

RATIONAL POINTS ON INTERSECTIONS OF CUBIC AND QUADRIC HYPERSURFACES

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Abstract We investigate the Hasse principle for complete intersections cut out by a quadric hypersurface and a cubic hypersurface defined over the rational numbers.

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1. Introduction

Suppose that we are given a pair of forms $C, Q \in \mathbb{Q}[x_1, \dots, x_n]$, with C cubic and Q quadratic, whose common zero locus defines a complete intersection $X \subset \mathbb{P}^{n-1}$ defined over \mathbb{Q} . The primary goal of this paper is to establish the existence of \mathbb{Q} -rational points on X under the mildest possible hypotheses.

One of the few results in the literature that specifically treats pairs of cubic and quadratic forms appears in work of Wooley [24, 25]. This deals with the special case in which C and Q are both diagonal, so that

$$C = a_1x_1^3 + \dots + a_nx_n^3, \quad Q = b_1x_1^2 + \dots + b_nx_n^2,$$

for integers a_i, b_i , with the b_i not all sharing the same sign. Assuming that $n \geq 13$, it follows from the main result in [25] that $X(\mathbb{Q})$ is non-empty provided only that $X(\mathbb{R}) \neq \emptyset$, with at least seven a_i non-zero.

In our work we wish to handle general forms C, Q in so far as is possible. All of the results that we obtain pertain to complete intersections $X \subset \mathbb{P}^{n-1}$ cut out by a cubic hypersurface $C = 0$ and a quadric hypersurface $Q = 0$, both defined over \mathbb{Q} .

One way to produce rational points on X is first to find a large-dimensional linear space on the quadric $Q = 0$, which is defined over \mathbb{Q} . One is then led to the simpler problem of

finding rational points on the intersection of the cubic hypersurface $C = 0$ with the linear space. Let us call a d -dimensional linear space $\Lambda \subset \mathbb{P}^{n-1}$ a d -plane. Let $Q \in \mathbb{Q}[x_1, \dots, x_n]$ be a quadratic form. For each prime p the quadric $Q = 0$ contains a \mathbb{Q}_p -rational d -plane, provided that

$$n \geq 5 + 2d.$$

The case $d = 0$ corresponds to the well-known fact that every quadratic form in at least five variables is isotropic over \mathbb{Q}_p . The general case follows from inserting this fact into work of Leep [18, Corollary 2.4 (ii)]. Moreover, the quadric $Q = 0$ contains a real d -plane, provided that $d \leq n - 1 - \max(r, s)$, where (r, s) is the signature of Q . The existence of a d -plane in the quadric everywhere locally is enough to ensure the existence of a \mathbb{Q} -rational d -plane Λ contained in the quadric, by the Hasse principle for linear spaces on quadratic forms (see the proof of [5, Theorem 2], for example). As soon as $d \geq 13$ we may apply the main result in work of Heath-Brown [17], which shows that $C = 0$ has a rational point on Λ , giving a rational point on X . Finally, it is clear that we may take $d = 13$ whenever $n \geq 31$ and $n - \max(r, s) \geq 14$. We record this observation as follows.

Theorem 1.1. *Suppose that $n \geq 31$ and Q has signature (r, s) , with $\max(r, s) \leq n - 14$. Then $X(\mathbb{Q}) \neq \emptyset$.*

It is worthwhile noting that, when working over totally imaginary number fields k , the assumption on the signature of the quadratic form can be removed. Appealing to work of Pleasants [21], which is valid for cubic forms in at least 16 variables over any number field, one concludes that $X(k) \neq \emptyset$ provided only that $n \geq 5 + 2(16 - 1) = 35$.

Our next results are established using the Hardy–Littlewood circle method directly. We write X_{sm} for the smooth locus of points on X . Recall that the smooth Hasse principle is said to hold for a family of such varieties when the existence of a point in $X_{\text{sm}}(\mathbf{A}) = X_{\text{sm}}(\mathbb{R}) \times \prod_p X_{\text{sm}}(\mathbb{Q}_p)$, where \mathbf{A} denotes the adèles, is enough to ensure the existence of a smooth \mathbb{Q} -rational point in X . Given a form $F \in \mathbb{Q}[x_1, \dots, x_n]$, we define the h -invariant $h(F)$ to be the least positive integer h such that F can be written identically as

$$A_1 B_1 + \dots + A_h B_h,$$

for forms $A_i, B_i \in \mathbb{Q}[x_1, \dots, x_n]$ of positive degree. Taking $R = 5, r_3 = r_2 = 1$, and $k = 3$ in work of Schmidt [22, Theorem II], we obtain the smooth Hasse principle for X provided that $h(C) \geq 480$ and $h(Q) \geq 30$. We note here that one clearly has $\text{rank}(Q) \leq 2h(Q)$ for any quadratic form, so that it suffices to have $h(C) \geq 480$ and $\text{rank}(Q) \geq 59$. With this in mind we state the following result.

Theorem 1.2. *Write $\text{rank}(Q) = \rho$. Then the smooth Hasse principle holds for X provided that*

$$(h(C) - 32)(\rho - 4) > 128.$$

In particular, it suffices to have $\min(h(C), \rho) \geq 37$.

If C is non-singular then the smooth Hasse principle holds for X provided that

$$(n - 32)(\rho - 4) > 128.$$

There is an old result of Birch [3] which establishes the smooth Hasse principle for complete intersections $V \subset \mathbb{P}^{n-1}$ cut out by forms F_1, \dots, F_R of equal degree d , provided that the inequality

$$n - \dim V^* > R(R + 1)(d - 1)2^{d-1}$$

holds, where V^* is the affine variety cut out by the condition

$$\text{rank}(\nabla F_i)_{1 \leq i \leq R} < R.$$

It is not entirely clear how this method could be adapted to handle a system of forms of differing degree, since the process of Weyl differencing involved in the proof eradicates the presence of the lower degree forms. A satisfactory treatment of this issue is a key ingredient in Theorem 1.2. Schmidt encounters the same problem in the work [22] cited above, and deals with it in a simpler but less effective manner. When the exponential sums involved have only one variable, the “final coefficient lemma” (see Baker [2, §4.2]) gives very good results. However, this relies ultimately on the use of strong bounds for complete exponential sums, which are not available when one has several variables.

When X is assumed to be non-singular we will show in Corollary 3.2 that the cubic form C can be taken to be non-singular with the quadratic form Q having rank $\rho \geq n - 1$. Theorem 1.2 therefore implies that the Hasse principle holds for non-singular X provided that $n \geq 37$. The following result improves on this further.

Theorem 1.3. *Suppose that X is non-singular, with $n \geq 29$. Then $X(\mathbb{Q}) \neq \emptyset$ if and only if $X(\mathbb{R}) \neq \emptyset$.*

Theorem 1.3 establishes the Hasse principle for non-singular X , with $n \geq 29$. The issue of determining when X has p -adic points for every prime p is of considerable interest in its own right. Artin’s conjecture would imply that it is sufficient to have $n > 3^2 + 2^2 = 13$. Indeed it has been shown by Zahid [26] that an arbitrary intersection $X : C = Q = 0$ with $n > 13$ has $X(\mathbb{Q}_p) \neq \emptyset$ for every prime $p > 293$. However, if $n \geq 29$ we can in fact recycle the proof of Theorem 1.1 to deduce that the quadric hypersurface $Q = 0$ contains a \mathbb{Q}_p -rational projective space of dimension at least $[(n - 5)/2]$. The existence of a point in $X(\mathbb{Q}_p)$ is then assured by an old result of Lewis [19], which shows that the cubic $C = 0$ has a \mathbb{Q}_p -rational point on any \mathbb{Q}_p -rational projective linear space of dimension 9 or more.

Our proofs of Theorems 1.2 and 1.3 are based on the Hardy–Littlewood circle method. We will give an overview of the proof in §2. As is usual with the circle method, our arguments show not only that $X(\mathbb{Q})$ is non-empty, but may even be developed to establish weak approximation. Moreover, we can prove a variant of Theorem 1.3 which applies to singular X . Suppose that $\sigma \geq -1$ is the dimension of the singular locus of X , with the convention that $\sigma = -1$ if and only if X is non-singular. Then an argument based on Bertini’s theorem can be used to show that the smooth Hasse principle holds for X , provided that $n \geq 30 + \sigma$. We leave the details of both of these remarks to the reader.

To state our remaining result, we need to introduce some more terminology. If $F \in K[x_1, \dots, x_n]$ for some field K , then we define the *order* of F to be the minimal non-negative integer m such that there exists a matrix $\mathbf{T} \in \text{GL}_n(K)$ with the property that

in $F(\mathbf{T}(x_1, \dots, x_n))$ only m of the variables x_1, \dots, x_n occur with a non-zero coefficient. It is a familiar fact that the order of F does not change if K is replaced by an extension of K . If $Q \in \mathbb{Q}[x_1, \dots, x_n]$ is a quadratic form, then we call a pair of cubic forms $C_1, C_2 \in \mathbb{Q}[x_1, \dots, x_n]$ Q -equivalent if there exists a linear form $L \in \mathbb{Q}[x_1, \dots, x_n]$ such that

$$C_1 - C_2 = LQ.$$

It is easily checked that this indeed defines an equivalence relation on the set of rational cubic forms, and that the set of zeros of the intersection $C = Q = 0$ does not change if one replaces C by another cubic form that is Q -equivalent to C . Finally, for a fixed quadratic form $Q \in \mathbb{Q}[x_1, \dots, x_n]$ and cubic form $C_1 \in \mathbb{Q}[x_1, \dots, x_n]$, we define the Q -order $\text{ord}_Q(C_1)$ of C_1 to be the minimal order amongst all cubic forms C_2 that are Q -equivalent to C_1 . We are now ready to reveal the following result.

Theorem 1.4. *Suppose that $n \geq 49$ and $\text{ord}_Q(C) \geq 17$, and that $X_{\text{sm}}(\mathbb{R}) \neq \emptyset$. Then $X(\mathbb{Q}) \neq \emptyset$.*

The hypothesis $\text{ord}_Q(C) \geq 17$ in the previous theorem can be weakened to $\text{ord}_Q(C) \geq 14$, provided that we impose the additional assumption that, for any cubic form that is Q -equivalent to C , the rational points on the corresponding cubic surface are dense in the locus of real points.

Simple considerations show that Theorem 1.4 could not be true without some sort of assumption on the Q -order of C . We assume that $n \geq 49$, in order to fall within the range of the theorem. Let $m \leq n$, and suppose that $C \in \mathbb{Q}[x_1, \dots, x_m]$ is a cubic form for which $C = 0$ has no \mathbb{Q} -rational point. In particular, C must be non-degenerate, so that m is the order of C . Let X be the variety cut out by C and the quadratic form

$$Q(x_1, \dots, x_n) = -x_m^2 + x_{m+1}^2 + \dots + x_n^2.$$

It is clear that $X_{\text{sm}}(\mathbb{R}) \neq \emptyset$ and $\text{ord}_Q(C) = m$. Any rational point on X would lie on $C = 0$, so that $x_1 = \dots = x_m = 0$. Then $Q = 0$ implies that $x_{m+1} = \dots = x_n = 0$, whence in fact $X(\mathbb{Q}) = \emptyset$. This example shows that if one had a version of Theorem 1.4 in which the condition on the Q -order of C were relaxed to $\text{ord}_Q(C) \geq 13$ then we would be able to deduce that any cubic over \mathbb{Q} in 13 variables has a non-trivial rational zero. In particular, any such improvement of Theorem 1.4 would lead to a corresponding sharpening of the result of [17].

Mordell [20] has constructed a non-degenerate cubic form C in 9 variables for which $C = 0$ has no \mathbb{Q}_p -point for some prime p , and hence has no point over \mathbb{Q} . This shows that, aside from extending the range for n , the best one can hope for in Theorem 1.4 is a reduction of the lower bound on the Q -order of C to $\text{ord}_Q(C) \geq 10$. Moreover, our example shows that it is really the Q -order of C that matters rather than the order, since we could replace C by $C + LQ$ for a linear form L , and in this way increase the order of the cubic form.

Notation. Throughout our work, \mathbb{N} will denote the set of positive integers. For any $\alpha \in \mathbb{R}$, we will follow common convention and write $e(\alpha) := e^{2\pi i \alpha}$ and $e_q(\alpha) := e^{2\pi i \alpha/q}$. The

parameter ε will always denote a small positive real number. We shall use $|\mathbf{x}|$ to denote the norm $\max |x_i|$ of a vector $\mathbf{x} = (x_1, \dots, x_n)$. All of the implied constants that appear in this work will be allowed to depend upon the coefficients of the forms C and Q under consideration, the number n of variables involved, and the parameter $\varepsilon > 0$. Any further dependence will be explicitly indicated by appropriate subscripts.

2. Overview of the paper

We have already established Theorem 1.1. In §3, we will collect together some geometric facts that will be used in the proof of Theorems 1.2–1.4. Theorems 1.2 and 1.3 will be established using the Hardy–Littlewood circle method. This will occupy the bulk of our paper (§§ 4–8). Finally, in §§ 9 and 10, we will turn to the proof of Theorem 1.4.

The aim of the present section is to survey the key ideas in the proof of Theorems 1.2 and 1.3. On multiplying through by a common denominator we can ensure that C and Q have coefficients in \mathbb{Z} . In both results the goal will be to establish an asymptotic formula for the quantity

$$N_\omega(X; P) := \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ C(\mathbf{x})=Q(\mathbf{x})=0}} \omega(\mathbf{x}/P), \tag{2.1}$$

as $P \rightarrow \infty$, for a suitably chosen function $\omega : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ with support in $(-1/2, 1/2)^n$. All of our weight functions will be compactly supported and infinitely differentiable. The starting point in the circle method is the identity

$$N_\omega(X; P) = \int_0^1 \int_0^1 S(\alpha_3, \alpha_2) d\alpha_3 d\alpha_2,$$

where

$$S(\alpha_3, \alpha_2) := \sum_{\mathbf{x} \in \mathbb{Z}^n} \omega(\mathbf{x}/P) e(\alpha_3 C(\mathbf{x}) + \alpha_2 Q(\mathbf{x})), \tag{2.2}$$

for any $\alpha_3, \alpha_2 \in \mathbb{R}$. The idea is then to divide the region $[0, 1]^2$ into a set of major arcs \mathfrak{M} and minor arcs \mathfrak{m} . In the usual way, we seek to prove an asymptotic formula

$$\iint_{\mathfrak{M}} S(\alpha_3, \alpha_2) d\alpha_3 d\alpha_2 \sim c_X P^{n-5}, \tag{2.3}$$

as $P \rightarrow \infty$, together with a satisfactory bound on the minor arcs

$$\iint_{\mathfrak{m}} S(\alpha_3, \alpha_2) d\alpha_3 d\alpha_2 = o(P^{n-5}). \tag{2.4}$$

Here, the constant c_X will be a product of local densities, which is positive when $X_{\text{sm}}(\mathbf{A})$ is non-empty.

For any pair α_3, α_2 , we will produce a simultaneous rational approximation $a_3/q, a_2/q$ using a two-dimensional version of Dirichlet’s approximation theorem. To describe this, we take positive integers Q_3, Q_2 satisfying

$$Q_3 := [P^{4/3}] \quad \text{and} \quad Q_2 := [P^{1/3}]. \tag{2.5}$$

Then, by the pigeon hole principle, there will be $\mathbf{a} = (a_3, a_2) \in \mathbb{Z}^2$ and $q \in \mathbb{N}$ such that $q \leq Q_3 Q_2$ and $\gcd(q, \mathbf{a}) = 1$, for which

$$\left| \alpha_3 - \frac{a_3}{q} \right| \leq \frac{1}{q Q_3} \quad \text{and} \quad \left| \alpha_2 - \frac{a_2}{q} \right| \leq \frac{1}{q Q_2}. \tag{2.6}$$

It will be convenient to write

$$\alpha_3 = \frac{a_3}{q} + \theta_3 \quad \text{and} \quad \alpha_2 = \frac{a_2}{q} + \theta_2.$$

Let $\delta \in (0, 1/3)$ be a parameter to be decided upon later (see (8.3)). We will take as major arcs

$$\mathfrak{M} := \bigcup_{q \leq P^\delta} \bigcup_{\substack{\mathbf{a} \pmod{q} \\ \gcd(q, \mathbf{a})=1}} \mathfrak{M}_{q, \mathbf{a}},$$

where

$$\mathfrak{M}_{q, \mathbf{a}} := \left\{ (\alpha_3, \alpha_2) \pmod{1} : \left| \alpha_i - \frac{a_i}{q} \right| \leq P^{-i+\delta}, \text{ for } i = 3, 2 \right\}.$$

It is easy to see that $\mathfrak{M}_{q, \mathbf{a}} \cap \mathfrak{M}_{q', \mathbf{a}'} = \emptyset$ whenever $\mathbf{a}/q \neq \mathbf{a}'/q'$, provided that P is taken to be sufficiently large. Moreover, each major arc is contained in the corresponding range given by (2.6).

Our treatment of (2.3) is relatively standard, and is the focus of § 8.

The minor arcs are defined to be $\mathfrak{m} = [0, 1]^2 \setminus \mathfrak{M}$. Thus they are defined by having either $q > P^\delta$ or $\max(|\theta_3|P^3, |\theta_2|P^2) > P^\delta$. Our estimation of $S(\alpha_3, \alpha_2)$ for $(\alpha_3, \alpha_2) \in \mathfrak{m}$ will differ according to the hypotheses placed on X . A common ingredient will be a more efficient version of Weyl differencing, which draws inspiration from the work of Birch [3], but which is specially adapted to systems of equations of differing degree. Suppose that

$$C(x_1, \dots, x_n) = \sum_{i, j, k=1}^n c_{ijk} x_i x_j x_k,$$

for integer coefficients c_{ijk} that are symmetric in the indices i, j, k . Define the bilinear forms

$$B_i(\mathbf{x}; \mathbf{y}) := 3! \sum_{j, k=1}^n c_{ijk} x_j y_k, \quad (1 \leq i \leq n).$$

Using two successive applications of Weyl differencing, as in Birch’s work, we can relate the size of the exponential sum $S(\alpha_3, \alpha_2)$ to the locus of integral points on the affine variety given by the simultaneous equations $B_i(\mathbf{x}; \mathbf{y}) = 0$, for $1 \leq i \leq n$. When C defines a smooth cubic hypersurface, or when $h(C)$ is sufficiently large, we shall be able to get good estimates for $S(\alpha_3, \alpha_2)$ unless α_3 happens to be close to a rational number with small denominator. If this occurs, then we shall use a single Weyl squaring, modified in a way motivated by van der Corput’s method so as to remove the effect of the cubic terms. This step marks a departure from the approach of Birch, which is completely insensitive to the quadratic form Q that appears in the sum. Our modified version of Weyl differencing is the subject of § 4, and is one of the more novel parts of the paper. The work in this section will ultimately suffice to establish Theorem 1.2 in § 5.

When it comes to establishing Theorem 1.3, for which X is assumed to be non-singular, the work in § 5 only allows us to establish an asymptotic formula for $N_\omega(X; P)$ when $n \geq 37$. Instead, in § 6, we shall produce a companion estimate for $S(\alpha_3, \alpha_2)$, which is based on Poisson summation. Once combined with the work in § 4, this will lead to an asymptotic formula for $N_\omega(X; P)$ when $n \geq 29$, as required for Theorem 1.3. One inconvenient feature of this combined attack is that, while both methods involve rational approximations to α_3 and α_2 , there is no *a priori* guarantee that the rational approximations occurring in the two methods are the same.

3. Geometric preliminaries

Let k be a field of characteristic zero. Suppose that $V \subset \mathbb{P}^{n-1}$ is a non-singular complete intersection of codimension r , whose homogeneous ideal in $k[\mathbf{x}] = k[x_1, \dots, x_n]$ is generated by r forms $F_1, \dots, F_r \in k[\mathbf{x}]$. Suppose that the maximum degree attained by any form is attained by F_1 . One has a great deal of freedom in the choice of F_1 , since one may equally take $F_1 + \sum_{1 < i \leq r} H_i F_i$ for any forms $H_i \in k[\mathbf{x}]$ such that $\deg H_i F_i = \deg F_1$. In this way it is reasonable to expect that one can always arrange for the leading form F_1 to be non-singular, provided that V itself is non-singular. This is made precise in the following result due to Aznar [1, § 2].

Lemma 3.1. *Let $V \subset \mathbb{P}^{n-1}$ be a non-singular complete intersection of codimension r , which is defined over a field k of characteristic zero. Then there is a system of generators $F_1, \dots, F_r \in k[\mathbf{x}]$ of the ideal of V , with*

$$\deg F_1 \geq \dots \geq \deg F_r,$$

such that the varieties

$$W_i : F_1 = \dots = F_i = 0, \quad (i \leq r),$$

are all non-singular.

