}^{(i-1)})$ by the assumptions on the position of c_j in $f_{j-1}^{(i)}$ stated above, and so $\mathrm{c}(d_j) \nsubseteq \mathrm{c}(f_j^{(i-1)})$. This determines also the overlap $\eta_j^{(i)}$ of $f_j^{(i-1)}$ and $f_j^{(i)}$ in $\tilde{f}_j$. Thus we may further consider also the overlap τ_j of the subwords $\eta_j^{(i)}$ and d_j in $f_j^{(i)}$ (it may again be empty, which happens if d_j only touches $\eta_j^{(i)}$). Hence it becomes clear that we have

$$\bar{f}_{j-1}^{(1)}\eta_{j-1}^{(2)}\bar{f}_{j-1}^{(2)}\eta_{j-1}^{(3)}\cdots\eta_{j-1}^{(i-1)}\bar{f}_{j-1}^{(i-1)}\eta_{j-1}^{(i)} \equiv_{\mathcal{K}} \bar{f}_j^{(1)}\eta_j^{(2)}\bar{f}_j^{(2)}\eta_j^{(3)}\cdots\eta_j^{(i-1)}\bar{f}_j^{(i-1)}\eta_j^{(i)}\tau_j^{n-1}\sigma_j,$$

since τ_j^{n-1} represents the inverse of the element represented by τ_j in the free group $X^*/{\equiv_{\mathcal{K}}}$. Notice also that

$$\mathrm{c}(\tau_j^{n-1}\sigma_j) \subseteq \mathrm{c}(\eta_{j-1}^{(i)}) \cup \mathrm{c}(\eta_j^{(i)}) \subseteq \mathrm{c}(v^{(i-1)}) \cap \mathrm{c}(v^{(i)}) = \mathrm{c}(w^{(i-1)}) \cap \mathrm{c}(w^{(i)}).$$

Thus, from the above-mentioned induction hypothesis with the given $i \in \{2, \dots, \ell\}$, we can now deduce that we have

$$\bar{v}^{(1)}r^{(2)}\bar{v}^{(2)}r^{(3)}\cdots r^{(i-1)}\bar{v}^{(i-1)}r^{(i)} \equiv_{\mathcal{K}} \bar{f}_j^{(1)}\eta_j^{(2)}\bar{f}_j^{(2)}\eta_j^{(3)}\cdots\eta_j^{(i-1)}\bar{f}_j^{(i-1)}\eta_j^{(i)}\tau_j^{n-1}\sigma_j g_{j-1}^{(i)}.$$

Therefore, in this case, we can put $g_j^{(i)} = \tau_j^{n-1}\sigma_j g_{j-1}^{(i)}$. The proof by induction on j is thus complete, and for $j = h$ we obtain the desired result, since $\tilde{f}_h = \tilde{w}$ and, for every $i \in \{2, \dots, \ell\}$, we can take $g^{(i)} = g_h^{(i)}$. □

Having in view the subsequent applications of the result that we have just proved, yet its following consequence is worth writing down.

Corollary 8.2. *Assume again the situation with the words $u \in X^*$ and $z \in X^*$ and with the selected raw k-nests $\tilde{v}$ of u and $\tilde{w}$ of z described in the paragraph preceding Lemma 8.1. Suppose now, in addition, that there is some $\nu \in \{1, \dots, \ell-1\}$ such that $v^{(1)} = w^{(1)}, \dots, v^{(\nu-1)} = w^{(\nu-1)}$ and $\bar{v}^{(\nu)} = \bar{w}^{(\nu)}$. If $\nu < \ell - 1$ then, for every $i \in \{\nu+2, \dots, \ell\}$, there exists a word $g^{(i)} \in X^*$ satisfying*

$$\mathrm{c}(g^{(i)}) \subseteq \mathrm{c}(v^{(i-1)}) \cap \mathrm{c}(v^{(i)}) = \mathrm{c}(w^{(i-1)}) \cap \mathrm{c}(w^{(i)})$$

such that

$$r^{(\nu+1)}\bar{v}^{(\nu+1)}r^{(\nu+2)}\cdots r^{(i-1)}\bar{v}^{(i-1)}r^{(i)} \equiv_{\mathcal{K}} \rho^{(\nu+1)}\bar{w}^{(\nu+1)}\rho^{(\nu+2)}\cdots\rho^{(i-1)}\bar{w}^{(i-1)}\rho^{(i)}g^{(i)}.$$

If $\nu = \ell - 1$ then $r^{(\ell)}\bar{v}^{(\ell)} \equiv_{\mathcal{K}} \rho^{(\ell)}\bar{w}^{(\ell)}$, that is, $v^{(\ell)} \equiv_{\mathcal{K}} w^{(\ell)}$.

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Proof. Before verifying these statements, note that if $\nu > 1$ then the equality $v^{(\nu-1)} = w^{(\nu-1)}$ yields that $r^{(\nu)} = \rho^{(\nu)}$. Now, if $\nu < \ell - 1$, then it is enough to use cancellation in the formula displayed in Lemma 8.1, and if $\nu = \ell - 1$, then the last statement follows by cancelling in $\tilde{v} \equiv_{\mathcal{K}} \tilde{w}$. □

9. Maľcev products of varieties of groups

Let H and K be arbitrary groups. By an extension of H by K we mean any group G possessing a normal subgroup Θ isomorphic to H such that the quotient group G/Θ is isomorphic to K. Let $\mathcal{P}$ and $\mathcal{Q}$ be any varieties of groups. We denote by $\mathcal{P}\circ\mathcal{Q}$ the class of all groups that are extensions of groups from $\mathcal{P}$ by groups from $\mathcal{Q}$. This class $\mathcal{P}\circ\mathcal{Q}$ is called the Maľcev product of the variety $\mathcal{P}$ by the variety $\mathcal{Q}$. It is well known that, for group varieties $\mathcal{P}$ and $\mathcal{Q}$, the Maľcev product $\mathcal{P}\circ\mathcal{Q}$ is a variety of groups again. It is also well known that this variety $\mathcal{P}\circ\mathcal{Q}$ is generated by the class $\mathcal{P}*\mathcal{Q}$ of all semidirect products of groups from $\mathcal{P}$ by groups from $\mathcal{Q}$. Thus, a binary operation on the collection of all varieties of groups arises, assigning to any two group varieties $\mathcal{P}$ and $\mathcal{Q}$ their Maľcev product $\mathcal{P}\circ\mathcal{Q}$. This binary operation is well known to be associative. For more information on the semigroup of varieties of groups which arises in this way, see the monograph [**11**] by Hanna Neumann.

Let again X be any (finite or infinite) set of variables. Let $X^{\circledast}$ be the free group on X. Remember that it consists of all reduced words over the alphabet $X \cup X^{-1}$, including the empty word 1, where X^{-1} is a disjoint copy of X containing for every variable $x \in X$ its counterpart x^{-1} treated as its inverse. Furthermore, in every variety $\mathcal{Q}$ of groups, there is a relatively free group on X which can be represented in the form $X^{\circledast}/{\approx_{\mathcal{Q}}}$ where $\approx_{\mathcal{Q}}$ is the congruence on $X^{\circledast}$ consisting of all pairs (s,t) of words $s,t \in X^{\circledast}$ such that the identity $s \doteq t$ holds in the variety $\mathcal{Q}$.

We next remember what is the Cayley graph of the free group $X^{\circledast}/{\approx_{\mathcal{Q}}}$. Generally, it is a graph of the same kind as were the graphs introduced in § 1. Concretely, it is the graph $\Gamma = \Gamma_{\mathcal{Q},X}$ whose sets $V(\Gamma)$ of vertices and $E(\Gamma)$ of edges look as follows. The vertices in $V(\Gamma)$ are the elements of the group $X^{\circledast}/{\approx_{\mathcal{Q}}}$ itself, that is, $V(\Gamma)$ is just the set $X^{\circledast}/{\approx_{\mathcal{Q}}}$. In this connection, for every word $s \in X^{\circledast}$, we will denote briefly by $\overline{s}$ the element of the group $X^{\circledast}/{\approx_{\mathcal{Q}}}$ represented by the word s, that is, $\overline{s}$ will stand for the class of the congruence $\approx_{\mathcal{Q}}$ containing the word s. Then the set $E(\Gamma)$ consists of edges of the following kinds. For every word $s \in X^{\circledast}$ and for every variable $x \in X$, there is an edge of the form $(\overline{s}, x)$ which is directed from $\overline{s}$ to $\overline{sx}$, that is, we have $\alpha(\overline{s}, x) = \overline{s}$ and $\omega(\overline{s}, x) = \overline{sx}$. Furthermore, for every word $s \in X^{\circledast}$ and for every variable $x \in X$, there is also an edge of the form $(\overline{sx}, x^{-1})$ which is directed from $\overline{sx}$ to $\overline{sxx^{-1}} = \overline{s}$, so that $\alpha(\overline{sx}, x^{-1}) = \overline{sx}$ and $\omega(\overline{sx}, x^{-1}) = \overline{s}$. The set $E(\Gamma)$ thus can be written as the disjoint union $E(\Gamma) = E^{+}(\Gamma) \cup E^{-}(\Gamma)$ where $E^{+}(\Gamma)$ is the set of all edges of the form $(\overline{s}, x)$, for all $s \in X^{\circledast}$ and $x \in X$, and $E^{-}(\Gamma)$ is the set of all edges of the form $(\overline{sx}, x^{-1})$, again for all $s \in X^{\circledast}$ and $x \in X$. Notice also that the set $E^{-}(\Gamma)$ is, in fact, a disjoint copy of the set $E^{+}(\Gamma)$, since the edge $(\overline{sx}, x^{-1})$ of $E^{-}(\Gamma)$ can be viewed as the counterpart of the edge $(\overline{s}, x)$ of $E^{+}(\Gamma)$, for every $s \in X^{\circledast}$ and $x \in X$.

Having thus defined the Cayley graph Γ of the free group $X^{\circledast}/\approx_{\mathcal{Q}}$ in such a way that its set $E(\Gamma)$ of edges is the union of the set $E^{+}(\Gamma)$ and its disjoint copy $E^{-}(\Gamma)$, we may now adopt the following point of view to this situation. For every $s \in X^{\circledast}$ and $x \in X$, we can interpret the edge $(\overline{sx}, x^{-1})$ of $E^{-}(\Gamma)$ as the inverse $(\overline{s}, x)^{-1}$ of the edge $(\overline{s}, x)$ of $E^{+}(\Gamma)$. Having this interpretation in mind, we may consider the free group $E^{+}(\Gamma)^{\circledast}$ on the set $E^{+}(\Gamma)$, which therefore consists of all reduced words over the alphabet $E(\Gamma) = E^{+}(\Gamma) \cup E^{-}(\Gamma)$, including the empty word $\mathtt{1}$. Let now $\mathcal{P}$ be another variety of groups. Then there exists a relatively free group in this variety on the set $E^{+}(\Gamma)$ which can be represented in the form $E^{+}(\Gamma)^{\circledast}/\boldsymbol{\approx}_{\mathcal{P}}$ where, this time, $\boldsymbol{\approx}_{\mathcal{P}}$ is the congruence on $E^{+}(\Gamma)^{\circledast}$ consisting of all pairs of words in $E^{+}(\Gamma)^{\circledast}$ corresponding to identities valid in the variety $\mathcal{P}$. For later use, we will adopt a simplified notation for the elements of this free group in $\mathcal{P}$. For every word $\boldsymbol{w} \in E^{+}(\Gamma)^{\circledast}$, we will denote by $\overline{\overline{\boldsymbol{w}}}$ the element of the group $E^{+}(\Gamma)^{\circledast}/\boldsymbol{\approx}_{\mathcal{P}}$ represented by the word $\boldsymbol{w}$, that is, $\overline{\overline{\boldsymbol{w}}}$ will stand for the class of the congruence $\boldsymbol{\approx}_{\mathcal{P}}$ on $E^{+}(\Gamma)^{\circledast}$ containing the word $\boldsymbol{w}$.

The group $X^{\circledast}/\approx_{\mathcal{Q}}$ acts on itself by multiplication on the left. This gives rise to a left action of this group on its Cayley graph Γ. Since vertices of Γ are just the elements of $X^{\circledast}/\approx_{\mathcal{Q}}$, in order to specify this left action, we have only to determine it on the edges of Γ. This is done, for every $r, s \in X^{\circledast}$ and $x \in X$, by putting $\overline{r}(\overline{s}, x) = (\overline{rs}, x)$ and $\overline{r}(\overline{sx}, x^{-1}) = (\overline{rsx}, x^{-1})$. We thus obtain a left action of the group $X^{\circledast}/\approx_{\mathcal{Q}}$ on the set $E(\Gamma) = E^{+}(\Gamma) \cup E^{-}(\Gamma)$. This left action then can be uniquely extended to a left action of $X^{\circledast}/\approx_{\mathcal{Q}}$ on the free group $E^{+}(\Gamma)^{\circledast}$ described above. Note that, in this way, $X^{\circledast}/\approx_{\mathcal{Q}}$ acts on $E^{+}(\Gamma)^{\circledast}$ by automorphisms on the left. Since the congruence $\boldsymbol{\approx}_{\mathcal{P}}$ on the free group $E^{+}(\Gamma)^{\circledast}$ is fully invariant, it hence follows that it is compatible with the mentioned left action. Hence this left action induces an action of the group $X^{\circledast}/\approx_{\mathcal{Q}}$ on the group $E^{+}(\Gamma)^{\circledast}/\boldsymbol{\approx}_{\mathcal{P}}$ by automorphisms on the left. This left action is given, for every $r \in X^{\circledast}$ and $\boldsymbol{w} \in E^{+}(\Gamma)^{\circledast}$, by the formula $\overline{r}\,\overline{\overline{\boldsymbol{w}}} = \overline{\overline{\overline{r}\boldsymbol{w}}}$. Thus we may consider the semidirect product $E^{+}(\Gamma)^{\circledast}/\boldsymbol{\approx}_{\mathcal{P}} * X^{\circledast}/\approx_{\mathcal{Q}}$ determined by this left action. It is fairly well known that then the free group in the Mal′cev product $\mathcal{P}\circ\mathcal{Q}$ on the set X, which can be represented in the form $X^{\circledast}/\approx_{\mathcal{P}\circ\mathcal{Q}}$, can be embedded in the above semidirect product. The exact description of this embedding is given below. Let us now digress for a while to give the appropriate references for this result.

The embedding theorem just mentioned is due to Smel′kin. It is obtained in [**12**] as a direct consequence of a yet more general theorem of this kind which is deduced in § 2 of [**12**]. Another short proof of this more general theorem is provided in [**13**], and this theorem itself is also quoted in the monograph [**11**] by Hanna Neumann. However, this general embedding theorem is phrased in [**12**] and in [**13**] in different terms than those reviewed above. The concept of verbal product of a family of groups relative to a given variety of groups and the successive concept of verbal wreath product of groups appear in the formulation of the mentioned embedding theorem in [**12**] and in [**13**]. Nevertheless, after penetrating these concepts (the treatise of verbal products in [**11**] may be helpful), one is led to the conclusion that the aforementioned consequence of the embedding theorem of Smel′kin is identical with the description of free groups in the above Mal′cev product $\mathcal{P}\circ\mathcal{Q}$ which is stated below in terms of semidirect products of groups. (The key observation is that the verbal product of a family of free groups from a group variety $\mathcal{P}$ relative to this variety $\mathcal{P}$ is actually a free group in the variety $\mathcal{P}$ again, and the set of free generators of this group is assembled from the sets of free generators of individual groups of the given family. Besides, the two versions of the embedding theorem just mentioned are, in fact, identical up to the left–right duality, since in [**12**] and in [**13**], right actions of groups are used, while in this paper, as is the custom now, we use left actions of groups, like those named in the previous paragraph.)

Lately, the development in the theory of regular semigroups has led essentially independently to a number of other embedding theorems of a similar kind as above. In some instances, even more general theorems than the one quoted above have been obtained. See § 7 of the paper [**9**] by Jones and Trotter for further information. This once, however, these embedding theorems have been obtained in terms of semidirect products of regular semigroups and the notion of Cayley graphs of relatively free groups has often been used in this context. In particular, § 7.1 of the paper [**9**] just cited contains a result which yields the embedding theorem discussed above in the form we need here directly as a special case. This required embedding theorem is stated in the next paragraph. Note that, with some effort, this theorem can also be recognized as a quite special case of the main result obtained in the paper [**4**] by Auinger and Polák.

Thus, let us turn back to the announced embedding of the free group in $\mathcal{P}\circ\mathcal{Q}$ on X into the above semidirect product of the free group in $\mathcal{P}$ on $E^+(\Gamma)$ by the free group in $\mathcal{Q}$ on X. The promised embedding theorem can be stated as follows. Recall that, in order to specify the mentioned embedding, it suffices to determine it on the generators of the free group in $\mathcal{P}\circ\mathcal{Q}$. According to the main result in § 7.1 of [**9**], this is done in the following way. For every variable $x \in X$, the generator of $X^{\circledast}/{\approx_{\mathcal{P}\circ\mathcal{Q}}}$ represented by x is sent to the element $(\overline{\overline{(\overline{1}, x)}}, \overline{x})$ of the semidirect product $E^+(\Gamma)^{\circledast}/{\boldsymbol{\approx}_{\mathcal{P}}} * X^{\circledast}/{\approx_{\mathcal{Q}}}$. Then the embedding theorem in question asserts that this assignment extends (in a unique way) to an injective homomorphism of the group $X^{\circledast}/{\approx_{\mathcal{P}\circ\mathcal{Q}}}$ into the semidirect product $E^+(\Gamma)^{\circledast}/{\boldsymbol{\approx}_{\mathcal{P}}} * X^{\circledast}/{\approx_{\mathcal{Q}}}$. Note that then the inverse of the generator of $X^{\circledast}/{\approx_{\mathcal{P}\circ\mathcal{Q}}}$ represented by the variable x must be sent by this homomorphism to the inverse of the element $(\overline{\overline{(\overline{1}, x)}}, \overline{x})$, that is, it is sent to the element $(\overline{\overline{(\overline{1}, x^{-1})}}, \overline{x^{-1}})$ of the mentioned semidirect product.

In order to specify the mentioned embedding explicitly on arbitrary elements of the group $X^{\circledast}/{\approx_{\mathcal{P}\circ\mathcal{Q}}}$, we introduce the following concepts. We may consider paths in the Cayley graph Γ of $X^{\circledast}/{\approx_{\mathcal{Q}}}$ defined in the same way as paths in arbitrary graphs in § 1. That is, by a path in Γ we mean any finite sequence $\boldsymbol{\upsilon}$ of consecutive edges from $E(\Gamma)$. Notice that then non-empty paths in Γ are particular words in the alphabet $E(\Gamma) = E^+(\Gamma) \cup E^-(\Gamma)$, so that, after possible reduction, they form certain elements of the free group $E^+(\Gamma)^{\circledast}$. If the first edge of a non-empty path $\boldsymbol{\upsilon}$ in the mentioned Cayley graph Γ begins at the vertex $\overline{s}$ for some $s \in X^{\circledast}$ and if the last edge of this path ends at the vertex $\overline{t}$ for some $t \in X^{\circledast}$, then we say that $\boldsymbol{\upsilon}$ is a path in Γ from $\overline{s}$ to $\overline{t}$. Besides, for every vertex $\overline{r}$ from $V(\Gamma)$, where $r \in X^{\circledast}$, there is an empty path $1_{\overline{r}}$ in Γ from $\overline{r}$ to $\overline{r}$. Now, for every word $f \in X^{\circledast}$, we define the path $\boldsymbol{\pi}(f)$ in Γ from $\overline{1}$ to $\overline{f}$ in the following

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way. If f is the empty word $\mathtt{1}$, then we put $\boldsymbol{\pi}(\mathtt{1}) = \mathtt{1}_{\overline{1}}$. Note that we may identify this empty path with the identity of the free group $E^+(\Gamma)^{\circledast}$, that is, with the empty word in $E^+(\Gamma)^{\circledast}$. If $f = y_1 y_2 \cdots y_h$ where h is a positive integer and $y_1, y_2, \ldots, y_h \in X \cup X^{-1}$, then we put

$$\boldsymbol{\pi}(f) = (\overline{1}, y_1)(\overline{y_1}, y_2)(\overline{y_1 y_2}, y_3) \cdots (\overline{y_1 y_2 \cdots y_{h-1}}, y_h).$$

Now, it can be verified directly that, for every word $f \in X^{\circledast}$, the embedding of the group $X^{\circledast}/\approx_{\mathcal{P}\circ\mathcal{Q}}$ into the semidirect product $E^+(\Gamma)^{\circledast}/\approx_{\mathcal{P}} * X^{\circledast}/\approx_{\mathcal{Q}}$ which has been determined above sends the element of $X^{\circledast}/\approx_{\mathcal{P}\circ\mathcal{Q}}$ represented by the word f to the element $(\overline{\overline{\boldsymbol{\pi}(f)}}, \overline{f})$ of that semidirect product. Hence it follows that, for arbitrary words $f, g \in X^{\circledast}$, we have $f \approx_{\mathcal{P}\circ\mathcal{Q}} g$ if and only if $(\overline{\overline{\boldsymbol{\pi}(f)}}, \overline{f}) = (\overline{\overline{\boldsymbol{\pi}(g)}}, \overline{g})$. This fact can also be stated as follows. For any words $f, g \in X^{\circledast}$, we have

$$f \approx_{\mathcal{P}\circ\mathcal{Q}} g \quad \text{if and only if} \quad f \approx_{\mathcal{Q}} g \quad \text{and} \quad \boldsymbol{\pi}(f) \approx_{\mathcal{P}} \boldsymbol{\pi}(g).$$

In this way, the word problem for the free groups in the Mal′cev product $\mathcal{P}\circ\mathcal{Q}$ is reduced to the word problems for the free groups in the varieties $\mathcal{Q}$ and $\mathcal{P}$. Note that we need the solution of the word problem for the free groups in $\mathcal{Q}$ in order to construct properly the paths $\boldsymbol{\pi}(f)$ and $\boldsymbol{\pi}(g)$ in the Cayley graph Γ of the free group $X^{\circledast}/\approx_{\mathcal{Q}}$. Thus we need the solution of both word problems just mentioned in order to verify the last condition displayed above.

