New Formulae for Combined Spherical Triangles

Tsung-Hsuan Hsieh, Shengzheng Wang, Wei Liu and Jiansen Zhao

(*Merchant Marine College, Shanghai Maritime University, Shanghai, China*)
(E-mail: zxxie@shmtu.edu.cn)

Spherical trigonometry formulae are widely adopted to solve various navigation problems. However, these formulae only express the relationships between the sides and angles of a single spherical triangle. In fact, many problems may involve different types of spherical shapes. If we can develop the different formulae for specific spherical shapes, it will help us solve these problems directly. Thus, we propose two types of formulae for combined spherical triangles. The first set are the formulae of the divided spherical triangle, and the second set are the formulae of the spherical quadrilateral. By applying the formulae of the divided spherical triangle, waypoints on a great circle track can be obtained directly without finding the initial great circle course angle in advance. By applying the formulae of the spherical quadrilateral, the astronomical vessel position can be yielded directly from two celestial bodies, and the calculation process concept is easier to comprehend. The formulae we propose can not only be directly used to solve corresponding problems, but also expand the spherical trigonometry research field.

1. INTRODUCTION.   Spherical trigonometry formulae are widely applied to solve the various spherical geometry problems in different fields such as navigation, aviation, geodesy and astronomy. The main spherical trigonometry formulae are the formulae of a right spherical triangle, the formulae of a quadrantal spherical triangle, the law of sines, the law of cosines for the sides, the law of cosines for the angles, the half-angle formulae, the half-side formulae, the four-part formulae, the five-part formulae, Delambre's analogies, Napier's analogies, Cagnoli's formulae, etc. Many textbooks have presented different approaches to derive these formulae, for example: Clough-Smith (1978), Todhunter (1886) and Smart (1977) provided a geometric method for gnomonic projection to derive the law of cosines for the sides. Green (1985) introduced a vector method to derive the law of cosines for the sides. Murray (1908) and Wentworth and Smith (1915) used the formulae of a right spherical triangle to derive the law of cosines for the sides. We can acquire these formulae and related knowledge from many textbooks or on websites such as MathWorld (1999) and Wikipedia (2001).
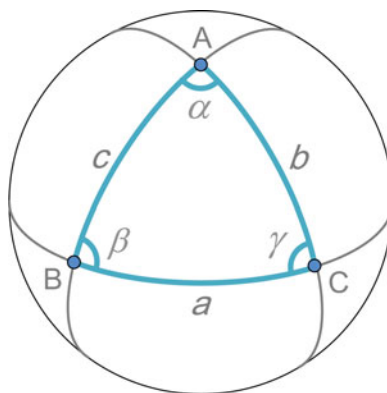
Figure 1. Illustration of the single spherical triangle.

In fact, these spherical trigonometry formulae are all used to express the relationships between the six variables of a single spherical triangle. The six variables include the three sides (a, b, and c) and three angles (α, β, and γ) as shown in Figure 1. For example, the variable relationships of the law of cosines for the sides are the opposite relations between three sides and one angle. The variable relationships of the four-part formulae are the adjacent relations between two sides and two angles. The advantage of these formulae is that they are convenient to apply. If the given variables match the precondition of a particular formula, the unknown variables can be directly obtained by using that formula. For instance, when two sides and their included angle are given, the third side can be directly obtained by using the law of cosines for the sides.

However, the current spherical trigonometry formulae only express the relationships between the sides and angles of a single spherical triangle. Many problems may involve different types of spherical shapes such as spherical quadrilaterals and spherical polygons, which cannot be directly solved by adopting single spherical triangle formulae. Therefore, we propose two types of formulae for combined spherical triangles, which are the formulae of the divided spherical triangle and the formulae of the spherical quadrilateral.