Proof. To be precise, Aznar works with $k = \mathbb{C}$, but the adaptation to arbitrary fields of characteristic zero is straightforward. We give the proof here for the sake of completeness. We argue by induction on i , the case $i = 0$ being trivial.

Now let i be such that $1 \leq i \leq r$. Fix a system of generators

$$F_1, \dots, F_{i-1}, G_i, \dots, G_r \in k[\mathbf{x}]$$

for the ideal of V , with

$$\deg F_1 \geq \dots \geq \deg F_{i-1} \geq \deg G_i \geq \dots \geq \deg G_r,$$

such that the varieties $W_1, \dots, W_{i-1} \subset \mathbb{P}^{n-1}$ are all non-singular. Suppose that $d_k = \deg G_k$, for $i \leq k \leq r$. Let us write

$$f_0 = G_i, \quad f_{j,k} = x_j^{d_i - d_k} G_k,$$

for $1 \leq j \leq n$ and $i < k \leq r$. This gives a system

$$\mathbf{f} = (f_0, f_{1,i+1}, \dots, f_{n,r})$$

of $N = 1 + n(r - i)$ forms in $k[\mathbf{x}]$ of degree d_i . The set of points in W_{i-1} for which $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ precisely coincides with the non-singular variety V . We let $U = W_{i-1} \setminus V$. Consider the morphism

$$\pi : U \rightarrow \mathbb{P}^{N-1},$$

given by $[\mathbf{x}] \mapsto [\mathbf{f}(\mathbf{x})]$. Then Bertini’s theorem (see Harris [12, Theorem 17.6], for example) reveals that for a general hyperplane $H \subset \mathbb{P}^{N-1}$ the fibre $\pi^{-1}(H)$ is non-singular. This means that, for a general choice of $\lambda_0, \lambda_{j,k} \in k$, the degree d_i form

$$F_i = \lambda_0 G_i + \sum_{\substack{1 \leq j \leq n \\ i < k \leq r}} \lambda_{j,k} x_j^{d_i - d_k} G_k$$

is defined over k , and $U \cap \{F_i = 0\}$ is non-singular. This implies that

$$W_i : F_1 = \dots = F_i = 0$$

is non-singular, since V is non-singular. The induction hypothesis therefore follows, which completes the proof of the lemma. □

We apply this result to the complete intersection in Theorem 1.3 to deduce the following consequence.

Corollary 3.2. *Let $X \subset \mathbb{P}^{n-1}$ be a non-singular complete intersection, cut out by a cubic hypersurface and a quadric hypersurface, both defined over \mathbb{Q} . Then there exists a non-singular cubic form $C \in \mathbb{Z}[\mathbf{x}]$ and a diagonal quadratic form $Q \in \mathbb{Z}[\mathbf{x}]$ of rank at least $n - 1$, such that X is given by $C = Q = 0$.*

Proof. Taking $k = \mathbb{Q}$ in Lemma 3.1 ensures the existence of a non-singular cubic form $C \in \mathbb{Q}[\mathbf{x}]$ and a quadratic form $Q \in \mathbb{Q}[\mathbf{x}]$ such that X is given by $C = Q = 0$. After a non-singular rational change of variables, we may further assume that Q is diagonal. By multiplying through by a common denominator we can ensure that C and Q are both defined over \mathbb{Z} .

Showing that $\text{rank}(Q) \geq n - 1$ is equivalent to showing that the quadric hypersurface $Q = 0$ in \mathbb{P}^{n-1} must have a singular locus of dimension less than 1. But if the singular locus had positive dimension, its intersection with the cubic hypersurface $C = 0$ would be non-empty, and every point in it would be a singular point of X . This contradicts the non-singularity of X , which thereby completes the proof. □

One of the hallmarks of Theorem 1.4 is that it applies to very general complete intersections $X \subset \mathbb{P}^{n-1}$ cut out by a cubic hypersurface $C = 0$ and a quadric hypersurface $Q = 0$. Let us define $h_Q(C)$ to be the minimal value of $h(C + LQ)$ as L varies over all linear forms defined over \mathbb{Q} . We remark at once that $\text{ord}_Q(C) \geq h_Q(C)$ and

$$h_Q(C) \leq h(C) \leq h_Q(C) + 1. \tag{3.1}$$

We will require easily checked criteria on the defining forms which are sufficient to ensure that X is absolutely irreducible. This is the purpose of the following result.

Lemma 3.3. *Let $X \subset \mathbb{P}^{n-1}$ be a variety cut out by a cubic hypersurface $C = 0$ and a quadric hypersurface $Q = 0$, both defined over \mathbb{Q} . Assume that $\text{rank}(Q) \geq 5$, that $\text{ord}_Q(C) \geq 4$, and that $h_Q(C) \geq 2$. Then X is an absolutely irreducible variety of codimension 2 and degree 6.*

We begin by showing that the lemma applies under the hypotheses of Theorem 1.2. The condition $\text{rank}(Q) \geq 5$ is automatically met. For the first part of the theorem, which requires $h(C) \geq 33$, the remaining conditions of Lemma 3.3 are clearly met, since $\text{ord}_Q(C) \geq h_Q(C) \geq 32$, by (3.1). For the second part of the theorem, which requires C to be non-singular and $n \geq 33$, we claim that $\text{ord}_Q(C) \geq 4$ and $h_Q(C) \geq 2$. Indeed, if $h_Q(C) = 1$, then C takes the shape $L_1Q + L_2Q_2$ for suitable linear forms L_1, L_2 and a quadratic form Q_2 , all defined over \mathbb{Q} . Since $n \geq 33$, the intersection $L_1 = L_2 = Q = Q_2 = 0$ is non-empty and produces a singular point of $C = 0$. Alternatively, if we had $\text{ord}_Q(C) \leq 3$, then we could take C to have the shape $C_1(x_1, x_2, x_3) + LQ$, for a suitable linear form L and a suitable cubic form C_1 , both defined over \mathbb{Q} . Again, since $n \geq 33$, we could find a singular point of $C = 0$ by considering the intersection $x_1 = x_2 = x_3 = L = Q = 0$. This shows that the X considered in Theorem 1.2 are absolutely irreducible under the hypotheses presented there.

For Theorem 1.3, we see from Corollary 3.2 that we will have

$$\text{rank}(Q) \geq n - 1 \geq 28 > 5.$$

Moreover, a variety $Q = L'Q' = 0$ will have singular points wherever $Q = L' = Q' = 0$. Thus if X is non-singular we must have $h_Q(C) \geq 2$. Similarly, a variety $Q(x_1, \dots, x_n) = C'(x_1, x_2, x_3) = 0$ will have singular points wherever $Q(0, 0, 0, x_4, \dots, x_n) = 0$, so that if X is non-singular we will have $\text{ord}_Q(C) \geq 4$. It follows that the lemma applies for Theorem 1.3. Finally, for Theorem 1.4, the lemma will apply unless $h_Q(C) \leq 1$ or $\text{rank}(Q) \leq 4$.

Proof of Lemma 3.3. Under the hypotheses of the lemma, the forms C and Q share no common factor of positive degree. Hence X is pure dimensional. Suppose that X decomposes into irreducible components $Z_1 \cup \dots \cup Z_t$. It follows from Bézout’s theorem (in the form given by Fulton [11, Example 8.4.6]) that

$$\text{deg}(Z_1) + \dots + \text{deg}(Z_t) \leq 6.$$

Each Z_i is an irreducible codimension 1 divisor on the quadric hypersurface $Q = 0$. Let Z be one of these components. Since $\text{rank}(Q) \geq 5$, by hypothesis, it follows from Klein’s theorem (see Hartshorne [13, Part II, Ex. 6.5(d)]) that there is an irreducible hypersurface $W \subset \mathbb{P}^{n-1}$ such that Z is the intersection of W with the quadric $Q = 0$, with multiplicity 1. But then a further application of Bézout’s theorem (see [11, § 8.4]) implies that $\text{deg}(Z)$ must be even.

In order to conclude the proof of the lemma it clearly suffices to show that Z cannot have degree 2. Suppose, for a contradiction, that Z is quadratic. Then Klein’s theorem shows that Z is given by $L = Q = 0$, say, where L is a linear form defined over $\overline{\mathbb{Q}}$. It follows that C must take the shape $LR + \tilde{L}Q$, where \tilde{L} and R are linear and quadratic forms, respectively, defined over $\overline{\mathbb{Q}}$. Indeed, if k is the minimal field of definition for $L = 0$

then we may choose R and \tilde{L} in such a way that they too are defined over k . Thus if $k = \mathbb{Q}$ we will have $h_Q(C) \leq 1$, contrary to assumption. If k is a quadratic extension of \mathbb{Q} then Z and its quadratic conjugate will be distinct components of X , and there will therefore be a third component of degree 2, which must be defined over \mathbb{Q} . We may then deduce as above that $h_Q(C) \leq 1$. We cannot have $[k : \mathbb{Q}] > 3$ since the number of components Z_i is at most 3, so that we are left with the case in which k is cubic.

Let $L = L_1, L_2$, and L_3 be the three conjugates of L , and write $C = L_i R_i + \tilde{L}_i Q$ accordingly. Thus $LR + \tilde{L}Q = L_2 R_2 + \tilde{L}_2 Q$, so that $LR = 0$ whenever $L_2 = Q = 0$. However, the variety $L_2 = Q = 0$ is absolutely irreducible, since $\text{rank}(Q) \geq 5$, and it follows that one or other of L and R must vanish whenever $L_2 = Q = 0$. The only hyperplane containing $L_2 = Q = 0$ is the obvious one $L_2 = 0$, so in the first case L and L_2 must be proportional. This however is impossible, since we have eliminated the case in which the hyperplane $L = 0$ is defined over \mathbb{Q} . Thus R must vanish on $L_2 = Q = 0$, so that $R = L_2 L'_2 + c_2 Q$ for some linear form L'_2 and constant c_2 , both defined over $\overline{\mathbb{Q}}$.

In the same way we will have $R = L_3 L'_3 + c_3 Q$, say. Then

$$(c_2 - c_3)Q = (R - L_2 L'_2) - (R - L_3 L'_3) = L_3 L'_3 - L_2 L'_2.$$

Since $\text{rank}(Q) \geq 5$, this can happen only when $c_2 = c_3$. We will write $c = c_2 = c_3$ for this common value. We then have $L_3 L'_3 = L_2 L'_2$, and since L_2 and L_3 are not proportional, by the argument above, we see that $L'_3 = \gamma L_2$ for some constant γ . Thus $R = \gamma L_2 L_3 + c Q$, so that $C = \gamma L_1 L_2 L_3 + (c L_1 + \tilde{L}_1) Q$.

We may now write $C = \gamma N + M Q$, where $N = L_1 L_2 L_3$ is a cubic norm form, defined over \mathbb{Q} , and γ and M are a constant and a linear form, respectively, both over $\overline{\mathbb{Q}}$. Since N and Q have no common factor, this representation must be unique, so that in fact γ and M are defined over \mathbb{Q} . We then deduce that $\text{ord}_Q(C) \leq \text{ord}_Q(\gamma N) \leq 3$, contrary to our hypotheses. The lemma therefore follows. □

To deal with the local solubility conditions in Theorem 1.4, we will also need some information about varieties over local fields. The authors are grateful to Professor Salberger for drawing their attention to a reference for the following fact.

Lemma 3.4. *Let k be \mathbb{R} or a finite extension of a p -adic field \mathbb{Q}_p . Let V be an absolutely irreducible projective variety defined over k with a smooth k -point. Then $V(k)$ is dense in V under the Zariski topology.*

Proof. This follows on applying work of Colliot-Thélène, Coray, and Sansuc [6, Lemme 3.1.2] to the open Zariski-dense subset of V which is non-singular. □

4. Weyl differencing

In this section, we will use the Weyl differencing approach to give bounds for $S(\alpha_3, \alpha_2)$, defined in (2.2). Our overall strategy will be to assume that $S(\alpha_3, \alpha_2)$ is large, and to deduce that α_3 has a good approximation by a rational number with small denominator. Using this information, we then go on to show that α_2 must also have a good approximation by a rational number with small denominator. The first phase of the argument will apply Weyl’s method to $|S(\alpha_3, \alpha_2)|^4$. In contrast, the second phase will use

$|S(\alpha_3, \alpha_2)|^2$, and will incorporate an idea related to van der Corput’s method. The reader will see that in the first stage it is only the cubic form $C(\mathbf{x})$ which is relevant, while in the second stage it is primarily the quadratic form $Q(\mathbf{x})$ which features.

For the first phase of the argument we write $h = n$ if the form C is non-singular, and otherwise take $h = h(C)$. Notice that for Theorems 1.2 and 1.3 we must have $h \geq 29$, as we henceforth assume. We now define $T_3 = T_3(\alpha_3, \alpha_2) \in \mathbb{R}_{>0} \cup \{\infty\}$ by setting

$$|S(\alpha_3, \alpha_2)| = P^n T_3^{-h}. \tag{4.1}$$

We then call on Lemma 1 of Davenport and Lewis [8]. We will require a version with some trivial modifications, as we will explain. Let $R > 1$, and define

$$L(R) := \#\{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^{2n} : |\mathbf{x}| < R, |\mathbf{y}| < R, B_i(\mathbf{x}; \mathbf{y}) = 0 \forall i \leq n\}.$$

Then, if $\varepsilon > 0$ is given, the lemma, suitably modified, shows that either

$$L(R) > R^{2n} P^{-\varepsilon} T_3^{-4h}, \tag{4.2}$$

or there exists a positive integer $s \ll R^2$ such that $\|s\alpha_3\| < P^{-3}R^2$. In order to obtain the result in this form we must remove the weight $\omega(\mathbf{x}/P)$ by partial summation. We must also verify that the proof of the lemma still applies when the exponents θ and κ for which $R = P^\theta$ and $T_3 = P^{\kappa/h}$ are not necessarily constant. Davenport and Lewis require that $0 < \theta < 1$. However, if $R \geq P$ then it is always true that $\|s\alpha_3\| < P^{-3}R^2$ for some positive integer $s \leq R^2$, by Dirichlet’s approximation theorem. Finally, the reader will need to verify that the proof still goes through for sums of $e(\alpha_3 C(\mathbf{x}) + \alpha_2 Q(\mathbf{x}))$, as opposed to the terms $e(\alpha\varphi(\mathbf{x}))$ (involving a cubic polynomial $\varphi(\mathbf{x})$) considered by Davenport and Lewis.

We now present two alternative estimates for $L(R)$. First, for any form C , we can use Lemma 3 of Davenport and Lewis [8], which states that $L(R) \ll R^{2n-h(C)}$. On the other hand, if C is non-singular, we use Lemma 3 of Heath-Brown [14], which shows that there are $O(R^r)$ integer vectors in the region $|\mathbf{x}| < R$ such that the solution set

$$\{\mathbf{y} \in \mathbb{R}^n : B_i(\mathbf{x}; \mathbf{y}) = 0 \forall i \leq n\}$$

is $(n - r)$ -dimensional. The set will therefore contain $O(R^{n-r})$ integer vectors in the region $|\mathbf{y}| < R$, and we deduce that $L(R) \ll R^n$, on summing for $0 \leq r \leq n$. Thus $L(R) \ll R^{2n-h}$ in this case too, since we have defined $h = n$ when C is non-singular.

It now follows that, if we choose $R = P^\varepsilon T_3^4$, then (4.2) must fail, if P is large enough. We must therefore have an integer $s \ll R^2$ for which $\|s\alpha_3\| < P^{-3}R^2$. We may therefore write

$$\alpha_3 = \frac{b_3}{s} + \varphi_3, \tag{4.3}$$

with $b_3 \in \mathbb{Z}$ and $s|\varphi_3| < P^{-3}R^2$. Thus $s(1 + P^3|\varphi_3|) \ll R^2$, and on replacing ε by $\varepsilon/2$ we conclude as follows.

Lemma 4.1. *Let $\varepsilon > 0$ be given, and define T_3 by (4.1). Then there is a positive integer s such that (4.3) holds with $\gcd(s, b_3) = 1$ and*

$$s(1 + P^3|\varphi_3|) \ll P^\varepsilon T_3^8.$$

We should emphasise at this point that, as remarked in §2, we cannot assume that we have $b_3/s = a_3/q$, for the approximation (2.6).

We turn now to our second application of Weyl’s method. We shall suppose that (4.3) holds, where we think of both s and φ_3 as being small in suitable senses, and we write

$$f(\mathbf{x}) = \alpha_3 C(\mathbf{x}) + \alpha_2 Q(\mathbf{x})$$

for brevity. Then

$$\begin{aligned} S(\alpha_3, \alpha_2) &= \sum_{\mathbf{x} \in \mathbb{Z}^n} \omega(\mathbf{x}/P) e(f(\mathbf{x})) \\ &= \sum_{\mathbf{u} \pmod{s}} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \mathbf{x} \equiv \mathbf{u} \pmod{s}}} \omega(\mathbf{x}/P) e(f(\mathbf{x})). \end{aligned}$$

Cauchy’s inequality yields

$$\begin{aligned} |S(\alpha_3, \alpha_2)|^2 &\leq s^n \sum_{\mathbf{u} \pmod{s}} \left| \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \mathbf{x} \equiv \mathbf{u} \pmod{s}}} \omega(\mathbf{x}/P) e(f(\mathbf{x})) \right|^2 \\ &= s^n \sum_{\substack{\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n \\ \mathbf{x} \equiv \mathbf{y} \pmod{s}}} \omega(\mathbf{y}/P) \omega(\mathbf{x}/P) e(f(\mathbf{y}) - f(\mathbf{x})) \\ &\leq s^n \sum_{|\mathbf{z}| < P/s} \left| \sum_{\mathbf{x} \in \mathbb{Z}^n} \omega_0(\mathbf{x}/P) e(f(\mathbf{x} + s\mathbf{z}) - f(\mathbf{x})) \right|, \end{aligned}$$

where $\omega_0(\mathbf{x}) = \omega_0(\mathbf{x}, \mathbf{z}) = \omega(\mathbf{x} + sP^{-1}\mathbf{z})\omega(\mathbf{x})$. Although this remains true even when $s > P$, it is sensible to impose the condition $s \leq P$ for the time being.