We will next be occupied with locally finite varieties of groups. It follows straightforwardly from the above embeddability result that if $\mathcal{P}$ and $\mathcal{Q}$ are any locally finite varieties of groups, then the Mal′cev product $\mathcal{P}\circ\mathcal{Q}$ is a locally finite variety of groups again. Besides, if this is the case, then there exist positive integers m and n such that the identity $x^m \doteq \mathtt{1}$ holds in $\mathcal{P}$ and the identity $x^n \doteq \mathtt{1}$ holds in $\mathcal{Q}$. Then it is obvious that the identity $x^{mn} \doteq \mathtt{1}$ holds in $\mathcal{P}\circ\mathcal{Q}$.

If $\mathcal{Q}$ is a locally finite variety of groups, then, as before, $\mathcal{Q}$ can be viewed as a variety of monoids. Its relatively free group on the given set X of variables then can be represented in the form $X^*/{\equiv_{\mathcal{Q}}}$ where X^* is the free monoid on X and $\equiv_{\mathcal{Q}}$ is the congruence on X^* consisting of all pairs of words from X^* which form identities valid in $\mathcal{Q}$. The construction of the Cayley graph Γ of the free group $X^*/{\equiv_{\mathcal{Q}}}$ then can be adapted as follows. Its set of vertices $V(\Gamma)$ is the set $X^*/{\equiv_{\mathcal{Q}}}$ and its set of edges $E(\Gamma)$ now consists merely of edges of the form $(\overline{\varsigma}, x)$ for arbitrary $\varsigma \in X^*$ and $x \in X$, where, this time, $\overline{\varsigma}$ stands for the class of the congruence $\equiv_{\mathcal{Q}}$ containing the word ς. As above, such an edge $(\overline{\varsigma}, x)$ is directed from $\overline{\varsigma}$ to $\overline{\varsigma x}$. One can now forget about the other kind of edges which previously formed inverses of the edges just described. Henceforth, we will treat the Cayley graph Γ of the free group $X^*/{\equiv_{\mathcal{Q}}}$ in this manner. If $\mathcal{P}$ is another locally finite variety of groups, then its relatively free group on our present set $E(\Gamma)$ can be represented in the form $E(\Gamma)^*/{\equiv_{\mathcal{P}}}$ where $E(\Gamma)^*$ is the free monoid on $E(\Gamma)$ and $\equiv_{\mathcal{P}}$ is the congruence on $E(\Gamma)^*$ consisting of all pairs of words from $E(\Gamma)^*$ corresponding to identities valid in $\mathcal{P}$. Then it is possible to conceive a left action of the group $X^*/{\equiv_{\mathcal{Q}}}$ on the group $E(\Gamma)^*/{\equiv_{\mathcal{P}}}$ which arises in an analogous way as above. Thus one also gets the semidirect product $E(\Gamma)^*/{\equiv_{\mathcal{P}}} * X^*/{\equiv_{\mathcal{Q}}}$ determined by this left action. Then the free group in the Mal′cev product $\mathcal{P}\circ\mathcal{Q}$ on X, which can now be represented in the form $X^*/{\equiv_{\mathcal{P}\circ\mathcal{Q}}}$, can be embedded in this semidirect product analogously as above. Notice that, for arbitrary words $F \in X^*$, one can construct the paths $\boldsymbol{\pi}(F)$ in the Cayley graph Γ of $X^*/\equiv_{\mathcal{Q}}$ in the same way as above. But now, in these paths, only the edges of our present set $E(\Gamma)$ may occur. Then the embedding of the group $X^*/\equiv_{\mathcal{P}\circ\mathcal{Q}}$ into the semidirect product $E(\Gamma)^*/\equiv_{\mathcal{P}} * X^*/\equiv_{\mathcal{Q}}$ can be fully specified in the same fashion as above and the description of the congruence $\equiv_{\mathcal{P}\circ\mathcal{Q}}$ on X^* in terms of the congruences $\equiv_{\mathcal{Q}}$ on X^* and $\equiv_{\mathcal{P}}$ on $E(\Gamma)^*$ is of the same form as the one displayed above.

10. Directed and undirected graphs

Remember that in § 1 we have defined a graph Γ to be a structure consisting of a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of edges together with two mappings $\alpha, \omega : E(\Gamma) \to V(\Gamma)$ assigning to every edge $e \in E(\Gamma)$ its beginning $\alpha(e)$ and its end $\omega(e)$. That is, as yet, by a graph Γ we have meant, in fact, a directed graph. Henceforth, however, we will have to deal also with undirected graphs. Such a graph Δ consists of a set $V(\Delta)$ of vertices and a set $E(\Delta)$ of edges together with a mapping $\iota : E(\Delta) \to \binom{V(\Delta)}{1} \cup \binom{V(\Delta)}{2}$ where $\binom{V(\Delta)}{1}$ and $\binom{V(\Delta)}{2}$ are, respectively, the sets of all one-element and two-element subsets of $V(\Delta)$. The mapping ι assigns to every edge $e \in E(\Delta)$ the set of endpoints of e. Every directed graph Γ can be converted into an undirected graph $\bar{\Gamma}$ by putting $V(\bar{\Gamma}) = V(\Gamma)$ and $E(\bar{\Gamma}) = E(\Gamma)$ and by introducing the corresppnding mapping ι by the formula: for every edge $e \in E(\Gamma)$, $\iota(e) = \{\alpha(e), \omega(e)\}$.

Remember next that in § 1, by a path in a directed graph Γ we have meant any finite sequence of consecutive edges from $E(\Gamma)$, that is, any sequence $p = e_1 e_2 \cdots e_m$ where m is a positive integer and $e_1, e_2, \ldots, e_m \in E(\Gamma)$ are edges such that, for every $i \in \{2, \ldots, m\}$, $\omega(e_{i-1}) = \alpha(e_i)$. If $\alpha(e_1) = v$ and $\omega(e_m) = w$, then we have said that p is a path in Γ from v to w. In addition, for every vertex $v \in V(\Gamma)$, we have added an empty path $\mathtt{1}_v$ from v to v. For certainty, such paths in a directed graph Γ are also-called the directed paths in Γ.

Analogously, in every undirected graph Δ, we may consider the undirected paths which are introduced in the standard way. Thus, by a path in an undirected graph Δ we mean any finite sequence of adjacent edges from $E(\Delta)$, that is, any sequence $q = f_1 f_2 \cdots f_m$ where m is a positive integer and $f_1, f_2, \ldots, f_m \in E(\Delta)$ are edges for which there exist vertices $v_0, v_1, \ldots, v_m \in V(\Delta)$ such that, for every $i \in \{1, \ldots, m\}$, $\iota(f_i) = \{v_{i-1}, v_i\}$. Then q is said to be an undirected path in Δ from v_0 to v_m. Again, for every vertex $v \in V(\Delta)$, we add also an empty path $\mathtt{1}_v$ from v to v.

As usual, in every undirected graph Δ, we may consider its maximal connected parts which are called the connected components of Δ. That is, we may consider the partition of the set $V(\Delta)$ into non-empty mutually disjoint classes $V_1(\Delta)$, $V_2(\Delta)$, …, $V_\nu(\Delta)$ such that, for every $j \in \{1, \ldots, \nu\}$ and for any vertices $v, w \in V_j(\Delta)$, there exists a path in Δ from v to w, and for any $j, j' \in \{1, \ldots, \nu\}$ such that $j \neq j'$ and for arbitrary vertices $v \in V_j(\Delta)$ and $w \in V_{j'}(\Delta)$, there exists no path in Δ from v to w. Then, for every edge $e \in E(\Delta)$, there exists $j \in \{1, \ldots, \nu\}$ such that $\iota(e) \subseteq V_j(\Delta)$. Thus, for every $j \in \{1, \ldots, \nu\}$, we let $E_j(\Delta)$ be the set of all edges $e \in E(\Delta)$ such that $\iota(e) \subseteq V_j(\Delta)$. Then the sets $E_1(\Delta)$, $E_2(\Delta)$, …, $E_\nu(\Delta)$ represent a partition of the set $E(\Delta)$ into mutually disjoint classes. Now, for every $j \in \{1, \ldots, \nu\}$, we may consider the undirected graph Δ_j such that $V(\Delta_j) = V_j(\Delta)$ and $E(\Delta_j) = E_j(\Delta)$, where the corresponding mapping ι on the set $E(\Delta_j)$ is inherited from the whole graph Δ. Then the graphs $\Delta_1, \Delta_2, \ldots, \Delta_\nu$ are the connected components of the graph Δ.

Let Γ be any directed graph. Let us convert it into an undirected graph $\bar{\Gamma}$ in the way described above. Let $\bar{\Gamma}_1, \bar{\Gamma}_2, \ldots, \bar{\Gamma}_\nu$ be the connected components of $\bar{\Gamma}$. Then we may transform these undirected graphs back into directed ones as follows. For every $j \in \{1, \ldots, \nu\}$, we let Γ_j be the directed graph such that $V(\Gamma_j) = V(\bar{\Gamma}_j)$ and $E(\Gamma_j) = E(\bar{\Gamma}_j)$, where the respective mappings α, ω on the set $E(\Gamma_j)$ are inherited from the whole directed graph Γ. Then we say that the directed graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_\nu$ are the connected components of Γ.

Let again Γ be a directed graph and let $\bar{\Gamma}$ be the undirected graph obtained from Γ by converting it in the same way as before. Let $q = f_1 f_2 \cdots f_m$ be any non-empty undirected path in $\bar{\Gamma}$ where m is a positive integer and $f_1, f_2, \ldots, f_m$ are edges from $E(\Gamma)$, together with the underlying sequence $v_0, v_1, \ldots, v_m$ of vertices from $V(\Gamma)$ such that, for every $i \in \{1, \ldots, m\}$, $\iota(f_i) = \{v_{i-1}, v_i\}$. Then we say that q is an undirected path in the directed graph Γ from v_0 to v_m. If, in addition, none of the edges $f_1, f_2, \ldots, f_m$ is a loop, then whenever $i \in \{1, \ldots, m\}$ is such that $\alpha(f_i) = v_{i-1}$ and $\omega(f_i) = v_i$, we say that the edge f_i is directed concordantly with the path q, while for every $i \in \{1, \ldots, m\}$ such that $\alpha(f_i) = v_i$ and $\omega(f_i) = v_{i-1}$, we say that the edge f_i is directed discordantly to the path q.

The following observation will be needed soon.

Lemma 10.1. *Let Γ be any directed graph and let Δ be a directed graph such that $V(\Delta) = V(\Gamma)$, $E(\Delta) \subseteq E(\Gamma)$ and the mappings α, ω pertinent to the graph Δ coincide on the set $E(\Delta)$ with the mappings α, ω pertinent to the graph Γ. Let $\Delta_1, \Delta_2, \ldots, \Delta_\nu$ be all pairwise distinct connected components of the graph Δ. Let $p = e_1 e_2 \cdots e_m$ be a non-empty directed path in Γ from v to w where $v \in V(\Delta_j)$ and $w \in V(\Delta_{j'})$ for some $j, j' \in \{1, \ldots, \nu\}$ such that $j \neq j'$. Then, for any integer $r > 1$, there exists an edge $f \in E(\Gamma) - E(\Delta)$ which is not a loop such that the number of occurrences of f in the path p is not divisible by r.*

Proof. Let $V = V(\Delta_j)$ for the given $j \in \{1, \ldots, \nu\}$ and let W be the union of the sets $V(\Delta_{j''})$ for all $j'' \in \{1, \ldots, j-1, j+1, \ldots, \nu\}$. Then $v \in V$ and $w \in W$. Let Λ be the directed graph such that $V(\Lambda) = V(\Gamma)$ and $E(\Lambda)$ consists of all edges $e \in E(\Gamma)$ such that either $\alpha(e), \omega(e) \in V$ or $\alpha(e), \omega(e) \in W$, where α, ω are the mappings pertinent to the graph Γ. By restricting these mappings to the just defined set $E(\Lambda)$, we obtain the corresponding mappings α, ω pertinent to the graph Λ. Note that obviously $E(\Delta) \subseteq E(\Lambda)$. Hence V forms the set of vertices of one of the connected components of the graph Λ and $v \in V$, while w belongs to the set of vertices of another connected component of Λ. Now turn to the directed path $p = e_1 e_2 \cdots e_m$ in Γ from v to w. Let $f_1, \ldots, f_\varkappa$ be all pairwise distinct edges occurring in this path p such that $\alpha(f_1) \in V$, $\omega(f_1) \in W$, $\ldots$, $\alpha(f_\varkappa) \in V$, $\omega(f_\varkappa) \in W$. Since $v \in V$ and $w \in W$, we have certainly $\varkappa > 0$. Further let $g_1, \ldots, g_\lambda$ be all pairwise distinct edges occurring in the path p (if there are

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any) such that $\alpha(g_1) \in W$, $\omega(g_1) \in V$, ..., $\alpha(g_\lambda) \in W$, $\omega(g_\lambda) \in V$. Then, of course, $f_1, \dots, f_\varkappa \in E(\Gamma) - E(\Delta)$ and $g_1, \dots, g_\lambda \in E(\Gamma) - E(\Delta)$. Let s be the total number of occurrences of the edges $f_1, \dots, f_\varkappa$ in the path p and let t be the total number of occurrences of the edges $g_1, \dots, g_\lambda$ in the path p. Since p is a directed path in Γ from v to w where $v \in V$ and $w \in W$, we clearly have $s - t = 1$. If the numbers of occurrences of individual edges from among $f_1, \dots, f_\varkappa$ in p were all divisible by r, then r would divide s. Likewise, if the numbers of occurrences of individual edges from among $g_1, \dots, g_\lambda$ in p were all divisible by r, then r would divide t. But this would contradict the equality $s - t = 1$, as $r > 1$. Thus, indeed, either there is an edge f among $f_1, \dots, f_\varkappa$ whose number of occurrences in the path p is not divisible by r, or there is an edge g among $g_1, \dots, g_\lambda$ whose number of occurrences in the path p is not divisible by r. □

11. The locality of the pseudovariety DG

In this section, we intend to prove the following fact.

Theorem 11.1. *The pseudovariety of finite monoids* **DG** *is local.*

Proof. Recall that by ℓ**DG** we denote the pseudovariety of all finite categories all of whose local monoids belong to the monoid pseudovariety **DG**. Let C be any non-empty finite category in ℓ**DG**. According to the notes at the end of § 1, in order to verify that the monoid pseudovariety **DG** is local, we need to find a finite monoid M in **DG** such that the category C divides M.

We have seen in Proposition 5.1 that there exist positive integers k and n and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq 1$ such that the category C belongs to the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$. Clearly, we may assume that $k > 2$ and that $n \geqslant 4k$. We have further seen in Proposition 5.2 that this variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ is locally finite.

Consider now C merely as a graph and take the free category C^* on C, that is, let C^* be the free category on the underlying graph of the category C. Then the free category on C relative to the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ can be represented in the form $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ where $\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}$ is the congruence on C^* consisting of all pairs (s,t) of coterminal paths in C such that $s \doteq t$ is a path identity over C which is satisfied in $\mathcal{W}_{n,k}(\mathcal{K})$. Since the category C, that is to say, the graph C is finite and the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ is locally finite, it follows that the relatively free category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ is finite. Besides, this category clearly belongs to the pseudovariety of categories ℓ**DG**. Moreover, the identity graph mapping $id_C : C \to C$ can be uniquely extended to the canonical homomorphism of categories $\varepsilon_C : C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}} \to C$. This homomorphism is a quotient homomorphism of categories.

Let p be any prime number. Consider the variety $\mathcal{A}_p$ of all abelian groups of exponent p. For every positive integer j, we will denote by $\mathcal{A}_p^j$ the Mal'cev product of j copies of the variety $\mathcal{A}_p$. Let further μ be the total number of all edges in the category C, that is, let μ be the number of elements of the set $E(C)$. Put $h = 2^\mu k^\mu$ and consider the Mal'cev product of group varieties $\mathcal{A}_p^h \circ \mathcal{K}$. Since all varieties of groups appearing in this product are locally finite, we know from our notes on the Mal'cev products of varieties of this

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kind stated in § 9 that then the variety of groups $\mathcal{A}_p^h \circ \mathcal{K}$ itself is locally finite. In addition, it satisfies the identity $x^{p^h n} \simeq 1$. Consider next the variety of monoids $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$. From Proposition 3.2 we know that then this variety of monoids is also locally finite. Let $E(C)^*$ be the free monoid on the set $E(C)$ of all edges of C. Notice, in passing, that then non-empty paths in C can be viewed as particular elements of $E(C)^*$. Furthermore, the free monoid on $E(C)$ relative to the monoid variety $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$ then can be represented in the form $E(C)^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ where $\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ is the congruence on $E(C)^*$ consisting of all pairs of words from $E(C)^*$ which constitute identities valid in $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$. Since the set $E(C)$ is finite and the variety of monoids $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$ is locally finite, it follows that the relatively free monoid $E(C)^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ is finite. Besides, this monoid clearly belongs to the pseudovariety of monoids **DG**. Now, in view of the note at the end of the previous paragraph, we will be done if we show that the category $C^*/\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}$ divides the monoid $E(C)^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$.

For this purpose, let us consider the congruence $\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ restricted to the hom-sets of the free category C^*. That is, more precisely, let us take the equivalence relation on the underlying graph of C^* consisting of all pairs (s, t) of coterminal paths in C such that $s \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} t$, in which formula, the empty paths on the vertices of C are treated as the empty word in $E(C)^*$. In this way, a congruence on C^* arises, and one can form the corresponding quotient category. We will denote this quotient category simply by $C^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$. Having in mind the fact that monoids can be viewed as categories with a single vertex, we may further consider the obvious homomorphism of free categories $\phi_C : C^* \to E(C)^*$ sending every non-empty path in C to itself and assigning the empty word in $E(C)^*$ to the empty paths at all vertices of C. Clearly, this homomorphism then induces a homomorphism of the quotient categories

$$\varphi_C : C^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \to E(C)^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} .$$

Moreover, φ_C is obviously a faithful homomorphism of categories. In order to complete our task, it therefore remains to provide a quotient homomorphism of the category $C^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ onto the category $C^*/\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}$. This homomorphism would send every vertex of C to itself. As far as edges of these categories are concerned, the natural choice would be, for every path s in C, to send the class of the restricted congruence $\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ on C^* containing s to the class of the congruence $\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}$ on C^* containing s. However, in order to make sure that this can be done so correctly, one has to prove first that, for arbitrary coterminal paths s, t in C,

$$s \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} t \quad \text{implies} \quad s \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} t. \tag{$*$}$$

This will be done in the subsequent text. Once this is confirmed, the above-mentioned assignment of classes of $\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}$ on C^* to classes of $\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ on C^* will be seen to be defined properly, and thus it will obviously determine a homomorphism of categories

$$\psi_C : C^*/\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \to C^*/\equiv_{\mathcal{W}_{n,k}(\mathcal{K})} .$$

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In addition, ψ_C will be a quotient homomorphism of categories. Thus, altogether we will eventually obtain that

$$C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}} \prec E(C)^*/{\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}},$$

as required. □

Proof of (∗). Consider arbitrary coterminal paths s, t in C such that $s \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} t$. Hence it follows that $\mathrm{c}(s) = \mathrm{c}(t)$, that is, both paths s and t contain the same edges from $E(C)$. If these paths are both empty, then they are equal since there is only one empty path on every vertex of C. Thus we may further assume that s, t are non-empty coterminal paths in C. Then s, t are words in $E(C)^*$. Since $p^h n \geqslant 4k$, from the notes at the beginning of § 6 we know that $\mathcal{U}_{4k} \subseteq \mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$, whence we get that the congruence $\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ is a subset of the congruence $\equiv_{\mathcal{U}_{4k}}$ on $E(C)^*$. Hence we obtain that $s \equiv_{\mathcal{U}_{4k}} t$. Consider now the collection of all truly maximal subwords of s having the $2k$-factorization property. We have seen in § 7 that these subwords can be grouped into maximal chains of consecutively overlapping subwords, that is, they are aligned into the so-called maximal $2k$-chains in s. As such, they form properly intersecting $2k$-sequences in s, and the segments of the word s composed of the subwords constituting separate maximal $2k$-chains of s are called the maximal genuine $2k$-nests in s. Thus let $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\ldots$, $\tilde{v}^{[m]}$ be all maximal genuine $2k$-nests in s arranged according to their occurrence in s from left to right. Then m is a non-negative integer and the word s itself is of the form

$$s = \varsigma_0 \tilde{v}^{[1]} \varsigma_1 \tilde{v}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{v}^{[m]} \varsigma_m,$$

where $\varsigma_0, \varsigma_1, \ldots, \varsigma_m \in E(C)^*$ are certain segments of s and the segments $\varsigma_1, \varsigma_2, \ldots, \varsigma_{m-1}$ are non-empty. Since, as we have seen above, $s \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} t$ entails $s \equiv_{\mathcal{U}_{4k}} t$, from Lemma 7.2 (with $2k$ in place of k) we know that the word t has thus been deduced $4k$-tamely from the word s with respect to the collection $\tilde{v}^{[1]}, \tilde{v}^{[2]}, \ldots, \tilde{v}^{[m]}$ of all maximal genuine $2k$-nests in s. Consequently, there are raw $2k$-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$ in t which originated from the maximal genuine $2k$-nests in s during this derivation of t from s. Remember briefly that these raw $2k$-nests in t are assembled from certain properly intersecting $2k$-sequences in t which came from the properly intersecting $2k$-sequences in s constituting the respective maximal genuine $2k$-nests in s, as it has been explained in § 7. The word t itself then is of the form

$$t = \varsigma_0 \tilde{w}^{[1]} \varsigma_1 \tilde{w}^{[2]} \varsigma_2 \cdots \varsigma_{m-1} \tilde{w}^{[m]} \varsigma_m,$$

where $\varsigma_0, \varsigma_1, \ldots, \varsigma_m$ are the same segments as above. Note that this entails that, for every $\epsilon \in \{1, 2, \ldots, m\}$, the $2k$-nests $\tilde{v}^{[\epsilon]}$ and $\tilde{w}^{[\epsilon]}$ are coterminal paths in C. Moreover, in the same way as in § 8, we hence get that the following relationships hold:

$$\tilde{v}^{[1]} \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tilde{w}^{[1]},\ \tilde{v}^{[2]} \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tilde{w}^{[2]},\ \ldots,\ \tilde{v}^{[m]} \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tilde{w}^{[m]}.$$