2. FORMULAE OF THE DIVIDED SPHERICAL TRIANGLE. In the case of the divided spherical triangle, the spherical triangle may be divided from one angle or one side as shown in Figure 2. In the given divided angle condition, the spherical triangle ($\widehat{\triangle}ABC$) is divided from the angle α into two spherical triangles ($\widehat{\triangle}ABX$ and $\widehat{\triangle}AXC$). When the two sides (c and b) and the divided angles (α, α_1 and α_2) are given, the other angles (β and γ) can be obtained by using the four-part formulae, and the side a can be obtained by using the law of cosines for the sides. However, none of formulae of the single spherical triangle can directly find the side x (which we called the common side). Similarly, in the given divided side condition, the spherical triangle ($\widehat{\triangle}ABC$) is divided from the side a into two spherical triangles ($\widehat{\triangle}ABX$ and $\widehat{\triangle}AXC$). When the two sides (c and b) and the divided sides (a, a_1 and a_2) are given, none of formulae of the single spherical triangle can directly find the common side (x). Therefore, we propose the formulae of the divided spherical triangle as follows.
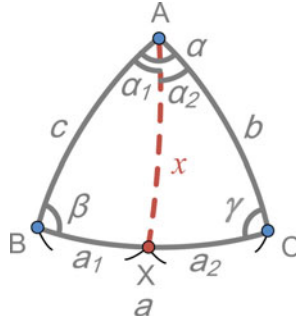
Figure 2.   Illustration of the divided spherical triangle.

First, list the four-part formulae of the two spherical triangles ($\widehat{\triangle}$ABC and $\widehat{\triangle}$ABX) in Figure 2:

$$\tan \beta = \frac{\sin \alpha}{\sin c \cot b - \cos c \cos \alpha} \tag{1}$$

$$\tan \beta = \frac{\sin \alpha_1}{\sin c \cot x - \cos c \cos \alpha_1} \tag{2}$$

Substitute Equation (1) into Equation (2), and rearrange the formula:

$$\tan x = \frac{\sin \alpha}{\sin \alpha_1 \cot b + \sin \alpha_2 \cot c} \tag{3}$$

Then, list the law of cosines for the sides of the two spherical triangles ($\widehat{\triangle}$ABC and $\widehat{\triangle}$ABX) in Figure 2:

$$\cos \beta = \frac{\cos b - \cos c \cos a}{\sin c \sin a} \tag{4}$$

$$\cos \beta = \frac{\cos x - \cos c \cos a_1}{\sin c \sin a_1} \tag{5}$$

Substitute Equation (4) into Equation (5), and rearrange the formula:

$$\cos x = \frac{\sin a_1 \cos b + \sin a_2 \cos c}{\sin a} \tag{6}$$

As a result, Equation (3) is the formula of the common side (x) in the given divided angle condition, and Equation (6) is the formula of the common side (x) in the given divided side condition.

3.   FORMULAE OF THE SPHERICAL QUADRILATERAL.   In the case of the spherical quadrilateral, its shape may be concave or convex. For example, in the concave shape condition as shown in Figure 3(a), the two spherical triangles ($\widehat{\triangle}$ABC and $\widehat{\triangle}$ACD), which have the common side b (that is, side d), combine to one spherical quadrilateral ($\widehat{\triangle}$ABCD).
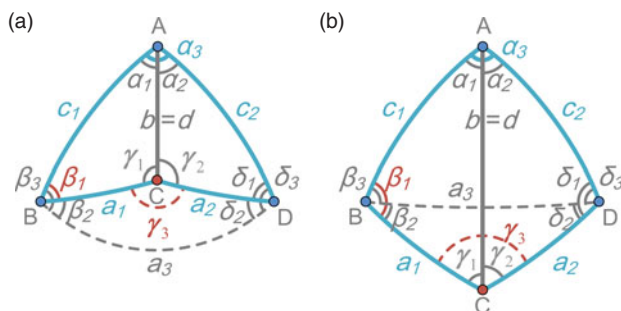
Figure 3.   Illustration of the spherical quadrilateral.

When the four sides (c_1, a_1, c_2 and a_2) and the combined angle (α_3) are given, the side a_3 can be obtained by using the law of cosines for the sides. However, none of formulae of the single spherical triangle can directly find the angle γ_3 and the angle β_1. Similarly, in the convex shape condition as shown in Figure 3(b), none of formulae of the single spherical triangle can directly find the angle γ_3 and the angle β_1, either. Thus, we propose the formulae which can directly obtain the auxiliary angle (γ_3) and the outer angles (β_1) as follows.