Since $C(\mathbf{x} + s\mathbf{z}) - C(\mathbf{x})$ is automatically divisible by s , we see that

$$e(f(\mathbf{x} + s\mathbf{z}) - f(\mathbf{x})) = e(\varphi_3\{C(\mathbf{x} + s\mathbf{z}) - C(\mathbf{x})\} + \alpha_2\{Q(\mathbf{x} + s\mathbf{z}) - Q(\mathbf{x})\}).$$

We now set

$$g(\mathbf{x}) = g(\mathbf{x}, \mathbf{z}) = \varphi_3\{C(\mathbf{x} + s\mathbf{z}) - C(\mathbf{x})\},$$

and conclude that

$$|S(\alpha_3, \alpha_2)|^2 \leq s^n \sum_{|\mathbf{z}| < P/s} \left| \sum_{\mathbf{x} \in \mathbb{Z}^n} \omega_0(\mathbf{x}/P) e(g(\mathbf{x}) + s\alpha_2 \nabla Q(\mathbf{z}) \cdot \mathbf{x}) \right|.$$

By the Poisson summation formula, the inner sum is

$$P^n \sum_{\mathbf{m} \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \omega_0(\mathbf{t}) e(g(P\mathbf{t}) + Ps\alpha_2 \nabla Q(\mathbf{z}) \cdot \mathbf{t} - P\mathbf{m} \cdot \mathbf{t}) d\mathbf{t}.$$

The integrals may be estimated by the multidimensional “first derivative bound”; see Heath-Brown [16, Lemma 10], for example. One has

$$|\nabla(g(P\mathbf{t}) + Ps\alpha_2 \nabla Q(\mathbf{z}) \cdot \mathbf{t} - P\mathbf{m} \cdot \mathbf{t})| \geq \lambda$$

on $\text{supp}(\omega_0)$, with

$$\lambda = P|s\alpha_2 \nabla Q(\mathbf{z}) - \mathbf{m}| + O(P^3|\varphi_3|).$$

The second-order and third-order derivatives are $O(P^3|\varphi_3|)$, and all higher-order derivatives vanish. It therefore follows from [16, Lemma 10] that

$$\int_{\mathbb{R}^n} \omega_0(\mathbf{t}) e(g(P\mathbf{t}) + P s \alpha_2 \nabla Q(\mathbf{z}) \cdot \mathbf{t} - P \mathbf{m} \cdot \mathbf{t}) d\mathbf{t} \ll_A (P|s\alpha_2 \nabla Q(\mathbf{z}) - \mathbf{m}|)^{-A}$$

for any fixed $A > 0$, whenever $P|s\alpha_2 \nabla Q(\mathbf{z}) - \mathbf{m}| \gg P^3|\varphi_3|$. In particular, if $\varepsilon \in (0, 1)$ is given, and

$$\|s\alpha_2 \nabla Q(\mathbf{z})\| \geq P^{-1+\varepsilon}(1 + P^3|\varphi_3|), \tag{4.4}$$

then

$$\begin{aligned} \sum_{\mathbf{x} \in \mathbb{Z}^n} \omega_0(\mathbf{x}/P) e(g(\mathbf{x}) + s\alpha_2 \nabla Q(\mathbf{z}) \cdot \mathbf{x}) &\ll P^n \sum_{\mathbf{m} \in \mathbb{Z}^n} (P|s\alpha_2 \nabla Q(\mathbf{z}) - \mathbf{m}|)^{-A} \\ &\ll 1 \end{aligned}$$

provided that we choose P sufficiently large and take $A > (n + 1)/\varepsilon$. Of course if (4.4) fails then we may estimate the sum trivially as $O(P^n)$. We therefore deduce that

$$|S(\alpha_3, \alpha_2)|^2 \ll s^n \#S_1 + s^n P^n \#S_2$$

with

$$S_1 = \{\mathbf{z} \in \mathbb{Z}^n : |\mathbf{z}| < P/s\}$$

and

$$S_2 = \{\mathbf{z} \in \mathbb{Z}^n : |\mathbf{z}| < P/s, \|s\alpha_2 \nabla Q(\mathbf{z})\| \leq P^{-1+\varepsilon}(1 + P^3|\varphi_3|)\}.$$

We may omit the term $s^n \#S_1$ from the above estimate since it is at most $O(P^n)$, while S_2 contains at least the element $\mathbf{z} = \mathbf{0}$. If $|\mathbf{z}| < P/s$, one has $|\nabla Q(\mathbf{z})| \leq cP/s$, for some constant $c = c(Q)$. We now recall the notation

$$\rho = \text{rank}(Q)$$

introduced earlier. Thus the values $\nabla Q(\mathbf{z})$ are restricted to a vector space of dimension ρ . Given \mathbf{w} , the equation $\mathbf{w} = \nabla Q(\mathbf{z})$ has $O((P/s)^{n-\rho})$ integral solutions \mathbf{z} with $|\mathbf{z}| < P/s$, and we conclude that

$$\#S_2 \ll (P/s)^{n-\rho} \mathcal{N}^\rho,$$

where

$$\mathcal{N} = \#\{w \in \mathbb{Z} : |w| \leq cP/s, \|s\alpha_2 w\| \leq P^{-1+\varepsilon}(1 + P^3|\varphi_3|)\}.$$

We therefore have

$$|S(\alpha_3, \alpha_2)|^2 \ll P^{2n-\rho} s^\rho \mathcal{N}^\rho.$$

We now define $T_2 = T_2(\alpha_3, \alpha_2)$ by setting

$$|S(\alpha_3, \alpha_2)| = P^n T_2^{-\rho}, \tag{4.5}$$

whence

$$T_2^2 \gg P/(s\mathcal{N}). \tag{4.6}$$

Naturally, our next task is to estimate \mathcal{N} . Generally, if

$$\mathcal{W} = \{w \in \mathbb{Z} : |w| \leq W, \|\mu w\| \leq \xi\},$$

then $\#\mathcal{W}$ is at most the number of points of the lattice

$$\Lambda = \{(W^{-1}u, \xi^{-1}(\mu u - v)) : (u, v) \in \mathbb{Z}^2\}$$

lying in the unit square. The determinant of the lattice is $(W\xi)^{-1}$, and so the number of points is $O(1 + W\xi + \sigma^{-1})$, where σ is the first successive minimum of the lattice. From the definition of σ we see that there will be a non-zero point $(u, v) \in \mathbb{Z}^2$ such that $|u| \leq \sigma W$ and $|\mu u - v| \leq \sigma \xi$.

Thus in our situation we find that

$$\mathcal{N} \ll 1 + P^\varepsilon s^{-1}(1 + P^3|\varphi_3|) + \sigma^{-1},$$

so that either $\mathcal{N} \ll 1 + P^\varepsilon s^{-1}(1 + P^3|\varphi_3|)$ or $\sigma \leq \mathcal{N}^{-1}$. In the former case, (4.6) yields

$$T_2^2 \gg \min\left(\frac{P}{s}, \frac{P^{1-\varepsilon}}{1 + P^3|\varphi_3|}\right) \gg \frac{P^{1-\varepsilon}}{s + P^3|\varphi_3|}.$$

In the latter case, (4.6) shows that there is a non-zero point (u, v) with

$$|u| \leq cP/(s\mathcal{N}) \ll T_2^2$$

and

$$|s\alpha_2 u - v| \leq P^{-1+\varepsilon}(1 + P^3|\varphi_3|)/\mathcal{N} \ll P^{-2+\varepsilon}s(1 + P^3|\varphi_3|)T_2^2.$$

If there is any such point (u, v) for which $u = 0$, then $v \neq 0$, whence we must have $P^{-2+\varepsilon}s(1 + P^3|\varphi_3|)T_2^2 \gg 1$. But in that case we may take $u = 1$, and we will automatically have $\|s\alpha_2 u\| \ll P^{-2+\varepsilon}s(1 + P^3|\varphi_3|)T_2^2$. Thus we can assume with no loss of generality that there is a solution in which $u \neq 0$. We now summarise our findings as follows.

Lemma 4.2. *Define*

$$|S(\alpha_3, \alpha_2)| = P^n T_2^{-\rho},$$

and suppose that (4.3) holds with $\gcd(s, b_3) = 1$. Then for any fixed $\varepsilon > 0$ one of the following must happen:

(i) *there is a positive integer $u \ll T_2^2$ such that*

$$\|su\alpha_2\| \ll P^{-2+\varepsilon}s(1 + P^3|\varphi_3|)T_2^2;$$

or

(ii) *we have*

$$T_2^2 \gg \frac{P^{1-\varepsilon}}{s + P^3|\varphi_3|}.$$

Note that we assumed that $s \leq P$ during the proof. However, the result is trivial when $s \geq P$, since we then have $T_2^2 \gg 1 \gg P/s$.

5. Minor arc contribution: the Weyl bound

In this section, we will see what can be said about the size of the minor arc integral (2.4) on the basis of Lemmas 4.1 and 4.2. For convenience, we write

$$I(\mathfrak{m}) := \iint_{\mathfrak{m}} S(\alpha_3, \alpha_2) d\alpha_3 d\alpha_2.$$

We begin by considering values α_3 for which case (i) of Lemma 4.2 holds.

Our first move is to show that on the minor arcs T_3 (and hence also T_2) cannot be too small. Lemma 4.1 and case (i) of Lemma 4.2 produce positive integers s and u such that

$$su \ll P^\varepsilon T_3^8 T_2^2.$$

Moreover, there will be integers b_3, b_2 for which

$$|su\alpha_3 - ub_3| = su|\varphi_3| \ll uP^{-3+\varepsilon} T_3^8 \ll P^{-3+\varepsilon} T_3^8 T_2^2$$

and

$$|su\alpha_2 - b_2| = \|su\alpha_2\| \ll P^{-2+\varepsilon} s(1 + P^3|\varphi_3|)T_2^2 \ll P^{-2+2\varepsilon} T_3^8 T_2^2.$$

It follows that we would have $su \leq P^\delta$ and

$$\begin{aligned} \left| \alpha_3 - \frac{b_3 u}{su} \right| &\leq |su\alpha_3 - ub_3| \leq P^{-3+\delta}, \\ \left| \alpha_2 - \frac{b_2}{su} \right| &\leq |su\alpha_2 - b_2| \leq P^{-2+\delta}, \end{aligned}$$

if $T_3^8 T_2^2 \leq P^{\delta-3\varepsilon}$ say, with P sufficiently large. Thus, if $(\alpha_3, \alpha_2) \in \mathfrak{m}$, we must have $T_3^8 T_2^2 \geq P^{\delta-3\varepsilon}$. It is clear from (4.1) and (4.5) that

$$T_2 = T_3^{h/\rho}. \tag{5.1}$$

We therefore deduce that

$$T_3 \geq P^{\delta\rho/(16\rho+4h)} \tag{5.2}$$

provided that $\varepsilon \leq \delta/6$, as we henceforth assume.

In estimating the minor arc integral $I(\mathfrak{m})$, it will be convenient to consider the contribution $I_{t_3}(\mathfrak{m})$, say, from all pairs α_3, α_2 for which T_3 lies in a dyadic range

$$t_3 < T_3 \leq 2t_3.$$

In view of (5.2), we may assume that $t_3 \geq P^{\delta\rho/(16\rho+4h)}$. Moreover, (5.1) implies that

$$t_3^{h/\rho} < T_2 \leq (2t_3)^{h/\rho}.$$

We put $t_2 = t_3^{h/\rho}$.

We proceed to consider the contribution to $I_{t_3}(\mathfrak{m})$ from all pairs α_3, α_2 for which the first alternative of Lemma 4.2 holds. We begin by considering the measure of the available $\alpha_2 \in (0, 1]$. For each positive integer $u \ll t_2^2$, there will be an integer $v \ll su$ such that

$$|su\alpha_2 - v| \ll P^{-2+\varepsilon} s(1 + P^3|\varphi_3|)t_2^2.$$

Thus the total measure for the values of α_2 will be

$$\ll \sum_{u \ll t_2^2} \sum_{v \ll su} (su)^{-1} P^{-2+\varepsilon} s(1 + P^3|\varphi_3|)t_2^2 \ll P^{-2+\varepsilon} s(1 + P^3|\varphi_3|)t_2^4.$$

According to Lemma 4.1, we will have $s(1 + P^3|\varphi_3|) \ll P^\varepsilon t_3^8$, so that the above is $O(P^{-2+2\varepsilon} t_3^8 t_2^4)$. We may calculate the available measure for α_3 in much the same way, given that $s \ll P^\varepsilon t_3^8$ and $|\varphi_3| \ll P^{-3+\varepsilon} s^{-1} t_3^8$, by Lemma 4.1. This yields

$$\begin{aligned} \text{meas}\{\alpha_3 : s(1 + P^3|\varphi_3|) \ll P^\varepsilon t_3^8\} &\ll \sum_{s \ll P^\varepsilon t_3^8} \sum_{v \ll s} P^{-3+\varepsilon} s^{-1} t_3^8 \\ &\ll P^\varepsilon t_3^8 \cdot P^{-3+\varepsilon} t_3^8 \\ &= P^{-3+2\varepsilon} t_3^{16}. \end{aligned} \tag{5.3}$$

Returning to our estimation of the contribution to $I_3(\mathbf{m})$ from the first case of Lemma 4.2, we obtain the overall contribution

$$\ll P^n t_3^{-h} \cdot P^{-2+2\varepsilon} t_3^8 t_2^4 \cdot P^{-3+2\varepsilon} t_3^{16} \ll P^{n-5+4\varepsilon} t_3^{-h+4h/\rho+24}.$$

If $(h - 24)(\rho - 4) > 96$ then $h - 4h/\rho - 24 \geq 1/\rho$. Thus, if we sum over all relevant dyadic ranges for $t_3 \geq P^{\delta\rho/(16\rho+4h)}$, we will obtain an overall contribution

$$\ll P^{n-5+4\varepsilon-\delta/(16\rho+4h)},$$

which is satisfactory if we choose ε sufficiently small. We record our conclusions as follows.

Lemma 5.1. *The contribution to $I(\mathbf{m})$ arising from pairs α_3, α_2 for which the first alternative of Lemma 4.2 holds is $o(P^{n-5})$, provided that*

$$(h - 24)(\rho - 4) > 96.$$

Turning to the contribution to $I_3(\mathbf{m})$ from those pairs α_3, α_2 for which the second alternative of Lemma 4.2 holds, we first consider the situation when $t_3 \geq P^{3/19}$. According to (5.3), the available set of values for α_3 has measure $O(P^{-3+2\varepsilon} t_3^{16})$, but there is no restriction on the values of α_2 . It follows that the contribution to the minor arc integral is

$$\ll P^n t_3^{-h} \cdot P^{-3+2\varepsilon} t_3^{16}. \tag{5.4}$$

We proceed to sum over dyadic values $t_3 \geq P^{3/19}$ to obtain a total

$$\ll P^{n-3+2\varepsilon-3(h-16)/19} \leq P^{n-5+2\varepsilon-1/19},$$

provided that $h \geq 29$. This gives us the following result.

Lemma 5.2. *Suppose that $h \geq 29$. Then the contribution to $I(\mathbf{m})$ arising from pairs α_3, α_2 for which the second case of Lemma 4.2 holds, and $T_3 \geq P^{3/19}$, is $o(P^{n-5})$.*

The simplest way to handle the remaining case is to combine the inequalities $s(1 + P^3|\varphi_3|) \ll P^\varepsilon T_3^8$ and $T_2^2 \gg P^{1-\varepsilon}/(s + P^3|\varphi_3|)$ from Lemma 4.1 and part (ii) of Lemma 4.2, respectively, to deduce that

$$T_3^8 T_2^2 \gg P^{1-2\varepsilon} \frac{s + sP^3|\varphi_3|}{s + P^3|\varphi_3|} \geq P^{1-2\varepsilon}.$$

Then (5.1) implies that $T_3^{8+2h/\rho} \gg P^{1-2\varepsilon}$. We therefore see from the bound (5.4) that the total contribution to the minor arc integral is $O(P^{n-\psi})$, with

$$\psi = 3 - 2\varepsilon + (h - 16) \frac{1 - 2\varepsilon}{8 + 2h/\rho}.$$

By taking ε sufficiently small, we can make $\psi > 5$ provided that

$$h - 16 > 2(8 + 2h/\rho).$$

This gives us the following lemma, which is exactly what we need for Theorem 1.2.

Lemma 5.3. *Suppose that $(h - 32)(\rho - 4) > 128$. Then*

$$I(\mathbf{m}) = o(P^{n-5}).$$

An alternative way to deal with the case $T_3 \leq P^{3/19}$ is to use an analysis based on the Poisson summation formula. We will carry this out in § 6. It is an essential feature of the method that one uses simultaneous rational approximations a_3/q and a_2/q to α_3 and α_2 , as given by (2.6).

We will want to know whether the approximation a_3/q corresponds to the approximation b_3/s given by (4.3). However, if $b_3/s \neq a_3/q$, then

$$\frac{1}{sq} \leq \left| \frac{a_3}{q} - \frac{b_3}{s} \right| \leq \left| \alpha_3 - \frac{a_3}{q} \right| + \left| \alpha_3 - \frac{b_3}{s} \right| \leq \frac{1}{qQ_3} + |\varphi_3|.$$

It would then follow from Lemma 4.1 and (2.5) that

$$\begin{aligned} 1 &\leq s/Q_3 + sq|\varphi_3| \\ &\ll P^\varepsilon T_3^8 (Q_3^{-1} + P^{-3} Q_3 Q_2) \\ &\ll P^{24/19+\varepsilon} \cdot P^{-4/3} \\ &\leq P^{-4/57+\varepsilon}, \end{aligned}$$

provided that $T_3 \leq P^{3/19}$. This will produce a contradiction if ε is small enough and P is large enough, thereby proving that $a_3/q = b_3/s$.

We record this conclusion as follows.

Lemma 5.4. *Suppose that $h \geq 29$ and $T_3 \leq P^{3/19}$. Then we will have $a_3/q = b_3/s$ if P is large enough.*

6. Poisson summation

In this section, we suppose that $X \subset \mathbb{P}^{n-1}$ is non-singular. By Corollary 3.2 we may assume that the cubic form C is non-singular and that Q takes the shape

$$Q(\mathbf{x}) = \sum_{i=1}^n d_i x_i^2, \tag{6.1}$$

with $d_1, \dots, d_n \in \mathbb{Z}$ such that $d_1 \cdots d_{n-1} \neq 0$. Thus Q has rank at least $n - 1$. We are now ready to begin our analysis of the exponential sums

$$S(\alpha_3, \alpha_2) = \sum_{\mathbf{x} \in \mathbb{Z}^n} \omega(\mathbf{x}/P) e(\alpha_3 C(\mathbf{x}) + \alpha_2 Q(\mathbf{x})),$$

for $\alpha_3, \alpha_2 \in \mathbb{R}$, based on an application of Poisson summation.

We will assume throughout this section that $\alpha_3 = a_3/q + \theta_3$ and $\alpha_2 = a_2/q + \theta_2$, as in § 2. Thus $\mathbf{a} = (a_3, a_2) \in \mathbb{Z}^2$ and $q \in \mathbb{Z}$ satisfy

$$1 \leq a_3, a_2 \leq q \leq Q_3 Q_2, \quad \gcd(q, \mathbf{a}) = 1, \tag{6.2}$$

and $\boldsymbol{\theta} = (\theta_3, \theta_2) \in \mathbb{R}^2$ satisfies

$$|\theta_i| \leq q^{-1} Q_i^{-1}, \quad (i = 3, 2). \tag{6.3}$$

We recall that Q_3, Q_2 are positive integers given by (2.5). Our first step involves introducing complete exponential sums modulo q . The following result is standard.

Lemma 6.1. *We have*

$$S(\alpha_3, \alpha_2) = \frac{P^n}{q^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} S(\mathbf{a}, q; \mathbf{m}) I(\theta_3 P^3, \theta_2 P^2; q^{-1} P \mathbf{m}),$$

where

$$S(\mathbf{a}, q; \mathbf{m}) := \sum_{\mathbf{y} \pmod{q}} e_q(a_3 C(\mathbf{y}) + a_2 Q(\mathbf{y}) + \mathbf{m} \cdot \mathbf{y}), \tag{6.4}$$

$$I(\boldsymbol{\gamma}; \mathbf{z}) := \int_{\mathbb{R}^n} \omega(\mathbf{x}) e(\gamma_3 C(\mathbf{x}) + \gamma_2 Q(\mathbf{x}) - \mathbf{z} \cdot \mathbf{x}) d\mathbf{x}. \tag{6.5}$$

Proof. Write $\mathbf{x} = \mathbf{y} + q\mathbf{z}$, for $\mathbf{y} \pmod{q}$. Then we obtain

$$\begin{aligned} S(\alpha_3, \alpha_2) &= \sum_{\mathbf{y} \pmod{q}} e_q(a_3 C(\mathbf{y}) + a_2 Q(\mathbf{y})) \\ &\quad \times \sum_{\mathbf{z} \in \mathbb{Z}^n} \omega((\mathbf{y} + q\mathbf{z})/P) e(\theta_3 C(\mathbf{y} + q\mathbf{z}) + \theta_2 Q(\mathbf{y} + q\mathbf{z})). \end{aligned}$$

The statement of the lemma follows from an application of Poisson summation, followed by an obvious change of variables. □

We begin by analysing the complete exponential sums $S(\mathbf{a}, q; \mathbf{m})$ given by (6.4), for $\gcd(q, \mathbf{a}) = 1$ and $\mathbf{m} \in \mathbb{Z}^n$. They satisfy the multiplicativity property

$$S(\mathbf{a}, rs; \mathbf{m}) = S(\mathbf{a}_s, r; \mathbf{m})S(\mathbf{a}_r, s; \mathbf{m}), \quad \text{for } \gcd(r, s) = 1, \tag{6.6}$$

where

$$\mathbf{a}_t := (t^2 a_3, t a_2).$$

The proof of this fact is standard (see [4, Lemma 10], for example). In view of this, it will suffice to analyse $S(\mathbf{a}, q; \mathbf{m})$ for prime power values of q .

It will be convenient to give a separate treatment of the moduli q that are built from prime divisors of a_3 . Recall the shape (6.1) that Q takes, with $d_1 \cdots d_{n-1} \neq 0$. For a given prime p , we let p^v be the largest power of p dividing any of $2d_1, \dots, 2d_{n-1}$. Since Q is fixed, we will have $p^v \ll 1$. We can now state our result.

Lemma 6.2. *Suppose that $p^{1+v} \mid a_3$, and let $r \geq 1$. Then, for any $\mathbf{m} \in \mathbb{Z}^n$, we have*

$$S(\mathbf{a}, p^r; \mathbf{m}) \ll p^{r(n+1)/2}.$$

Proof. Since $p \mid a_3$, we may assume that $p \nmid a_2$. Let $S = S(x)$ be the sum

$$\sum_{x_1, \dots, x_{n-1} \pmod{p^r}} e_{p^r}(a_2 Q(x_1, \dots, x_{n-1}, x) + a_3 C(x_1, \dots, x_{n-1}, x)).$$

We will show that $S \ll p^{r(n-1)/2}$ for every x , which will suffice. Our approach is based on applying Weyl’s method to $|S|^2$. This gives

$$|S|^2 \leq \sum_{y_1, \dots, y_{n-1} \pmod{p^r}} \left| \sum_{x_1, \dots, x_{n-1} \pmod{p^r}} e_{p^r}(f) \right|, \tag{6.7}$$

where $f = f(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-1})$ has the shape

$$2a_2 \sum_{i=1}^{n-1} d_i x_i y_i + p^{1+v} \sum_{i=1}^{n-1} y_i g_i(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-1}),$$

since $p^{1+v} \mid a_3$. Here, the g_i are suitable polynomials defined over \mathbb{Z} .