This shows that, in order to verify the implication in (∗), we have to deduce from these relationships the following ones:

$$\tilde{v}^{[1]} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}^{[1]},\ \tilde{v}^{[2]} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}^{[2]},\ \ldots,\ \tilde{v}^{[m]} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}^{[m]}.$$

This requirement can more concisely be stated as follows. Let $\tilde{v}$ be any of the maximal genuine $2k$-nests $\tilde{v}^{[1]}$, $\tilde{v}^{[2]}$, $\ldots$, $\tilde{v}^{[m]}$ of s and let $\tilde{w}$ be the respective raw $2k$-nest among the above raw $2k$-nests $\tilde{w}^{[1]}$, $\tilde{w}^{[2]}$, $\ldots$, $\tilde{w}^{[m]}$ of t. Then $\tilde{v}$ and $\tilde{w}$ are coterminal paths in C and, in order to establish the implication in $(*)$, we have to prove that

$$\tilde{v} \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tilde{w} \quad \text{implies} \quad \tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}. \tag{$**$}$$

This will be accomplished in the subsequent text. Hence, in view of the previous considerations, it will ensue that $s \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} t$, as desired. □

Proof of $(**)$. Suppose that $\tilde{v} \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tilde{w}$ holds. Just as above, this yields that the raw $2k$-nest $\tilde{w}$ has thus been deduced $4k$-tamely from the maximal genuine $2k$-nest $\tilde{v}$. Besides, since the group variety $\mathcal{A}_p^h \circ \mathcal{K}$ is a subvariety of the monoid variety $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$, the congruence $\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ is a subset of the congruence $\equiv_{\mathcal{A}_p^h \circ \mathcal{K}}$ on $E(C)^*$. Hence we obtain also that $\tilde{v} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} \tilde{w}$ holds. Next remember that, for some positive integer ℓ, the maximal genuine $2k$-nest $\tilde{v}$ is composed of ℓ truly maximal subwords $v^{(1)}, \ldots, v^{(\ell)}$ of s having the $2k$-factorization property which form a properly intersecting $2k$-sequence in s, and consequently, in view of the notes in § 7, the raw $2k$-nest $\tilde{w}$ is composed of ℓ maximal subwords $w^{(1)}, \ldots, w^{(\ell)}$ of t which form a properly intersecting $2k$-sequence in t. Recall also that then we have $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, $\ldots$, $\mathrm{c}(v^{(\ell)}) = \mathrm{c}(w^{(\ell)})$. In addition, from Lemma 7.3 we know that $\ell < 2^{\mu} k^{\mu}$, that is, we have $\ell < h$.

We will prove the implication in $(**)$ essentially by induction on ℓ. Thus assume first that $\ell = 1$. Then the maximal genuine $2k$-nest $\tilde{v}$ consists only of one truly maximal subword v of s having the $2k$-factorization property. The raw $2k$-nest $\tilde{w}$ then also consists only of one maximal subword w. This subword w alone forms a properly intersecting $2k$-sequence in t, and hence it also has the $2k$-factorization property. Since, at the same time, v and w are coterminal paths in C, so that $\alpha(v) = \alpha(w)$ and $\omega(v) = \omega(w)$, it hence follows that there is a loop u in C on the vertex $\alpha(v) = \alpha(w)$ such that $\mathrm{c}(u) = \mathrm{c}(v) = \mathrm{c}(w)$. From the path identities defining the variety of categories $\mathcal{W}_{p^h n,2k}$ which appeared in § 4 we know that then we have

$$v \equiv_{\mathcal{W}_{p^h n,2k}} u^{p^h n} v \quad \text{and} \quad w \equiv_{\mathcal{W}_{p^h n,2k}} u^{p^h n} w.$$

Once again, since the paths v and w have the $2k$-factorization property, it follows that, for every edge e in the set $\mathrm{c}(v) = \mathrm{c}(w)$, there exists a path γ_e in C such that $\alpha(\gamma_e) = \alpha(v) = \alpha(w)$, $\omega(\gamma_e) = \alpha(e)$ and $\mathrm{c}(\gamma_e) \subseteq \mathrm{c}(v) = \mathrm{c}(w)$, and there exists also a path δ_e in C such that $\alpha(\delta_e) = \omega(e)$, $\omega(\delta_e) = \alpha(v) = \alpha(w)$ and $\mathrm{c}(\delta_e) \subseteq \mathrm{c}(v) = \mathrm{c}(w)$. In addition, it is possible to choose these paths in such a way that whenever e, f are edges in the set $\mathrm{c}(v) = \mathrm{c}(w)$ such that $\alpha(e) = \alpha(f)$, then $\gamma_e = \gamma_f$, and whenever e, f are edges in $\mathrm{c}(v) = \mathrm{c}(w)$ such that $\omega(e) = \omega(f)$, then $\delta_e = \delta_f$. It is also natural to let γ_e be the empty path on $\alpha(v) = \alpha(w)$ whenever the edge e in $\mathrm{c}(v) = \mathrm{c}(w)$ is such that $\alpha(e) = \alpha(v) = \alpha(w)$, and to let δ_e be the empty path on $\alpha(v) = \alpha(w)$ whenever the edge e in $\mathrm{c}(v) = \mathrm{c}(w)$ is such that $\omega(e) = \alpha(v) = \alpha(w)$. Assume that the paths γ_e, δ_e have been so chosen for all edges e in $\mathrm{c}(v) = \mathrm{c}(w)$. Furthermore, for the same reason as above, for every edge e in $\mathrm{c}(v) = \mathrm{c}(w)$, there exists some edge f in $\mathrm{c}(v) = \mathrm{c}(w)$ such that $\omega(f) = \alpha(e)$. Then

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we denote by β_e the path δ_f. Note that, under the assumption given above, the path β_e does not depend on the choice of the edge f satisfying $\omega(f) = \alpha(e)$, for every edge e in $\mathrm{c}(v) = \mathrm{c}(w)$. Now consider the substitution σ assigning to every edge e in $\mathrm{c}(v) = \mathrm{c}(w)$ the loop $\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$ on the vertex $\alpha(v) = \alpha(w)$, and consider yet the substitution τ assigning to every edge e in $\mathrm{c}(v) = \mathrm{c}(w)$ the loop $u^{p^h n}\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$ on the same vertex $\alpha(v) = \alpha(w)$. These substitutions σ, τ can, of course, be naturally extended to arbitrary paths in C containing only edges from the set $\mathrm{c}(v) = \mathrm{c}(w)$. Besides, take any path χ in C such that $\alpha(\chi) = \alpha(v) = \alpha(w)$, $\omega(\chi) = \omega(v) = \omega(w)$ and $\mathrm{c}(\chi) \subseteq \mathrm{c}(v) = \mathrm{c}(w)$, and denote by η the path $\chi(\delta_g\chi)^{p^h n-1}$ where g is the last edge in v (the last edge in w would do the same job since $\omega(v) = \omega(w)$). Then, from the path identities which are valid in the variety of categories $\mathcal{W}_{p^h n}$ according to Corollary 4.3, we obtain that

$$u^{p^h n}v \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\sigma(v)\eta \quad \text{and} \quad u^{p^h n}w \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\sigma(w)\eta,$$

where $\sigma(v)$ and $\sigma(w)$ are loops in C on $\alpha(v) = \alpha(w)$, and $\sigma(v)\eta$ and $\sigma(w)\eta$ are paths in C from $\alpha(v) = \alpha(w)$ to $\omega(v) = \omega(w)$. Using the mentioned path identities valid in $\mathcal{W}_{p^h n}$ once again, we further obtain that

$$u^{p^h n}\sigma(v)\eta \equiv_{\mathcal{W}_{p^h n}} \tau(v)\eta \quad \text{and} \quad u^{p^h n}\sigma(w)\eta \equiv_{\mathcal{W}_{p^h n}} \tau(w)\eta,$$

where $\tau(v)$ and $\tau(w)$ are loops in C on $\alpha(v) = \alpha(w)$ composed of the loops of the form $u^{p^h n}\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$ on $\alpha(v) = \alpha(w)$ for all edges e in $\mathrm{c}(v) = \mathrm{c}(w)$. However, according to the consequences of the path identities valid in $\mathcal{W}_{p^h n}$ that we have deduced in § 4 in the text preceding Proposition 4.4, the following is true. In the quotient category $C^*/\equiv_{\mathcal{W}_{p^h n}}$, the edges that are congruence classes of the loops in C of the above form $u^{p^h n}\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$, for arbitrary edges e in $\mathrm{c}(v) = \mathrm{c}(w)$, all lie in the same maximal subgroup of the local monoid of this quotient category at the vertex $\alpha(v) = \alpha(w)$. Namely, they are all elements of the maximal subgroup of this local monoid containing the idempotent that is represented by the loop $u^{p^h n}$. Of course, the same statement is then true of the quotient category $C^*/\equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$. According to the path identities defining the variety of categories $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$ which appeared in § 5, maximal subgroups of the local monoids of the quotient category $C^*/\equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}$ all lie in the group variety $\mathcal{A}_p^h \circ \mathcal{K}$. But we have seen in the previous paragraph that $\tilde{v} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} \tilde{w}$, which means, in the present situation, that $v \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} w$. Applying the substitution τ, we hence get that both loops $\tau(v)$ and $\tau(w)$ represent the same element of the maximal subgroup of the local monoid of the last quotient category at the vertex $\alpha(v) = \alpha(w)$ containing the idempotent represented by the loop $u^{p^h n}$. However, this shows that we have

$$\tau(w) \equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tau(w)$$

and hence

$$\tau(w)\eta \equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tau(w)\eta.$$

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Summing up all formulae displayed above in this paragraph, we hence eventually deduce that

$$v \equiv_{\mathcal{W}_{p^h n,2k}(\mathcal{A}_p^h \circ \mathcal{K})} w.$$

But this clearly entails that $v \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} w$, that is, $\tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}$, as required.

Assume next that $\ell > 1$. Consider the truly maximal subword $v^{(1)}$ of s having the $2k$-factorization property which stands at the beginning of the maximal genuine $2k$-nest $\tilde{v}$ of s. Since $v^{(1)}$ is a path in C, it hence follows that there is a loop u in C on the vertex $\alpha(v^{(1)}) = \alpha(\tilde{v})$ such that $\mathrm{c}(u) = \mathrm{c}(v^{(1)})$. Then, from the path identities defining the variety of categories $\mathcal{W}_{p^h n,2k}$ which appeared in § 4, we know that we have

$$v^{(1)} \equiv_{\mathcal{W}_{p^h n,2k}} u^{p^h n} v^{(1)}.$$

For the same reasons, that is, because the subword $v^{(1)}$ has the $2k$-factorization property and it is a path in C, it turns out that, for every edge e in the set $\mathrm{c}(v^{(1)})$, there exists a path λ_e in C such that $\alpha(\lambda_e) = \omega(e)$, $\omega(\lambda_e) = \alpha(e)$ and $\mathrm{c}(\lambda_e) \subseteq \mathrm{c}(v^{(1)})$. Then $e\lambda_e$ is a loop in C on the vertex $\alpha(e)$. Note that, according to the loop identities defining the variety of categories $\mathcal{W}_{p^h n}$, the loop $(e\lambda_e)^{p^h n}$ then represents an idempotent in the local monoid of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n}}}$ at the vertex $\alpha(e)$. Now, for every edge e in the set $\mathrm{c}(v^{(1)})$, let us denote by ζ_e the path $\lambda_e(e\lambda_e)^{p^h n-1}$. Then, of course, $(e\lambda_e)^{p^h n} = e\zeta_e$, $\alpha(\zeta_e) = \omega(e)$ and $\omega(\zeta_e) = \alpha(e)$. Consider further also the maximal subword $w^{(1)}$ of t appearing at the beginning of the properly intersecting $2k$-sequence which gives rise to the raw $2k$-nest $\tilde{w}$ of t. Then $w^{(1)}$ is a path in C, $\alpha(v^{(1)}) = \alpha(\tilde{v}) = \alpha(\tilde{w}) = \alpha(w^{(1)})$ and $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. Let $w^{(1)} = f_1 f_2 \cdots f_\varkappa$ where $\varkappa$ is a positive integer and $f_1, f_2, \ldots, f_\varkappa$ are consecutive edges in C. Then put $\Im(w^{(1)}) = \zeta_{f_\varkappa}\zeta_{f_{\varkappa-1}} \ldots \zeta_{f_1}$. It is obvious that then $\Im(w^{(1)})$ is a directed path in C from $\omega(w^{(1)})$ to $\alpha(w^{(1)}) = \alpha(\tilde{w}) = \alpha(\tilde{v})$ and that $\mathrm{c}(\Im(w^{(1)})) = \mathrm{c}(w^{(1)}) = \mathrm{c}(v^{(1)})$. Consider next the path $w^{(1)}\Im(w^{(1)})v^{(1)}$ in C. Namely, using again the path identities which are valid in the variety of categories $\mathcal{W}_{p^h n}$ according to Corollary 4.3, we consecutively obtain that

$$u^{p^h n} v^{(1)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n} f_1 \zeta_{f_1} v^{(1)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n} f_1 f_2 \zeta_{f_2} \zeta_{f_1} v^{(1)} \equiv_{\mathcal{W}_{p^h n}} \cdots$$
$$\cdots \equiv_{\mathcal{W}_{p^h n}} u^{p^h n} w^{(1)} \Im(w^{(1)}) v^{(1)},$$

since, as it has been said above, the loops $f_1\zeta_{f_1}$, $f_2\zeta_{f_2}$, $\ldots$, $f_\varkappa\zeta_{f_\varkappa}$ in C represent idempotents in the local monoids of the category $C^*/{\equiv_{\mathcal{W}_{p^h n}}}$ at the vertices $\alpha(f_1), \alpha(f_2), \ldots, \alpha(f_\varkappa)$, respectively, and $\mathrm{c}(f_1\zeta_{f_1}) \subseteq \mathrm{c}(u)$, $\mathrm{c}(f_2\zeta_{f_2}) \subseteq \mathrm{c}(u)$, $\ldots$, $\mathrm{c}(f_\varkappa\zeta_{f_\varkappa}) \subseteq \mathrm{c}(u)$.

Consider next also the truly maximal subword $v^{(2)}$ of s having the $2k$-factorization property which stands at the second place in the maximal $2k$-chain constituting the maximal genuine $2k$-nest $\tilde{v}$ of s, and the maximal subword $w^{(2)}$ of t appearing at the second place in the properly intersecting $2k$-sequence which gives rise to the raw $2k$-nest $\tilde{w}$ of t. Remember that then $v^{(2)}$ and $w^{(2)}$ are paths in C and that $\mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$. Moreover, then there exist paths $r^{(2)}$ and $\rho^{(2)}$ in C such that $\mathrm{c}(r^{(2)}) \subseteq \mathrm{c}(v^{(1)}) \cap \mathrm{c}(v^{(2)})$ and $\mathrm{c}(\rho^{(2)}) \subseteq \mathrm{c}(w^{(1)}) \cap \mathrm{c}(w^{(2)})$ which are, respectively,

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the overlaps of the paths $v^{(1)}$, $v^{(2)}$ in s and $w^{(1)}$, $w^{(2)}$ in t. It means that there exist paths $\hat{v}^{(1)}$, $\hat{v}^{(2)}$ and $\hat{w}^{(1)}$, $\hat{w}^{(2)}$ in C satisfying $\mathrm{c}(\hat{v}^{(1)}) = \mathrm{c}(v^{(1)})$, $\mathrm{c}(\hat{v}^{(2)}) = \mathrm{c}(v^{(2)})$ and $\mathrm{c}(\hat{w}^{(1)}) = \mathrm{c}(w^{(1)})$, $\mathrm{c}(\hat{w}^{(2)}) = \mathrm{c}(w^{(2)})$ such that $v^{(1)} = \hat{v}^{(1)}r^{(2)}$, $v^{(2)} = r^{(2)}\hat{v}^{(2)}$ and $w^{(1)} = \hat{w}^{(1)}\rho^{(2)}$, $w^{(2)} = \rho^{(2)}\hat{w}^{(2)}$. Then the path $\hat{v}^{(1)}r^{(2)}\hat{v}^{(2)}$ is an initial segment of the path $\tilde{v}$ and the path $\hat{w}^{(1)}\rho^{(2)}\hat{w}^{(2)}$ is an initial segment of the path $\tilde{w}$. Furthermore, since $\tilde{v} \equiv_{\mathcal{V}_{p^h n, 4k}(\mathcal{A}_p^h \circ \mathcal{K})} \tilde{w}$ and the raw $2k$-nest $\tilde{w}$ has thus been deduced $4k$-tamely from the maximal genuine $2k$-nest $\tilde{v}$, according to Lemma 8.1, there exists a word $q^{(2)} \in E(C)^*$ satisfying $\mathrm{c}(q^{(2)}) \subseteq \mathrm{c}(v^{(1)}) \cap \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(1)}) \cap \mathrm{c}(w^{(2)})$ such that $\hat{v}^{(1)}r^{(2)} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} \hat{w}^{(1)}\rho^{(2)}q^{(2)}$, that is, $v^{(1)} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} w^{(1)}q^{(2)}$. Since $\Im(w^{(1)})w^{(1)}$ obviously represents the identity in the relatively free group $E(C)^*/{\equiv_{\mathcal{A}_p^h \circ \mathcal{K}}}$, because this group satisfies the identity $x^{p^h n} \doteq 1$, it hence immediately follows that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} q^{(2)}$. Herewith we also get, in particular, that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p} q^{(2)}$.

Consider now the directed graph Δ with $V(\Delta) = V(C)$ and $E(\Delta) = \mathrm{c}(v^{(1)}) \cap \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(1)}) \cap \mathrm{c}(w^{(2)})$. The mappings α, ω pertaining to Δ are obtained by restricting the initial mappings α, ω pertaining to C from $E(C)$ to $E(\Delta)$. Now we claim that the vertices $\omega(w^{(1)})$ and $\omega(v^{(1)})$ occur both in the same connected component of the graph Δ. In order to see that this is really the case, notice that $\Im(w^{(1)})v^{(1)}$ is a directed path in C from $\omega(w^{(1)})$ to $\omega(v^{(1)})$. If the vertices $\omega(w^{(1)})$ and $\omega(v^{(1)})$ appeared in different connected components of Δ, then, by Lemma 10.1, there would exist an edge $f \in E(C) - E(\Delta)$ whose number of occurrences in the path $\Im(w^{(1)})v^{(1)}$ would not be divisible by p. But this circumstance would contradict the facts that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p} q^{(2)}$ and $\mathrm{c}(q^{(2)}) \subseteq E(\Delta)$. Thus, indeed, both vertices $\omega(w^{(1)})$ and $\omega(v^{(1)})$ occur in the same connected component of Δ. Denote this connected component of Δ by Δ_1. Besides, there may be further connected components $\Delta_2, \dots, \Delta_\nu$ of Δ. Since $\omega(w^{(1)}), \omega(v^{(1)}) \in V(\Delta_1)$, there exists an undirected path $\hbar$ in Δ_1 from $\omega(w^{(1)})$ to $\omega(v^{(1)})$. This path $\hbar$ is thus composed only of edges from the set $E(\Delta_1)$, which is a subset of the set $\mathrm{c}(v^{(1)}) \cap \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(1)}) \cap \mathrm{c}(w^{(2)})$. We may clearly assume that none of the edges in $\hbar$ is a loop. If we replace in this path $\hbar$ every edge g which is directed discordantly to the path $\hbar$ with the directed path ζ_g, we obtain from $\hbar$ a directed path θ in C going from $\omega(w^{(1)})$ to $\omega(v^{(1)})$. This directed path θ will thus be composed only of edges from the set $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$.