First, list the law of cosines for the sides of the two spherical triangles ($\widehat{\triangle}ABD$ and $\widehat{\triangle}CBD$) in Figure 3(a):

$$\cos a_3 = \cos c_1 \cos c_2 + \sin c_1 \sin c_2 \cos \alpha_3 \tag{7}$$

$$\cos a_3 = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos \gamma_3 \tag{8}$$

Substitute Equation (7) into Equation (8), and rearrange the formula to derive the formula of the auxiliary angle (γ_3):

$$\cos \gamma_3 = \frac{\cos c_1 \cos c_2 - \cos a_1 \cos a_2 + \sin c_1 \sin c_2 \cos \alpha_3}{\sin a_1 \sin a_2} \tag{9}$$

Then, list the four-part formulae of the two spherical triangles ($\widehat{\triangle}ABD$ and $\widehat{\triangle}CBD$) in Figure 3(a):

$$\tan \beta_3 = \frac{\sin \alpha_3}{\sin c_1 \cot c_2 - \cos c_1 \cos \alpha_3} \tag{10}$$

$$\tan \beta_2 = \frac{\sin \gamma_3}{\sin a_1 \cot a_2 - \cos a_1 \cos \gamma_3} \tag{11}$$

Use Equation (10), (11), and the addition formula of trigonometric functions to derive the formula of the outer angle (β_1), where $\beta_1 = \beta_3 - \beta_2$:

$$\tan \beta_1 = \frac{\sin \alpha_3(\sin a_1 \cot a_2 - \cos a_1 \cos \gamma_3) - \sin \gamma_3(\sin c_1 \cot c_2 - \cos c_1 \cos \alpha_3)}{(\sin c_1 \cot c_2 - \cos c_1 \cos \alpha_3)(\sin a_1 \cot a_2 - \cos a_1 \cos \gamma_3) + \sin \alpha_3 \sin \gamma_3} \tag{12}$$

Note that in the convex shape condition as shown in Figure 3(b), the formula of the auxiliary angle (γ_3) is the same formula as Equation (9), but the formula of the outer angle

(β_1), where $\beta_1 = \beta_3 + \beta_2$, should be substituted as follows:

$$\tan \beta_1 = \frac{\sin \alpha_3(\sin a_1 \cot a_2 - \cos a_1 \cos \gamma_3) + \sin \gamma_3(\sin c_1 \cot c_2 - \cos c_1 \cos \alpha_3)}{(\sin c_1 \cot c_2 - \cos c_1 \cos \alpha_3)(\sin a_1 \cot a_2 - \cos a_1 \cos \gamma_3) - \sin \alpha_3 \sin \gamma_3} \quad (13)$$

Hence, Equations (9) and (12) are the formulae of the auxiliary angle (γ_3) and the outer angle (β_1) in the concave shape condition, and Equations (9) and (13) are the formulae of the auxiliary angle (γ_3) and the outer angle (β_1) in the convex shape condition.

Since the spherical quadrilateral can also be seen as the divided spherical triangle when the auxiliary angle (γ_3) is $180°$, the formulae of the divided spherical triangle can be derived from the formulae of the spherical quadrilateral as follows:

First, substitute $180°$ into the auxiliary angle (γ_3) in Equation (12) and Equation (13), and rearrange the formula:

$$\tan \beta_1 = \frac{\sin \alpha_3}{(\sin c_1 \cot c_2 - \cos c_1 \cos \alpha_3)} \quad (14)$$

Then, list the four-part formulae of the spherical triangle ($\widehat{\triangle}ABC$) in Figure 3(a) and Figure 3(b):

$$\tan \beta_1 = \frac{\sin \alpha_1}{(\sin c_1 \cot b - \cos c_1 \cos \alpha_1)} \quad (15)$$

Substitute Equation (14) into Equation (15), and rearrange the formula:

$$\tan b = \frac{\sin \alpha_3}{\sin \alpha_1 \cot c_2 + \sin \alpha_2 \cot c_1} \quad (16)$$

Through derivation, it can be verified that Equation (16) is equal to Equation (3) which we proposed in Section 2.