Suppose now that we have an exponent $h \leq r - v - 1$ such that $p^h \mid y_1, \dots, y_{n-1}$, but some y_i is not divisible by p^{h+1} . Let us suppose without loss of generality that $p^{h+1} \nmid y_1$. Writing $x_1 = s + p^{r-h-v-1}t$, with s running modulo $p^{r-h-v-1}$ and t modulo p^{h+v+1} , one finds that

$$f \equiv 2a_2 d_1 y_1 p^{r-h-v-1} t + f_0(s; x_2, \dots, x_{n-1}; y_1, \dots, y_{n-1}) \pmod{p^r},$$

for some integral polynomial f_0 . It follows that the sum over t vanishes unless p^{h+v+1} divides $2a_2 d_1 y_1$. However, this latter condition would contradict the facts that $p \nmid a_2$, $p^{v+1} \nmid 2d_1$, and $p^{h+1} \nmid y_1$.

We therefore deduce that the inner sum of (6.7) vanishes unless p^{r-v} divides each of y_1, \dots, y_{n-1} . There are therefore $p^v \ll 1$ choices for each of these, and for each such choice the inner sum has modulus at most $p^{r(n-1)}$. We then deduce that $|S|^2 \ll p^{r(n-1)}$, and the lemma follows. \square

We are now ready to begin in earnest our treatment of the exponential sum $S(\mathbf{a}, q; \mathbf{m})$ for $q \in \mathbb{N}$. Let us write $q = q_0q_1q_2$, where

$$q_0 = \prod_{\substack{p^e \parallel q \\ p^{1+v} \nmid a_3}} p^e, \quad q_2 = \prod_{\substack{p^e \parallel q, e \geq 3 \\ p^{1+v} \nmid a_3}} p^e. \tag{6.8}$$

Thus q_1 is cube-free, and $\gcd(q_1q_2, a_3)$ divides $\prod_p p^v$, which in turn divides $2 \prod_{i=1}^{n-1} d_i$, where d_i are the coefficients of Q . It follows that $\gcd(q_1q_2, a_3) \ll 1$.

Lemma 6.2 and (6.6) will suffice to deal with the sum associated to the modulus q_0 . The cube-free modulus q_1 will be handled via the following result.

Lemma 6.3. *Let $\varepsilon > 0$. Suppose that q is cube-free, and is a product of primes p for which $p^{1+v} \nmid a_3$. Then, for any $\mathbf{m} \in \mathbb{Z}^n$, we have*

$$S(\mathbf{a}, q; \mathbf{m}) \ll q^{n/2+\varepsilon}.$$

Proof. By (6.6) it will suffice to show that $S(\mathbf{a}, p^r; \mathbf{m}) \ll p^{rn/2}$, for $r \in \{1, 2\}$ and each prime with $p^{1+v} \nmid a_3$. The result is trivial for the finitely many primes with $v \neq 0$. Indeed we may assume that $p \gg 1$, where the implied constant is taken large enough to ensure that C is non-singular modulo p . When $r = 2$, the result therefore follows from work of Heath-Brown [15]. Suppose next that $r = 1$. We wish to apply the estimate

$$\sum_{\mathbf{x} \in \mathbb{F}_p^n} e_p(f(\mathbf{x})) \ll_{d,n} p^{n/2},$$

of Deligne [9], which applies to any polynomial f over \mathbb{F}_p of degree d , in n variables, whose leading homogeneous part is smooth modulo p . Taking $f(\mathbf{x}) = a_3C(\mathbf{x}) + a_2Q(\mathbf{x}) + \mathbf{m} \cdot \mathbf{x}$, we get $S(\mathbf{a}, p; \mathbf{m}) \ll p^{n/2}$, as required. \square

It is now time to turn our attention to the cube-full modulus q_2 , with $\gcd(q_2, a_3) \ll 1$. Our next goal is the following variant of [14, Lemma 14].

Lemma 6.4. *Let $\varepsilon > 0$, and let $\mathbf{m}_0 \in \mathbb{R}^n$. Suppose that q is cube-full, with $\gcd(q, a_3) \ll 1$. Then we have*

$$\sum_{|\mathbf{m} - \mathbf{m}_0| \leq V} |S(\mathbf{a}, q; \mathbf{m})| \ll q^{n/2+\varepsilon} \{V^n + q^{n/3}\},$$

for any $V \geq 1$.

For the proof we will modify parts of the argument from Browning and Heath-Brown [4, § 5]. We will be fairly brief, since the changes necessary are minor, if somewhat tedious. We will write our cube-full modulus q as $q = c^2d$, with d square-free.

First, in analogy to [4, Lemma 11], one may show that

$$|S(\mathbf{a}, q; \mathbf{m})| \leq (c^2d)^{n/2} \sum_{\substack{\mathbf{u} \pmod{c} \\ c | (\nabla g(\mathbf{u}) + \mathbf{m})}} M_d(\mathbf{u})^{1/2}, \tag{6.9}$$

where

$$g(\mathbf{u}) = a_3C(\mathbf{u}) + a_2Q(\mathbf{u})$$

and

$$M_d(\mathbf{u}) = \#\{\mathbf{x} \pmod{d} : \nabla^2 g(\mathbf{u}) \cdot \mathbf{x} \equiv \mathbf{0} \pmod{d}\}.$$

Corresponding to the sum $\mathcal{S}(V, a; \mathbf{m}_0, c, d)$ in [4, Eq. (5.9)], we define

$$\mathcal{S}(V) = \mathcal{S}(V, a_3, a_2; \mathbf{m}_0, c, d) := \sum_{|\mathbf{m}-\mathbf{m}_0| \leq V} \sum_{\substack{\mathbf{a} \pmod{c} \\ c | (\nabla g(\mathbf{a}) + \mathbf{m})}} M_d(\mathbf{a})^{1/2}.$$

We would like to adapt [4, Lemma 16] to our present situation. Note that [4, §5] is concerned with exponential sums associated to general cubic polynomials g for which the cubic part is non-singular and $\|g\|_P = \|P^{-3} f(Px_1, \dots, Px_n)\| \leq H$ for some parameter H . In our setting, one may verify that it is possible to replace H by 1 in the various estimates of [4]. A number of trivial adjustments need to be made, since we have $\gcd(c^2d, a_3) \ll 1$, rather than $\gcd(c^2d, a) = 1$.

Note that [4, Lemma 13] can be applied directly with $H \ll 1$, since it pertains only to the cubic part C of g , where $\|g_0\|_P = \|g_0\| \ll 1$. Moreover, [4, Lemma 14] can also be applied, with $D \ll 1$. Turning to the analogue of [4, Lemma 16], which relies on [4, Lemmas 13 and 14], the proof goes through unchanged, with $D \ll 1$ and $H \ll 1$. For any $\varepsilon > 0$, this leads to the estimate

$$\mathcal{S} \ll q^\varepsilon V^n \left(1 + \frac{c^2d}{V^3}\right)^{n/2}.$$

But then, taking

$$V_1 = V + (c^2d)^{1/3},$$

we have

$$\mathcal{S}(V) \leq \mathcal{S}(V_1) \ll q^\varepsilon V_1^n \left(1 + \frac{c^2d}{V_1^3}\right)^{n/2} \ll q^\varepsilon (V^n + (c^2d)^{n/3}).$$

Lemma 6.4 now follows from (6.9).

We next turn to the analysis of the exponential integral

$$\begin{aligned} I &= I(\theta_3 P^3, \theta_2 P^2; q^{-1} P \mathbf{m}) \\ &= \int_{\mathbb{R}^n} \omega(\mathbf{x}) e(\theta_3 P^3 C(\mathbf{x}) + \theta_2 P^2 Q(\mathbf{x}) - q^{-1} P \mathbf{m} \cdot \mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

For this, it will be convenient to write

$$f(\mathbf{x}) = \theta_3 P^3 C(\mathbf{x}) + \theta_2 P^2 Q(\mathbf{x}).$$

We will proceed by adapting the proof of [4, Lemma 6], noting that our weight function ω belongs to the class of weight functions considered therein. Let $\nu \in \mathbb{R}$ be a parameter in the range $0 < \nu \leq 1$, to be chosen in due course. We decompose I into an average of integrals over subregions of size at most ν . It follows from [16, Lemma 2] that there exists an infinitely differentiable weight function $w_\nu(\mathbf{x}, \mathbf{y}) : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$, such that

$$\omega(\mathbf{x}) = \nu^{-n} \int_{\mathbb{R}^n} w_\nu\left(\frac{\mathbf{x}-\mathbf{y}}{\nu}, \mathbf{y}\right) \, d\mathbf{y}.$$

Moreover, $\text{supp}(w_\nu) \subseteq [-1, 1]^n \times \text{supp}(\omega)$. Then, on making this substitution into I , and writing $\mathbf{x} = \mathbf{y} + \nu\mathbf{u}$, we obtain

$$\begin{aligned} |I| &= \nu^{-n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} w_\nu(\nu^{-1}(\mathbf{x} - \mathbf{y}), \mathbf{y}) e(f(\mathbf{x}) - q^{-1}P\mathbf{m}\cdot\mathbf{x}) d\mathbf{x}d\mathbf{y} \right| \\ &\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} w_\nu(\mathbf{u}, \mathbf{y}) e(f(\mathbf{y} + \nu\mathbf{u}) - \nu q^{-1}P\mathbf{m}\cdot\mathbf{u}) d\mathbf{u} \right| d\mathbf{y} \\ &= \int_{\text{supp}(\omega)} |K(\mathbf{y})| d\mathbf{y}, \end{aligned} \tag{6.10}$$

say.

Let us write $F(\mathbf{u}) = f(\mathbf{y} + \nu\mathbf{u}) - \nu q^{-1}P\mathbf{m}\cdot\mathbf{u}$, for fixed \mathbf{y} . It is clear that $f(\mathbf{y} + \nu\mathbf{u}) \ll \Theta$ for any $(\mathbf{y}, \mathbf{u}) \in \text{supp}(\omega) \times [-1, 1]^n$, where we put

$$\Theta = 1 + |\theta_3|P^3 + |\theta_2|P^2. \tag{6.11}$$

For such (\mathbf{y}, \mathbf{u}) , it follows that the k th-order derivatives of $F(\mathbf{u})$ are all $O_k(\nu^k\Theta)$, for $k \geq 2$. Likewise, one finds that

$$\nabla F(\mathbf{u}) = \nu \nabla f(\mathbf{y}) - \nu q^{-1}P\mathbf{m} + O(\nu^2\Theta).$$

Let $R \geq 1$, and suppose that $|\nabla f(\mathbf{y}) - q^{-1}P\mathbf{m}| \geq \nu^{-1}R$. Then it follows that there exists a constant $c(n) > 0$ such that $|\nabla F(\mathbf{u})| \gg R$, provided that

$$R \geq c(n)\nu^2\Theta.$$

We will take $\nu = \Theta^{-1/2}$, so that $0 < \nu \leq 1$. An application of [16, Lemma 10] now reveals that $K(\mathbf{y}) \ll_N R^{-N}$ for any $N \geq 1$, when $R \geq c(n)$. Inserting this into (6.10) gives

$$I \ll_N R^{-N} + \text{meas } \mathcal{S}(R)$$

for any $N \geq 1$ and any $R \geq c(n)$, where we have written

$$\mathcal{S}(R) = \left\{ \mathbf{y} \in \text{supp}(\omega) : |\nabla f(\mathbf{y}) - q^{-1}P\mathbf{m}| \leq R\sqrt{\Theta} \right\}.$$

If we choose $R = P^\varepsilon$ with some fixed $\varepsilon > 0$, then R^{-N} can be made smaller than any given negative power of P , via a suitable choice of N . This leads to the following result.

Lemma 6.5. *Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be given. Then*

$$I(\theta_3P^3, \theta_2P^2; q^{-1}P\mathbf{m}) \ll_N P^{-N} + \text{meas } \mathcal{S}(P^\varepsilon).$$

Alternatively, if $|\mathbf{m}| \geq c\Theta q/P$ with a suitably large constant c , then $|\nabla f(\mathbf{y})| \leq \frac{1}{2}q^{-1}P|\mathbf{m}|$ for $\mathbf{y} \in \text{supp}(\omega)$. Thus, if we take

$$R = \frac{1}{3}q^{-1}P|\mathbf{m}|\Theta^{-1/2},$$

say, then $\mathcal{S}(R)$ will be empty. Moreover, if $|\mathbf{m}| \geq P^{\varepsilon-1}q\Theta$ for some positive $\varepsilon < 1$, then we will have

$$\frac{R}{(P|\mathbf{m}|)^{\varepsilon/3}} = \frac{(P|\mathbf{m}|)^{1-\varepsilon/3}}{3q\sqrt{\Theta}} \geq \frac{P^{\varepsilon(1-\varepsilon/3)}\Theta^{1/2-\varepsilon/3}}{3q^{\varepsilon/3}} \geq \frac{P^{\varepsilon(1-\varepsilon/3)}}{3P^{2\varepsilon/3}} \geq \frac{1}{3},$$

since $\Theta \geq 1$ and $q \leq P^2$. It follows that $R \gg (P|\mathbf{m}|)^{\varepsilon/3}$ whenever $|\mathbf{m}| \geq P^{\varepsilon-1}q\Theta$. This leads to the following conclusion.

Lemma 6.6. *Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be given. Then*

$$I(\theta_3 P^3, \theta_2 P^2; q^{-1} P \mathbf{m}) \ll_N P^{-N} |\mathbf{m}|^{-N}$$

whenever $|\mathbf{m}| \geq P^{\varepsilon-1} q \Theta$.

We are now ready to deduce a final estimate for the exponential sum $S(\alpha_3, \alpha_2)$ in (2.2), for any $(\alpha_3, \alpha_2) \in \mathbb{R}^2$. We suppose as before that $\alpha_i = a_i/q + \theta_i$, with $\mathbf{a} = (a_3, a_2) \in \mathbb{Z}^2$ and $q \in \mathbb{Z}$ satisfying (6.2) and $\boldsymbol{\theta} = (\theta_3, \theta_2) \in \mathbb{R}^2$ satisfying (6.3). Here, $Q_3, Q_2 \in \mathbb{N}$ are given by (2.5).

Our starting point is Lemma 6.1. We use Lemma 6.6 to handle the tail of the summation over \mathbf{m} , so that

$$S(\alpha_3, \alpha_2) \ll 1 + \frac{P^n}{q^n} \sum_{|\mathbf{m}| \leq P^{\varepsilon-1} q \Theta} |S(\mathbf{a}, q; \mathbf{m})| \cdot |I(\theta_3 P^3, \theta_2 P^2; q^{-1} P \mathbf{m})|$$

for any fixed $\varepsilon > 0$. Next, we employ the multiplicativity property (6.6) in conjunction with Lemmas 6.2 and 6.3 to show that the second term is

$$\ll \frac{P^{n+\varepsilon}}{q^n} q_0^{(n+1)/2} q_1^{n/2} \sum_{|\mathbf{m}| \leq P^{\varepsilon-1} q \Theta} |S(\mathbf{a}_{q_0 q_1}, q_2; \mathbf{m})| \cdot |I(\theta_3 P^3, \theta_2 P^2; q^{-1} P \mathbf{m})|.$$

We then use Lemma 6.5, which shows that this is

$$\ll 1 + \frac{P^{n+\varepsilon}}{q^n} q_0^{(n+1)/2} q_1^{n/2} \int_{\text{supp}(\omega)} \left(\sum_{|\mathbf{m} - \mathbf{m}_0(\mathbf{y})| \leq V} |S(\mathbf{a}_{q_0 q_1}, q_2; \mathbf{m})| \right) d\mathbf{y},$$

where

$$\mathbf{m}_0(\mathbf{y}) = q(\theta_3 P^2 \nabla C(\mathbf{y}) + \theta_2 P \nabla Q(\mathbf{y}))$$

and

$$V = P^{\varepsilon-1} q \sqrt{\Theta}.$$

Finally, Lemma 6.4 produces the bound

$$S(\alpha_3, \alpha_2) \ll 1 + \frac{P^{n+\varepsilon}}{q^n} q_0^{(n+1)/2} q_1^{n/2} q_2^{n/2+\varepsilon} \{V^n + q_2^{n/3}\}.$$

We have therefore established the following result, on redefining ε .

Lemma 6.7. *Let $\varepsilon > 0$, and let q_0, q_2 , and Θ be defined as in (6.8) and (6.11). Then we have*

$$S(\alpha_3, \alpha_2) \ll q_0^{1/2} P^\varepsilon \{q^{n/2} \Theta^{n/2} + P^n q^{-n/2} q_2^{n/3}\}.$$

We should comment at this point that the first term on the right is more or less what one would hope for. The second term on the right could probably be improved, but it suffices for our purposes. When q and Θ are both of order 1, so that (α_3, α_2) lies in the major arcs, we expect that $S(\alpha_3, \alpha_2)$ is approximately $P^n q^{-n} S(\mathbf{a}, q, \mathbf{0}) I(0, 0; \mathbf{0})$. This corresponds to the second term on the right in Lemma 6.7. However, when Θ is a little larger than 1, we would expect to have a non-trivial bound for $I(\theta_3 P^3, \theta_2 P^2; \mathbf{0})$, and the lemma does not take any account of this.

7. Minor arc contribution: Theorem 1.3

In this section, we will combine the work of §§5 and 6 so as to handle the minor arcs for Theorem 1.3. Thus we will assume that $h = n \geq 29$ and that $\rho \geq n - 1$. Lemma 5.1 then gives a satisfactory result under the first alternative of Lemma 4.2, provided that $n \geq 29$. Indeed we see that one cannot hope to handle the case $n = 28$, even when $\rho = n$. Moreover, Lemma 5.2 gives a satisfactory result under the second alternative of Lemma 4.2 when $T_3 \geq P^{3/19}$. We therefore investigate the second alternative of Lemma 4.2, under the assumption that $T_3 \leq P^{3/19}$ and $n \geq 29$. Furthermore, it follows from Lemma 5.4 that we may proceed under the assumption that $b_3/s = a_3/q$.

We will assume that T_3 lies in a dyadic range $t_3 < T_3 \leq 2t_3$, with

$$P^{\delta\rho/(20n)} \leq P^{\delta\rho/(16\rho+4h)} \leq t_3 \leq P^{3/19}, \tag{7.1}$$

where the lower bound comes from (5.2). It follows from (5.1) that $t_2 < T_2 \leq 2^{h/\rho}t_2$, with $t_2 = t_3^{h/\rho}$.

