BULL. AUSTRAL. MATH. SOC. 47H15, 34H05, 47B44, 47D05
VOL. 10 (1974), 31-37.

Nonlinear differential equations in reflexive Banach spaces

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Let X be a reflexive Banach space and $\{A(t) \mid t \in [0, T]\}$ be a family of weakly continuous operators which map X to X. Conditions are provided which guarantee the existence and the uniqueness to the Cauchy initial value problem $u'(t) + A(t)u(t) = 0$; $u(0) = x$.

1. Introduction

In this paper we shall be concerned with the existence of solutions to the Cauchy initial value problem,

(1.1) $$u'(t) + A(t)u(t) = 0 \ ; \quad u(0) = x \ ,$$

where $\{A(t) \mid t \in [0, T]\}$ is a family of operators which map a reflexive Banach space X to itself. Basically we require that the operator $A(\cdot) : [0, T] \times X \to X$ be weakly continuous and that for each $t \in [0, T]$ the operator $A(t)$ satisfy a modified accretivity condition. In [1] Browder provides a local solution to (1.1) in case X is a complex Hilbert space. More recently, Diaz and Weinacht [3] and Medeiros [11] discuss the uniqueness of solutions to (1.1) in Hilbert spaces; Goldstein in [6] extends their results to general Banach spaces; and Chow and Schuur [2] guarantee local existence to (1.1) in case X is a separable reflexive, Banach space. In [5] the author establishes the global existence of solutions to (1.1) in case $A(t) \equiv A$ is accretive.

Received 20 August 1973.

https://doi.org/10.1017/S0004972700040594 Published online by Cambridge University Press

2. Preliminaries

Throughout this paper X will denote a Banach space and $\|\cdot\|$ will be its norm. The dual space of X will be X^*.

DEFINITION 2.1. Let X be a Banach space, then the duality map $F : X \to 2^{X^*}$ is defined in the following manner: if $x \in X$, then $x^* \in F(x)$ iff $x^*(x) = \|x\|^2$ and $\|x^*\| = \|x\|$.

In general the duality map is not single valued; however, in [7], Kato shows that if X is a Banach space having uniformly convex dual X^*, then the duality map is uniformly continuous on bounded subsets of X.

The following definition makes clear our notions of operator continuity.

DEFINITION 2.2. Let $\{A(t) \mid t \in [0, T]\}$ be a family of operators which map X to X. Then $\{A(t) \mid t \in [0, T]\}$ is said to be *weakly continuous* provided that $t_n \to t_0$ and $x_n \rightharpoonup x_0$ imply $A(t_n)x_n \rightharpoonup A(t_0)x_0$. If $t_n \to t_0$ and $x_n \to x_0$ implies $A(t_n)x_n \rightharpoonup A(t_0)x_0$, then $\{A(t) \mid t \in [0, T]\}$ is said to be *demi-continuous*.

We now define an accretive operator and give two useful characterizations of accretive operators.

DEFINITION 2.3. Let X be a Banach space and A an operator mapping a subset of X to X; then A is said to be accretive provided (2.4), $\|x+\lambda Ax-(y+\lambda Ay)\| \geq \|x-y\|$ whenever $x, y \in D(A)$ and $\lambda \geq 0$.

Although Definition 2.3 is easily stated it is difficult to apply. In [8] Kato shows that an operator is accretive if $\langle Ax-Ay, f\rangle \geq 0$ for $x, y \in D(A)$ and some $f \in F(x-y)$ where F is the duality map. An accretive operator A is said to be *strongly accretive* provided that $\langle Ax-Ay, f\rangle \geq 0$ for all $f \in F(x-y)$. It is easily shown that weakly continuous accretive operators are strongly accretive. Martin [10] shows that if A is strongly accretive, then

(2.4) $$\lim_{h\to 0^+} \left(\|x-y-h(Ax-Ay)\|-\|x-y\|\right)/h \leq 0 \quad \text{for all } x, y \in D(A) .$$

We now make precise our notion of strong solutions to the Cauchy problem.

DEFINITION 2.5. A function $u(\cdot) : [0, T] \to X$ is said to be a strong solution to the Cauchy problem

(2.6) $$u'(t) + A(t)u(t) = 0 \; ; \; u(0) = x ,$$

provided that u is Lipschitz continuous on $[0, T]$, $u(0) = x$, u is strongly differentiable almost everywhere and $u'(t) + A(t)u(t) = 0$ for $t \in [0, T]$ almost everywhere.