4. APPLYING THE FORMULAE OF THE COMBINED SPHERICAL TRIANGLES.

4.1. *Finding the waypoints on the great circle track.* When the position of the departure point (F) and the destination point (T) are given, the latitude of the departure point (φ_F), the difference of longitude between the departure point and the destination point ($\Delta\lambda_{FT}$), and the latitude of the destination point (φ_T) will be known. Steering along a number of waypoints, which are on the great circle track from the departure point to the destination point, can give a shorter distance. Thus, the main issue of the great circle track is to obtain the positions of the waypoints. There are two kinds of conditions: The first one is given the difference of longitude between the departure point and the waypoint ($\Delta\lambda_{FX}$), to obtain the latitude of the waypoint (φ_X) as shown in Figure 4(a). The second one is given the great circle distance from the departure point to the waypoint (D_{FX}) and the great circle distance from the waypoint to the destination point (D_{XT}), to obtain the latitude of the waypoint (φ_X) and the difference of longitude between the departure point and the waypoint ($\Delta\lambda_{FX}$) as shown in Figure 4(b). These two kinds of problems do not match the preconditions of any formula of the single spherical triangle, but can use the formulae of the divided spherical triangle which we proposed in Section 2 using the following steps:
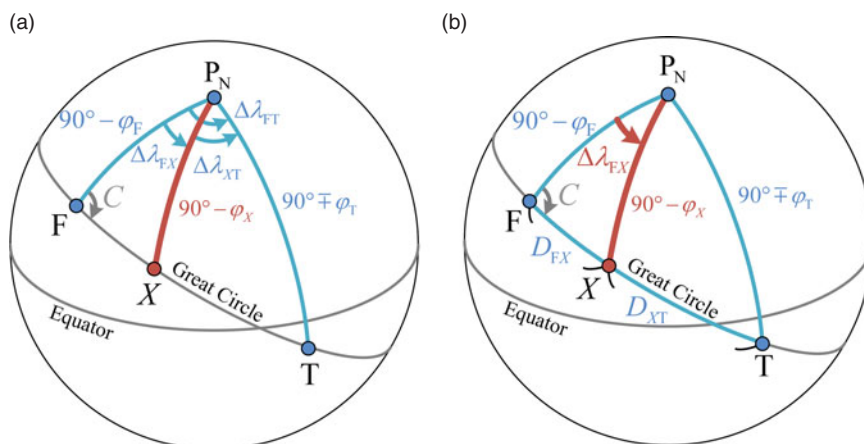
Figure 4.   Finding the waypoints on the great circle track.

In the first given condition, by using Equation (3) we can directly obtain the latitude of the waypoint (φ_X):

$$\tan \varphi_X = \frac{\sin \Delta \lambda_{FX} \tan \varphi_T + \sin \Delta \lambda_{XT} \tan \varphi_F}{\sin \Delta \lambda_{FT}} \tag{17}$$

In the second given condition, by using Equation (6) we can directly obtain the latitude of the waypoint (φ_X):

$$\sin \varphi_X = \frac{\sin D_{FX} \sin \varphi_T + \sin D_{XT} \sin \varphi_F}{\sin D_{FT}} \tag{18}$$

Then, the difference of longitude between the departure point and the waypoint ($\Delta \lambda_{FX}$) can be obtained by using the law of cosines, and it can be converted into the longitude of the waypoint (λ_X).

$$\cos \Delta \lambda_{FX} = \frac{\cos D_{FX} - \sin \varphi_F \sin \varphi_X}{\cos \varphi_F \cos \varphi_X} \tag{19}$$

We now take a practical example to demonstrate how to use the proposed formulae to find the waypoints on the great circle track. A ship is proceeding from Cape Town (33°53·3′S, 18°23·1′E) to New York (40°27·1′N, 73°49·4′W). In the first given condition, the difference of longitude between the departure point and the waypoint ($\Delta \lambda_{FX}$) is given (the value is 8°23·1′). In the second given condition, the great circle distance from the departure point to destination point (D_{FT}) and the great circle distance from the departure point to the waypoint (D_{FX}) are given (the values are 6,762.7 and 600 nautical miles). As shown in Table 1, using Equations (17), (18) and (19), we can respectively find the positions of the waypoints (X) in two kinds of given conditions.