We can rapidly dispose of the case in which $s \leq P^3|\varphi_3|$. If this happens, then the second part of Lemma 4.2, together with Lemma 4.1, yields

$$1 \ll P^{2+\varepsilon}t_2^2|\varphi_3| \ll P^{2+\varepsilon}t_3^{2n/(n-1)} \cdot P^{\varepsilon-3}t_3^8s^{-1},$$

whence

$$1 \ll P^{-2+4\varepsilon}t_3^{16+4n/(n-1)}s^{-2}.$$

We now write \mathcal{P}_1 for the set of α_3 for which $s \leq P^3|\varphi_3|$ and the value of T_3 lies in our dyadic range $t_3 < T_3 \leq 2t_3$. Then, if $\alpha_3 \in \mathcal{P}_1$, we have

$$S(\alpha_3, \alpha_2) \ll P^n t_3^{-n} \ll P^{n-2+4\varepsilon} t_3^{16+4n/(n-1)-n} s^{-2}.$$

Moreover, since $\varphi_3 \ll P^{\varepsilon-3}t_3^8s^{-1}$, we have

$$\int_{\mathcal{P}_1} s^{-2} d\alpha_3 \ll \sum_{s \ll P^\varepsilon t_3^8} s^{-2} \sum_{b_3 \pmod{s}} P^{\varepsilon-3} t_3^8 s^{-1} \ll P^{\varepsilon-3} t_3^8.$$

The contribution to the minor arc integral is therefore

$$\ll P^{n-2+4\varepsilon} t_3^{16+4n/(n-1)-n} \cdot P^{\varepsilon-3} t_3^8.$$

In view of (7.1), this provides a satisfactory bound if $n > 24 + 4n/(n - 1)$, so that $n \geq 29$ will be sufficient. Thus in applying Lemma 4.2 we may henceforth take

$$s \gg P^{1-\varepsilon}t_2^{-2}. \tag{7.2}$$

We proceed to examine the case in which the first term on the right in Lemma 6.7 dominates the second. Thus we will assume that

$$S(\alpha_3, \alpha_2) \ll q_0^{1/2} P^\varepsilon q^{n/2} \Theta^{n/2}.$$

It follows from (2.5), (2.6), and (6.11) that $q\Theta \ll P^{5/3}$, whence

$$S(\alpha_3, \alpha_2) \ll q_0^{1/2} P^{5n/6+\varepsilon}. \tag{7.3}$$

We now consider the contribution from the set \mathcal{P}_2 of pairs (α_3, α_2) for which (7.3) holds and $t_3 < T_3 \leq 2t_3$. It will be convenient to write $I_3(\mathbf{m})$ for the corresponding part of the minor arc integral. From (4.1), we will have

$$S(\alpha_3, \alpha_2) \ll P^n t_3^{-n}. \tag{7.4}$$

As in (5.3), the measure of the available set of points α_3 is $O(P^{2\varepsilon-3}t_3^{16})$, whence

$$\text{meas}(\mathcal{P}_2) \ll P^{2\varepsilon-3}t_3^{16}. \tag{7.5}$$

Thus we certainly have

$$I_3(\mathbf{m}) \ll P^{2\varepsilon-3}t_3^{16} \cdot P^n t_3^{-n}. \tag{7.6}$$

For an alternative bound we consider two cases. We give ourselves a parameter $Q_0 \geq 1$, to be chosen shortly, and consider separately the ranges $q_0 \leq Q_0$ and $q_0 \geq Q_0$. If $q_0 \leq Q_0$, the bounds (7.3) and (7.5) show that the contribution to $I_3(\mathbf{m})$ is

$$\ll P^{2\varepsilon-3}t_3^{16} \cdot Q_0^{1/2} P^{5n/6+\varepsilon}. \tag{7.7}$$

To investigate the second alternative, we consider the subset, \mathcal{P}_3 say, of \mathcal{P}_2 for which the corresponding value of q_0 is at least Q_0 . Using the facts that α_3 and α_2 satisfy (2.6), and that q_0 is given by (6.8), we have

$$\text{meas}(\mathcal{P}_3) \ll \sum_{\substack{q \leq Q_3 Q_2 \\ q_0 \geq Q_0}} \sum_{a_3} \sum_{a_2} (q Q_3)^{-1} (q Q_2)^{-1} \ll P^{-5/3} \sum_{\substack{q \leq Q_3 Q_2 \\ q_0 \geq Q_0}} \sum_{a_3} q^{-1}.$$

If we define

$$q_3 = \prod_{p|q_0} p,$$

then $q_3 \mid a_3$, so that there are $O(q/q_3)$ available values for a_3 , and we deduce that

$$\text{meas}(\mathcal{P}_3) \ll P^{-5/3} \sum_{\substack{q \leq Q_3 Q_2 \\ q_0 \geq Q_0}} q_3^{-1} \ll \sum_{q_0 \geq Q_0} q_3^{-1} q_0^{-1}.$$

However, a standard estimation via Rankin's method shows that

$$\begin{aligned} \sum_{q_0 \geq Q_0} q_3^{-1} q_0^{-1} &\leq Q_0^{\varepsilon-1} \sum_{q_0 \geq Q_0} q_3^{-1} q_0^{-\varepsilon} \\ &\leq Q_0^{\varepsilon-1} \sum_{q_0=1}^{\infty} q_3^{-1} q_0^{-\varepsilon} \\ &= Q_0^{\varepsilon-1} \prod_p \{1 + p^{-1-\varepsilon} + p^{-1-2\varepsilon} + p^{-1-3\varepsilon} + \dots\} \\ &\ll Q_0^{\varepsilon-1}. \end{aligned}$$

We therefore deduce that

$$\text{meas}(\mathcal{P}_3) \ll P^{2\varepsilon} Q_0^{-1},$$

so that the estimate (7.4) shows that the contribution to $I_{t_3}(\mathbf{m})$ is

$$\ll P^{2\varepsilon} Q_0^{-1} \cdot P^n t_3^{-n}.$$

Comparing this bound with (7.7), we see that we have an estimate

$$I_{t_3}(\mathbf{m}) \ll P^{2\varepsilon-3} t_3^{16} \cdot Q_0^{1/2} P^{5n/6+\varepsilon} + P^{2\varepsilon} Q_0^{-1} \cdot P^n t_3^{-n},$$

for any $Q_0 \geq 1$, covering both cases $q_0 \leq Q_0$ and $q_0 \geq Q_0$. We choose

$$Q_0 = 1 + P^{n/9+2} t_3^{-2n/3-32/3}$$

so as to balance the two terms approximately, and then deduce that

$$I_{t_3}(\mathbf{m}) \ll P^{3\varepsilon} \{P^{5n/6-3} t_3^{16} + P^{8n/9-2} t_3^{(32-n)/3}\}.$$

We now combine this with (7.6) to produce

$$I_{t_3}(\mathbf{m}) \ll P^{3\varepsilon} \{\min(P^{n-3} t_3^{16-n}, P^{5n/6-3} t_3^{16}) + \min(P^{n-3} t_3^{16-n}, P^{8n/9-2} t_3^{(32-n)/3})\}.$$

If $n \geq 16$, then we use the inequality $\min(A, B) \leq A^{16/n} B^{(n-16)/n}$ to conclude that

$$\min(P^{n-3} t_3^{16-n}, P^{5n/6-3} t_3^{16}) \leq P^{n-5-(n-28)/6}.$$

If $16 \leq n \leq 32$, we use $\min(A, B) \leq A^{(32-n)/(2n-16)} B^{(3n-48)/(2n-16)}$ to deduce that

$$\min(P^{n-3} t_3^{16-n}, P^{8n/9-2} t_3^{(32-n)/3}) \leq P^{n-5-\eta}$$

with

$$\eta = \frac{(n-29)(n-8)+8}{6(n-8)}.$$

These bounds make it clear that $I_{t_3}(\mathbf{m}) \ll P^{n-5-\varepsilon}$ for a small $\varepsilon > 0$, when $29 \leq n \leq 32$; and if $n \geq 33$, then

$$\begin{aligned} \min(P^{n-3} t_3^{16-n}, P^{8n/9-2} t_3^{(32-n)/3}) &\leq P^{8n/9-2} t_3^{(32-n)/3} \\ &\leq P^{8n/9-2} \\ &\leq P^{n-5-2/3}. \end{aligned}$$

We may therefore conclude as follows.

Lemma 7.1. *If $h = n$ and $\rho \geq n - 1$, then the contribution to the minor arc integral when the first term on the right in Lemma 6.7 dominates the second will be $o(P^{n-5})$, provided that $n \geq 29$.*

We turn now to the pairs (α_3, α_2) for which the second term on the right in Lemma 6.7 dominates the first, so that

$$S(\alpha_3, \alpha_2) \ll q_0^{1/2} P^{n+\varepsilon} q^{-n/2} q_2^{n/3}.$$

We now recall that we may assume that $a_3/q = b_3/s$ with $\gcd(s, b_3) = 1$. Thus in particular we will have $s \mid q$. In view of the definitions in (6.8), we therefore deduce that

$$\begin{aligned} q_0^{1/2} q^{-n/2} q_2^{n/3} &= q_0^{(1-n)/2} q_1^{-n/2} q_2^{-n/6} \\ &\leq (q_0 q_1 q_2)^{-(n+4)/6} q_2^{2/3} \\ &= q^{-(n+4)/6} q_2^{2/3} \\ &\leq s^{-(n+4)/6} q_2^{2/3} \end{aligned}$$

as soon as $n \geq 4$. Moreover, if $p^e \parallel q_2$, then we have $p^e \mid qb_3 = a_3s$ and $p^{1+v} \nmid a_3$, whence $p^e \mid sp^v$. It follows that $q_2 \mid Ds$, with

$$D = 2 \prod_{i=1}^{n-1} d_i$$

for the coefficients d_i in (6.1). Now let \mathcal{P}_4 be the set of pairs (α_3, α_2) for which the relevant values of s, q_2 , and T_3 lie in given dyadic ranges $S < s \leq 2S, Q_2 < q_2 \leq 2Q_2$, and $t_3 < T_3 \leq 2t_3$, so that

$$S(\alpha_3, \alpha_2) \ll P^{n+\varepsilon} S^{-(n+4)/6} Q_2^{2/3}$$

on \mathcal{P}_4 . Since $q_2 \mid Ds$ and $D \ll 1$, there are $O(S/q_2)$ choices for s , given q_2 . We have $s|q_3| \ll P^{\varepsilon-3} t_3^8$ by Lemma 4.1, and so we may calculate that

$$\begin{aligned} \text{meas}(\mathcal{P}_4) &\ll \sum_{Q_2 < q_2 \leq 2Q_2} \sum_{S < s \leq 2S} \sum_{b_3 \pmod{s}} P^{\varepsilon-3} s^{-1} t_3^8 \\ &\ll \sum_{Q_2 < q_2 \leq 2Q_2} P^{\varepsilon-3} t_3^8 S q_2^{-1} \\ &\ll P^{\varepsilon-3} t_3^8 S Q_2^{-2/3}, \end{aligned}$$

since q_2 runs over cube-full numbers. This yields the bound

$$\int_{\mathcal{P}_4} |S(\alpha_3, \alpha_2)| d\alpha_3 d\alpha_2 \ll P^{n-3+\varepsilon} t_3^8 S^{-(n-2)/6}. \tag{7.8}$$

Alternatively, (7.4) produces

$$\int_{\mathcal{P}_4} |S(\alpha_3, \alpha_2)| d\alpha_3 d\alpha_2 \ll P^n t_3^{-n} \text{meas}(\mathcal{P}_4) \ll P^{n-3+\varepsilon} t_3^{8-n} S.$$

We may combine these to give

$$\begin{aligned} &\int_{\mathcal{P}_4} |S(\alpha_3, \alpha_2)| d\alpha_3 d\alpha_2 \\ &\ll P^{n-3+\varepsilon} \min\left(t_3^8 S^{-(n-2)/6}, t_3^{8-n} S\right) \\ &\ll P^{n-3+\varepsilon} \left(t_3^8 S^{-(n-2)/6}\right)^{6/(n+4)} \left(t_3^{8-n} S\right)^{(n-2)/(n+4)} \\ &= P^{n-3+\varepsilon} t_3^{-\kappa_1}, \end{aligned}$$

with

$$\kappa_1 = \frac{n^2 - 10n - 32}{n + 4}.$$

We may also couple (7.8) with (7.2) to produce a bound

$$\ll P^{n-3-(n-2)/6+n\epsilon} t_3^8 t_2^{(n-2)/3} \ll P^{n-3-(n-2)/6+n\epsilon} t_3^{\kappa_2},$$

with

$$\kappa_2 = 8 + \frac{(n-2)n}{3n-3}.$$

Comparing this with the previous bound, we deduce that

$$\begin{aligned} & \int_{\mathcal{P}_2} |S(\alpha_3, \alpha_2)| d\alpha_3 d\alpha_2 \\ & \ll P^{n\epsilon} \min \left(P^{n-3} t_3^{-\kappa_1}, P^{n-3-(n-2)/6} t_3^{\kappa_2} \right) \\ & \leq P^{n\epsilon} \min \left(P^{n-3} t_3^{-\kappa_1} \right)^{(n-14)/(n-2)} \left(P^{n-3-(n-2)/6} t_3^{\kappa_2} \right)^{12/(n-2)} \\ & = P^{n-5+n\epsilon} t_3^{-\kappa} \end{aligned}$$

with

$$\begin{aligned} \kappa &= \frac{n-14}{n-2} \kappa_1 - \frac{12}{n-2} \kappa_2 \\ &= \left(\frac{n-14}{n-2} \right) \left(\frac{n^2-10n-32}{n+4} \right) - \frac{12}{n-2} \left(8 + \frac{(n-2)n}{3n-3} \right). \end{aligned}$$

A slightly unpleasant calculation confirms that $\kappa > 0$ whenever $n \geq 28$. This completes our treatment of the minor arcs, which we summarise as follows.

Lemma 7.2. *If $h = n \geq 29$ and $\rho \geq n - 1$, then (2.4) holds.*

This will suffice for our application to Theorem 1.3.

8. Major arc contribution

The purpose of this section is to complete the proof of Theorems 1.2 and 1.3, by establishing (2.3) under suitable hypotheses on \mathfrak{M} and the forms C and Q . In what follows, we will put $h = n$ if C is non-singular and $h = h(C)$ otherwise. Moreover, we continue to adopt the notation

$$\rho = \text{rank}(Q),$$

for the rank of the quadratic form Q . By Corollary 3.2, we have $\rho \geq n - 1$ when the intersection $C = Q = 0$ is non-singular.

It is now time to reveal the weight functions ω that we shall use in the definition (2.1) of our counting function

$$N_\omega(X; P) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ C(\mathbf{x})=Q(\mathbf{x})=0}} \omega(\mathbf{x}/P).$$

There is nothing to prove unless the variety X contains a non-singular real point. Consequently, we let $\mathbf{x}_0 \in \mathbb{R}^n$ be a non-zero vector such that $C(\mathbf{x}_0) = Q(\mathbf{x}_0) = 0$ and

$\nabla C(\mathbf{x}_0)$ is not proportional to $\nabla Q(\mathbf{x}_0)$. We will find it convenient to work with a weight function that forces us to count points lying very close to \mathbf{x}_0 . For any $\xi \in (0, 1]$, we define the function $\omega : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ by

$$\omega(\mathbf{x}) := \nu(\xi^{-1} \|\mathbf{x} - \mathbf{x}_0\|),$$

where $\|\mathbf{y}\| = \sqrt{y_1^2 + \dots + y_n^2}$ and

$$\nu(x) = \begin{cases} e^{-1/(1-x^2)}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

We will require ξ to be sufficiently small, with $1 \ll \xi \leq 1$. It is clear that ω is infinitely differentiable, and that it is supported on the region $\|\mathbf{x} - \mathbf{x}_0\| \leq \xi$. Moreover, there exist constants $c_j > 0$ depending only on j and ξ such that

$$\max \left\{ \left| \frac{\partial^{j_1 + \dots + j_n} \omega(\mathbf{x})}{\partial^{j_1} x_1 \dots \partial^{j_n} x_n} \right| : \mathbf{x} \in \mathbb{R}^n, j_1 + \dots + j_n = j \right\} \leq c_j,$$

for each integer $j \geq 0$.

We are now ready to begin our analysis of the exponential sums $S(\alpha_3, \alpha_2)$ on the set of major arcs \mathfrak{M} defined in § 2, for $\delta \in (0, \frac{1}{3})$. Let us define

$$S(\mathbf{a}, q) := \sum_{\mathbf{y} \pmod{q}} e_q(a_3 C(\mathbf{y}) + a_2 Q(\mathbf{y})),$$

for $\mathbf{a} = (a_3, a_2)$ with $\gcd(q, \mathbf{a}) = 1$. Our work in this section will lead us to study the truncated singular series

$$\mathfrak{S}(R) = \sum_{q \leq R} \frac{1}{q^n} \sum_{\substack{\mathbf{a} \pmod{q} \\ \gcd(q, \mathbf{a})=1}} S(\mathbf{a}, q), \tag{8.1}$$

for any $R > 1$. We put $\mathfrak{S} = \lim_{R \rightarrow \infty} \mathfrak{S}(R)$, whenever this limit exists. Next, let

$$\mathfrak{J}(R) = \int_{-R}^R \int_{-R}^R \int_{\mathbb{R}^n} \omega(\mathbf{x}) e(\gamma_3 C(\mathbf{x}) + \gamma_2 Q(\mathbf{x})) d\mathbf{x} d\gamma_3 d\gamma_2, \tag{8.2}$$

for any $R > 0$. We put $\mathfrak{J} = \lim_{R \rightarrow \infty} \mathfrak{J}(R)$, whenever the limit exists. The main aim of this section is to establish the following result.

Lemma 8.1. *Assume that $(h - 24)(\rho - 4) > 96$. Then the singular series \mathfrak{S} and the singular integral \mathfrak{J} are absolutely convergent. Moreover, if we choose*

$$\delta = 1/8, \tag{8.3}$$

then there is a positive constant Δ such that

$$\iint_{\mathfrak{M}} S(\alpha_3, \alpha_2) d\alpha_3 d\alpha_2 = \mathfrak{S} \mathfrak{J} P^{n-5} + O(P^{n-5-\Delta}).$$

Accepting Lemma 8.1 for the time being, let us indicate how it can be used to complete the proof of Theorems 1.2 and 1.3. In the context of Theorem 1.3, for which $h = n \geq 29$ and $\rho \geq n - 1$, we combine Lemma 7.2 and Lemma 8.1 to deduce that

$$N_\omega(X; P) = \mathfrak{S}\mathfrak{J}P^{n-5} + o(P^{n-5}),$$

as $P \rightarrow \infty$, with both \mathfrak{S} and \mathfrak{J} absolutely convergent. The same asymptotic formula holds when $(h(C) - 32)(\rho - 4) > 128$, as in Theorem 1.2. This can be seen by combining Lemma 5.3 with Lemma 8.1.

In order to complete the proof of Theorems 1.2 and 1.3, we are required to show that $\mathfrak{S}\mathfrak{J} > 0$ whenever $X_{\text{sm}}(\mathbf{A}) \neq \emptyset$. Indeed, if $\mathfrak{S}\mathfrak{J} > 0$ for any $[\mathbf{x}_0] \in X_{\text{sm}}(\mathbb{R})$, then it will follow from our asymptotic formula for $N_\omega(X; P)$ that $X(\mathbb{Q})$ is Zariski dense in X , whence the existence of a point in $X_{\text{sm}}(\mathbb{Q})$ is assured. The proof that $\mathfrak{S} > 0$ follows a standard line of reasoning, as in [3, Lemma 7.1], and makes use of the fact that \mathfrak{S} is absolutely convergent. To show that $\mathfrak{J} > 0$, it will suffice to show that $\mathfrak{J}(R) \gg 1$ for sufficiently large values of R . This again is standard, and will follow from an easy adaptation of work of Heath-Brown [14, § 10] on the corresponding problem for a single cubic form. The only difference lies in the choice of weights used and the fact that we now have a complete intersection of codimension 2, but neither of these alters the nature of the proof. Performing the integrations over γ_3 and γ_2 , and writing $\mathbf{x} = \mathbf{x}_0 + \mathbf{y}$, it follows from (8.2) that

$$\begin{aligned} \mathfrak{J}(R) &= \int_{\mathbb{R}^n} \omega(\mathbf{x}) \frac{\sin(2\pi RC(\mathbf{x})) \sin(2\pi RQ(\mathbf{x}))}{\pi^2 C(\mathbf{x})Q(\mathbf{x})} d\mathbf{x} \\ &= \int_{\mathbb{R}^n} \nu(\xi^{-1} \|\mathbf{y}\|) \frac{\sin(2\pi RC(\mathbf{x}_0 + \mathbf{y})) \sin(2\pi RQ(\mathbf{x}_0 + \mathbf{y}))}{\pi^2 C(\mathbf{x}_0 + \mathbf{y})Q(\mathbf{x}_0 + \mathbf{y})} d\mathbf{y}. \end{aligned}$$

Let $a_i = \partial C / \partial x_i(\mathbf{x}_0)$ and $b_i = \partial Q / \partial x_i(\mathbf{x}_0)$ for $1 \leq i \leq n$. We may assume without loss of generality that $a_1 b_2 - a_2 b_1 \neq 0$. The need for $\xi > 0$ to be sufficiently small emerges through an application of the inverse function theorem. Since $\|\mathbf{y}\| \leq \xi$, if we write

$$\begin{aligned} z_3 &= C(\mathbf{x}_0 + \mathbf{y}) = a_1 y_1 + \dots + a_n y_n + P_2(\mathbf{y}) + P_3(\mathbf{y}), \\ z_2 &= Q(\mathbf{x}_0 + \mathbf{y}) = b_1 y_1 + \dots + b_n y_n + Q_2(\mathbf{y}), \end{aligned}$$

for forms P_i of degree i and Q_2 of degree 2, then $z_3, z_2 \ll \xi$, and we can invert this expression to represent y_1 and y_2 as a power series in $z_3, z_2, y_3, \dots, y_n$, if ξ is sufficiently small. We refer the reader to [14] for the remainder of the argument.

To prove Lemma 8.1, we begin by recalling that $q \leq P^\delta$, that we have $\mathbf{a} = (a_3, a_2)$ with $\text{gcd}(q, \mathbf{a}) = 1$, and that $(\alpha_3, \alpha_2) \in \mathfrak{M}_{\mathbf{a}, q}$, with $\alpha_i = a_i/q + \theta_i$, for $i = 3, 2$. We will use the argument of [3, Lemma 5.1] to show that

$$S(\alpha_3, \alpha_2) = q^{-n} P^n S(\mathbf{a}, q) I(\theta_3 P^3, \theta_2 P^2; \mathbf{0}) + O(P^{n-1+2\delta}), \tag{8.4}$$

where $S(\mathbf{a}, q)$ is given above and I is given by (6.5).