LEMMA 2.7. *Let X be a Banach space and g be a function from the number interval (a, b) to X. Define $p(t) = \|g(t)\|$ for $t \in [a, b]$; then if $g'^{+}(t)$ exists, $p'^{+}(t)$ exists and*

$$p'^{+}(t) = \lim_{h \to 0^+} \|g(t)+hg'^{+}(t)\| - \|g(t)\| \; /h .$$

3. Existence of solutions

The following lemma provides a local solution to Definition 2.5.

LEMMA 3.1. *Let X be a reflexive Banach space and suppose that $\{A(t) \mid t \in [0, T]\}$ is a weakly continuous family of operators which map X to X; then there is a finite interval $[0, T_0]$ such that the Cauchy problem has a strong solution on $[0, T_0]$.*

Proof. Let $x \in X$. By virtue of the weak continuity of $\{A(t) \mid t \in [0, T]\}$ there exist T_1, R and $K_1 > 0$ such that if $0 \le t \le T_1$ and $y \in S_R(x)$ then $\|A(t)y\| \le K$. Choose $T_0 = \min\{R/K, T_1\}$. Let $\varepsilon_n \downarrow 0$. We shall recursively define a sequence of functions which solve the approximate equations

(3.2) $$u_n'(t) + A(t)u_n(t-\varepsilon_n) = 0 \; ; \; u(0) = x \; ;$$

$$u_n(t) = \begin{cases} x \text{ if } t < 0 , \\ x - \int_0^t A(s)u_n(s-\varepsilon_n)ds \text{ if } t \in [j\varepsilon_n, (j+1)\varepsilon_n] , \\ \qquad j = 0, \ldots, [T_0/\varepsilon_n] - 1 . \end{cases}$$

We argue that $u_n(t) \in S_R(x)$. If $t \in [0, \varepsilon_n]$ then $\|u_n(t)-x\| \le t \sup_{s \in [0,T]} \|A(s)x\| \le (R/K)K = R$. If we assume the desired

https://doi.org/10.1017/S0004972700040594 Published online by Cambridge University Press

result for $t \in [0, j\varepsilon_n]$ and consider $t \in [0, (j+1)\varepsilon_n]$, we have

$$\|u_n(t)-x\| \le \left\|\int_0^t A(s)u_n(s-\varepsilon_n)ds\right\| \le t\max\{\|A(s)u_n(s-\varepsilon_n)\| \mid s \in [0, (j+1)\varepsilon_n]\} .$$

By observing that $\|u_n(t)-u_n(\tau)\| \le \int_\tau^t \|A(s)u_n(s-\varepsilon_n)\|ds \le |t-\tau|K$ we see that the sequence is uniformly Lipschitz continuous in t .

We now claim that there is a subsequence $\{u_{n'}(t)\}$ of $\{u_n(t)\}$ such that $\{u_{n'}(t)\}$ converges weakly to a Lipschitz continuous function $\{u(t)\}$. The argument of Lemma 2.1, [5], is directly applicable to establish this convergence.

Since $u_n(t-\varepsilon_n) \rightharpoonup u(t)$, $A(t)u_n(t-\varepsilon_n) \rightharpoonup A(t)u(t)$. If $f \in X^*$ we take limits of the equation,

$$(u_n(t), f) = (x, f) - \int_0^t (A(s)u_n(s-\varepsilon_n), f)ds$$

to obtain

$$(3.3) \quad (u(t), f) = (x, f) - \int_0^t (A(s)u(s), f)ds \quad \text{for} \quad t \in [0, T_0] .$$

Applying standard techniques to (3.3) yields

$$u(t) = x - \int_0^t A(s)u(s)ds \quad \text{for} \quad t \in [0, T_0] ,$$

and hence that

$$du(t)/dt + A(t)u(t) = 0 \quad \text{for} \quad t \in [0, T_0] \quad \text{almost everywhere.}$$

We now place further conditions on $\{A(t) \mid t \in [0, T]\}$ which allow us to extend the local solution of Lemma 3.1.

THEOREM 1. *Let X be a reflexive Banach space and suppose that $\{A(t) \mid t \in [0, T]\}$ is a weakly continuous family of operators which maps X to X . Further assume that for each $t \in [0, T]$ the operator $A(t) + (1/t)I$ is accretive. Then there is a strong solution the Cauchy initial value problem, Definition* 2.5, *on* $[0, T]$.