Now let us turn back to the fact that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} q^{(2)}$ for some word $q^{(2)}$ satisfying $\mathrm{c}(q^{(2)}) \subseteq E(\Delta)$, which we have deduced above. We have seen that $\Im(w^{(1)})v^{(1)}$ is a directed path in C from $\omega(w^{(1)})$ to $\omega(v^{(1)})$, where $\omega(w^{(1)})$ and $\omega(v^{(1)})$ are both vertices of the connected component Δ_1 of Δ. Notice that the group variety $\mathcal{A}_p^h \circ \mathcal{K}$ can be viewed as the Mal'cev product $\mathcal{A}_p \circ (\mathcal{A}_p^{h-1} \circ \mathcal{K})$. Thus, according to the notes on the free groups in the Mal'cev products of group varieties which are contained in § 9, the condition $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^h \circ \mathcal{K}} q^{(2)}$ is equivalent to the conditions $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^{h-1} \circ \mathcal{K}} q^{(2)}$ and $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{A}_p} \boldsymbol{\pi}(q^{(2)})$, where $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)})$ and $\boldsymbol{\pi}(q^{(2)})$ are the paths in the Cayley graph Γ of the relatively free group $E(C)^*/{\equiv_{\mathcal{A}_p^{h-1} \circ \mathcal{K}}}$ determined, respectively, by the words $\Im(w^{(1)})v^{(1)}$ and $q^{(2)}$, as described in § 9. Recall from that section that the set $V(\Gamma)$ of vertices of the Cayley graph Γ consists just of the elements of the group $E(C)^*/{\equiv_{\mathcal{A}_p^{h-1} \circ \mathcal{K}}}$ itself. We have adopted the simplified notation $\bar{\varsigma}$ for the class of the congruence $\equiv_{\mathcal{A}_p^{h-1} \circ \mathcal{K}}$ containing the word ς, for any $\varsigma \in E(C)^*$. The set $E(\Gamma)$ of edges

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of the Cayley graph $\mathit{\Gamma}$ consists of pairs of the form $(\overline{\varsigma}, e)$, for arbitrary $\varsigma \in E(C)^*$ and $e \in E(C)$. Every edge $(\overline{\varsigma}, e)$ is directed from the vertex $\overline{\varsigma}$ to the vertex $\overline{\varsigma e}$. For the empty word $\mathtt{1}$ of $E(C)^*$, we have defined $\boldsymbol{\pi}(\mathtt{1})$ to be the empty path $\mathtt{1}_{\overline{\mathtt{1}}}$ on $\overline{\mathtt{1}}$, and for every non-empty word $\varpi = g_1 g_2 \cdots g_\lambda$, where λ is a positive integer and $g_1, g_2, \ldots, g_\lambda \in E(C)$, we have defined $\boldsymbol{\pi}(\varpi)$ to be the path in $\mathit{\Gamma}$ from $\overline{\mathtt{1}}$ to $\overline{\varpi}$ given by the formula

$$\boldsymbol{\pi}(\varpi) = (\overline{\mathtt{1}}, g_1)(\overline{g_1}, g_2)(\overline{g_1 g_2}, g_3) \cdots (\overline{g_1 g_2 \cdots g_{\lambda-1}}, g_\lambda).$$

Note that such paths $\boldsymbol{\pi}(\varpi)$ are, in fact, particular elements of the free monoid $E(\mathit{\Gamma})^*$ and that the empty path $\boldsymbol{\pi}(\mathtt{1})$ can be viewed as the identity of $E(\mathit{\Gamma})^*$. In addition, $E(\mathit{\Gamma})^*/\equiv_{\mathcal{A}_p}$ then is the free group on the set $E(\mathit{\Gamma})$ in the variety $\mathcal{A}_p$ of all abelian groups of exponent p.

Concentrate now on the condition $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{A}_p} \boldsymbol{\pi}(q^{(2)})$. This condition is satisfied if and only if, for every edge $(\overline{\varsigma}, e)$ of the Cayley graph $\mathit{\Gamma}$ of the group $E(C)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$, the number of occurrences of $(\overline{\varsigma}, e)$ in the path $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)})$ is congruent modulo p to the number of occurrences of $(\overline{\varsigma}, e)$ in the path $\boldsymbol{\pi}(q^{(2)})$. Now look over which edges of $\mathit{\Gamma}$ may actually occur in the path $\boldsymbol{\pi}(q^{(2)})$. Since $\mathrm{c}(q^{(2)}) \subseteq E(\mathit{\Delta})$, there may be only edges of the form $(\overline{\varsigma}, e)$ where $\varsigma \in E(\mathit{\Delta})^*$ and $e \in E(\mathit{\Delta})$. As $E(\mathit{\Delta}) \subseteq E(C)$, we may consider the subgroup $E(\mathit{\Delta})^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$ of the group $E(C)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$ and the Cayley graph $\mathit{\Omega}$ of this subgroup $E(\mathit{\Delta})^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. This Cayley graph $\mathit{\Omega}$ forms, in fact, a part of the Cayley graph $\mathit{\Gamma}$, since its vertices are just the elements of the subgroup $E(\mathit{\Delta})^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$ and its edges are those we have specified last. Thus $\boldsymbol{\pi}(q^{(2)})$ is actually a path in $\mathit{\Omega}$. Next take into account the connected component $\mathit{\Delta}_1$ of $\mathit{\Delta}$ and its set of edges $E(\mathit{\Delta}_1)$. Since $E(\mathit{\Delta}_1) \subseteq E(\mathit{\Delta})$, we may again consider the subgroup $E(\mathit{\Delta}_1)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$ of the group $E(\mathit{\Delta})^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. Now consider the left cosets of the subgroup $E(\mathit{\Delta}_1)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$ in the group $E(\mathit{\Delta})^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. These left cosets form a partition of the set $V(\mathit{\Omega})$ of vertices of the Cayley graph $\mathit{\Omega}$. One of these cosets is the subgroup $E(\mathit{\Delta}_1)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$ itself. For any edge $(\overline{\varsigma}, e)$ of $\mathit{\Omega}$, it is the case that both endpoints $\overline{\varsigma}$ and $\overline{\varsigma e}$ of $(\overline{\varsigma}, e)$ lie in the same left coset of the mentioned partition if and only if $e \in E(\mathit{\Delta}_1)$. Vertices of $\mathit{\Omega}$ together with edges $(\overline{\varsigma}, e)$ of $\mathit{\Omega}$ such that $e \in E(\mathit{\Delta}_1)$ thus form a subgraph $\mathit{\Upsilon}$ of $\mathit{\Omega}$ such that the connected components of $\mathit{\Upsilon}$ have for their sets of vertices exactly the left cosets of the mentioned partition. Edges of $\mathit{\Omega}$ having their endpoints in different cosets of this partition are exactly those of the form $(\overline{\varsigma}, e)$ where $\varsigma \in E(\mathit{\Delta})^*$ and $e \in E(\mathit{\Delta}) - E(\mathit{\Delta}_1)$, that is, $e \in E(\mathit{\Delta}_2) \cup \cdots \cup E(\mathit{\Delta}_\nu)$ where $\mathit{\Delta}_2, \ldots, \mathit{\Delta}_\nu$ are the other connected components of $\mathit{\Delta}$, if there are any.

Now take note of those edges $(\overline{\varsigma}, e)$ of $\mathit{\Gamma}$ where $e \in E(\mathit{\Delta}_2) \cup \cdots \cup E(\mathit{\Delta}_\nu)$ which occur in the paths $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)})$ and $\boldsymbol{\pi}(q^{(2)})$. If such an edge $(\overline{\varsigma}, e)$ appears in the path $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)})$, then, by the definition of this path, there exists a proper initial segment ϱ of $\Im(w^{(1)})v^{(1)}$ such that ϱe is also an initial segment of $\Im(w^{(1)})v^{(1)}$ and such that $\varsigma \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} \varrho$, that is, $\overline{\varsigma} = \overline{\varrho}$. Then, since $\Im(w^{(1)})v^{(1)}$ is a directed path in C beginning at $\omega(w^{(1)})$, which is a vertex of $\mathit{\Delta}_1$, its initial segment ϱ is a directed path in C from $\omega(w^{(1)})$ to $\alpha(e)$, which in turn is a vertex of one of the components $\mathit{\Delta}_2, \ldots, \mathit{\Delta}_\nu$ of $\mathit{\Delta}$, since $e \in E(\mathit{\Delta}_2) \cup \cdots \cup E(\mathit{\Delta}_\nu)$. Then, however, by Lemma 10.1, there is an edge $f \in E(C) - E(\mathit{\Delta})$ whose number of occurrences in ϱ is not divisible by p. Since $\mathcal{A}_p$ is

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a subvariety of the group variety $\mathcal{A}_p^{h-1}\circ\mathcal{K}$, it hence follows that $\overline{\varrho}$ is not an element of the subgroup $E(\Delta)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. Since $\overline{\varsigma} = \overline{\varrho}$, the same is true of the element $\overline{\varsigma}$. Thus we may conclude that the edges $(\overline{\varsigma}, e)$ of Γ with $e \in E(\Delta_2) \cup \cdots \cup E(\Delta_\nu)$ which occur in the path $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)})$ have the property that $\overline{\varsigma}$ is not an element of the subgroup $E(\Delta)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. Since $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{A}_p} \boldsymbol{\pi}(q^{(2)})$, this entails that, for every edge $(\overline{\varsigma}, e)$ of Γ such that $e \in E(\Delta_2) \cup \cdots \cup E(\Delta_\nu)$ and $\overline{\varsigma}$ is an element of the subgroup $E(\Delta)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$, the number of occurrences of this edge in the path $\boldsymbol{\pi}(q^{(2)})$ must be divisible by p. We claim that this causes that $\overline{q^{(2)}}$ is an element of the subgroup $E(\Delta_1)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. In other words, we state that then the path $\boldsymbol{\pi}(q^{(2)})$ must have its end in the connected component of the subgraph Υ of the Cayley graph Ω whose set of vertices consists of the elements of the subgroup $E(\Delta_1)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. Yet otherwise stated, the path $\boldsymbol{\pi}(q^{(2)})$ then must end in the same connected component of Υ where it has begun. Indeed, suppose that this were not the case, so that the path $\boldsymbol{\pi}(q^{(2)})$ ended in another connected component of Υ. Then, using Lemma 10.1 once again, but this time applied to the Cayley graph Ω and its subgraph Υ, we would come to the existence of an edge $(\overline{\varsigma}, e)$ in the set $E(\Omega) - E(\Upsilon)$ whose number of occurrences in the path $\boldsymbol{\pi}(q^{(2)})$ is not divisible by p. However, since this edge $(\overline{\varsigma}, e)$ belongs to $E(\Omega)$, $\overline{\varsigma}$ is an element of the subgroup $E(\Delta)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$, and as this edge $(\overline{\varsigma}, e)$ is not an element of $E(\Upsilon)$, we get that $e \in E(\Delta_2) \cup \cdots \cup E(\Delta_\nu)$. But this contradicts our previous conclusion, according to which the number of occurrences of such an edge $(\overline{\varsigma}, e)$ in the path $\boldsymbol{\pi}(q^{(2)})$ should be divisible by p. This verifies that, indeed, $\overline{q^{(2)}}$ is an element of the subgroup $E(\Delta_1)^*/\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}$. That is, there exists a word $b \in E(\Delta_1)^*$ such that $q^{(2)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} b$. Since $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^{h}\circ\mathcal{K}} q^{(2)}$, it hence follows that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} b$ for some word $b \in E(\Delta_1)^*$.

In order to exploit this piece of knowledge, remember that $v^{(1)}$ is a truly maximal subword of s having the $2k$-factorization property and that it is a path in C. Hence it follows that, for every edge e in the set $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, there exists a path γ_e in C such that $\alpha(\gamma_e) = \omega(w^{(1)})$, $\omega(\gamma_e) = \alpha(e)$ and $\mathrm{c}(\gamma_e) \subseteq \mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, and there exists also a path δ_e in C such that $\alpha(\delta_e) = \omega(e)$, $\omega(\delta_e) = \omega(w^{(1)})$ and $\mathrm{c}(\delta_e) \subseteq \mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. Again, it is possible to choose these paths in such a way that whenever e, f are edges in the set $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ such that $\alpha(e) = \alpha(f)$, then $\gamma_e = \gamma_f$, and whenever e, f are edges in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ such that $\omega(e) = \omega(f)$, then $\delta_e = \delta_f$. It is also an evident choice to let γ_e be the empty path on $\omega(w^{(1)})$ whenever the edge e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ is such that $\alpha(e) = \omega(w^{(1)})$, and to let δ_e be the empty path on $\omega(w^{(1)})$ whenever the edge e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ is such that $\omega(e) = \omega(w^{(1)})$. In addition, for edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ such that $\alpha(e) \in V(\Delta_1)$, it is possible to construct the paths γ_e in the following way. Since, in such a situation, $\omega(w^{(1)}), \alpha(e) \in V(\Delta_1)$ and Δ_1 is a conected component of Δ, there is an undirected path c_e in Δ_1 from $\omega(w^{(1)})$ to $\alpha(e)$. One can assume that no edge in c_e is a loop. Then, if we replace in this path c_e every edge g which is directed discordantly to the path c_e with the directed path ζ_g, we obtain from c_e the required directed path γ_e from $\omega(w^{(1)})$ to $\alpha(e)$. Similarly, for edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ such that $\omega(e) \in V(\Delta_1)$, it is possible to select the paths δ_e in the following way. Since then $\omega(e), \omega(w^{(1)}) \in V(\Delta_1)$ and, as noted before, Δ_1 is a conected component of Δ, there is an undirected path d_e

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in Δ_1 from $\omega(e)$ to $\omega(w^{(1)})$. Again, one can assume that no edge in d_e is a loop. Then, if we replace in this path d_e every edge g which is directed discordantly to the path d_e with the directed path ζ_g, we obtain from d_e the required directed path δ_e from $\omega(e)$ to $\omega(w^{(1)})$. Thus assume that the paths γ_e, δ_e have been chosen in accordance with all these requirements, for all edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. Furthermore, for the same reason as at the beginning of this paragraph, for every edge e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, there exists some edge f in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ such that $\omega(f) = \alpha(e)$. Then we denote by β_e the path δ_f. Note that, owing to one of the assumptions given above, the path β_e does not depend on the choice of the edge f satisfying $\omega(f) = \alpha(e)$, for every edge e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. Now consider the substitution σ assigning to every edge e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ the loop $\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$ on the vertex $\omega(w^{(1)})$. Note that this substitution can be naturally extended to arbitrary paths in C containing only edges from the set $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, and it can be further extended also to arbitrary words from $E(C)^*$ containing only edges from the set just mentioned. Yet remember that $\omega(w^{(1)}), \omega(v^{(1)}) \in V(\Delta_1)$ and that we have already constructed above a directed path θ from $\omega(w^{(1)})$ to $\omega(v^{(1)})$ containing only edges from the set $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, using a similar procedure as the one applied here to construct the paths γ_e or δ_e for those edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ which satisfy, respectively, $\alpha(e) \in V(\Delta_1)$ or $\omega(e) \in V(\Delta_1)$. Now denote by η the path $\theta(\delta_g\theta)^{p^h n-1}$ where g stands for the last edge in the path $v^{(1)}$. Then, from the path identities which are valid in the variety of categories $\mathcal{W}_{p^h n}$ according to Corollary 4.3, we obtain that

$$u^{p^h n} w^{(1)} \Im(w^{(1)}) v^{(1)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n} w^{(1)} \sigma(\Im(w^{(1)}) v^{(1)})\eta.$$

Using once again the facts that $v^{(1)}$ is a path in C, it has the $2k$-factorization property, and $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, we find out that there is also a loop z in C on the vertex $\omega(w^{(1)})$ such that $\mathrm{c}(z) = \mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. Then consider yet the substitution τ assigning to every edge e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ the loop $z^{p^h n}\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$ on the vertex $\omega(w^{(1)})$, as above. This substitution can likewise be naturally extended to arbitrary paths in C containing only edges from the set $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, or even to arbitrary words from $E(C)^*$ containing only edges from this set. Then, using the path identities which are valid in $\mathcal{W}_{p^h n}$ in view of Corollary 4.3 once again, we further obtain that

$$u^{p^h n} w^{(1)} \sigma(\Im(w^{(1)}) v^{(1)})\eta \equiv_{\mathcal{W}_{p^h n}} u^{p^h n} w^{(1)} \tau(\Im(w^{(1)}) v^{(1)})\eta,$$

where $\tau(\Im(w^{(1)})v^{(1)})$ is a loop in C on $\omega(w^{(1)})$ composed of the loops of the form $z^{p^h n}\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$ on $\omega(w^{(1)})$ for all edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. However, according to the consequences of the path identities valid in $\mathcal{W}_{p^h n}$ that we have deduced in § 4 in the text preceding Proposition 4.4, the following holds true. In the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n}}}$, the edges that are congruence classes of the loops in C of the above form $z^{p^h n}\gamma_e(\beta_e\gamma_e)^{p^h n-1}e\delta_e$, for arbitrary edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$, all lie in the same maximal subgroup of the local monoid of this quotient category at the vertex $\omega(w^{(1)})$. To be specific, they are all elements of the maximal subgroup containing the idempotent that is represented by the loop $z^{p^h n}$. The same statement holds true, of course, also of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})}}$. But, according to the path identities defining the variety of categories $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$ which appeared in § 5, maximal subgroups of the local monoids of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})}}$ all lie in the group variety $\mathcal{A}_p^{h-1}\circ\mathcal{K}$. However, in the previous paragraph, we have come to the conclusion that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} b$ for some word $b \in E(\varDelta_1)^*$. Hence, applying the substitution τ, we obtain that $\tau(\Im(w^{(1)})v^{(1)})$ and $\tau(b)$ are both loops in C on the vertex $\omega(w^{(1)})$ and that they represent the same element of one of the maximal subgroups of the local monoid at $\omega(w^{(1)})$ of the last quotient category. Namely, these two loops both represent the same element of the maximal subgroup containing the idempotent represented by the loop $z^{p^h n}$. But this means that we have

$$\tau(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tau(b),$$

and hence

$$u^{p^h n}w^{(1)}\tau(\Im(w^{(1)})v^{(1)})\eta \equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} u^{p^h n}w^{(1)}\tau(b)\eta.$$

Then, using the path identities which are valid in $\mathcal{W}_{p^h n}$ in view of Corollary 4.3 once more, we further get that

$$u^{p^h n}w^{(1)}\tau(b)\eta \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}w^{(1)}\sigma(b)\eta.$$

Since the maximal subword $w^{(1)}$ of $\tilde{w}$ has the right weak $2k$-factorization property, it certainly has the k-factorization property. Besides, we also have $\mathrm{c}(w^{(1)}) = \mathrm{c}(v^{(1)}) = \mathrm{c}(u)$. Thus, having in view the path identities defining the variety of categories $\mathcal{W}_{p^h n,k}$ which appeared in § 4, we can still simplify the path on the right-hand side of the above formula, as we then have

$$u^{p^h n}w^{(1)}\sigma(b)\eta \equiv_{\mathcal{W}_{p^h n,k}} w^{(1)}\sigma(b)\eta.$$

Now, summing up all formulae displayed throughout the whole previous text concerning the case $\ell > 1$ (except the one describing the path $\boldsymbol{\pi}(\varpi)$), we hence obtain that

$$v^{(1)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} w^{(1)}\sigma(b)\eta. \tag{\#}$$

Note that here $w^{(1)}\sigma(b)\eta$ is a path in C from $\alpha(v^{(1)}) = \alpha(w^{(1)})$ to $\omega(v^{(1)})$. Further remember once again that b is some word in $E(\varDelta_1)^*$, and so $\sigma(b)\eta$ is a directed path in C from $\omega(w^{(1)})$ to $\omega(v^{(1)})$ which, according to the above definitions in this paragraph, is composed only of the edges $e \in E(\varDelta_1)$ and of the directed paths ζ_e for arbitrary edges $e \in E(\varDelta_1)$.