To see this, we write $\mathbf{x} = \mathbf{y} + q\mathbf{z}$ in (2.2), where \mathbf{y} runs over a complete set of residues modulo q , giving

$$S(\alpha_3, \alpha_2) = \sum_{\mathbf{y} \pmod{q}} e_q(a_3 C(\mathbf{y}) + a_2 Q(\mathbf{y})) \sum_{\mathbf{z} \in \mathbb{Z}^n} f(\mathbf{z}), \tag{8.5}$$

with

$$f(\mathbf{z}) = \omega \left(\frac{\mathbf{y} + q\mathbf{z}}{P} \right) e(\theta_3 C(\mathbf{y} + q\mathbf{z}) + \theta_2 Q(\mathbf{y} + q\mathbf{z})).$$

We now want to replace the summation over \mathbf{z} by an integration. If $\mathbf{t} \in [0, 1]^n$, then

$$f(\mathbf{z} + \mathbf{t}) = f(\mathbf{z}) + O \left(\max_{\mathbf{u} \in [0, 1]^n} |\nabla f(\mathbf{z} + \mathbf{u})| \right).$$

Hence

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f(\mathbf{z}) d\mathbf{z} - \sum_{\mathbf{z} \in \mathbb{Z}^n} f(\mathbf{z}) \right| &\ll \text{meas}(\mathcal{B}) \max_{\mathbf{z} \in \mathcal{B}} |\nabla f(\mathbf{z})| \\ &\ll \left(\frac{P}{q} \right)^n (q/P + q|\theta_3|P^2 + q|\theta_2|P) \\ &= q^{1-n} P^{n-1} + |\theta_3|q^{1-n} P^{n+2} + |\theta_2|q^{1-n} P^{n+1}, \end{aligned}$$

where \mathcal{B} is an n -dimensional cube with sides of order $1 + P/q \leq 2P/q$. Substituting this into (8.5), and making the change of variables $P\mathbf{u} = \mathbf{y} + q\mathbf{z}$, we therefore deduce that

$$\begin{aligned} S(\alpha_3, \alpha_2) &= q^{-n} P^n S(\mathbf{a}, q) I(\theta_3 P^3, \theta_2 P^2; \mathbf{0}) \\ &\quad + O(qP^{n-1} + |\theta_3|qP^{n+2} + |\theta_2|qP^{n+1}). \end{aligned} \tag{8.6}$$

This completes the proof of (8.4), since $|\theta_i| \leq P^{-i+\delta}$ and $q \leq P^\delta$ on the major arcs.

Using (8.4), and noting that the major arcs have measure $O(P^{-5+5\delta})$, it is now easy to deduce that

$$\iint_{\mathfrak{M}} S(\alpha_3, \alpha_2) d\alpha_3 d\alpha_2 = P^{n-5} \mathfrak{S}(P^\delta) \mathfrak{J}(P^\delta) + O(P^{n-6+7\delta}), \tag{8.7}$$

where $\mathfrak{S}(P^\delta)$ is given by (8.1), and $\mathfrak{J}(P^\delta)$ is given by (8.2).

We proceed to use (8.4) in conjunction with our Weyl estimates, namely Lemmas 4.1 and 4.2, to bound $S(\mathbf{a}, q)$, with the aim of proving the following result.

Lemma 8.2. *Let $\varepsilon > 0$ be given. If h and ρ are both positive, then*

$$S(\mathbf{a}, q) \ll q^{n+\varepsilon} \left(\frac{q}{\gcd(q, a_3)} \right)^{-h/8} \tag{8.8}$$

and

$$S(\mathbf{a}, q) \ll q^{n+\varepsilon} \gcd(q, a_3)^{-\rho/2}. \tag{8.9}$$

Proof. To prove this, we reverse our normal point of view, and think of q as given and of P as being large in terms of q . Specifically, it will suffice to take

$$P = q^{8n}. \tag{8.10}$$

When $\theta_3 = \theta_2 = 0$, we have $I(0, 0; \mathbf{0}) \gg 1$, whence (8.4) yields

$$\begin{aligned} S(\mathbf{a}, q) &\ll 1 + q^n P^{-n} |S(a_3/q, a_2/q)| \\ &= 1 + q^n T_3^{-h} = 1 + q^n T_2^{-\rho}, \end{aligned} \tag{8.11}$$

since (8.10) shows that $P^{1-\delta} \gg q^n$.

We proceed to apply Lemma 4.1, bearing in mind that the integer s is not necessarily equal to q . Thus we have $a_3/q = b_3/s + \varphi_3$ and

$$s(1 + P^3|\varphi_3|) \ll P^\varepsilon T_3^8. \tag{8.12}$$

If $a_3/q \neq b_3/s$, then $|\varphi_3| \geq (sq)^{-1}$, whence

$$T_3^8 \gg P^{3-\varepsilon} s |\varphi_3| \geq P^{3-\varepsilon} q^{-1}.$$

Then, taking $\varepsilon < 1$, we see that (8.11) leads to the estimate

$$S(\mathbf{a}, q) \ll 1 + q^n T_3^{-1} \ll 1 + q^n \cdot P^{(\varepsilon-3)/8} q^{1/8} \ll 1 + q^{n+1} P^{-1/4} \ll 1,$$

in view of (8.10). This is more than sufficient for the lemma, and so we henceforth assume that $a_3/q = b_3/s$ and that $\varphi_3 = 0$. Thus $sa_3 = qb_3$ with $\gcd(s, b_3) = 1$, whence $s = q/\gcd(q, a_3)$. Moreover, (8.12) reduces to $s \ll P^\varepsilon T_3^8$, so that $T_3 \gg P^{-\varepsilon/8} (q/\gcd(q, a_3))^{1/8}$. Inserting this into (8.11) leads to the estimate

$$S(\mathbf{a}, q) \ll 1 + q^n \cdot P^{\varepsilon h/8} \left(\frac{q}{\gcd(q, a_3)} \right)^{-h/8}.$$

This is suitable for (8.8), given our choice (8.10), on redefining ε .

To obtain (8.9), we apply Lemma 4.2, which either shows that

$$T_2^2 \gg P^{1-\varepsilon} s^{-1} \geq P^{1-\varepsilon} q^{-1},$$

or produces a positive integer $u \ll T_2^2$ for which

$$\|sua_2/q\| \ll P^{-2+\varepsilon} s T_2^2. \tag{8.13}$$

If the first alternative holds then, taking $\varepsilon < 1/2$, we find that (8.11) produces a bound

$$S(\mathbf{a}, q) \ll 1 + q^n T_2^{-1} \ll 1 + q^n \cdot P^{(\varepsilon-1)/2} q^{1/2} \ll 1 + q^{n+1} P^{-1/4} \ll 1,$$

in view of (8.10). Again, this is more than sufficient for the lemma, and so we examine the second alternative.

If $q \nmid sua_2$, the bound (8.13) would imply that $q^{-1} \ll P^{-2+\varepsilon} s T_2^2$, so that

$$T_2^2 \gg P^{2-\varepsilon} (sq)^{-1} \geq P^{2-\varepsilon} q^{-2}.$$

Just as above, this would produce an acceptable estimate:

$$S(\mathbf{a}, q) \ll 1 + q^n T_2^{-1} \ll 1 + q^n \cdot P^{(\varepsilon-2)/2} q \ll 1 + q^{n+1} P^{-1/4} \ll 1.$$

On the other hand, if $q \mid sua_2$ then $\gcd(q, a_3) \mid ua_2$, since we have $s = q/\gcd(q, a_3)$, as noted above. Recalling that $\gcd(q, a_3, a_2) = 1$, we deduce that $\gcd(q, a_3) \mid u$, so that $\gcd(q, a_3) \leq u \ll T_2^2$. Thus (8.11) yields

$$S(\mathbf{a}, q) \ll 1 + q^n \gcd(q, a_3)^{-\rho/2},$$

as required for (8.9). This completes the proof of the lemma. □

We can now handle the singular series. Let

$$A(q) = \sum_{\substack{\mathbf{a} \pmod{q} \\ \gcd(q, \mathbf{a})=1}} |S(\mathbf{a}, q)|.$$

Then we have

$$A(q) \ll q \sum_{a_3 \pmod{q}} q^{n+\varepsilon} \min \left(\left(\frac{q}{\gcd(q, a_3)} \right)^{-h/8}, \gcd(q, a_3)^{-\rho/2} \right).$$

There are at most q/d values of a_3 for which $\gcd(q, a_3) = d$, and each one contributes a total

$$\begin{aligned} &\ll q^{n+1+\varepsilon} \min((q/d)^{-h/8}, d^{-\rho/2}) \\ &\ll q^{n+1+\varepsilon} ((q/d)^{-h/8})^{(4\rho+8)/(4\rho+h)} (d^{-\rho/2})^{(h-8)/(4\rho+h)} \\ &= q^{n+1+\varepsilon-\xi} d \end{aligned}$$

with

$$\xi = \frac{h(\rho+2)}{8\rho+2h}.$$

It follows that

$$A(q) \ll \sum_{d|q} qd^{-1} \cdot q^{n+1+\varepsilon-\xi} d \ll q^{n+2+2\varepsilon-\xi},$$

so that the singular series is absolutely convergent when $\xi > 3$, and

$$\mathfrak{S}(R) = \mathfrak{S} + O(R^{2\varepsilon-(\xi-3)}).$$

Since $\xi > 3$ when $(h-24)(\rho-4) > 96$, the claim in Lemma 8.1 follows.

We now estimate the exponential integral $I(\boldsymbol{\gamma}; \mathbf{0})$, for general values of $\boldsymbol{\gamma} = (\gamma_3, \gamma_2)$.

Lemma 8.3. *We have $I(\boldsymbol{\gamma}; \mathbf{0}) \ll 1$ for any $\boldsymbol{\gamma}$. Moreover, if h and ρ are positive, and if $\varepsilon \in (0, 1/8)$, then*

$$I(\boldsymbol{\gamma}; \mathbf{0}) \ll |\gamma_3|^{-h/8} |\boldsymbol{\gamma}|^\varepsilon \tag{8.14}$$

and

$$I(\boldsymbol{\gamma}; \mathbf{0}) \ll \left(\frac{|\gamma_2|}{1+|\gamma_3|} \right)^{-\rho/2} |\boldsymbol{\gamma}|^\varepsilon. \tag{8.15}$$

Proof. The estimate $I(\boldsymbol{\gamma}; \mathbf{0}) \ll 1$ is trivial. Moreover, it implies both (8.14) and (8.15) when $|\boldsymbol{\gamma}| \leq 1$. We assume henceforth that $|\boldsymbol{\gamma}| > 1$, and follow an argument analogous to that used for Lemma 8.2.

Taking $a_3 = a_2 = 0$ and $q = 1$ in (8.6), and setting $\alpha_3 = P^{-3}\gamma_3$ and $\alpha_2 = P^{-2}\gamma_2$, we deduce that

$$\begin{aligned} I(\boldsymbol{\gamma}; \mathbf{0}) &= P^{-n} S(\alpha_3, \alpha_2) + O(P^{-1}|\boldsymbol{\gamma}|) \\ &= T_3^{-h} + O(P^{-1}|\boldsymbol{\gamma}|) \\ &= T_2^{-\rho} + O(P^{-1}|\boldsymbol{\gamma}|), \end{aligned}$$

for any $P \geq 1$. We will choose P to be large, given by

$$P = |\boldsymbol{\gamma}|^{2n(2n+8)}, \tag{8.16}$$

so that

$$I(\boldsymbol{\gamma}; \mathbf{0}) = T_3^{-h} + O(|\boldsymbol{\gamma}|^{-n}) = T_2^{-\rho} + O(|\boldsymbol{\gamma}|^{-n}). \tag{8.17}$$

We now need estimates for the quantities T_3 and T_2 . We begin by applying Lemma 4.1, which shows that

$$P^{-3}\gamma_3 = \alpha_3 = \frac{b_3}{s} + \varphi_3$$

with

$$s(1 + P^3|\varphi_3|) \ll P^\varepsilon T_3^8. \tag{8.18}$$

If $b_3 \neq 0$, then

$$s^{-1} \leq |b_3|s^{-1} \leq P^{-3}|\gamma_3| + |\varphi_3| \leq P^{-3}|\boldsymbol{\gamma}|(1 + P^3|\varphi_3|).$$

It follows that $s(1 + P^3|\varphi_3|)|\boldsymbol{\gamma}| \geq P^3$, whence

$$T_3^8 \gg P^{3-\varepsilon}|\boldsymbol{\gamma}|^{-1} \gg |\boldsymbol{\gamma}|^{8n}$$

for $\varepsilon < 1$, in view of our choice (8.16) of P . The estimates (8.14) and (8.15) then follow from (8.17), for the case $b_3 \neq 0$.

We therefore assume that $b_3 = 0$, and hence that $\varphi_3 = P^{-3}\gamma_3$. To prove (8.14), we observe that (8.18) yields

$$T_3^8 \gg P^{-\varepsilon}s|\gamma_3| \gg P^{-\varepsilon}|\gamma_3|.$$

Inserting this into (8.17) leads to the bound

$$I(\boldsymbol{\gamma}; \mathbf{0}) \ll P^{h\varepsilon}|\gamma_3|^{-h/8} + |\boldsymbol{\gamma}|^{-n}.$$

The relation (8.16) allows us to replace $P^{h\varepsilon}$ by $|\boldsymbol{\gamma}|^\varepsilon$ on redefining ε , and (8.14) follows.

We turn now to the estimate (8.15), for which we use Lemma 4.2. This tells us that either

$$T_2^2 \gg \frac{P^{1-\varepsilon}}{s + P^3|\varphi_3|}$$

or that there is a positive integer $u \ll T_2^2$ for which

$$\|su\alpha_2\| \ll P^{-2+\varepsilon}s(1 + P^3|\varphi_3|)T_2^2. \tag{8.19}$$

In the first case, we have

$$P^{1-\varepsilon} \ll T_2^2(s + P^3|\varphi_3|) \ll T_2^2 \cdot P^\varepsilon T_3^8,$$

by (8.18). Thus, if $\varepsilon < 1/4$, we will have

$$P^{1/2} \ll T_2^2 T_3^8 = T_3^{2h/\rho+8} \leq T_3^{2n+8}.$$

Our choice (8.16) then shows that $T_3 \geq |\boldsymbol{\gamma}|^n$, so that (8.15) follows from (8.17).

If the second alternative (8.19) holds, we can write

$$\alpha_2 = \frac{b_2}{su} + \varphi_2$$

with

$$\varphi_2 \ll u^{-1} P^{-2+\varepsilon} (1 + P^3 |\varphi_3|) T_2^2. \tag{8.20}$$

If $b_2 \neq 0$, then

$$(su)^{-1} \leq |b_2|(su)^{-1} \leq P^{-2} |\gamma_2| + |\varphi_2|,$$

so that (8.18) yields

$$P^2 \ll su |\gamma_2| + su P^2 |\varphi_2| \ll P^\varepsilon T_3^8 \cdot T_2^2 |\boldsymbol{\gamma}| + P^{2\varepsilon} T_3^8 T_2^2.$$

This produces $P^2 \ll P T_3^8 T_2^2$ on taking $\varepsilon < 1/2$ and using the crude bound $|\boldsymbol{\gamma}| \leq P^{1/2}$ from (8.16). We can then deduce (8.15) just as in the previous paragraph.

We are left with the case in which $b_2 = 0$, so that $P^{-2} \gamma_2 = \alpha_2 = \varphi_2$. Since $\varphi_3 = P^{-3} \gamma_3$, it follows from (8.20) that

$$\gamma_2 \ll P^\varepsilon (1 + |\gamma_3|) T_2^2.$$

Thus (8.17) produces

$$I(\boldsymbol{\gamma}; \mathbf{0}) \ll P^{\varepsilon\rho/2} \left(\frac{|\gamma_2|}{1 + |\gamma_3|} \right)^{-\rho/2} + |\boldsymbol{\gamma}|^{-n}.$$

The first term on the right dominates the second, and we may replace $P^{\varepsilon\rho/2}$ by $|\boldsymbol{\gamma}|^\varepsilon$ after redefining ε , in view of our choice (8.16) of P . This establishes (8.15), thereby completing our treatment of Lemma 8.3. □

We are now ready to show that the singular integral converges. We have

$$\mathfrak{J} - \mathfrak{J}(R) = \iint_{|\boldsymbol{\gamma}| \geq R} I(\boldsymbol{\gamma}; \mathbf{0}) d\boldsymbol{\gamma}, \tag{8.21}$$

and we split the region of integration into two parts, to use the two estimates of Lemma 8.3. When $|\gamma_2| \leq |\gamma_3|^{1+h/(4\rho)}$ and $|\boldsymbol{\gamma}| \geq R$, we have

$$I(\boldsymbol{\gamma}; \mathbf{0}) \ll |\gamma_3|^{-h/8+\varepsilon}$$

and $|\gamma_3| \geq R^{4\rho/(h+4\rho)}$. The corresponding contribution to (8.21) is then

$$\ll \int_{R^{4\rho/(h+4\rho)}}^\infty x^{1+h/(4\rho)} x^{-h/8+\varepsilon} dx \ll R^{-\mu+\varepsilon},$$

with

$$\mu = \frac{h\rho - 16\rho - 2h}{2h + 8\rho}.$$

Similarly, when $|\gamma_2| \geq |\gamma_3|^{1+h/(4\rho)}$ and $|\boldsymbol{\gamma}| \geq R$, we have

$$I(\boldsymbol{\gamma}; \mathbf{0}) \ll (1 + |\gamma_3|)^{\rho/2} |\gamma_2|^{-\rho/2+\varepsilon}$$

and $|\gamma_2| \geq R$. In this case, the contribution to (8.21) is

$$\ll \int_R^\infty x^{-\rho/2+\varepsilon} \int_0^{x^{4\rho/(h+4\rho)}} (1 + y)^{\rho/2} dy dx$$

$$\begin{aligned} &\ll \int_R^\infty x^{-\rho/2+\varepsilon} x^{(1+\rho/2)4\rho/(h+4\rho)} dx \\ &\ll R^{-\mu+\varepsilon}, \end{aligned}$$

with the same μ as before. Thus we have absolute convergence when $\mu > 0$, or equivalently when $(h - 16)(\rho - 2) > 32$. This suffices for Lemma 8.1.

To complete the proof of the lemma, it remains to show that we can replace the truncated singular series and integral in (8.7) by their limits, with an acceptable error. This is clear, however, since we have shown that \mathfrak{S} and \mathfrak{J} are finite, and differ from $\mathfrak{S}(R)$ and $\mathfrak{J}(R)$ respectively by negative powers of R .

9. Proof of Theorem 1.4

In this section, we will establish Theorem 1.4, subject to various lemmas, all of which we will delay proving until the next section. These will involve the parameters $n, \rho = \text{rank}(Q)$, $\text{ord}_Q(C)$, and the h -invariants $h(C)$ and $h_Q(C)$. The latter, in particular, satisfy the inequalities $h_Q(C) \leq h(C) \leq h_Q(C) + 1$, as recorded in (3.1). The reader should note that in Theorem 1.2 one can replace C by $C + LQ$ for a generic L , and hence, somewhat surprisingly, it is the *maximal* value of $h(C + LQ)$ which is of relevance there.

We begin by recording some basic deductions about the above parameters. We may assume that

$$\rho \geq n - 13 \geq 36, \tag{9.1}$$

because otherwise Q vanishes on a \mathbb{Q} -rational 13-plane, and we can conclude as in the proof of Theorem 1.1. We will always have $h_Q(C) \leq \text{ord}_Q(C)$, and indeed

$$h_Q(C) \leq \text{ord}_Q(C) - 1, \quad \text{if } \text{ord}_Q(C) \geq 14. \tag{9.2}$$

To see this, suppose that $C = C(x_1, \dots, x_m)$ with $m = \text{ord}_Q(C) \geq 14$, after a suitable change of variable. Then, by the result of Heath-Brown [17], the form C has a non-trivial rational zero, which we may take to be $(0, \dots, 0, 1)$. We can then write

$$C = x_1 Q_1(x_1, \dots, x_m) + \dots + x_{m-1} Q_{m-1}(x_1, \dots, x_m),$$

which shows that $h_Q(C) \leq m - 1$.

We may also eliminate the case in which $h_Q(C) = 1$, which would mean that one could take C to factor as LQ' , say, over \mathbb{Q} . If this were to happen, then a smooth real point on $C = Q = 0$ would lie either on $Q = L = 0$ or $Q = Q' = 0$. In the first case, the Hasse–Minkowski theorem suffices to complete the proof, since $n \geq 49 \geq 6$. In the second case, we apply Lemma 9.2 below, using the fact that $n \geq 49 \geq 9$. If some combination $aQ + bQ'$ were to have rank at most 4, then $b \neq 0$ by (9.1). However, $bC + aLQ = L(aQ + bQ')$ would have order at most 5, giving $\text{ord}_Q(C) \leq 5$, in contradiction to our hypotheses. Thus the conditions needed for Lemma 9.2 do indeed hold. In what follows, we will therefore be able to assume that $h_Q(C) \geq 2$, and hence, via Lemma 3.3, that X is absolutely irreducible.