In fact, a number of recent textbooks adopt the formulae of the right spherical triangle to obtain waypoints on the great circle track (Bowditch, 2017; Cutler, 2004; Royal Navy, 2008). This method is called the indirect approach because the position of the vertex must be found in advance. Although the follow-up studies claim that some new formulae which they proposed are the direct approach, they still have to calculate the initial great circle

Table 1.   Results of calculating waypoints by using the formulae of the divided spherical triangle.

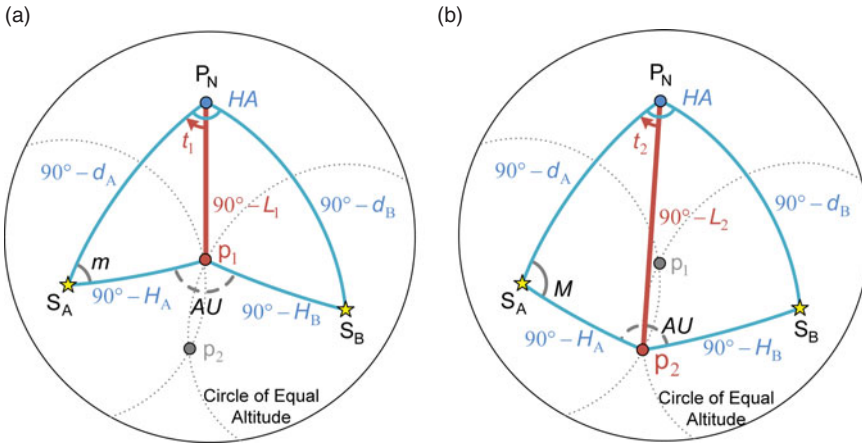| Equation	Input	Output	Solution
17	$\Delta\lambda_{FX} = 8\cdot385°$, $\varphi_T = 40\cdot452°$(N), $\Delta\lambda_{XT} = 83\cdot823°$, $\varphi_F = -33\cdot888°$(S), $\Delta\lambda_{FT} = 92\cdot208°$	$\varphi_X = -28\cdot540°$	$X\begin{pmatrix}28°32\cdot4'S \\ 10°00\cdot0'E\end{pmatrix}$
18	$D_{FX} = 10°$, $\varphi_T = 40\cdot452°$(N), $D_{XT} = 102\cdot712°$, $\varphi_F = -33\cdot888°$(S), $D_{FT} = 112\cdot712°$	$\varphi_X = -27\cdot872°$	$X\begin{pmatrix}27°52\cdot3'S \\ 9°04\cdot0'E\end{pmatrix}$
19	$D_{FX} = 10°$, $\varphi_F = -33\cdot888°$(S), $\varphi_X = -27\cdot872°$(S)	$\Delta\lambda_{FX} = 9\cdot319°$	

Figure 5.   Finding the astronomical vessel position of two celestial bodies.

course angle (*C*) first (Chen, 2016; Chen et al., 2015; 2014; Nastro and Tancredi, 2010). In contrast, using the formulae of the divided spherical triangle can directly yield waypoints without calculating the initial great circle course angle (*C*). Some great circle track studies have also presented this type of method (Holm, 1972; Miller et al., 1991; Tseng and Chang, 2014).