Recall now that we have denoted by $r^{(2)}$ and $\rho^{(2)}$ the paths in C which are, respectively, the overlaps of the paths $v^{(1)}$, $v^{(2)}$ in $\tilde{v}$ and $w^{(1)}$, $w^{(2)}$ in $\tilde{w}$, and that we have denoted by $\hat{v}^{(1)}$, $\hat{v}^{(2)}$ and $\hat{w}^{(1)}$, $\hat{w}^{(2)}$ the paths in C for which $v^{(1)} = \hat{v}^{(1)}r^{(2)}$, $v^{(2)} = r^{(2)}\hat{v}^{(2)}$ and $w^{(1)} = \hat{w}^{(1)}\rho^{(2)}$, $w^{(2)} = \rho^{(2)}\hat{w}^{(2)}$, so that then the path $\hat{v}^{(1)}r^{(2)}\hat{v}^{(2)}$ is an initial segment of $\tilde{v}$ and the path $\hat{w}^{(1)}\rho^{(2)}\hat{w}^{(2)}$ is an initial segment of $\tilde{w}$. Thus, multiplying both paths in the formula (#) displayed above by $\hat{v}^{(2)}$ on the right, we hence get that

$$\hat{v}^{(1)}r^{(2)}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \hat{w}^{(1)}\rho^{(2)}\sigma(b)\eta\hat{v}^{(2)}.$$

We have already noticed that the maximal subword $w^{(1)} = \hat{w}^{(1)}\rho^{(2)}$ of $\tilde{w}$ has the k-factorization property and that we have $\mathrm{c}(w^{(1)}) = \mathrm{c}(v^{(1)}) = \mathrm{c}(u)$, where u is the loop in C on the vertex $\alpha(w^{(1)}) = \alpha(v^{(1)})$ that we have used before. Furthermore, since the maximal subword $v^{(2)} = r^{(2)}\hat{v}^{(2)}$ of $\tilde{v}$ has the $2k$-factorization property and $\mathrm{c}(r^{(2)}) \subsetneqq \mathrm{c}(v^{(2)})$, the final segment $\hat{v}^{(2)}$ of $v^{(2)}$ has certainly the left weak $2k$-factorization property. Therefore, this word $\hat{v}^{(2)}$ also has surely the k-factorization property. As $k > 2$, similarly as before, this entails that there exists a loop ϑ in C on the vertex $\alpha(\hat{v}^{(2)}) = \omega(v^{(1)})$ such that $\mathrm{c}(\vartheta) = \mathrm{c}(\hat{v}^{(2)}) = \mathrm{c}(v^{(2)})$. Thus, in view of the path identities defining the variety of categories $\mathcal{W}_{p^h n,k}$ which appeared in § 4, we have

$$\hat{w}^{(1)}\rho^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)} \quad \text{and} \quad \hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} \vartheta^{p^h n}\hat{v}^{(2)}$$

and hence

$$\hat{w}^{(1)}\rho^{(2)}\sigma(b)\eta\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}\sigma(b)\eta\vartheta^{p^h n}\hat{v}^{(2)}.$$

For the same reasons as above, that is, because the subword $\hat{v}^{(2)}$ has the k-factorization property, where $k > 2$, and it is a path in C, it turns out that, for every edge e in the set $\mathrm{c}(v^{(2)}) = \mathrm{c}(\hat{v}^{(2)})$, there exists a path χ_e in C such that $\alpha(\chi_e) = \omega(e)$, $\omega(\chi_e) = \alpha(e)$ and $\mathrm{c}(\chi_e) \subseteq \mathrm{c}(v^{(2)})$. Then $e\chi_e$ is a loop in C on the vertex $\alpha(e)$. Besides, in view of the loop identities defining the variety of categories $\mathcal{W}_{p^h n}$, the loop $(e\chi_e)^{p^h n}$ then represents an idempotent in the local monoid of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n}}}$ at the vertex $\alpha(e)$. Now, for every edge e in the set $\mathrm{c}(v^{(2)})$, let us denote by ξ_e the path $\chi_e(e\chi_e)^{p^h n-1}$. Then, of course, $(e\chi_e)^{p^h n} = e\xi_e$, $\alpha(\xi_e) = \omega(e)$ and $\omega(\xi_e) = \alpha(e)$. Next remember from the previous paragraph that the segment $\sigma(b)\eta$ appearing in the path on the right-hand side of the last formula displayed above is a path which is composed of the edges $e \in E(\varDelta_1)$ and of the paths ζ_e for arbitrary edges $e \in E(\varDelta_1)$. Now we are going to show that if we replace in this segment $\sigma(b)\eta$ every occurrence of any path ζ_e with the path ξ_e, for all edges $e \in E(\varDelta_1)$, then we obtain from the path on the right-hand side of the above formula another path which, however, will be congruent modulo $\equiv_{\mathcal{W}_{p^h n}}$ to the path appearing now on the right-hand side of that formula.

Thus let $\sigma(b)\eta = a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}a_\kappa$ where κ is a non-negative integer, $e_1, e_2, \dots, e_\kappa \in E(\varDelta_1)$ and $a_0, a_1, \dots, a_\kappa \in E(\varDelta_1)^*$. Then the path on the right-hand side of the last formula displayed in the previous paragraph is of the form $u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)}$. Since $e_\kappa\xi_{e_\kappa}$ is a loop on the vertex $\alpha(e_\kappa) = \omega(\zeta_{e_\kappa})$ representing an idempotent in the local monoid of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n}}}$ at that vertex, and as $\mathrm{c}(e_\kappa\xi_{e_\kappa}) \subseteq \mathrm{c}(\vartheta)$ and $\mathrm{c}(a_\kappa) \subseteq \mathrm{c}(\vartheta)$, using the path identities which are valid in the variety of categories $\mathcal{W}_{p^h n}$ according to Corollary 4.3, we obtain that

$$\begin{aligned} &u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)} \\ &\qquad\qquad \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}e_\kappa\xi_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)}. \end{aligned}$$

Furthermore, since $\zeta_{e_\kappa}e_\kappa = (\lambda_{e_\kappa}e_\kappa)^{p^h n}$ by the definition of the path ζ_{e_κ}, so that $\zeta_{e_\kappa}e_\kappa$ is a loop on the vertex $\omega(e_\kappa) = \alpha(\xi_{e_\kappa})$ representing an idempotent in the local

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monoid of the quotient category $C^*/\equiv_{\mathcal{W}_{p^h n}}$ at that vertex, and as $\mathrm{c}(\zeta_{e_\kappa} e_\kappa) \subseteq \mathrm{c}(u)$ and $\mathrm{c}(\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}) \subseteq \mathrm{c}(u)$, according to the path identities which are valid in the variety of categories $\mathcal{W}_{p^h n}$ in view of Corollary 4.3, we also get that

$$u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}e_\kappa\xi_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\xi_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)},$$

which together with the previous formula yields that

$$u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\xi_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)}.$$

Proceeding in this way further step by step from the right to the left in the segment between a_0 and a_κ, we eventually obtain that

$$u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\zeta_{e_1}a_1\zeta_{e_2}a_2\cdots a_{\kappa-1}\zeta_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}a_0\xi_{e_1}a_1\xi_{e_2}a_2\cdots a_{\kappa-1}\xi_{e_\kappa}a_\kappa\vartheta^{p^h n}\hat{v}^{(2)},$$

which verifies the assertion stated at the end of the previous paragraph. In order to write this last formula somewhat more concisely, we introduce yet the following notations. Remember that, earlier in this text, we have defined the directed path θ going from $\omega(w^{(1)})$ to $\omega(v^{(1)})$. Afterwards, we have also defined the directed paths γ_e and δ_e going, respectively, from $\omega(w^{(1)})$ to $\alpha(e)$ and from $\omega(e)$ to $\omega(w^{(1)})$, for all edges $e \in \mathrm{c}(v^{(1)})$. These definitions have had particular form for edges e satisfying, respectively, $\alpha(e) \in V(\Delta_1)$ or $\omega(e) \in V(\Delta_1)$. Especially this has concerned the edges $e \in E(\Delta_1)$. Return now to these definitions and replace in them everywhere the paths ζ_g with the paths ξ_g, for all edges $g \in E(\Delta_1)$. In this way, we obtain a directed path θ' going from $\omega(w^{(1)})$ to $\omega(v^{(1)})$ such that $\mathrm{c}(\theta') \subseteq \mathrm{c}(v^{(2)})$, and we further get directed paths γ'_e and δ'_e going, respectively, from $\omega(w^{(1)})$ to $\alpha(e)$ and from $\omega(e)$ to $\omega(w^{(1)})$ such that $\mathrm{c}(\gamma'_e), \mathrm{c}(\delta'_e) \subseteq \mathrm{c}(v^{(2)})$, for all edges $e \in \mathrm{c}(v^{(1)})$ satisfying, respectively, $\alpha(e) \in V(\Delta_1)$ or $\omega(e) \in V(\Delta_1)$. Further, for every edge $e \in E(\Delta_1)$, we put $\beta'_e = \delta'_f$ where, as before, $f \in \mathrm{c}(v^{(1)})$ is any edge such that $\omega(f) = \alpha(e)$. Again, the path β'_e does not depend on the choice of the edge f, for any edge $e \in E(\Delta_1)$. Then we let η' be the path $\theta'(\delta'_g\theta')^{p^h n-1}$ where g stands for the last edge in the path $v^{(1)}$. Furthermore, we now consider the substitution σ' assigning to every edge $e \in E(\Delta_1)$ the loop $\gamma'_e(\beta'_e\gamma'_e)^{p^h n-1}e\delta'_e$ on the vertex $\omega(w^{(1)})$. Of course, this substitution can be naturally extended to arbitrary words in $E(\Delta_1)^*$. Then the last formula displayed above can be written concisely in the form

$$u^{p^h n}\hat{w}^{(1)}\rho^{(2)}\sigma(b)\eta\vartheta^{p^h n}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\vartheta^{p^h n}\hat{v}^{(2)}.$$

We have already seen above, in a formula displayed in the middle of the previous paragraph, that we have $\hat{w}^{(1)}\rho^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} u^{p^h n}\hat{w}^{(1)}\rho^{(2)}$ and $\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} \vartheta^{p^h n}\hat{v}^{(2)}$. Hence, as regards the path on the right-hand side of the last formula displayed above, it follows that

$$u^{p^h n}\hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\vartheta^{p^h n}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} \hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\hat{v}^{(2)}.$$

Thus, this formula and the last formula displayed in the previous paragraph together make it possible to deduce from the preceding formula displayed above that we have

$$\hat{w}^{(1)}\rho^{(2)}\sigma(b)\eta\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}} \hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\hat{v}^{(2)}. \tag{\#\#}$$

Lastly, this formula together with the first formula displayed in the previous paragraph yields that

$$\hat{v}^{(1)}r^{(2)}\hat{v}^{(2)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\hat{v}^{(2)}.$$

The segment $\sigma'(b)\eta'$ of the path on the right-hand side of this formula has now the property that $\mathrm{c}(\sigma'(b)\eta') \subseteq \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$.

Let now $o^{(2)}$ be the longest initial segment of the path $\sigma'(b)\eta'$ such that $\mathrm{c}(o^{(2)}) \subseteq \mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ (it may also be empty). Denote by $\varpi^{(1)}$ the path $\hat{w}^{(1)}\rho^{(2)}o^{(2)}$ and by $\varpi^{(2)}$ the path $\rho^{(2)}\sigma'(b)\eta'\hat{v}^{(2)}$. Then $\rho^{(2)}o^{(2)}$ is the overlap of $\varpi^{(1)}$ and $\varpi^{(2)}$ in the above path $\hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\hat{v}^{(2)}$. The word $\varpi^{(1)}$ has the right weak $2k$-factorization property, since its initial segment $w^{(1)} = \hat{w}^{(1)}\rho^{(2)}$ has this property, and the word $\varpi^{(2)}$ has the left weak $2k$-factorization property, since, as we have seen before, its final segment $\hat{v}^{(2)}$ has this property. Let us now denote by $\tilde{\varpi}$ the path in C which one obtains from $\tilde{v}$ by replacing its initial segment $v^{(1)} = \hat{v}^{(1)}r^{(2)}$ with the path $\hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta' = w^{(1)}\sigma'(b)\eta'$. Then, from the last formula displayed in the previous paragraph it follows immediately that

$$\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{\varpi}.$$

Moreover, by inspecting the whole derivation of the path $\tilde{\varpi}$ from the path $\tilde{v}$, as it has been carried out in the preceding sequence of paragraphs, we find out that the word $\tilde{\varpi}$ has thus been deduced k-tamely from the maximal genuine $2k$-nest $\tilde{v}$ (which can be viewed also as a k-nest in this context, since the consecutively overlapping truly maximal subwords $v^{(1)}, \dots, v^{(\ell)}$ having the $2k$-factorization property which together form this maximal genuine $2k$-nest are, of course, also maximal subwords having the k-factorization property). Therefore, by our findings made in § 7, the word $\tilde{\varpi}$ itself must have the form of a raw k-nest composed of a properly intersecting sequence of ℓ maximal subwords relative to the k-factorization property. However, it turns out that they are exactly the maximal subwords $\varpi^{(1)}$, $\varpi^{(2)}$ and $v^{(3)}, \dots, v^{(\ell)}$ which arise in the course of the above k-tame derivation of $\tilde{\varpi}$ from $\tilde{v}$ and which form the properly intersecting k-sequence constituting the raw k-nest $\tilde{\varpi}$. Remember that, by the definition of the words $\varpi^{(1)}$ and $\varpi^{(2)}$, we have $\mathrm{c}(\varpi^{(1)}) = \mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$ and $\mathrm{c}(\varpi^{(2)}) = \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$. In addition, we have seen above that $\varpi^{(1)}$ has the right weak $2k$-factorization property and $\varpi^{(2)}$ has the left weak $2k$-factorization property. Of course, $v^{(3)}, \dots, v^{(\ell)}$ have all along the $2k$-factorization property.

Furthermore, by our assumption, we have $\tilde{v} \equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h\circ\mathcal{K})} \tilde{w}$, and the raw $2k$-nest $\tilde{w}$ has thus been deduced $4k$-tamely from the maximal genuine $2k$-nest $\tilde{v}$. In the previous paragraph, we have seen that $\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{\varpi}$ and that the raw k-nest $\tilde{\varpi}$ has thus been deduced k-tamely from the maximal genuine $2k$-nest $\tilde{v}$ (which can again be viewed merely as a k-nest). But this relationship between $\tilde{v}$ and $\tilde{\varpi}$ is symmetric, so that we also

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get that $\tilde{\varpi} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{v}$ and that $\tilde{v}$ has thus been deduced k-tamely from $\tilde{\varpi}$. Since the relation $\equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})}$ is evidently a subset of the congruence $\equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})}$ on $E(C)^*$, we can conclude from the above conditions that, altogether, we have

$$\tilde{\varpi} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}.$$

Moreover, it is clear that, in this way, the raw $2k$-nest $\tilde{w}$ has been deduced k-tamely from the raw k-nest $\tilde{\varpi}$ (the raw $2k$-nest $\tilde{w}$ can also be viewed only as a k-nest, since the properly intersecting $2k$-sequence of maximal subwords $w^{(1)}, \ldots, w^{(\ell)}$ which together constitute this raw $2k$-nest is obviously also a properly intersecting k-sequence). The additional point is that the maximal subword $\varpi^{(1)}$ appearing at the beginning of the properly intersecting k-sequence mentioned in the previous paragraph which gives rise to the raw k-nest $\tilde{\varpi}$ is of the form $\varpi^{(1)} = \hat{w}^{(1)}\rho^{(2)}o^{(2)}$, and the maximal subword $w^{(1)}$ appearing at the beginning of the just mentioned properly intersecting $2k$-sequence which gives rise to the raw $2k$-nest $\tilde{w}$ is of the form $w^{(1)} = \hat{w}^{(1)}\rho^{(2)}$. This prompt us to consider the paths $\tilde{\varpi}'$ and $\tilde{w}'$ obtained from $\tilde{\varpi}$ and $\tilde{w}$, respectively, by omitting from these paths their common initial segment $\hat{w}^{(1)}$. Then $\tilde{\varpi}'$ and $\tilde{w}'$ are coterminal paths in C. Moreover, the words $\tilde{\varpi}'$ and $\tilde{w}'$ have evidently the form of raw k-nests composed, respectively, of the maximal subwords occurring in the properly intersecting k-sequences $\varpi^{(2)}, v^{(3)}, \ldots, v^{(\ell)}$ and $w^{(2)}, \ldots, w^{(\ell)}$. It is easy to realize that these sequences of maximal subwords are actually properly intersecting k-sequences. Besides, since the group variety $\mathcal{A}_p^{h-1}\circ\mathcal{K}$ is a subvariety of the monoid variety $\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$, from $\tilde{\varpi} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}$ we obtain that $\tilde{\varpi} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} \tilde{w}$. Thus, using cancellation, we hence get that $\tilde{\varpi}' \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} \tilde{w}'$. However, not only this consequence can be gained from here. As $\tilde{\varpi} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}$ and the raw k-nest $\tilde{w}$ has thus been deduced k-tamely from the raw k-nest $\tilde{\varpi}$, we see that the other devices that we have prepared so far can now be newly applied, since the assumptions needed to do so are satisfied. The way how this can be done will be made clear subsequently.

Recall that we are carrying this proof essentially by induction on ℓ. The case $\ell = 1$ has already been settled formerly. Next we will exemplify our approach by showing how the proof can be accomplished if $\ell = 2$. Then the raw k-nest $\tilde{\varpi}$ is composed of two maximal subwords $\varpi^{(1)}$ and $\varpi^{(2)}$ and the raw k-nest $\tilde{w}$ is composed of two maximal subwords $w^{(1)}$ and $w^{(2)}$. Besides, we know that then $\tilde{\varpi} = \hat{w}^{(1)}\varpi^{(2)}$ and $\tilde{w} = \hat{w}^{(1)}w^{(2)}$. Moreover, returning to our notation that we have introduced in the previous paragraph, we see that, in the present case, $\tilde{\varpi}' = \varpi^{(2)}$ and $\tilde{w}' = w^{(2)}$. We have seen in the previous paragraph that $\tilde{\varpi}' \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} \tilde{w}'$, which means, in the present situation, that $\varpi^{(2)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} w^{(2)}$. Remember yet that now both words $\varpi^{(2)}$ and $w^{(2)}$ have the left weak $2k$-factorization property, and hence they have the k-factorization property. Besides, $\varpi^{(2)}$ and $w^{(2)}$ are also coterminal paths in C and $\mathrm{c}(\varpi^{(2)}) = \mathrm{c}(w^{(2)})$. Now we may apply essentially the same procedure to these words $\varpi^{(2)}$ and $w^{(2)}$ that we have used before in the case $\ell = 1$ to deal with the words v and w. The difference is only that now the varieties of categories $\mathcal{W}_{p^h n,k}$ and $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$ will appear in this procedure instead of the varieties $\mathcal{W}_{p^h n,2k}$ and $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h}\circ\mathcal{K})$, respectively. Therefore, applying this procedure, we eventually obtain that $\varpi^{(2)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} w^{(2)}$. Multiplying the paths in this formula by $\hat{w}^{(1)}$ on the left, we hence get, in this case, that $\tilde{\varpi} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}$. Previously, we have deduced that $\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{\varpi}$. Hence, altogether we obtain that $\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}$. But this again clearly entails that $\tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}$, as required.

Now we return to the considerations in the last paragraph but one and we indicate how the proof can be accomplished in the general case, for any positive integer ℓ. In view of what has already been done, we may next assume that $\ell > 2$. The idea is that, in that case, we may proceed further essentially in the same way as before when we started to treat the case $\ell > 1$. The difference is that now, instead of the paths $\tilde{v}$ and $\tilde{w}$ and their maximal subwords $v^{(1)}$ and $w^{(1)}$ which appear at the beginning of the properly intersecting $2k$-sequences $v^{(1)}, \ldots, v^{(\ell)}$ and $w^{(1)}, \ldots, w^{(\ell)}$ that constitute, respectively, the $2k$-nests $\tilde{v}$ and $\tilde{w}$, we will consider further the paths $\tilde{\varpi}'$ and $\tilde{w}'$ which, as we have seen in the last paragraph but one, have the form of raw k-nests, and we will continue by treating their maximal subwords $\varpi^{(2)}$ and $w^{(2)}$ appearing at the beginning of the properly intersecting k-sequences $\varpi^{(2)}, v^{(3)}, \ldots, v^{(\ell)}$ and $w^{(2)}, \ldots, w^{(\ell)}$ which constitute, respectively, these raw k-nests $\tilde{\varpi}'$ and $\tilde{w}'$. Recall that the subwords $\varpi^{(2)}$ and $w^{(2)}$ have actually the weak $2k$-factorization property; the subword $\varpi^{(2)}$ has, in fact, even the left weak $2k$-factorization property. Thus these subwords have both also the k-factorization property. Besides, we have assumed that $k > 2$. This shows that all considerations that we have carried out previously when we started to deal with the case $\ell > 1$ can now be repeated with the exception that the variety of categories $\mathcal{W}_{p^h n,2k}$ must now be replaced with the variety $\mathcal{W}_{p^h n,k}$. Thus, in particular, similarly as before, we come to the directed path $\Im(w^{(2)})$ in C from $\omega(w^{(2)})$ to $\alpha(w^{(2)})$ such that $\mathrm{c}(\Im(w^{(2)})) = \mathrm{c}(w^{(2)}) = \mathrm{c}(\varpi^{(2)})$. Consider next also the maximal subwords $v^{(3)}$ and $w^{(3)}$ which appear at the second place in the properly intersecting k-sequences given above. These subwords are, of course, paths in C. Recall again that the subword $v^{(3)}$ has actually the $2k$-factorization property and that the subword $w^{(3)}$ has the weak $2k$-factorization property. Moreover, then there exist paths $r^{(3)}$ and $\rho^{(3)}$ in C which are, respectively, the overlaps of the paths $\varpi^{(2)}$, $v^{(3)}$ in $\tilde{\varpi}$ and $w^{(2)}$, $w^{(3)}$ in $\tilde{w}$. Note that $r^{(3)}$ is, in fact, the overlap of the paths $\hat{v}^{(2)}$, $v^{(3)}$, and hence it is identical with the overlap of the paths $v^{(2)}$, $v^{(3)}$ in $\tilde{v}$. Remember that then $\mathrm{c}(r^{(3)}) \subseteq \mathrm{c}(v^{(2)}) \cap \mathrm{c}(v^{(3)})$ and $\mathrm{c}(\rho^{(3)}) \subseteq \mathrm{c}(w^{(2)}) \cap \mathrm{c}(w^{(3)})$. Let $\hat{v}^{(3)}$ and $\hat{w}^{(3)}$ be the paths in C such that $v^{(3)} = r^{(3)}\hat{v}^{(3)}$ and $w^{(3)} = \rho^{(3)}\hat{w}^{(3)}$. Further, let $\bar{v}^{(2)}$ and $\bar{w}^{(2)}$ be the paths in C such that $v^{(2)} = r^{(2)}\bar{v}^{(2)}r^{(3)}$ and $w^{(2)} = \rho^{(2)}\bar{w}^{(2)}\rho^{(3)}$. Note that then $\hat{v}^{(2)} = \bar{v}^{(2)}r^{(3)}$ and $\hat{w}^{(2)} = \bar{w}^{(2)}\rho^{(3)}$. Then, by the definition of the path $\tilde{\varpi}$, the path $\hat{w}^{(1)}\rho^{(2)}\sigma'(b)\eta'\bar{v}^{(2)}r^{(3)}\hat{v}^{(3)}$ is an initial segment of this path $\tilde{\varpi}$. Clearly, the path $\hat{w}^{(1)}\rho^{(2)}\bar{w}^{(2)}\rho^{(3)}\hat{w}^{(3)}$ is an initial segment of the path $\tilde{w}$. Recall that, in the first of these paths, we have $\varpi^{(1)} = \hat{w}^{(1)}\rho^{(2)}o^{(2)}$

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and $\varpi^{(2)} = \rho^{(2)}\sigma'(b)\eta'\bar{v}^{(2)}r^{(3)}$. Additionally, we have seen in the last paragraph but one that $\tilde{\varpi} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}$ and that the raw k-nest $\tilde{w}$ has thus been deduced k-tamely from the raw k-nest $\tilde{\varpi}$. Therefore, in view of Corollary 8.2, there exists a word $q^{(3)} \in E(C)^*$ satisfying $\mathrm{c}(q^{(3)}) \subseteq \mathrm{c}(v^{(2)}) \cap \mathrm{c}(v^{(3)}) = \mathrm{c}(w^{(2)}) \cap \mathrm{c}(w^{(3)})$ such that $\rho^{(2)}\sigma'(b)\eta'\bar{v}^{(2)}r^{(3)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} \rho^{(2)}\bar{w}^{(2)}\rho^{(3)}q^{(3)}$, that is, $\varpi^{(2)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} w^{(2)}q^{(3)}$. As before, by the construction of the path $\Im(w^{(2)})$, we know that $\Im(w^{(2)})w^{(2)}$ represents the identity in the relatively free group $E(C)^*/{\equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}}}$. Consequently, from the last formula obtained above we readily conclude that $\Im(w^{(2)})\varpi^{(2)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} q^{(3)}$.