Our strategy for the proof of Theorem 1.4 is now to combine two basic arguments, one of which covers the case in which $h_Q(C) \leq n - 13$ whereas the other deals with larger values

of $h_Q(C)$. We begin by discussing the second of these, which is more straightforward. The idea is to apply Theorem 1.2, which will require us to have smooth solutions for every completion of \mathbb{Q} . A smooth real solution is provided by our hypothesis, and we will then require the following lemma to give us suitable p -adic solutions.

Lemma 9.1. *If $\text{ord}_Q(C) \geq 4$, $h_Q(C) \geq 2$, and $\rho \geq 23$, then we have $X_{\text{sm}}(\mathbb{Q}_p) \neq \emptyset$ for every prime p .*

We will prove this in the next section. The conditions given are sufficient for our purposes but are probably not optimal.

The conditions of the lemma are amply met, in view of (9.1), and Theorem 1.2 completes the argument if $h_Q(C) \geq n - 12$, since then

$$(h_Q(C) - 32)(\rho - 4) \geq (n - 44)(n - 17) \geq 5 \times 32 > 128,$$

via a further application of (9.1).

We will henceforth assume that $h_Q(C) \leq n - 13$, and we will replace C by $C + LQ$ so that $h_Q(C) = h(C) = h$, say. Then, after a suitable non-singular linear change of variables, we can write

$$\mathbf{x} = (x_1, \dots, x_n) = (u_1, \dots, u_h; v_1, \dots, v_s)$$

where $s = n - h$, so that C and Q take the shapes

$$C(\mathbf{x}) = A(\mathbf{u}) + \sum_{j=1}^s v_j D_j(\mathbf{u}) + \sum_{i=1}^h u_i B_i(\mathbf{v}) \tag{9.3}$$

and

$$Q(\mathbf{x}) = R(\mathbf{u}; \mathbf{v}) + S(\mathbf{v}).$$

Here, $A(\mathbf{u})$ is a cubic form, while $D_j(\mathbf{u})$, $B_i(\mathbf{v})$, $R(\mathbf{u}; \mathbf{v})$, and $S(\mathbf{v})$ are quadratic forms, such that $R(\mathbf{u}; \mathbf{v})$ contains no quadratic terms in \mathbf{v} . We remark at once that if $\text{rank}(S) < n - h$ then there is a vector $\mathbf{v}_0 \in \mathbb{Q}^s - \{\mathbf{0}\}$ such that $S(\mathbf{v}_0) = 0$. Thus $C(\mathbf{0}, \mathbf{v}_0) = Q(\mathbf{0}, \mathbf{v}_0) = 0$, so that our system has a non-trivial rational zero. We may therefore assume that $\text{rank}(S) = n - h$ from now on. We can then apply a suitable linear transformation so as to reduce $Q(\mathbf{x})$ to the form

$$Q(\mathbf{x}) = R(\mathbf{u}) + S(\mathbf{v}),$$

while leaving $C(\mathbf{x})$ in the shape (9.3), but with new forms A , D_j , and B_i .

In what follows, it will also be useful to adopt the notation

$$C_{\mathbf{a}}(t, v_1, \dots, v_s) := A(\mathbf{a})t^2 + \left\{ \sum_{j=1}^s D_j(\mathbf{a})v_j \right\} t + \sum_{i=1}^h a_i B_i(\mathbf{v}), \tag{9.4}$$

$$Q_{\mathbf{a}}(t, v_1, \dots, v_s) := Q(t\mathbf{a}, \mathbf{v}) = R(\mathbf{a})t^2 + S(\mathbf{v}).$$

Note that both $Q_{\mathbf{a}}$ and $C_{\mathbf{a}}$ are quadratic forms in t, v_1, \dots, v_s . If we can show that there is a non-zero vector $\mathbf{a} \in \mathbb{Q}^h$ such that the forms $Q_{\mathbf{a}}$ and $C_{\mathbf{a}}$ have a common rational zero (t_0, \mathbf{v}_0) , then C and Q will have the common zero $(t_0 \mathbf{a}, \mathbf{v}_0)$, which will complete the the proof.

Here we will employ the following result, which will be an easy corollary of Theorem A of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [7].

Lemma 9.2. *Let f, g be quadratic forms over the rationals in $m \geq 9$ variables, and suppose that the equations $f = g = 0$ have a smooth solution over \mathbb{R} , and that every form in the rational pencil has rank at least 5. Then the forms have a common rational zero.*

Note that in applying Lemma 9.2 we will have forms in $s + 1$ variables, where

$$s + 1 = n - h + 1 \geq 14.$$

We call a non-zero real vector $\mathbf{a} \in \mathbb{R}^h$ *good* if the system of equations

$$Q_{\mathbf{a}}(t, v_1, \dots, v_s) = C_{\mathbf{a}}(t, v_1, \dots, v_s) = 0$$

has a non-singular real zero. We shall then prove the following result.

Lemma 9.3. *If $n - h \geq 5$ and $\text{ord}_{\mathcal{O}}(C) \geq \max(h + 1, 4)$, then the set of $[\mathbf{a}] \in \mathbb{P}^{h-1}(\mathbb{Q})$ such that \mathbf{a} is good is Zariski dense.*

In our case, we have $n - h \geq 13$ and

$$\text{ord}_{\mathcal{O}}(C) \geq \max(h + 1, 17),$$

by (9.2) and the hypotheses of Theorem 1.4.

If there is any good rational \mathbf{a} for which every form in the rational pencil generated by $Q_{\mathbf{a}}$ and $C_{\mathbf{a}}$ has rank at least 5, then Theorem 1.4 will follow from Lemma 9.2. We therefore proceed on the alternative assumption that for every good $\mathbf{a} \in \mathbb{Q}^r$ there is a form $\alpha C_{\mathbf{a}} + \beta Q_{\mathbf{a}}$ with $(\alpha, \beta) \in \mathbb{Q}^2 - \{(0, 0)\}$, having rank at most 4. We will prove the following lemma.

Lemma 9.4. *Suppose that $n - h \geq 13$ and that there is a Zariski-dense set of $[\mathbf{a}] \in \mathbb{P}^{h-1}(\mathbb{Q})$ for each of which there is a form $\alpha C_{\mathbf{a}} + \beta Q_{\mathbf{a}}$ with $(\alpha, \beta) \in \mathbb{Q}^2 - \{(0, 0)\}$, having rank at most 4. Then, after replacing C by $C + LQ$ for a suitable linear form L defined over \mathbb{Q} , and after making a suitable linear change of variables, we may write $C(\mathbf{x})$ in the shape*

$$C(\mathbf{x}) = C(\mathbf{u}, \mathbf{v}) = \sum_{1 \leq i \leq j \leq H} u_i u_j L_{ij}(\mathbf{u}, \mathbf{v}),$$

with linear forms L_{ij} defined over \mathbb{Q} , and with $H = h + 4$.

The reader should notice that our vectors \mathbf{u} and \mathbf{v} now have different lengths from before.

We now define $Q_{\mathbf{a}}(t, \mathbf{v})$ as previously, and set

$$L_{\mathbf{a}}(t, \mathbf{v}) = \sum_{1 \leq i \leq j \leq H} a_i a_j L_{ij}(t\mathbf{a}, \mathbf{v}).$$

Thus a rational solution (t, \mathbf{v}) of $Q_{\mathbf{a}}(t, \mathbf{v}) = L_{\mathbf{a}}(t, \mathbf{v}) = 0$ produces a corresponding point $[t\mathbf{a}, \mathbf{v}]$ on X . We now have the following result, which plays a similar role to Lemma 9.3, but is much easier to prove.

Lemma 9.5. *In the situation of Lemma 9.4, assume that at least one linear form L_{ij} depends explicitly on \mathbf{v} . Then there is a non-empty Zariski-open set of $[\mathbf{a}] \in \mathbb{P}^{H-1}(\mathbb{Q})$ such that the equations $Q_{\mathbf{a}}(t, \mathbf{v}) = L_{\mathbf{a}}(t, \mathbf{v}) = 0$ have a real solution.*

Let us now show how to proceed under the assumption of Lemma 9.5. The equations

$$Q_{\mathbf{a}}(t, \mathbf{v}) = L_{\mathbf{a}}(t, \mathbf{v}) = 0$$

describe the intersection of a quadric hypersurface with a hyperplane. In general, such an intersection will have a rational point whenever there is a real point, as long as $\text{rank}(Q_{\mathbf{a}}) \geq 6$. However,

$$Q_{\mathbf{a}}(t, \mathbf{v}) = R(\mathbf{a})t^2 + S(\mathbf{v})$$

with new quadratic forms R and S , where as before we may assume that $\text{rank}(S) = n - H$. Thus

$$\text{rank}(Q_{\mathbf{a}}) \geq \text{rank}(S) = n - H = n - (h + 4) \geq 9, \tag{9.5}$$

which suffices to complete the proof of Theorem 1.4.

It remains to consider the possibility that the assumption is not met in Lemma 9.5. Thus all of the linear forms L_{ij} are independent of \mathbf{v} , and so $C(\mathbf{x}) = C(\mathbf{u})$. It is enough to find a non-trivial rational zero of the system $C(\mathbf{u}) = 0$ and

$$Q_{\mathbf{u}}(1, \mathbf{v}) = R(\mathbf{u}) + S(\mathbf{v}) = 0,$$

where, as above, $\text{rank}(S) \geq 9$. Since X is absolutely irreducible, the same is true for C . Likewise, on making a suitable linear change of variables, we may assume that C is a non-degenerate cubic form in $H' \leq H$ variables. We will also assume, temporarily, that the locus of rational solutions to $C = 0$ is dense in the locus of real solutions. We call this the “real density hypothesis”, for convenience.

If S is indefinite or is singular, then it suffices to take $\mathbf{u} = \mathbf{0}$, and to solve $S(\mathbf{v}) = 0$ non-trivially over the rationals. This will certainly be possible, since S has rank at least 9. We therefore suppose that S is definite, and without loss of generality we take S to be positive definite.

We now make use of our assumption that there is a non-singular real zero of the system $C = Q = 0$ under consideration. The form R cannot vanish since Q has a non-trivial real zero and S is positive definite. Thus X cannot be contained in $R = 0$ and it follows from Lemma 3.4 that X has a real point $(\mathbf{u}_0, \mathbf{v}_0)$ with $R(\mathbf{u}_0) \neq 0$. Our assumption that S is positive definite then shows that we must have $R(\mathbf{u}_0) < 0$.

In view of the real density hypothesis, we can now find a rational zero \mathbf{u} of C sufficiently close to \mathbf{u}_0 that $R(\mathbf{u}) < 0$. Then, since $\text{rank}(S) \geq 9$, there will be a rational vector \mathbf{v} such that $S(\mathbf{v}) = -R(\mathbf{u})$. This produces a non-trivial rational point $[\mathbf{x}] = [(\mathbf{u}, \mathbf{v})]$ on X , thereby completing the proof in the second case, subject to the real density hypothesis.

Finally, we claim that the real density hypothesis holds if $\text{ord}_Q(C) \geq 17$. If $h \geq 14$, a straightforward modification of the main result in [17] establishes the desired conclusion (see [23, Lemma 1]). Alternatively, if $h \leq 13$, then it follows from our lower bound for $\text{ord}_Q(C)$ that

$$H' \geq \text{ord}_Q(C) \geq 17 \geq h + 4.$$

But then the claim follows from work of Swarbrick Jones [23, Lemma 2].

10. Proof of Lemmas 9.1–9.5

It remains to establish Lemmas 9.1–9.5, and we begin with the first of these. For the proof we work over \mathbb{Q}_p . The quadratic form Q may be written as a non-singular form in variables x_1, \dots, x_ρ , and it vanishes on a linear space of projective dimension at least $\lceil(\rho - 6)/2\rceil \geq 9$, in terms of these variables. Hence, as remarked upon in the introduction in connection with Theorem 1.3, the form C will vanish at a p -adic point P , which we see may be taken to be a non-singular point on $Q = 0$. If we choose coordinates so that $P = [1, 0, \dots, 0]$, our forms take the shape

$$C(\mathbf{x}) = x_1^2 L_1(x_2, \dots, x_n) + x_1 Q_1(x_2, \dots, x_n) + C_1(x_2, \dots, x_n)$$

and

$$Q(\mathbf{x}) = x_1 L_2(x_2, \dots, x_n) + Q_2(x_2, \dots, x_n).$$

Then L_2 cannot vanish identically, since P is a non-singular point on $Q = 0$. Moreover, if L_1 and L_2 are not proportional, then P is a smooth point on X . We may therefore assume that $L_1 = cL_2$. Thus, if $C' = C + LQ = C - cx_1Q$, we can write $C'(\mathbf{x})$ in the simpler shape

$$C'(\mathbf{x}) = x_1 Q_1(x_2, \dots, x_n) + C_1(x_2, \dots, x_n).$$

Since L_2 does not vanish identically, we can make a change of variables to replace L_2 by x_2 , say, so that $Q(\mathbf{x})$ becomes

$$\begin{aligned} Q(\mathbf{x}) &= x_1 x_2 + Q_2(x_2, \dots, x_n) \\ &= x_1 x_2 + x_2 L_3(x_2, \dots, x_n) + Q_3(x_3, \dots, x_n), \end{aligned}$$

say. Now replacing x_1 by $x_1 + L_3$ we further simplify Q to the shape $x_1 x_2 + Q_3(x_3, \dots, x_n)$. We then write

$$Q_1(x_2, \dots, x_n) = x_2 L_4(x_2, \dots, x_n) + Q_4(x_3, \dots, x_n)$$

and replace C' by $C' - L_4Q$ so that (renaming our forms)

$$\begin{aligned} C'(\mathbf{x}) &= x_1 Q_1(x_3, \dots, x_n) + C_1(x_2, \dots, x_n) \\ Q(\mathbf{x}) &= x_1 x_2 + Q_2(x_3, \dots, x_n). \end{aligned}$$

Consider the projection $X \rightarrow \mathbb{P}^{n-2}$ from the point $[1, 0, \dots, 0]$. The Zariski closure of the image of this rational map is the hypersurface

$$Y : \quad x_2 C_1(x_2, \dots, x_n) - Q_1(x_3, \dots, x_n) Q_2(x_3, \dots, x_n) = 0$$

in \mathbb{P}^{n-2} . In fact X and Y are birational to each other over \mathbb{Q} , the reverse map being given by

$$[x_2, \dots, x_n] \mapsto \begin{cases} [-Q_2/x_2, x_2, \dots, x_n], & \text{if } x_2 \neq 0, \\ [-C_1/Q_1, x_2, \dots, x_n], & \text{if } Q_1 \neq 0, \end{cases}$$

on the Zariski-open subset where $(x_2, Q_1) \neq (0, 0)$. Lemma 3.3 ensures that X is absolutely irreducible, and we therefore deduce that Y is also absolutely irreducible. Lemma 9.1 will follow if we are able to show that the p -adic points on X are Zariski

dense. For this it will suffice to show that the p -adic points on Y are Zariski dense. This will follow from Lemma 3.4 if we can show that Y has a non-singular p -adic point.

To verify the existence of a non-singular p -adic point on Y , we consider points with $x_2 = Q_2 = 0$. Such a point will be non-singular on Y provided that $\nabla Q_2 \neq \mathbf{0}$ and that Q_1 and C_1 are not both zero. However,

$$\text{rank}(Q_2) \geq \text{rank}(Q) - 2 = \rho - 2 \geq 21 \geq 5,$$

so that the p -adic zeros $[x_3, \dots, x_n]$ of Q_2 are Zariski dense on $Q_2 = 0$. In particular, we can choose a point where $\nabla Q_2 \neq \mathbf{0}$, and where Q_1 and C_1 are not both zero, unless both Q_1 and C_1 are multiples of Q_2 . However, if Q_2 divides C' we have $C' = L'Q_2$ for some linear form L' , and hence

$$C = L''Q + C' = L''Q + L'Q_2 = (L'' + L')Q - L'x_1x_2 = \bar{L}Q + L_1L_2L_3,$$

say. Here, L_1, L_2, L_3 and \bar{L} are linear forms defined over \mathbb{Q}_p . If \bar{L} were defined over \mathbb{Q} , then we would have $\text{ord}_Q(C) = \text{ord}_Q(L_1L_2L_3) \leq 3$, contrary to our hypotheses. Thus there is a field automorphism σ say, such that $\bar{L}^\sigma \neq \bar{L}$. Since $C^\sigma = C$, this yields

$$(\bar{L}^\sigma - \bar{L})Q = L_1^\sigma L_2^\sigma L_3^\sigma - L_1L_2L_3.$$

Changing variables, we may write $\bar{L}^\sigma - \bar{L} = x_1$, whence x_1Q has order at most 6. We claim in general that, for any form $F(x_1, \dots, x_n)$, the order of F is at most one more than the order of $x_1F(x_1, \dots, x_n)$. Given this claim, we would deduce that $\text{rank}(Q) \leq 7$, contrary to the hypothesis. Thus to complete the proof of Lemma 9.1 it is enough to establish the claim. However, this is easy, since if we can write

$$x_1F(x_1, \dots, x_n) = G(L_1, \dots, L_m)$$

with forms

$$L_i(x_1, \dots, x_n) = a_i x_1 + \bar{L}_i(x_2, \dots, x_n)$$

then $G(\bar{L}_1, \dots, \bar{L}_m)$ must vanish identically, and F will be a function of x_1 and $\bar{L}_1, \dots, \bar{L}_m$. This suffices for the claim.

The next result to prove is Lemma 9.2. Theorem A of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [7] tells us that an absolutely irreducible non-degenerate intersection of quadrics in $m \geq 9$ variables satisfies the smooth Hasse principle. Of course, if the intersection is degenerate there will trivially be a rational point (though not necessarily a smooth rational point). Thus we may assume that our intersection is non-degenerate. We claim that $\text{rank}(h) \geq 5$ for every form h in the pencil generated by f and g , either over \mathbb{Q} , or over some \mathbb{Q}_p . This follows from our hypotheses if h is proportional to a rational form. Otherwise, there is some field automorphism σ such that h^σ and h are not proportional. However, h^σ is also in the pencil generated by f and g . Now if $\text{rank}(h) \leq 4$ then $\text{rank}(h^\sigma) \leq 4$, so that the variety $h^\sigma = h = 0$ would be degenerate. This however is impossible given our previous assumption, since h^σ and h generate the same pencil as f and g . Our claim is therefore established. In particular, we now see that the intersection $f = g = 0$ will be absolutely irreducible, by [7, Lemma 1.11], so that the Hasse principle applies.

The variety $f = g = 0$ has a smooth real point by hypothesis, and we claim that there are smooth p -adic points for every prime p . This will suffice for the proof of the lemma.

To prove this, we note that for any prime p there is a p -adic point by the result of Demyanov [10], since $m \geq 9$. Clearly, we may assume that this point is a singular point, since otherwise the claim is immediate. Then, choosing coordinates so that the point in question is at $[1, 0, \dots, 0]$, the forms become $x_1 L_1(x_2, \dots, x_m) + f_1(x_2, \dots, x_m)$ and $x_1 L_2(x_2, \dots, x_m) + g_1(x_2, \dots, x_m)$. Here, the forms L_1 and L_2 cannot both vanish, since we are assuming that $f = g = 0$ is non-degenerate. Moreover, they must be proportional, since $[1, 0, \dots, 0]$ was assumed to be singular. Thus, after replacing the forms f and g by a suitable linear combination, and after making a further change of variables, we may take L_2 , say, to vanish, and take $L_1 = x_2$. Now, since $\text{rank}(g_1) \geq 5$ by what we proved above, we see that $g_1 = 0$ has a smooth p -adic zero. Its smooth p -adic zeros are therefore Zariski dense. Choosing such a zero with $x_2 \neq 0$, we may then set $x_1 = -x_2^{-1} f_1(x_2, \dots, x_m)$, obtaining a smooth point on $f = g = 0$. This establishes Lemma 9.2.

We turn now to the proof of Lemma 9.3. If it were the case that for every $\mathbf{a} \in \mathbb{R}^h$ there is a linear combination $C_{\mathbf{a}}(t, \mathbf{v}) + \lambda Q_{\mathbf{a}}(t, \mathbf{v})$ with rank at most 1, then it would be impossible for the variety $C_{\mathbf{a}}(t, \mathbf{v}) = Q_{\mathbf{a}}(t, \mathbf{v}) = 0$ to have a non-singular zero. We therefore begin by showing that this case cannot arise.