Proof. From the preceding lemma it is clear that there exists a local solution to Definition 2.5 on a maximal interval of existence $[0, T_0)$. We wish to argue that $T_0 < T$ leads to a contradiction. Let $0 < t_0 < T_0$ and define $p(t) = \|u(t)\|$. By virtue of equation (2.4) and Lemmas 2.7 and 3.1 we have

$$\begin{aligned} p'^{+}(t) = \lim_{h\to 0^+} \left(\|u(t)-hA(t)u(t) - u(t)\|\right)/h \\ \le \lim_{h\to 0^+} \left(\|u(t)-h\left(A(t)u(t)+(1/t)u(t)-A(t)0\right)\|-\|u(t)\|\right)/h \\ + (1/t)\|u(t)\| + \|A(t)0\| \\ \le \sup_{t\in[0,T]} \|A(t)0\| + (1/t)\|u(t)\| \\ \le (1/t)\|u(t)\| + M \quad \text{for some } M > 0 . \end{aligned}$$

Thus

$$\left((1/t)\|u(t)\|\right)'^{+} \le (1/t)M ; \tag{3.4}$$

integrating on (t_0, t) we have

$$(1/t)\|u(t)\| \le \left(1/t_0\right)\|u\left(t_0\right)\| + M' \quad \text{for some } M' .$$

Thus there is an $N > 0$ such that $\|u(t)\| < N$ for $t \in [0, T_0)$. Since $A(\cdot)$ maps bounded subsets of $[0, T] \times X$ to bounded subsets of X, there exists an N_1 such that $\int_0^t \|A(s)u(s)\|ds < N_1$ for $t \in [0, T_0)$. This implies that $\int_0^t A(s)u(s)ds$ exists for $t \in [0, T_0)$ and by virtue of the continuity of the integral we can define $u\left(T_0\right) = \lim_{t\to T_0} \int_0^t A(s)u(s)ds - x$. Lemma 3.1 can be applied to continue the solution $u(t)$ past T_0 and thereby contradict the definition of T_0. In [6] Goldstein insures the uniqueness of the solution $u(t)$.

If we require that X have uniformly convex dual and that each $A(t)$ is accretive we can relax the continuity requirement. The following theorem is an extension of a time independent result of Kato [7].

https://doi.org/10.1017/S0004972700040594 Published online by Cambridge University Press

THEOREM 2. *Let X be a Banach space such that X^* is uniformly convex and let $\{A(t) \mid t \in [0, T]\}$ be a family of demi-continuous operators such that all map bounded subsets of $[0, T] \times X$ to bounded subsets of X. Assume that for each $t \in [0, T]$, $A(t)$ is accretive; then there is a unique solution to* (2.6) *on* $[0, T]$.

Proof. If we provide a local solution to (2.6) we can apply the argument of Theorem 1 to extend the solution to $[0, T]$. Our local existence argument follows Kato [7]. Let $\varepsilon_n \downarrow 0$. Choosing $R, T_0, K > 0$ as in Lemma 3.1 we define $u_n(t)$ for $t \in [0, T_0]$ by equation (3.2). We observe that

$$d/dt\left(\|u_n(t)-u_m(t)\|^2\right) = -2\langle A(t)u_n(t-\varepsilon_n)-A(t)u_m(t-\varepsilon_m), F(u_n(t)-u_m(t))\rangle$$

where F is the duality map. Using the accretiveness of $A(t)$ we obtain

$$d/dt\left(\|u_n(t)-u_m(t)\|^2\right)$$
$$\leq -2\langle A(t)u_n(t-\varepsilon_n)-A(t)u_m(t-\varepsilon_m), F(u_n(t)-u_m(t))-F(u_n(t-\varepsilon_n)-u_m(t-\varepsilon_m))\rangle .$$

Since F is uniformly continuous the arguments of [7] and [8] are directly applicable to establish the uniform convergence of $u_n(t)$ to $u(t)$ on $[0, T_0]$. We apply the argument of Theorem 1 to see that $u(t)$ can be extended to a solution of (2.6) on $[0, T]$. The uniqueness of the solution follows from standard methods involving the accretiveness of $A(t)$.

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https://doi.org/10.1017/S0004972700040594 Published online by Cambridge University Press