Having thus clarified this particular circumstance, remember once more that we now assume that $\ell > 2$ and that we proceed further in this proof essentially by repeating the same considerations that we have made since we started to treat the case $\ell > 1$ until the moment when we have branched off in order to complete the case $\ell = 2$. However, as stated already in the previous paragraph, instead of the $2k$-nests $\tilde{v}$ and $\tilde{w}$, we now consider the raw k-nests $\tilde{\varpi}'$ and $\tilde{w}'$ which are composed, respectively, of the subwords of the properly intersecting k-sequences $\varpi^{(2)}, v^{(3)}, \dots, v^{(\ell)}$ and $w^{(2)}, \dots, w^{(\ell)}$. Further comments on this variation have been made in the preceding paragraph. It is now the matter of a routine check to verify that everything that has been carried out so far can essentially be accomplished again in this altered situation. The difference is, as it has already been mentioned, that instead of the variety of categories $\mathcal{W}_{p^h n,2k}$, the variety $\mathcal{W}_{p^h n,k}$ has to be used now from the beginning. Moreover, instead of the variety of groups $\mathcal{A}_p^h\circ\mathcal{K}$, it is the group variety $\mathcal{A}_p^{h-1}\circ\mathcal{K}$ which now has to be taken into account. In this connection, note that there is only one critical point in the considerations that we are now repeating in the currently modified situation which requires extra attention. Namely, it is the argument that has been updated towards the end of the previous paragraph which needs especial care. Everything else which follows afterwards can be taken over essentially without any amendments. Of course, since we have to deal with the group variety $\mathcal{A}_p^{h-1}\circ\mathcal{K}$, which can be viewed as the Mal'cev product $\mathcal{A}_p\circ(\mathcal{A}_p^{h-2}\circ\mathcal{K})$, the variety of categories $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$ has to be replaced, accordingly, with the variety $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})$. But, in the end, we come out with the variety $\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})$, either. Similarly, instead of the monoid variety $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h\circ\mathcal{K})$, we have to manage with the variety $\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$. But this limitation actually represents no obstacle in the further development of the arguments in the proof.

Thus let us try to summarize the principal conclusions that we come to in the course of the repetition of our preceding considerations, as we have specified it in the previous two paragraphs. Consider now the directed graph ∇ with $V(\nabla) = V(C)$ and $E(\nabla) = \mathrm{c}(v^{(2)}) \cap \mathrm{c}(v^{(3)}) = \mathrm{c}(w^{(2)}) \cap \mathrm{c}(w^{(3)})$, where the mappings α and ω pertaining to ∇ are obtained by restricting the mappings α and ω pertaining to C from $E(C)$ to $E(\nabla)$. Note that then $\mathrm{c}(q^{(3)}) \subseteq E(\nabla)$. Then it turns out again that the vertices $\omega(w^{(2)})$ and $\omega(\varpi^{(2)}) = \omega(v^{(2)})$ occur both in the same connected component of the graph ∇. Let us denote this connected component of ∇ by ∇_1. Furthermore, by the same arguments as before, it follows that then there exists a word $\mathfrak{b} \in E(\nabla_1)^*$ such that $q^{(3)} \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} \mathfrak{b}$. Then, since $\Im(w^{(2)})\varpi^{(2)} \equiv_{\mathcal{A}_p^{h-1}\circ\mathcal{K}} q^{(3)}$, altogether we get that $\Im(w^{(2)})\varpi^{(2)} \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} \mathfrak{b}$ for

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some word $\flat \in E(\nabla_1)^*$. We next exploit this piece of knowledge in quite the same way as before. Let $\bar{\sigma}$ and $\bar{\tau}$ be the substitutions assigning to every edge e in $\mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$ the loops in C on the vertex $\omega(w^{(2)})$ which are defined in entirely the same fashion as the loops assigned previously by the substitutions σ and τ to the edges e in $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)})$. We will not repeat the details. We only notice that all loops assigned by the substitution $\bar{\tau}$ to the edges e in $\mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$ begin with the power $z^{p^h n}$ of a loop z on the vertex $\omega(w^{(2)})$ such that $\mathrm{c}(z) = \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$. Further on, let $\bar{\eta}$ be the directed path in C going from $\omega(w^{(2)})$ to $\omega(v^{(2)})$ which is also constructed in exactly the same way as the previous directed path η going from $\omega(w^{(1)})$ to $\omega(v^{(1)})$. Recall that the loops assigned by the substitution $\bar{\tau}$ to the edges from the set $\mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$ all represent elements of the same maximal subgroup of the local monoid of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n}}}$ at the vertex $\omega(w^{(2)})$. Namely, they represent elements of the maximal subgroup containing the idempotent represented by the loop $z^{p^h n}$. The same statement holds true also of the quotient category $C^*/{\equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})}}$. Maximal subgroups of the local monoids of this quotient category, however, all belong to the group variety $\mathcal{A}_p^{h-2}\circ\mathcal{K}$. Hence, since we have $\Im(w^{(2)})\varpi^{(2)} \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} \flat$, applying the substitution $\bar{\tau}$, just as before, we obtain that

$$\bar{\tau}(\Im(w^{(2)})\varpi^{(2)}) \equiv_{\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \bar{\tau}(\flat).$$

Incorporating this fact into the respective preceding and accompanying calculations that we have not reproduced here since they are quite analogous to those we have carried out before, in the same way as previously when we have derived the formula (#), we now eventually come to the conclusion that

$$\varpi^{(2)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} w^{(2)}\bar{\sigma}(\flat)\bar{\eta}.$$

Recall yet that the segment $\bar{\sigma}(\flat)\bar{\eta}$ on the right-hand side of this formula satisfies $\mathrm{c}(\bar{\sigma}(\flat)\bar{\eta}) \subseteq \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$.

Remember once more that we have denoted by $r^{(3)}$ and $\rho^{(3)}$ the paths in C which are, respectively, the overlaps of the paths $v^{(2)}$, $v^{(3)}$ in $\tilde{v}$ and $w^{(2)}$, $w^{(3)}$ in $\tilde{w}$, and that we have denoted by $\bar{v}^{(2)}$, $\hat{v}^{(3)}$ and $\bar{w}^{(2)}$, $\hat{w}^{(3)}$ the paths in C for which $\hat{v}^{(2)} = \bar{v}^{(2)}r^{(3)}$, $v^{(3)} = r^{(3)}\hat{v}^{(3)}$ and $\hat{w}^{(2)} = \bar{w}^{(2)}\rho^{(3)}$, $w^{(3)} = \rho^{(3)}\hat{w}^{(3)}$. Now, multiplying both paths in the last formula displayed in the previous paragraph by $\hat{v}^{(3)}$ on the right, we obtain that

$$\varpi^{(2)}\hat{v}^{(3)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} w^{(2)}\bar{\sigma}(\flat)\bar{\eta}\hat{v}^{(3)}.$$

The next step we have to take is to modify the substitution $\bar{\sigma}$ on the edges from $E(\nabla_1)$ and the construction of the path $\bar{\eta}$ mentioned in the previous paragraph in just the same way that the substitution σ on the edges from $E(\Delta_1)$ and the path η have been changed formerly in order to yield the substitution σ' and the path η'. Proceeding in precisely the same way as then, we thus obtain from $\bar{\sigma}$ the substitution $\bar{\sigma}'$ and from $\bar{\eta}$ the path $\bar{\eta}'$.

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Furthermore, arguing exactly as before when we have been deducing the formula (##), this time we arrive at the conclusion that

$$w^{(2)}\bar{\sigma}(\flat)\bar{\eta}\hat{v}^{(3)} \equiv_{\mathcal{W}_{p^h n,k}} w^{(2)}\bar{\sigma}'(\flat)\bar{\eta}'\hat{v}^{(3)}.$$

Since, according to the above notes, we have $w^{(2)} = \rho^{(2)}\bar{w}^{(2)}\rho^{(3)}$, this formula together with the previous one yields that

$$\varpi^{(2)}\hat{v}^{(3)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \rho^{(2)}\bar{w}^{(2)}\rho^{(3)}\bar{\sigma}'(\flat)\bar{\eta}'\hat{v}^{(3)}.$$

Note that, by the definition of the modified substitution $\bar{\sigma}'$ and the modified construction of the path $\bar{\eta}'$, the segment $\bar{\sigma}'(\flat)\bar{\eta}'$ of the path on the right-hand side of this formula has now the property that $\mathrm{c}(\bar{\sigma}'(\flat)\bar{\eta}') \subseteq \mathrm{c}(v^{(3)}) = \mathrm{c}(w^{(3)})$.

Let now $o^{(3)}$ be the longest initial segment of the path $\bar{\sigma}'(\flat)\bar{\eta}'$ such that $\mathrm{c}(o^{(3)}) \subseteq \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$ (it may possibly be empty). Denote by $\mho^{(2)}$ the path $\rho^{(2)}\bar{w}^{(2)}\rho^{(3)}o^{(3)}$ and by $\mho^{(3)}$ the path $\rho^{(3)}\bar{\sigma}'(\flat)\bar{\eta}'\hat{v}^{(3)}$. Then $\mathrm{c}(\mho^{(2)}) = \mathrm{c}(v^{(2)}) = \mathrm{c}(w^{(2)})$, $\mathrm{c}(\mho^{(3)}) = \mathrm{c}(v^{(3)}) = \mathrm{c}(w^{(3)})$, and $\rho^{(3)}o^{(3)}$ is the overlap of $\mho^{(2)}$ and $\mho^{(3)}$ in the above path $\rho^{(2)}\bar{w}^{(2)}\rho^{(3)}\bar{\sigma}'(\flat)\bar{\eta}'\hat{v}^{(3)}$. The word $\mho^{(2)}$ has the weak $2k$-factorization property, since its initial segment $w^{(2)} = \rho^{(2)}\bar{w}^{(2)}\rho^{(3)}$ has this property, and the word $\mho^{(3)}$ has the left weak $2k$-factorization property, since, as it can be easily seen, its final segment $\hat{v}^{(3)}$ has this property. Let us now denote by $\tilde{\mho}'$ the path in C which one obtains from $\tilde{\varpi}'$ by replacing its initial segment $\varpi^{(2)}$ with the path $\rho^{(2)}\bar{w}^{(2)}\rho^{(3)}\bar{\sigma}'(\flat)\bar{\eta}' = w^{(2)}\bar{\sigma}'(\flat)\bar{\eta}'$, and let us denote by $\tilde{\mho}$ the path $\hat{w}^{(1)}\tilde{\mho}'$. Then, since $\tilde{\varpi}$ equals $\hat{w}^{(1)}\tilde{\varpi}'$, from the last formula displayed in the previous paragraph it readily follows that

$$\tilde{\varpi} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{\mho}.$$

Moreover, as before, by inspecting the whole derivation of the path $\tilde{\mho}$ from the path $\tilde{\varpi}$, as it has been outlined in the preceding two paragraphs, we again find out that the word $\tilde{\mho}$ has thus been deduced k-tamely from the raw k-nest $\tilde{\varpi}$. Therefore, the word $\tilde{\mho}$ itself must have the form of a raw k-nest composed of a properly intersecting sequence of ℓ maximal subwords relative to the k-factorization property. But, this time, it turns out that they are exactly the maximal subwords $w^{(1)}$, $\mho^{(2)}$, $\mho^{(3)}$ and $v^{(4)}, \ldots, v^{(\ell)}$ which arise in the course of the above k-tame derivation of $\tilde{\mho}$ from $\tilde{\varpi}$ and which form the properly intersecting k-sequence constituting the raw k-nest $\tilde{\mho}$. Furthermore, we have seen formerly that $\tilde{\varpi} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{w}$ and that, in this way, the raw $2k$-nest $\tilde{w}$ (which can again be viewed only as a k-nest) has been deduced k-tamely from the raw k-nest $\tilde{\varpi}$. In this paragraph, we have obtained that $\tilde{\varpi} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{\mho}$ and that the raw k-nest $\tilde{\mho}$ has thus been deduced k-tamely from the raw k-nest $\tilde{\varpi}$. Since this relationship between $\tilde{\varpi}$ and $\tilde{\mho}$ is symmetric, we hence also get that $\tilde{\mho} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{\varpi}$ and that $\tilde{\varpi}$ has thus been deduced k-tamely from $\tilde{\mho}$. Since the relation $\equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})}$ is evidently a subset of the congruence $\equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})}$ on $E(C)^*$, altogether we get that

$$\tilde{\mho} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{w}$$

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and that, in this way, the raw $2k$-nest $\tilde{w}$ (which can be viewed as a k-nest) has been deduced k-tamely from the raw k-nest $\tilde{\mho}$. Additionally, the first two maximal subwords $w^{(1)}$ and $\mho^{(2)}$ occurring at the beginning of the above properly intersecting k-sequence which gives rise to the raw k-nest $\tilde{\mho}$ are of such a form that the path $\hat{w}^{(1)}\rho^{(2)}\bar{w}^{(2)}\rho^{(3)}$ appears as an initial segment of the raw k-nest $\tilde{\mho}$. However, the same path appears also as an initial segment of the raw $2k$-nest $\tilde{w}$. As before, this prompt us to consider the paths $\tilde{\mho}''$ and $\tilde{w}''$ obtained from $\tilde{\mho}$ and $\tilde{w}$, respectively, by omitting from these paths their common initial segment $\hat{w}^{(1)}\rho^{(2)}\bar{w}^{(2)}$. Then $\tilde{\mho}''$ and $\tilde{w}''$ are coterminal paths in C. Moreover, the words $\tilde{\mho}''$ and $\tilde{w}''$ have evidently again the form of raw k-nests composed, respectively, of the maximal subwords occurring in the properly intersecting k-sequences $\mho^{(3)}, v^{(4)}, \dots, v^{(\ell)}$ and $w^{(3)}, \dots, w^{(\ell)}$, and the maximal subwords $\mho^{(3)}$ and $w^{(3)}$ appearing at the beginning of these k-sequences have, respectively, the left weak $2k$-factorization property and the weak $2k$-factorization property. Therefore, both these subwords have the k-factorization property. Thus, it appears that we are in the position when we can again repeat our former considerations, this time with the raw k-nests $\tilde{\mho}''$ and $\tilde{w}''$ which are composed of the subwords in the two properly intersecting k-sequences just mentioned, instead of the $2k$-nests $\tilde{v}$ and $\tilde{w}$ with which we have initially started our reasonings. Remember once more, in this context, that we have shown above that $\tilde{\mho} \equiv_{\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{w}$ and that the raw k-nest $\tilde{w}$ has thus been deduced k-tamely from the raw k-nest $\tilde{\mho}$. This makes it possible to apply again the tools we have prepared before and we have already used while treating the previous situation with the raw k-nests $\tilde{\varpi}'$ and $\tilde{w}'$, for instance. Note also, in passing, that the group variety $\mathcal{A}_p^{h-2}\circ\mathcal{K}$ is a subvariety of the monoid variety $\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})$, and so, from the last condition just quoted, we also obtain that $\tilde{\mho} \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} \tilde{w}$, whence, using cancellation, we get that $\tilde{\mho}'' \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} \tilde{w}''$. Furthermore, we have deduced formerly that $\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})} \tilde{\varpi}$, and we have seen above that $\tilde{\varpi} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{\mho}$. In both instances, the second k-nest has thus been deduced k-tamely from the first one. Hence altogether we get that

$$\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2}\circ\mathcal{K})} \tilde{\mho}$$

and that the raw k-nest $\tilde{\mho}$ has thus been deduced k-tamely from the maximal genuine $2k$-nest $\tilde{v}$. These notes should already make it clear how this proof has to be completed for arbitrary $\ell > 2$.

Thus, the conclusions we can draw from the reasonings we have carried out so far can be summarized as follows. If $\ell = 3$ then the raw k-nest $\tilde{\mho}$ is composed of three properly intersecting maximal subwords $w^{(1)}$, $\mho^{(2)}$, $\mho^{(3)}$ and the raw $2k$-nest $\tilde{w}$ is composed of three properly intersecting maximal subwords $w^{(1)}$, $w^{(2)}$, $w^{(3)}$. Moreover, in this situation, we have $\tilde{\mho}'' = \mho^{(3)}$, $\tilde{w}'' = w^{(3)}$ and $\tilde{\mho} = \hat{w}^{(1)}\rho^{(2)}\bar{w}^{(2)}\mho^{(3)}$, $\tilde{w} = \hat{w}^{(1)}\rho^{(2)}\bar{w}^{(2)}w^{(3)}$. We have seen near the end of the previous paragraph that $\tilde{\mho}'' \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} \tilde{w}''$, which means, in the present case, that $\mho^{(3)} \equiv_{\mathcal{A}_p^{h-2}\circ\mathcal{K}} w^{(3)}$. Hence, just in the same way as in the cases $\ell = 1$

and $\ell = 2$, we deduce, in the present case, that we have $\mho^{(3)} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2} \circ \mathcal{K})} w^{(3)}$, and therefore $\tilde{\mho} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2} \circ \mathcal{K})} \tilde{w}$. Since, at the end of the previous paragraph, we have also seen that $\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2} \circ \mathcal{K})} \tilde{\mho}$, together this yields that $\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-2} \circ \mathcal{K})} \tilde{w}$. This again clearly entails that $\tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}$, as needed. If $\ell > 3$ then we again repeat our former considerations, this time with the raw k-nests $\tilde{\mho}''$ and $\tilde{w}''$, as indicated in the previous paragraph, coming thus to conclusions which are quite analogous to those appearing at the close of the previous paragraph. It is now clear that we can continue further in this manner until we reach the number of $\ell - 1$ repetitions, and then we can end our calculations by the last step which is entirely similar to what we have done above in this paragraph in the case $\ell = 3$. Thus, in the general case, for any admissible positive integer ℓ, we eventually come to the finding that

$$\tilde{v} \equiv_{\mathcal{W}_{p^h n,k}(\mathcal{A}_p^{h-\ell+1} \circ \mathcal{K})} \tilde{w}.$$

Remember here from the first paragraph of the proof of (∗∗) that $\ell < h$. Thus this formula again entails that $\tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}$, as desired. □

12. The locality of the pseudovarieties DH for appropriate group pseudovarieties H

Let **H** be any pseudovariety of finite groups. Let **DH** be the class of all finite monoids all of whose regular $\mathcal{D}$-classes are groups from **H**. Then, clearly, **DH** is a pseudovariety of finite monoids. Remember that by ℓ**DH** we then denote the class of all finite categories all of whose local monoids belong to **DH**. Then, of course, ℓ**DH** is a pseudovariety of finite categories.

Similarly as in the case of varieties of groups, for any pseudovarieties **P** and **Q** of finite groups, we denote by **P**∘**Q** the class of all groups that are extensions of groups from **P** by groups from **Q**. Then, of course, **P**∘**Q** is a class of finite groups and it is again called the Mal'cev product of the pseudovariety **P** by the pseudovariety **Q**. Moreover, as it is the case for varieties of groups, **P**∘**Q** is again a pseudovariety of finite groups.

In the previous section, we have proved that the monoid pseudovariety **DG** is local. However, analysing this proof, we easily realize that, in this way, we have actually proved the locality of the monoid pseudovarieties **DH** for a large family of pseudovarieties **H** of finite groups. Namely, denoting by $\mathbf{A}_p$ the pseudovariety of all finite abelian groups of exponent p, for every prime number p, we thus get the following result.

Theorem 12.1. *Let* **H** *be any pseudovariety of finite groups having the property that, for some prime number* p, $\mathbf{A}_p \circ \mathbf{H} \subseteq \mathbf{H}$. *Then the pseudovariety of finite monoids* **DH** *is local.*

Proof. In order to verify this statement, take any non-empty finite category C in ℓ**DH**. This time, we need to find a finite monoid M in **DH** such that the category C divides M. As before, we use the fact that there exist positive integers k and n and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \simeq 1$ such that the category C belongs to the

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variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$. Returning to the methods used in Proposition 5.1 which provide such positive integers k and n and such a variety $\mathcal{K}$ of groups, we remember that $\mathcal{K}$ was obtained as the variety of groups generated by all maximal subgroups of all local monoids of the category C. As C is now from the pseudovariety of categories $\ell\mathbf{DH}$, it means that $\mathcal{K}$ is a variety of groups generated by several finite groups from the group pseudovariety $\mathbf{H}$. Thus finitely generated groups from $\mathcal{K}$ are finite and they belong to $\mathbf{H}$. Besides, as before, we may assume that $k > 2$ and that $n \geqslant 4k$. Recall also that the above variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ is itself locally finite by Proposition 5.2.