Lemma 10.1. *Suppose that $n - h \geq 5$, and that*

$$\text{ord}_Q(C) \geq \max(h + 1, 3).$$

Then either $X(\mathbb{Q}) \neq \emptyset$, or there is at least one non-zero $\mathbf{a} \in \mathbb{Q}^h$ such that every linear combination $C_{\mathbf{a}}(t, \mathbf{v}) + \lambda Q_{\mathbf{a}}(t, \mathbf{v})$ with $\lambda \in \overline{\mathbb{Q}}$ has rank 2 or more.

Proof. For the proof, we write $Q(\mathbf{u}, \mathbf{v}) = R(\mathbf{u}) + S(\mathbf{v})$ as before, with $\text{rank}(S) = n - h$. We will assume for a contradiction that for every rational \mathbf{a} there is some λ for which $C_{\mathbf{a}}(t, \mathbf{v}) + \lambda Q_{\mathbf{a}}(t, \mathbf{v})$ has rank at most 1. In particular, for any j between 1 and h , we may define \mathbf{a} by taking $a_i = 0$ for $i \neq j$ and $a_j = 1$. Then, setting $t = 0$, we see that $B_j(\mathbf{v}) + \lambda_j S(\mathbf{v})$ has rank at most 1, in the notation (9.4). In the same way, for distinct positive integers $j, k \leq h$, we may take $a_i = 0$ for $i \neq j, k$ and $a_j = a_k = 1$, finding that $B_j(\mathbf{v}) + B_k(\mathbf{v}) + \lambda_{j,k} S(\mathbf{v})$ has rank at most 1. This produces equations

$$B_j(\mathbf{v}) + \lambda_j S(\mathbf{v}) = L_j(\mathbf{v})^2, \quad B_k(\mathbf{v}) + \lambda_k S(\mathbf{v}) = L_k(\mathbf{v})^2$$

and

$$B_j(\mathbf{v}) + B_k(\mathbf{v}) + \lambda_{j,k} S(\mathbf{v}) = L_{j,k}(\mathbf{v})^2.$$

Here, the coefficients λ and the linear forms L are defined over $\overline{\mathbb{Q}}$. By subtraction, we find that either $\lambda_j + \lambda_k = \lambda_{j,k}$, or that $\text{rank}(S) \leq 3$. Since we have assumed that $\text{rank}(S) = n - h \geq 5$, we deduce that $\lambda_j + \lambda_k = \lambda_{j,k}$, and then that $L_j^2 + L_k^2 = L_{j,k}^2$. This can happen only when L_j, L_k , and $L_{j,k}$ are proportional, allowing us to conclude that there is a non-zero linear form L_0 defined over $\overline{\mathbb{Q}}$, and constants $\mu_j \in \overline{\mathbb{Q}}$, such that

$$B_j(\mathbf{v}) + \lambda_j S(\mathbf{v}) = \mu_j L_0(\mathbf{v})^2$$

for every j . In fact, if $\lambda_j \notin \mathbb{Q}$, we can apply some non-trivial Galois automorphism σ to show that $B_j(\mathbf{v}) + \lambda_j^\sigma S(\mathbf{v}) = \mu_j^\sigma (L_0(\mathbf{v})^\sigma)^2$. Then, by subtraction, we see that $(\lambda_j - \lambda_j^\sigma) S(\mathbf{v})$

has rank at most 2, again contradicting our assumptions. Thus all the λ_j are in \mathbb{Q} , so that we may suppose L_0 and the μ_j to be defined over \mathbb{Q} .

Taking

$$L(\mathbf{x}) = \sum_{i=1}^h \lambda_i u_i,$$

we now replace $C(\mathbf{x})$ by $C' = C(\mathbf{x}) + L(\mathbf{x})Q(\mathbf{x})$. This new cubic may be written in the shape given by (9.3), with a different function $A(\mathbf{u})$, and with $B_i(\mathbf{v})$ replaced by $B'_i(\mathbf{v}) = B_i(\mathbf{v}) + \lambda_i S(\mathbf{v}) = \mu_i L_0(\mathbf{v})^2$. In particular, we will have $h(C') \leq h$, and since we chose our original cubic C to have $h(C) = h_Q(C)$ we see in fact that $h(C') = h_Q(C) = h$. For ease of notation we will just write C in place of C' henceforth, and assume that

$$B_i(\mathbf{v}) = \mu_i L_0(\mathbf{v})^2. \tag{10.1}$$

Now suppose that

$$C_{\mathbf{a}}(t, \mathbf{v}) + \lambda Q_{\mathbf{a}}(t, \mathbf{v}) = (\alpha t + J(\mathbf{v}))^2 \tag{10.2}$$

for some α and $J(\mathbf{v})$ defined over $\overline{\mathbb{Q}}$. Then, on comparing the terms not involving t , and using (10.1), we see that

$$\begin{aligned} J(\mathbf{v})^2 &= \sum_{j=1}^h a_j B_j(\mathbf{v}) + \lambda S(\mathbf{v}) \\ &= \left(\sum_{j=1}^h \mu_j a_j \right) L_0(\mathbf{v})^2 + \lambda S(\mathbf{v}). \end{aligned}$$

Using the fact that $\text{rank}(S) \geq 5$ once again, we conclude that $\lambda = 0$ and that $J(\mathbf{v})$ is proportional to $L_0(\mathbf{v})$, and hence equal to $\beta L_0(\mathbf{x})$ say.

We now expand (10.2) further, using (9.4). We then see from the linear term in t that

$$\sum_{j=1}^s D_j(\mathbf{a}) v_j = 2\alpha\beta L_0(\mathbf{v}). \tag{10.3}$$

Thus for every rational vector \mathbf{a} the linear form $\sum_j D_j(\mathbf{a}) v_j$ is proportional to $L_0(\mathbf{v})$. This can happen only when the quadratic forms D_j are all proportional to each other, of the shape $v_j D(\mathbf{a})$ say, with constants $v_j \in \mathbb{Q}$. This allows us to write

$$C(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}) + D(\mathbf{u})L'(\mathbf{v}) + \ell(\mathbf{u})L_0(\mathbf{v})^2$$

for suitable linear forms L' and ℓ defined over \mathbb{Q} , and indeed (10.3) shows that we may take $L'(\mathbf{v}) = L_0(\mathbf{v})$.

It follows that $C_{\mathbf{a}}(t, \mathbf{v}) = A(\mathbf{a})t^2 + D(\mathbf{a})tL_0(\mathbf{v}) + \ell(\mathbf{a})L_0(\mathbf{v})^2$, which must have rank at most 1 for every choice of $\mathbf{a} \in \mathbb{Q}^h$. If L_0 vanishes identically, or if $\ell(\mathbf{u})$ and $D(\mathbf{u})$ both vanish identically, then $C(\mathbf{x}) = A(\mathbf{u})$, which has order at most h , contrary to the hypothesis of Lemma 10.1. Thus $D(\mathbf{a})^2 = 4A(\mathbf{a})\ell(\mathbf{a})$ for any $\mathbf{a} \in \mathbb{Q}^h$, and then $D(\mathbf{u}) = 2\ell(\mathbf{u})\ell'(\mathbf{u})$ and $A(\mathbf{u}) = \ell(\mathbf{u})\ell'(\mathbf{u})^2$ for some linear form $\ell'(\mathbf{u})$ defined over \mathbb{Q} . However, in this case,

$$C(\mathbf{x}) = A(\mathbf{u}) + D(\mathbf{u})L_0(\mathbf{v}) + \ell(\mathbf{u})L_0(\mathbf{v})^2 = \ell(\mathbf{u})\{\ell'(\mathbf{u}) + L_0(\mathbf{v})\}^2,$$

which has order at most 2, again contradicting our hypotheses. This therefore establishes the lemma. □

The next stage in the proof of Lemma 9.3 is the following result.

Lemma 10.2. *Under the hypotheses of Lemma 10.1, either $X(\mathbb{Q}) \neq \emptyset$, or there is at least one non-zero $\mathbf{a} \in \mathbb{Q}^h$ such that the variety*

$$C_{\mathbf{a}} = Q_{\mathbf{a}} = 0$$

has a point $(t, \mathbf{v}) \in \overline{\mathbb{Q}}^{1+s}$ with $t \neq 0$, at which $\nabla C_{\mathbf{a}}$ and $\nabla Q_{\mathbf{a}}$ are not proportional.

Proof. By Lemma 10.1, we may choose \mathbf{a} so that every form in the pencil generated by $C_{\mathbf{a}}$ and $Q_{\mathbf{a}}$ has rank at least 2. As before, we may assume that $\text{rank}(S) = n - h \geq 5$, whence $\text{rank}(Q_{\mathbf{a}}) \geq 5$. We will show in general that, if $A(\mathbf{y})$ and $B(\mathbf{y})$ are quadratic forms such that $\text{rank}(A) \geq 5$, and such that every form in the pencil generated by A and B over $\overline{\mathbb{Q}}$ has rank at least 2, then $A = B = 0$ has a point with ∇A not proportional to ∇B , and lying off any given hyperplane $L(\mathbf{y}) = 0$. (In this general formulation the condition $t \neq 0$ corresponds to a requirement of the type $L(y_1, \dots, y_n) \neq 0$.) Without loss of generality we can take B with as small rank, r say, as possible. If $r \geq 3$, then the variety $A = B = 0$ is irreducible of degree 4 and codimension 2, and is not contained in the hyperplane $L = 0$. Since the variety $A = B = 0$ has projective dimension $n - 3 \geq n - h - 3 \geq 2$, there will be a non-empty Zariski-open set of points satisfying the conditions of the lemma.

We therefore assume that B has rank exactly 2, and write $B = x_1x_2$. Since L cannot be proportional to both x_1 and x_2 , we may assume that x_1 , say, is not proportional to L . We set $x_1 = 0$ and $L'(x_2, \dots, x_n) = L(0, x_2, \dots, x_n)$, and look for points on $A = x_1 = 0$ with $x_2L' \neq 0$ and such that ∇A is not proportional to $(1, 0, \dots, 0)$. However, $A' = A(0, x_2, x_3, \dots, x_n)$ has rank at least $\text{rank}(A) - 2 \geq 3$, and hence is an absolutely irreducible quadratic form. Moreover, at least one partial derivative $P_i = \partial A' / \partial x_i$ for $i = 2, \dots, n$ is not identically zero. Thus A' cannot divide $x_2L'P_i$, whence $A' = 0$ has a point at which $x_2L'P_i \neq 0$. This produces a point $(0, x_2, \dots, x_n)$ on $A = B = 0$ for which $L \neq 0$ and such that ∇A is not proportional to ∇B . This completes the proof of the lemma. □

We are now ready to complete the proof of Lemma 9.3. The variety $X \subset \mathbb{P}^{n-1}$ is defined by $C(\mathbf{u}, \mathbf{v}) = Q(\mathbf{u}, \mathbf{v}) = 0$, and it is absolutely irreducible, by Lemma 3.3. The points $[\mathbf{u}, \mathbf{v}]$ on X for which $[t, \mathbf{v}] = [1, \mathbf{v}]$ is a singular point of $C_{\mathbf{u}}(t, \mathbf{v}) = Q_{\mathbf{u}}(t, \mathbf{v}) = 0$ form a Zariski-closed subset of X , and by Lemma 10.2 it is a proper subset of X . We have assumed that X has a smooth real point, and by Lemma 3.4 the real points must be Zariski dense on X . Hence there is a Zariski-dense set of smooth real points $[\mathbf{u}, \mathbf{v}]$ of X , with $\mathbf{u} \neq \mathbf{0}$ and such that $[1, \mathbf{v}]$ is a smooth point of $C_{\mathbf{u}}(t, \mathbf{v}) = Q_{\mathbf{u}}(t, \mathbf{v}) = 0$. It follows in particular that there is a non-zero real \mathbf{u} such that $C_{\mathbf{u}}(t, \mathbf{v}) = Q_{\mathbf{u}}(t, \mathbf{v}) = 0$ has a smooth real point $[1, \mathbf{v}]$. Suppose now that \mathbf{a}_m is a sequence of rational points tending to \mathbf{u} in the real metric. Write $A(t, \mathbf{v})$ and $B(t, \mathbf{v})$ for the quadratic forms $C_{\mathbf{u}}(t, \mathbf{v})$ and $Q_{\mathbf{u}}(t, \mathbf{v})$, and write A_m, B_m for the corresponding forms when \mathbf{u} is replaced by \mathbf{a}_m . Then A_m and B_m tend to A and B , respectively. However, A and B have a smooth real zero at $[1, \mathbf{v}]$, whence it follows that A_m and B_m will also have a smooth real zero $[1, \mathbf{v}_m]$, say, if m is large enough. This suffices for the proof of Lemma 9.3. In particular, the rational points $[\mathbf{a}] \in \mathbb{P}^{h-1}$ obtained in this way cannot be restricted to a proper subvariety of \mathbb{P}^{h-1} , since the points $[\mathbf{u}]$ were Zariski dense.

Moving on to Lemma 9.4, we begin by observing that, if $\alpha C_{\mathbf{a}} + \beta Q_{\mathbf{a}}$ has rank at most 4, then, on setting $t = 0$, we must have

$$\text{rank} \left(\alpha \sum_{i=1}^h a_i B_i(\mathbf{v}) + \beta S(\mathbf{v}) \right) \leq 4.$$

Since $\text{rank}(S) = n - h \geq 13$, we will have $\alpha \neq 0$, and we may therefore assume that $\alpha = 1$. We now consider the variety

$$\mathcal{I} = \left\{ [u_1, \dots, u_h, \beta] \in \mathbb{P}^h : \text{rank} \left(\sum_{i=1}^h u_i B_i(\mathbf{v}) + \beta S(\mathbf{v}) \right) \leq 4 \right\}.$$

The projection $[u_1, \dots, u_h, \beta] \mapsto [u_1, \dots, u_h]$ is well defined on \mathcal{I} , since $[0, \dots, 0, 1] \notin \mathcal{I}$. Its image is Zariski dense in \mathbb{P}^{h-1} , and must therefore be the whole of \mathbb{P}^{h-1} , so that for every $[\mathbf{a}] \in \mathbb{P}^{h-1}$ there is a corresponding β such that

$$\text{rank} \left(\sum_{i=1}^h a_i B_i(\mathbf{v}) + \beta S(\mathbf{v}) \right) \leq 4. \tag{10.4}$$

It is possible indeed that this might still be true with the bound 4 replaced by some smaller number. We therefore define $\tau \leq 4$ as the smallest integer such that (10.4) is solvable for β , for all \mathbf{a} .

We now claim that, after replacing C by $C + LQ$ for a suitable linear form $L = L(\mathbf{u})$ defined over \mathbb{Q} , and after making a suitable linear change of variables among the u_i , we will have $\text{rank}(B_i) = \tau$ for $1 \leq i \leq h$. Moreover, it will remain true that for every \mathbf{a} there is a corresponding $\beta = \beta(\mathbf{a})$ with

$$\text{rank} \left(\sum_{i=1}^h a_i B_i(\mathbf{v}) + \beta S(\mathbf{v}) \right) \leq \tau.$$

To establish the claim, we first note that there is a Zariski-dense set of values of $[\mathbf{a}]$ such that the rank given above is actually equal to τ . Thus we may choose a linearly independent set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_h \in \mathbb{Q}^h$ with this property. Then, after a suitable change of variable among the u_i , we can suppose that

$$\text{rank}(B_i(\mathbf{v}) + \beta_i S(\mathbf{v})) = \tau, \quad (1 \leq i \leq h).$$

If β_i were irrational for some i , there would be a Galois automorphism σ such that $\beta_i^\sigma \neq \beta_i$. We would then have

$$\text{rank}(B_i(\mathbf{v}) + \beta_i^\sigma S(\mathbf{v})) = \tau,$$

whence $\text{rank}((\beta_i^\sigma - \beta_i)S(\mathbf{v})) \leq 2\tau$, by subtraction. This, however, is impossible, since we have $\beta_i^\sigma - \beta_i \neq 0$ and $\text{rank}(S) = n - h \geq 13$. Thus all the β_i must be rational. We then define

$$L(\mathbf{u}) = \sum_{i=1}^h \beta_i u_i,$$

and consider $C' = C + LQ$. The corresponding quadratic forms $B'_i(\mathbf{v})$ are now equal to $B_i(\mathbf{v}) + \beta_i S(\mathbf{v})$, and therefore have rank τ . The claim then follows.

To complete the argument, we take any index $i = 2, \dots, h$, and any $\mu \in \mathbb{Q}$. There is then a $\gamma_i \in \overline{\mathbb{Q}}$ such that

$$\text{rank}(B_1(\mathbf{v}) + \mu B_i(\mathbf{v}) + \gamma_i S(\mathbf{v})) \leq \tau.$$

However, B_1 and B_i both have rank τ , so that

$$\text{rank}(\gamma_i S(\mathbf{v})) \leq 3\tau \leq 12,$$

by subtraction. Since $\text{rank}(S) = n - h \geq 13$, this would give a contradiction unless $\gamma_i = 0$, as we now assume. It therefore follows that $\text{rank}(B_1 + \mu B_i) \leq \tau$ for every i , and for every choice of μ .

We proceed to make a change of variables among the v_j so as to make $B_1(\mathbf{v}) = B_1^*(v_1, \dots, v_\tau)$. We now claim that $B_i(0, \dots, 0, v_{\tau+1}, \dots, v_s)$ must vanish identically, for every i . If this were not the case, we could introduce a change of variable among $v_{\tau+1}, \dots, v_s$ so as to make $v_{\tau+1}^2$ appear with coefficient 1, in B_i . The $(\tau + 1) \times (\tau + 1)$ minor of $B_1 + \mu B_i$ corresponding to the first $\tau + 1$ rows and first $\tau + 1$ columns would then be a polynomial $P(\mu)$ say, with linear term $\mu \det(B_1^*)$. Since $\text{rank}(B_1^*) = \tau$, we have $\det(B_1^*) \neq 0$, so that $P(\mu)$ does not vanish identically. Thus there can be at most finitely many values of μ for which $P(\mu) = 0$. Taking any other value of μ produces a combination $B_1 + \mu B_i$ of rank strictly greater than τ , which is a contradiction. This establishes our claim.

We therefore see that $B_i(0, \dots, 0, v_{\tau+1}, \dots, v_s)$ vanishes identically, for every i , so that the forms B_1, \dots, B_h may be written in the shape $B_i(\mathbf{v}) = v_1 \ell_{i1}(\mathbf{v}) + \dots + v_\tau \ell_{i\tau}(\mathbf{v})$. Thus, if we relabel v_1, \dots, v_τ as $u_{h+1}, \dots, u_{h+\tau}$, we will be able to put $C(\mathbf{x})$ into the form

$$C(\mathbf{x}) = C(\mathbf{u}, \mathbf{v}) = \sum_{1 \leq i \leq j \leq H} u_i u_j L_{ij}(\mathbf{u}, \mathbf{v}),$$

with $H = h + \tau$. Lemma 9.4 then follows.

The proof of Lemma 9.5 is rather easy. Since at least one linear form $L_{ij}(\mathbf{u}, \mathbf{v})$ depends explicitly on \mathbf{v} , we can choose $\mathbf{a} \in \mathbb{Q}^H$ such that $L_{\mathbf{a}}(t, \mathbf{v})$ also explicitly depends on \mathbf{v} . In particular, the equation $L_{\mathbf{a}}(t, \mathbf{v}) = 0$ has solutions with $t \neq 0$, and they are Zariski dense amongst the set of all solutions. Hence, taking such a suitable $\mathbf{a} \in \mathbb{Q}^H$, we see that $\text{rank}(Q_{\mathbf{a}}) \geq \text{rank}(S) \geq 9$, as in (9.5). It follows that the variety $Q_{\mathbf{a}}(t, \mathbf{v}) = L_{\mathbf{a}}(t, \mathbf{v}) = 0$ will have a point of the form $[1, \mathbf{v}]$ over $\overline{\mathbb{Q}}$ which is non-singular in the sense that $\nabla Q_{\mathbf{a}}$ is not proportional to $\nabla L_{\mathbf{a}}$.

We now argue as in the final stages of the proof of Lemma 9.3. We have shown that there is a point $[\mathbf{u}, \mathbf{v}]$ on X such that $[1, \mathbf{v}]$ is a smooth point on $Q_{\mathbf{a}}(t, \mathbf{v}) = L_{\mathbf{a}}(t, \mathbf{v}) = 0$. There is therefore a non-empty Zariski-open subset of such points $[\mathbf{u}, \mathbf{v}]$. However, the variety $C = Q = 0$ is absolutely irreducible, and it has a smooth real point. The real points are therefore Zariski dense, by Lemma 3.4. We choose any such point with $\mathbf{u} \neq \mathbf{0}$, and such that $[1, \mathbf{v}]$ is a smooth point on $Q_{\mathbf{a}}(t, \mathbf{v}) = L_{\mathbf{a}}(t, \mathbf{v}) = 0$. Then, taking rational points \mathbf{a}_m converging to \mathbf{u} in the real topology, we may complete the argument as before.

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