As before, consider now again C merely as a graph and take the free category C^* on this graph C. We have seen that then the free category on C relative to the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ can be represented in the form $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$. Since the category C, that is to say, the graph C is finite and the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ is locally finite, the relatively free category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ is finite. Besides, since subgroups of the local monoids of this category belong to the group variety $\mathcal{K}$, in view of the above notes, they are members of the group pseudovariety $\mathbf{H}$. Thus the category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ this time also belongs to the pseudovariety of categories $\ell\mathbf{DH}$. In addition, the identity graph mapping $id_C : C \to C$ can again be uniquely extended to the canonical homomorphism of categories $\varepsilon_C : C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}} \to C$. This homomorphism is a quotient homomorphism of categories.

Consider now the variety $\mathcal{A}_p$ of all abelian groups of exponent p, for the prime number p given above. As before, let μ be the number of elements in the set $E(C)$ of all edges of C. Then put $h = 2^\mu k^\mu$ and consider the Mal′cev product of group varieties $\mathcal{A}_p^h \circ \mathcal{K}$. Again, since all varieties of groups in this product are locally finite, we already know that then the variety of groups $\mathcal{A}_p^h \circ \mathcal{K}$ itself is also locally finite. Besides, it satisfies the identity $x^{p^h n} \simeq 1$. As before, consider further the variety of monoids $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$. From Proposition 3.2 we know that then this variety of monoids is locally finite as well. Let $E(C)^*$ be the free monoid on the set $E(C)$ of all edges of C. We have seen that then the free monoid on $E(C)$ relative to the monoid variety $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$ can be represented in the form $E(C)^*/{\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}}$. Since the set $E(C)$ is finite and the variety of monoids $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$ is locally finite, the relatively free monoid $E(C)^*/{\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}}$ is finite. Besides, subgroups of this monoid belong to the group variety $\mathcal{A}_p^h \circ \mathcal{K}$. Since these subgroups are finite, in view of the above notes on finite groups in $\mathcal{K}$, and in accordance with the definition of the Mal′cev product of varieties of groups and pseudovarieties of groups, which is based on the notion of extensions of groups, we come to the conclusion that the mentioned subgroups actually belong to the group pseudovariety $\mathbf{A}_p^h \circ \mathbf{H}$. Since, by our assumption, we have $\mathbf{A}_p \circ \mathbf{H} \subseteq \mathbf{H}$, it hence follows that the subgroups of the monoid $E(C)^*/{\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}}$ are, in fact, members of the pseudovariety of groups $\mathbf{H}$. Therefore, this monoid itself belongs to the pseudovariety of monoids $\mathbf{DH}$.

Now, having in view the note at the end of the last paragraph but one, we again realize that all which remains to do is to show that the above category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ divides the monoid $E(C)^*/{\equiv_{\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})}}$ treated in the previous paragraph. But this is exactly what has been done in the proof presented in the previous section. Since the last monoid

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has now been shown to belong to **DH**, this verifies the statement given in the above theorem. □

Recall that a pseudovariety **H** of finite groups is said to be extension closed if it satisfies the condition $\mathbf{H} \circ \mathbf{H} \subseteq \mathbf{H}$. Examples of extension closed pseudovarieties of finite groups include the pseudovarieties $\mathbf{G}_p$ of all p-groups, for arbitrary prime numbers p, and the pseudovariety $\mathbf{G}_{\text{sol}}$ of all finite solvable groups. From the previous result it follows straightforwardly that, for any non-trivial extension closed pseudovariety **H** of finite groups, the pseudovariety of finite monoids **DH** is local. In particular, the monoid pseudovarieties $\mathbf{DG}_p$, for arbitrary prime numbers p, and $\mathbf{DG}_{\text{sol}}$ are local.

However, as we shall see further on, there are other natural examples of pseudovarieties **H** of finite groups having the property that the corresponding pseudovarieties of finite monoids **DH** are local, for which the locality of **DH** cannot be established by an application of the previous theorem. But a minor improvement of the proof proposed in the previous section makes it possible to obtain the following generalization of the previous result.

Theorem 12.2. *Let I be any non-empty set. For every $i \in I$, let $\mathbf{H}_i$ be a pseudovariety of finite groups and let p_i be a prime number such that the pseudovariety $\mathbf{H}_i$ satisfies the condition $\mathbf{A}_{p_i} \circ \mathbf{H}_i \subseteq \mathbf{H}_i$. Let $\mathbf{Q} = \bigvee_{i \in I} \mathbf{H}_i$ be the join of the pseudovarieties $\mathbf{H}_i$, for all $i \in I$, in the complete lattice of all pseudovarieties of finite groups. Then the pseudovariety of finite monoids* **DQ** *is local.*

Proof. In order to verify this strengthened statement, take again any non-empty finite category C in $\ell\mathbf{DQ}$. This once, we need to find a finite monoid M in **DQ** such that the category C divides M. Once again, we use the fact that there exist positive integers k and n and a finitely generated variety $\mathcal{K}$ of groups satisfying the identity $x^n \doteq 1$ such that the category C belongs to the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$. Returning once more to the method providing such positive integers k and n and such a variety $\mathcal{K}$ of groups, we again remember that $\mathcal{K}$ was obtained as the group variety generated by all maximal subgroups of all local monoids of the category C. As C is now from the pseudovariety of categories $\ell\mathbf{DQ}$, it means that $\mathcal{K}$ is a group variety generated by a finite collection of finite groups from the group pseudovariety **Q**. In fact, taking the direct product of this finite collection of finite groups, we can say that $\mathcal{K}$ is a group variety generated by one finite group G from the group pseudovariety **Q**. As $\mathbf{Q} = \bigvee_{i \in I} \mathbf{H}_i$, it means that there exists a non-empty finite subset $J \subseteq I$ and there exist finite groups $F_i \in \mathbf{H}_i$, for all $i \in J$, such that the finite group G divides the direct product of these finite groups F_i, for all $i \in J$. This incites us to redefine the variety $\mathcal{K}$ of groups by letting now $\mathcal{K}$ be the group variety generated by the finite family of finite groups F_i, for all $i \in J$. Otherwise stated, denoting by $\mathcal{L}_i$ the variety of groups generated by the finite group F_i, for every $i \in J$, we let $\mathcal{K}$ be the finite join $\bigvee_{i \in J} \mathcal{L}_i$ of these group varieties. Furthermore, we have also to modify the positive integer n by letting it to be henceforth the least common multiple of the previous value of n and the exponents of the finite groups F_i, for all $i \in J$. It is clear that then the current group variety $\mathcal{K}$ again satisfies the identity $x^n \doteq 1$. It is also obvious that then the category C belongs to the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$, with the updated value of

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the positive integer n and with the newly defined finitely generated variety $\mathcal{K}$ of groups. Notice yet that finitely generated groups from $\mathcal{K}$ are finite and that they belong to the finite join of group pseudovarieties $\bigvee_{i\in J}\mathbf{H}_i$. Likewise, for every $i \in J$, finitely generated groups from $\mathcal{L}_i$ are finite and they belong to the group pseudovariety $\mathbf{H}_i$. Besides, as before, we may still assume that $k > 2$ and that $n \geqslant 4k$. Recall also once more that the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ obtained in this way is itself again locally finite.

As before, consider once again C merely as a graph and take the free category C^* on this graph C. Then the free category on C relative to the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ can again be represented in the form $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$. Since the category C (viewed as a graph) is finite and the variety of categories $\mathcal{W}_{n,k}(\mathcal{K})$ is locally finite, we see, as above, that the relatively free category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ is finite. Besides, since subgroups of the local monoids of this category belong to the group variety $\mathcal{K}$, according to the above notes, they are now members of the join of group pseudovarieties $\bigvee_{i\in J}\mathbf{H}_i$, and, consequently, they belong to the pseudovariety of groups $\mathbf{Q}$. Therefore, this once, the category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ belongs to the pseudovariety of categories $\ell\mathbf{DQ}$. Furthermore, as above, the identity graph mapping $id_C : C \to C$ can be uniquely extended to the canonical homomorphism of categories $\boldsymbol{\varepsilon}_C : C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}} \to C$. This homomorphism is a quotient homomorphism of categories.

Consider, this time, the varieties $\mathcal{A}_{p_i}$ of all abelian groups of exponent p_i, for all prime numbers p_i where $i \in J$. As before, let again μ be the number of elements in the set $E(C)$ of all edges of C. Then put $h = 2^\mu k^\mu$ and, for every $i \in J$, consider the Mal'cev product of group varieties $\mathcal{A}_{p_i}^h \circ \mathcal{L}_i$. Once again, since all varieties of groups in this product are locally finite, we know that then the group variety $\mathcal{A}_{p_i}^h \circ \mathcal{L}_i$ is also locally finite, for every $i \in J$. Let now $\mathcal{M}$ be the join of group varieties $\bigvee_{i\in J}\mathcal{A}_{p_i}^h \circ \mathcal{L}_i$. Since $\mathcal{M}$ is the join of finitely many locally finite group varieties, it is itself a locally finite variety of groups, as well. Let $\wp$ be the product of the prime numbers p_i, for all $i \in J$. Then the variety of groups $\mathcal{M}$ clearly satisfies the identity $x^{\wp^h n} \simeq 1$. Besides, since $\mathcal{K}$ is the join of group varieties $\bigvee_{i\in J}\mathcal{L}_i$, it follows that $\mathcal{K}$ is a subvariety of $\mathcal{M}$. Consider next the variety of monoids $\mathcal{V}_{\wp^h n,4k}(\mathcal{M})$. We know that then this variety of monoids is also locally finite. Let $E(C)^*$ be the free monoid on the set $E(C)$ of all edges of C. Then, similarly as before, the free monoid on $E(C)$ relative to the monoid variety $\mathcal{V}_{\wp^h n,4k}(\mathcal{M})$ can be represented in the form $E(C)^*/{\equiv_{\mathcal{V}_{\wp^h n,4k}(\mathcal{M})}}$. Since the set $E(C)$ is finite and the variety of monoids $\mathcal{V}_{\wp^h n,4k}(\mathcal{M})$ is locally finite, we see, as before, that the relatively free monoid $E(C)^*/{\equiv_{\mathcal{V}_{\wp^h n,4k}(\mathcal{M})}}$ is finite. Moreover, subgroups of this monoid belong to the group variety $\mathcal{M}$. Let S be the direct product of all maximal subgroups of this relatively free monoid. Then also S is a finite group in $\mathcal{M}$. By the definition of the group variety $\mathcal{M}$, this means that, for every $i \in J$, there exists a group T_i in the Mal'cev product of group varieties $\mathcal{A}_{p_i}^h \circ \mathcal{L}_i$ such that the finite group S divides the direct product of the groups T_i, for all $i \in J$. In this situation, it is clear that the groups T_i may be assumed finitely generated, and therefore finite, for all $i \in J$, since the varieties of groups $\mathcal{A}_{p_i}^h \circ \mathcal{L}_i$ are all locally finite. We have seen that, for every $i \in J$, by the definition of the group variety $\mathcal{L}_i$, finite groups from $\mathcal{L}_i$ belong to the group pseudovariety $\mathbf{H}_i$. Consequently, in accordance with the definition of the Mal'cev product of varieties of

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groups and pseudovarieties of groups, we may conclude that, for every $i \in J$, the finite group T_i actually belongs to the group pseudovariety $\mathbf{A}_{p_i}^h \circ \mathbf{H}_i$. Since, by our assumption, we have $\mathbf{A}_{p_i} \circ \mathbf{H}_i \subseteq \mathbf{H}_i$, for all $i \in J$, it hence follows that, for each $i \in J$, the group T_i belongs to the pseudovariety of groups $\mathbf{H}_i$. Thus the above finite group S, which divides the direct product of these groups T_i, belongs to the pseudovariety of groups $\mathbf{Q}$. Since the subgroups of the above relatively free monoid $E(C)^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}}$ all divide the group S, they belong to the pseudovariety of groups $\mathbf{Q}$, as well. Therefore, this monoid itself belongs to the pseudovariety of monoids $\mathbf{DQ}$.

Now, having in view the note at the end of the last paragraph but one, we realize once again that, this time, we will be done if we show that the above category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$ divides the monoid $E(C)^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}}$ which has been dealt with in the previous paragraph. Since this monoid has been shown to belong to $\mathbf{DQ}$, this will verify the principal statement made above. Thus, similarly as in the previous section, consider now the congruence $\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}$ restricted to the hom-sets of the free category C^*. That is, more precisely, take the set of all pairs (s, t) of coterminal paths in C such that $s \equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})} t$ holds, where the empty paths on the vertices of C are treated as the empty word in $E(C)^*$. In this way, we again obtain a congruence on C^* and we can form the corresponding quotient category, which we will denote, as before, simply by $C^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}}$. Then, with this device at hand, just as in the previous section, we come to the obvious homomorphism of the quotient categories

$$\varphi_C : C^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}} \to E(C)^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}} .$$

This is, of course, a faithful homomorphism of categories. Now, in order to establish the above-mentioned division, it remains to provide a quotient homomorphism of the category $C^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}}$ onto the category $C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$. This homomorphism will send every vertex of C to itself. As far as edges of these categories are concerned, just as in the previous section, we have to prove first that, for arbitrary coterminal paths s, t in C,

$$s \equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})} t \quad \text{implies} \quad s \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} t. \tag{$\dagger$}$$

Once we have this confirmed, we can complete the definition of the proposed quotient homomorphism by sending, for every path s in C, the class of the restricted congruence $\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}$ on C^* containing the path s to the class of the congruence $\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}$ on C^* containing the path s. In this way, we will then obtain a quotient homomorphism

$$\psi_C : C^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}} \to C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}}$$

of the mentioned categories. Altogether, we will thus come to the finding that

$$C^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{K})}} \prec E(C)^*/{\equiv_{\mathcal{V}_{\wp h_{n,4k}}(\mathcal{M})}},$$

as required.

Proceeding further in the same way as in the previous section, we next reduce the task of checking the implication in (†) to the verification of the following condition. Let $\tilde{v}$ and

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$\tilde{w}$ be coterminal paths in C and let $\tilde{v}$ have the form of a maximal genuine $2k$-nest. Then we have to prove that

$$\tilde{v} \equiv_{\mathcal{V}_{\wp^h n,4k}(\mathcal{M})} \tilde{w} \quad \text{implies} \quad \tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}. \tag{††}$$

Thus suppose that $\tilde{v} \equiv_{\mathcal{V}_{\wp^h n,4k}(\mathcal{M})} \tilde{w}$. Since $\wp^h n \geqslant 4k$, from the beginning of § 6 we know that $\mathcal{U}_{4k} \subseteq \mathcal{V}_{\wp^h n,4k}(\mathcal{M})$. Therefore $\tilde{v} \equiv_{\mathcal{V}_{\wp^h n,4k}(\mathcal{M})} \tilde{w}$ entails that $\tilde{v} \equiv_{\mathcal{U}_{4k}} \tilde{w}$. Accordingly, from Lemma 7.2 (with $2k$ in place of k) we obtain that the word $\tilde{w}$ has thus been deduced $4k$-tamely from the maximal genuine $2k$-nest $\tilde{v}$. Consequently, the word $\tilde{w}$ itself then has the form of a raw $2k$-nest. Moreover, for some positive integer ℓ, the maximal genuine $2k$-nest $\tilde{v}$ is composed of ℓ truly maximal subwords $v^{(1)}, \dots, v^{(\ell)}$ having the $2k$-factorization property which form a properly intersecting $2k$-sequence in $\tilde{v}$, and, consequently, according to the notes in § 7, the raw $2k$-nest $\tilde{w}$ is composed of ℓ maximal subwords $w^{(1)}, \dots, w^{(\ell)}$ which form a properly intersecting $2k$-sequence in $\tilde{w}$. Recall once again that then $\mathrm{c}(v^{(1)}) = \mathrm{c}(w^{(1)}), \dots, \mathrm{c}(v^{(\ell)}) = \mathrm{c}(w^{(\ell)})$. Remember also that we know from Lemma 7.3 that $\ell < 2^\mu k^\mu$, that is, we have $\ell < h$.

The proof of the implication in (††) then proceeds further in the same manner as in the previous section. Of course, the prime number p will now be replaced with the product of prime numbers $\wp$. Instead of the group variety $\mathcal{A}_p^h \circ \mathcal{K}$ appearing in the arguments of the previous section, the group variety $\mathcal{M}$ defined above will now figure in our present arguments, and instead of the variety of categories $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^h \circ \mathcal{K})$ occurring in the previous section, the variety of categories $\mathcal{W}_{\wp^h n,4k}(\mathcal{M})$ will appear now. Thus, using the fact that $\tilde{v} \equiv_{\mathcal{V}_{\wp^h n,4k}(\mathcal{M})} \tilde{w}$ yields $\tilde{v} \equiv_{\mathcal{M}} \tilde{w}$, we see that if $\ell = 1$ then the proof can be accomplished proceeding entirely analogously as in the previous section. If $\ell > 1$ then, however, some of the arguments used previously must be somewhat updated now. In that case, quite analogously as in the previous section, we come to the directed path $\Im(w^{(1)})$ in C from $\omega(w^{(1)})$ to $\alpha(w^{(1)})$ such that $\mathrm{c}(\Im(w^{(1)})) = \mathrm{c}(w^{(1)}) = \mathrm{c}(v^{(1)})$ and we deduce from Lemma 8.1 the fact that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{M}} q^{(2)}$ for some word $q^{(2)} \in E(C)^*$ satisfying $\mathrm{c}(q^{(2)}) \subseteq \mathrm{c}(v^{(1)}) \cap \mathrm{c}(v^{(2)})$. Next consider the directed graph Δ with $V(\Delta) = V(C)$ and $E(\Delta) = \mathrm{c}(v^{(1)}) \cap \mathrm{c}(v^{(2)})$ defined in the same way as in the previous section. Since $\mathcal{M}$ is the join of group varieties $\bigvee_{i \in J} \mathcal{A}_{p_i}^h \circ \mathcal{L}_i$, it follows that, for any $i \in J$, the variety $\mathcal{A}_{p_i}$ is contained as a subvariety in $\mathcal{M}$, and so, from $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{M}} q^{(2)}$ we get that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_{p_i}} q^{(2)}$. Using this fact and arguing similarly as in the previous section, we again come to the conclusion that both vertices $\omega(w^{(1)})$ and $\omega(v^{(1)})$ occur in the same connected component of Δ, which we denote by Δ_1. The other connected components of Δ are denoted by $\Delta_2, \dots, \Delta_\nu$, if there are any.

The core of the proof, which has to be recast thoroughly, consists of the subsequent considerations. As $\mathcal{M}$ is the join of group varieties $\bigvee_{i \in J} \mathcal{A}_{p_i}^h \circ \mathcal{L}_i$, the congruence $\equiv_{\mathcal{M}}$ on $E(C)^*$ is the intersection of congruences $\bigcap_{i \in J} \equiv_{\mathcal{A}_{p_i}^h \circ \mathcal{L}_i}$. Therefore, from $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{M}} q^{(2)}$ it follows that, for every $i \in J$, we have $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_{p_i}^h \circ \mathcal{L}_i} q^{(2)}$. Notice that, for each $i \in J$, the group variety $\mathcal{A}_{p_i}^h \circ \mathcal{L}_i$ can be viewed as the Mal′cev product $\mathcal{A}_{p_i} \circ (\mathcal{A}_{p_i}^{h-1} \circ \mathcal{L}_i)$. Thus, according to the former notes on the free groups in the Mal′cev products of group varieties, for every $i \in J$, the condition $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_{p_i}^h \circ \mathcal{L}_i} q^{(2)}$ is

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equivalent to the conditions

$$\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i} q^{(2)} \quad \text{and} \quad \boldsymbol{\pi}_i(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{A}_{p_i}} \boldsymbol{\pi}_i(q^{(2)}),$$

where $\boldsymbol{\pi}_i(\Im(w^{(1)})v^{(1)})$ and $\boldsymbol{\pi}_i(q^{(2)})$ are the paths in the Cayley graph $\varGamma_i$ of the relatively free group $E(C)^*/{\equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i}}$ determined, respectively, by the words $\Im(w^{(1)})v^{(1)}$ and $q^{(2)}$. Now it is clear that, for every $i \in J$, the condition $\boldsymbol{\pi}_i(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{A}_{p_i}} \boldsymbol{\pi}_i(q^{(2)})$ can be employed in precisely the same way as it has been done with the condition $\boldsymbol{\pi}(\Im(w^{(1)})v^{(1)}) \equiv_{\mathcal{A}_p} \boldsymbol{\pi}(q^{(2)})$ in the proof contained in the previous section. Proceeding in this way, just as in the previous section, we arrive at the conclusion that, for each $i \in J$, there exists a word $b_i \in E(\varDelta_1)^*$ such that $q^{(2)} \equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i} b_i$. Remember now that $\mathrm{c}(q^{(2)}) \subseteq E(\varDelta)$ and denote by $\boldsymbol{b}$ the word in $E(\varDelta_1)^*$ obtained from $q^{(2)}$ by deleting from it all edges lying in the set $E(\varDelta)-E(\varDelta_1)$, that is, occurring in the set $E(\varDelta_2)\cup\cdots\cup E(\varDelta_\nu)$. Since $\equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i}$ is a fully invariant congruence on the free monoid $E(C)^*$, for each $i \in J$, from the formula $q^{(2)} \equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i} b_i$ we get that $\boldsymbol{b} \equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i} b_i$, using the substitution assigning the identity (that is, the empty word of $E(C)^*$) to all elements in the set $E(\varDelta_2)\cup\cdots\cup E(\varDelta_\nu)$. Elements in the set $E(\varDelta_1)$ are left unaltered by this substitution. Using transitivity of $\equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i}$, we hence deduce that $q^{(2)} \equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i} \boldsymbol{b}$ holds for all $i \in J$. Thus, if we denote by $\mathcal{N}$ the join of group varieties $\bigvee_{i\in J} \mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i$, we can conclude that $q^{(2)} \equiv_{\mathcal{N}} \boldsymbol{b}$, since the congruence $\equiv_{\mathcal{N}}$ on $E(C)^*$ is the intersection of congruences $\bigcap_{i\in J} \equiv_{\mathcal{A}_{p_i}^{h-1}\circ\mathcal{L}_i}$. Then, clearly, $\mathcal{K}$ is a subvariety of $\mathcal{N}$, and $\mathcal{N}$ is a subvariety of $\mathcal{M}$. Consequently, we have $\equiv_{\mathcal{M}} \subseteq \equiv_{\mathcal{N}}$. Therefore, from the formula $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{M}} q^{(2)}$ obtained in the previous paragraph and from the formula $q^{(2)} \equiv_{\mathcal{N}} \boldsymbol{b}$ deduced above we get that $\Im(w^{(1)})v^{(1)} \equiv_{\mathcal{N}} \boldsymbol{b}$ for a certain word $\boldsymbol{b} \in E(\varDelta_1)^*$.

From now on, we can resume advancing in accordance with the proof which has been presented in the previous section. Of course, the prime number p will stay replaced with the product of prime numbers $\wp$ all along. Furthermore, instead of the group variety $\mathcal{A}_p^{h-1}\circ\mathcal{K}$ appearing in the subsequent arguments of the proof in the previous section, the group variety $\mathcal{N}$ defined above will occur in the respective arguments of our present proof. Accordingly, instead of the variety of categories $\mathcal{W}_{p^h n,4k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$ appearing subsequently in the proof given in the previous section, the variety of categories $\mathcal{W}_{\wp^h n,4k}(\mathcal{N})$ will figure now. Later in the proof, nevertheless, this variety will be replaced with the variety $\mathcal{W}_{\wp^h n,k}(\mathcal{N})$, just as it has happened before. Likewise, instead of the monoid variety $\mathcal{V}_{p^h n,4k}(\mathcal{A}_p^h\circ\mathcal{K})$ appearing in the previous section, it is the monoid variety $\mathcal{V}_{\wp^h n,4k}(\mathcal{M})$ which will have its role to play now, and instead of the monoid variety $\mathcal{V}_{p^h n,k}(\mathcal{A}_p^{h-1}\circ\mathcal{K})$ emerging in the later considerations of the previous proof, the monoid variety $\mathcal{V}_{\wp^h n,k}(\mathcal{N})$ will appear in the respective considerations of the present proof. Thus, if $\ell = 2$ then, with these updates, the proof can be completed in exactly the same way as in the previous section. If $\ell > 2$ then, just as it has been described in the previous section, the proof proceeds further essentially by repeating the considerations we have carried out so far since we started to deal with the case $\ell > 1$. However, as before, this repetition has to be so conducted under somewhat modified circumstances. The details have been explicated in the respective paragraphs of the previous section. Of course, this repetition has also to be performed now with the amendments analogous to those specified in the preceding paragraphs of this last section. Further on, it has been explained at the end of the previous section that this procedure has to be repeated $\ell - 1$ times in this manner before it ends with the last step which is quite similar as in the case $\ell = 2$. Thus, in the general case, for arbitrary admissible positive integer ℓ, we eventually find out that

$$\tilde{v} \equiv_{\mathcal{W}_{\wp^h n,k}(\mathcal{P})} \tilde{w},$$

where $\mathcal{P}$ is the join of group varieties $\bigvee_{i\in J} \mathcal{A}_{p_i}^{h-\ell+1} \circ \mathcal{L}_i$. Recall once again that $\ell < h$. Hence it is clear that $\mathcal{K}$ is a subvariety of $\mathcal{P}$. Consequently, from the above formula we obtain that $\tilde{v} \equiv_{\mathcal{W}_{n,k}(\mathcal{K})} \tilde{w}$, as required. □

In particular, from the theorem we have just proved it follows that if **Q** is a join of a non-empty family of non-trivial extension closed pseudovarieties of finite groups, then the pseudovariety of finite monoids **DQ** is local. As an example of a pseudovariety of finite groups with this property, we can mention the pseudovariety $\mathbf{G}_{\mathrm{nil}}$ of all finite nilpotent groups, since it is the join of the pseudovarieties $\mathbf{G}_p$ of all p-groups where p runs over all prime numbers. Thus $\mathbf{DG}_{\mathrm{nil}}$ is a local pseudovariety of finite monoids.

Another example of a pseudovariety of finite groups to which the hypotheses of the last theorem apply is the class of all finite supersolvable groups. Recall that a group is supersolvable if it has a series of normal subgroups whose factors are cyclic (see the classic book [**6**] by Hall, for instance). The class of all finite supersolvable groups forms a pseudovariety of finite groups, which we denote by $\mathbf{G}_{\mathrm{ssol}}$. Quite recently, Auinger and Steinberg have proved in § 2 of [**5**] that $\mathbf{G}_{\mathrm{ssol}}$ is the join over all prime numbers p of the pseudovarieties $\mathbf{G}_p \circ \mathbf{A}_{p-1}$ where $\mathbf{A}_{p-1}$ is the pseudovariety of all finite abelian groups of exponent dividing $p - 1$. Consequently, by the last theorem, also $\mathbf{DG}_{\mathrm{ssol}}$ is a local pseudovariety of finite monoids.

13. The non-locality of the pseudovariety DAb

We conclude this paper with an example demonstrating that the specific assumptions about the pseudovarieties of finite groups related to the Mal′cev products of these pseudovarieties cannot be completely omitted from the results obtained in the previous section. Namely, we are going to show that, for the pseudovariety **Ab** of all finite abelian groups, the pseudovariety of finite monoids **DAb** is not local.

Consider the graph Γ having the diagram shown in Figure 1.

Take the free category Γ^* over this graph Γ. Next choose any prime number p and any integers n, k satisfying $n \geqslant k > 1$ such that p divides n. Remember that we have denoted by $\mathcal{A}_p$ the variety of all abelian groups of exponent p. Then $\mathcal{A}_p$ is a locally finite variety of groups, and so we may consider the variety of categories $\mathcal{W}_{n,k}(\mathcal{A}_p)$ introduced in § 5. We have seen in Proposition 5.2 that then $\mathcal{W}_{n,k}(\mathcal{A}_p)$ is again a locally finite variety of categories. Thus the free category on the finite graph Γ shown in Figure 1 relative to this variety $\mathcal{W}_{n,k}(\mathcal{A}_p)$ is finite. Note that this relatively free category can be represented in the form $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ where $\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}$ is the congruence on Γ^* consisting of all

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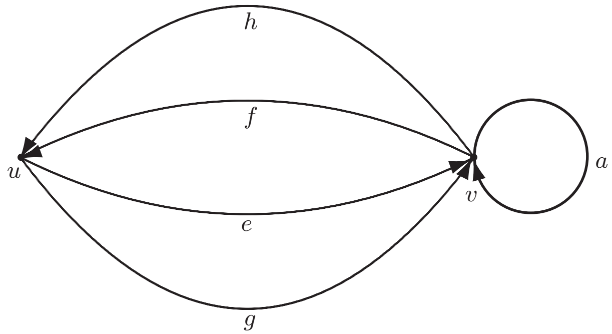

Figure 1.

pairs of coterminal paths in Γ which constitute path identities over Γ that are satisfied in $\mathcal{W}_{n,k}(\mathcal{A}_p)$. By the definition of the variety of categories $\mathcal{W}_{n,k}(\mathcal{A}_p)$, the relatively free category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ thus belongs to the pseudovariety of finite categories $\ell\mathbf{DAb}$, since it is itself finite. However, we will further see that this relatively free category does not divide any monoid in the pseudovariety of finite monoids **DAb**. This will establish the fact that this pseudovariety **DAb** is not local.

In order to prove this statement, we first show that the condition

$$(eaf)^n eafgh(gah)^n \not\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)} (eaf)^n efgah(gah)^n$$

holds, that is, we consecutively verify the fact that the two loops $(eaf)^n eafgh(gah)^n$ and $(eaf)^n efgah(gah)^n$ in the graph Γ in Figure 1 on the vertex u represent distinct elements of the local monoid at u of the quotient category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$. To this end, assume that r is any loop in Γ on u having the property that $(eaf)^n eafgh(gah)^n \equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)} r$. We claim that then there exist loops s, t in Γ on u such that $r = st$, $\mathrm{c}(s) = \{a, e, f\}$, $\mathrm{c}(t) = \{a, g, h\}$, and the number of occurrences of the edge a in t is divisible by p. As p divides n, this clearly entails that such a loop r must be distinct from the loop $(eaf)^n efgah(gah)^n$, confirming thus the condition displayed above.

According to the definition of the variety $\mathcal{W}_{n,k}(\mathcal{A}_p)$, the following statement holds. If $(eaf)^n eafgh(gah)^n \equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)} r$, then there exist a non-negative integer m, loops $q_0, q_1, \ldots, q_m$ in Γ on u such that $q_0 = (eaf)^n eafgh(gah)^n$, $q_m = r$, and for every $j \in \{1, \ldots, m\}$, there exist paths c_j, d_j in Γ satisfying $\alpha(c_j) = u$, $\omega(d_j) = u$ and non-empty paths w_j, ϖ_j in Γ satisfying $\omega(c_j) = \alpha(w_j) = \alpha(\varpi_j)$, $\alpha(d_j) = \omega(w_j) = \omega(\varpi_j)$ such that $q_{j-1} = c_j w_j d_j$, $q_j = c_j \varpi_j d_j$, and the coterminal paths w_j and ϖ_j satisfy one of the following conditions:

$\{w_j, \varpi_j\} = \{\eta^n, \eta^{2n}\}$	for some loop η in Γ on the vertex $\omega(c_j) = \alpha(d_j)$,
$\{w_j, \varpi_j\} = \{(\eta\vartheta)^n, (\vartheta\eta)^n\}$	for some loops η, ϑ in Γ on the vertex $\omega(c_j) = \alpha(d_j)$,

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$\{w_j, \varpi_j\} = \{\varsigma(z)^n, \varsigma(z)^{n+1}\}$	for some word z over some non-empty set X of variables such that the identity $z \doteq 1$ is satisfied in $\mathcal{A}_p$, and for some substitution ς assigning to every variable $x \in X$ some loop $\varsigma(x)$ in Γ on the vertex $\omega(c_j) = \alpha(d_j)$,
$\{w_j, \varpi_j\} = \{\tau_1 \ldots \tau_k, \sigma^n \tau_1 \cdots \tau_k\}$	for some loop σ in Γ and some paths $\tau_1, \ldots, \tau_k$ in Γ such that $\omega(c_j) = \alpha(\sigma) = \omega(\sigma) = \alpha(\tau_1)$, $\omega(\tau_1) = \alpha(\tau_2), \ldots, \omega(\tau_{k-1}) = \alpha(\tau_k)$, $\omega(\tau_k) = \alpha(d_j)$, and $\mathrm{c}(\sigma) = \mathrm{c}(\tau_1) = \cdots = \mathrm{c}(\tau_k)$.

Now, the above claim regarding the loop r can be verified by induction on m. Otherwise stated, one can check consecutively that the above claim on the loop r holds for all loops $q_0, q_1, \ldots, q_m$. For $q_0 = (eaf)^n eafgh(gah)^n$ it suffices to notice that both $(eaf)^n eaf$ and $gh(gah)^n$ are loops on u and that $\mathrm{c}((eaf)^n eaf) = \{a, e, f\}$ and $\mathrm{c}(gh(gah)^n) = \{a, g, h\}$. Of course, the number of occurrences of the edge a in $gh(gah)^n$ is divisible by p, as p divides n. Next assume that for some $j \in \{1, \ldots, m\}$, the loop q_{j-1} is of the form $q_{j-1} = st$ where s, t are loops on u such that $\mathrm{c}(s) = \{a, e, f\}$, $\mathrm{c}(t) = \{a, g, h\}$, and the number of occurrences of the edge a in t is divisible by p. Since the loops q_{j-1} and q_j are of the form $q_{j-1} = c_j w_j d_j$ and $q_j = c_j \varpi_j d_j$ where the coterminal paths w_j and ϖ_j satisfy one of the four conditions displayed above, it is fairly easy hence to deduce that the path w_j in its marked position in q_{j-1} must appear as a segment of either the loop s or the loop t. This follows from the form of the paths w_j and ϖ_j in each of the above four conditions, since $n \geqslant k > 1$. A simple checkup of the form of these paths in either of these four conditions then also reveals immediately that the loop q_j has properties analogous to those we have assumed to be satisfied by the loop q_{j-1}. That is to say, the above claim holds also for the loop q_j. In the end, we thus find out that this claim holds for the loop $q_m = r$, as stated above.

Having thus confirmed, in this way, the fact that the two loops $(eaf)^n eafgh(gah)^n$ and $(eaf)^n efgah(gah)^n$ on u in the graph Γ in Figure 1 represent, indeed, distinct elements of the local monoid at u of the quotient category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$, we can proceed to prove that this finite category does not divide any finite monoid in the pseudovariety **DAb**. For the sake of simplicity, for every path q in Γ, we will denote briefly by $[q]$ the class of the congruence $\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}$ containing this path q, that is, $[q]$ will stand for the element of the category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ represented by the path q. Now, by contradiction, suppose that there exist a finite monoid M in **DAb**, a finite category B, a faithful homomorphism $\varphi : B \to M$ and a quotient homomorphism $\psi : B \to \Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$. Since ψ is bijective on the vertices of B, the category B has exactly two vertices. Let $\bar{u} \in V(B)$ be that vertex for which $\psi(\bar{u}) = u$ and let $\bar{v} \in V(B)$ be that vertex for which $\psi(\bar{v}) = v$. Since ψ is surjective on the hom-sets of B, it is possible to select edges $\overline{e}, \overline{g} \in B(\bar{u}, \bar{v})$, $\overline{f}, \overline{h} \in B(\bar{v}, \bar{u})$ and $\overline{a} \in B(\bar{v}, \bar{v})$ such that $\psi(\overline{e}) = [e]$, $\psi(\overline{f}) = [f]$, $\psi(\overline{g}) = [g]$, $\psi(\overline{h}) = [h]$ and $\psi(\overline{a}) = [a]$. Then we may consider the elements $\varphi(\overline{e})$, $\varphi(\overline{f})$, $\varphi(\overline{g})$, $\varphi(\overline{h})$ and $\varphi(\overline{a})$ of the monoid M. Let κ be a positive integer such that, for every element $\mu \in M$, μ^κ is an idempotent of M. Then let us consider the elements $\varphi((\overline{eaf})^\kappa \overline{eafgh}(\overline{gah})^\kappa)$ and $\varphi((\overline{eaf})^\kappa \overline{efgah}(\overline{gah})^\kappa)$ of M. Since M is in **DAb**, and hence in **DG**, according to the defining property of the positive

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integer κ, this monoid M belongs to the variety of monoids $\mathcal{V}_\kappa$. Thus, by the results on the monoids in $\mathcal{V}_\kappa$ that we have deduced in § 2 in the text preceding Proposition 2.3, we know, in particular, that the elements $\varphi((\overline{eaf})^\kappa)\varphi(\overline{e})$, $\varphi((\overline{eaf})^\kappa)\varphi(\overline{f})$ and $\varphi((\overline{eaf})^\kappa)\varphi(\overline{a})$ of M all lie in the maximal subgroup of M containing the idempotent $\varphi((\overline{eaf})^\kappa)$ and that we have the equality

$$\varphi((\overline{eaf})^\kappa\overline{eaf}) = \varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi((\overline{eaf})^\kappa)\varphi(\overline{a})\varphi((\overline{eaf})^\kappa)\varphi(\overline{f}).$$

However, M is in **DAb**, and so its maximal subgroups are abelian. Therefore, we next obtain that

$$\begin{aligned}\varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi((\overline{eaf})^\kappa)\varphi(\overline{a})\varphi((\overline{eaf})^\kappa)\varphi(\overline{f})\\ = \varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi((\overline{eaf})^\kappa)\varphi(\overline{f})\varphi((\overline{eaf})^\kappa)\varphi(\overline{a}),\end{aligned}$$

and referring to the results on the monoids in $\mathcal{V}_\kappa$ established in § 2 once again, we further get that

$$\varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi((\overline{eaf})^\kappa)\varphi(\overline{f})\varphi((\overline{eaf})^\kappa)\varphi(\overline{a}) = \varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi(\overline{f})\varphi(\overline{a}).$$

Thus, summing up the above equalities, we conclude that

$$\varphi((\overline{eaf})^\kappa\overline{eaf}) = \varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi(\overline{f})\varphi(\overline{a}).$$

Besides, we have the obvious equality

$$\varphi(\overline{gh}(\overline{gah})^\kappa) = \varphi(\overline{g})\varphi(\overline{h})\varphi((\overline{gah})^\kappa).$$

From these two equalities we altogether get that

$$\varphi((\overline{eaf})^\kappa\overline{eafgh}(\overline{gah})^\kappa) = \varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi(\overline{f})\varphi(\overline{a})\varphi(\overline{g})\varphi(\overline{h})\varphi((\overline{gah})^\kappa).$$

Using the same arguments as above, we further see that the elements $\varphi(\overline{g})\varphi((\overline{gah})^\kappa)$, $\varphi(\overline{h})\varphi((\overline{gah})^\kappa)$ and $\varphi(\overline{a})\varphi((\overline{gah})^\kappa)$ all lie in the maximal subgroup of M containing the idempotent $\varphi((\overline{gah})^\kappa)$. Hence, proceeding dually, we deduce also the equality

$$\varphi((\overline{eaf})^\kappa)\varphi(\overline{e})\varphi(\overline{f})\varphi(\overline{a})\varphi(\overline{g})\varphi(\overline{h})\varphi((\overline{gah})^\kappa) = \varphi((\overline{eaf})^\kappa\overline{efgah}(\overline{gah})^\kappa).$$

From this and the previous equalities we eventually conclude that

$$\varphi((\overline{eaf})^\kappa\overline{eafgh}(\overline{gah})^\kappa) = \varphi((\overline{eaf})^\kappa\overline{efgah}(\overline{gah})^\kappa).$$

Since $\varphi((\overline{eaf})^\kappa)$ and $\varphi((\overline{gah})^\kappa)$ are idempotents of M, this last equality can still be rewritten in the form

$$\varphi((\overline{eaf})^{n\kappa}\overline{eafgh}(\overline{gah})^{n\kappa}) = \varphi((\overline{eaf})^{n\kappa}\overline{efgah}(\overline{gah})^{n\kappa}).$$

Now, since the homomorphism φ is faithful, it is injective on the hom-sets of the category B. Therefore, from this equality it follows that, in B, we have

$$(\overline{eaf})^{n\kappa}\overline{eafgh}(\overline{gah})^{n\kappa} = (\overline{eaf})^{n\kappa}\overline{efgah}(\overline{gah})^{n\kappa}.$$

Applying the quotient homomorphism ψ to the elements on both sides of this equality, we hence obtain that

$$\psi((\overline{eaf})^{n\kappa}\overline{eafgh}(\overline{gah})^{n\kappa}) = \psi((\overline{eaf})^{n\kappa}\overline{efgah}(\overline{gah})^{n\kappa}),$$

that is, we get the equality

$$[(eaf)^{n\kappa}eafgh(gah)^{n\kappa}] = [(eaf)^{n\kappa}efgah(gah)^{n\kappa}]$$

of the corresponding classes of the congruence $\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}$ on Γ^*. Since the quotient category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ satisfies the loop identity $x^n \simeq x^{2n}$, the congruence classes $[(eaf)^n]$ and $[(gah)^n]$ are idempotents of this category, which entails that the last equality of congruence classes in $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ can be simplified in the form

$$[(eaf)^n eafgh(gah)^n] = [(eaf)^n efgah(gah)^n].$$

But this means that the two loops $(eaf)^n eafgh(gah)^n$ and $(eaf)^n efgah(gah)^n$ represent the same element of the local monoid of the category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ at the vertex u. However, this contradicts what we have proved before in this section. To put it still in other terms, notice that the equality of congruence classes obtained lastly is equivalent to the condition

$$(eaf)^n eafgh(gah)^n \equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)} (eaf)^n efgah(gah)^n,$$

which negates the condition verified previously in this section. The contradiction thus obtained confirms that the category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ does not divide any monoid in **DAb**, so that the monoid pseudovariety **DAb** is not local.

Finally, notice, in connection with the considerations in the previous paragraph, that if we deal with an arbitrary finite category from the pseudovariety of categories g**DAb**, which category does divide a finite monoid from **DAb** (instead of treating the category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$), then the arguments in the previous paragraph can be reinterpreted as a verification of the fact that the pseudovariety of categories g**DAb** satisfies the path pseudoidentity

$$(eaf)^\omega eafgh(gah)^\omega \simeq (eaf)^\omega efgah(gah)^\omega$$

over the finite graph Γ drawn in Figure 1. The previous notes in this section then show that this path pseudoidentity does not hold in the pseudovariety of categories ℓ**DAb**, since the category $\Gamma^*/{\equiv_{\mathcal{W}_{n,k}(\mathcal{A}_p)}}$ does not satisfy it. This means that g**DAb** $\subsetneq$ ℓ**DAb**, which shows once again that the monoid pseudovariety **DAb** is not local.

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