4.2.   *Finding the astronomical vessel position of two celestial bodies.*   When the geographical positions and the observed altitudes of two celestial bodies are known, the main issue of the astronomical vessel position is to obtain the observer's position. Since the two circles of equal altitude may have two intersections, the obtained two possible positions of the observer need to be further judged to find the real astronomical vessel position. The judgement can be done through using the dead reckoning position. As shown in Figure 5(a) and Figure 5(b), the declination of two celestial bodies (d_A, d_B), the difference of hour angle between two celestial bodies (*HA*), and the observed altitude of two celestial bodies (H_A, H_B) are given to find the two possible positions of the observer (P_1, P_2), which include the latitude (L_1, L_2) and the meridian angle (t_1, t_2). Using the formulae of the spherical quadrilateral which was proposed in Section 3, this problem can be dealt with using the following steps.

First, use Equation (9) to obtain the auxiliary angle (*AU*):

$$\cos AU = \frac{\sin d_A \sin d_B - \sin H_A \sin H_B + \cos d_A \cos d_B \cos HA}{\cos H_A \cos H_B} \tag{20}$$

where AU is the auxiliary angle, d_A is the declination of celestial body A, d_B is the declination of celestial body B, H_A is the observed altitude of celestial body A, H_B is the observed altitude of celestial body B and HA is the difference of hour angle between two celestial bodies.

Next, by using Equation (12), we can obtain the parallactic angle (m) in Figure 5(a):

$$\tan m = \frac{\sin HA(\cos H_A \tan H_B - \sin H_A \cos AU) - \sin AU(\cos d_A \tan d_B - \sin d_A \cos HA)}{(\cos d_A \tan d_B - \sin d_A \cos HA)(\cos H_A \tan H_B - \sin H_A \cos AU) + \sin HA \sin AU}$$
(21)

When the parallactic angle (m) is obtained, the latitude (L_1) and the meridian angle (t_1) can be yielded by using the law of cosines for the sides and the four-part formulae respectively:

$$\sin L_1 = \sin d_A \sin H_A + \cos d_A \cos H_A \cos m$$
(22)

$$\tan t_1 = \frac{\sin m}{\cos d_A \tan H_A - \sin d_A \cos m}$$
(23)

Lastly, we use Equation (13) to obtain the parallactic angle (M) in Figure 5(b):

$$\tan M = \frac{\sin HA(\cos H_A \tan H_B - \sin H_A \cos AU) + \sin AU(\cos d_A \tan d_B - \sin d_A \cos HA)}{(\cos d_A \tan d_B - \sin d_A \cos HA)(\cos H_A \tan H_B - \sin H_A \cos AU) - \sin HA \sin AU}$$
(24)

When the parallactic angle (M) is obtained, the latitude (L_2) and the meridian angle (t_2) can also be obtained by using the following formulae:

$$\sin L_2 = \sin d_A \sin H_A + \cos d_A \cos H_A \cos M$$
(25)

$$\tan t_2 = \frac{\sin M}{\cos d_A \tan H_A - \sin d_A \cos M}$$
(26)

We now demonstrate how we use the proposed formulae to find the astronomical vessel position of two celestial bodies by taking a practical example. We use the stars *Capella* and *Alkaid* as the basis of observation. The geographical position of *Capella* is (45°58·4′N, 131°24·8′W), and its observed altitude is 15°19·3′. The geographical position of *Alkaid* is (49°25·7′N, 3°14·2′W), and its observed altitude is 77°34·9′. The dead reckoning position is (41°34·8′N, 017°00·5′W). As shown in Table 2, adopting Equations (20) to (26) can respectively find the two possible positions of the observer (P_1, P_2). The second possible position of the observer (P_2) is the real astronomical vessel position because its position is nearer to the position of dead reckoning. It should be noted that due to the property of the tangent function, $-\theta$ should be replaced as $180° - \theta$ in the calculation process when the meridian angle is negative ($t = -\theta$).

In fact, to find the astronomical vessel position of two celestial bodies, many studies have provided different approaches, such as the spherical triangle method (Chiesa and Chiesa, 1990; Gery, 1997; Karl, 2007; Pepperday, 1992; Pierros, 2018), the simultaneous equal-altitude equation method (Hsu et al., 2005), the complex analysis method (Stuart, 2009) and the rectangular coordinate method (Van Allen, 1981). Similarly, adopting the formulae of the spherical quadrilateral can also deal with this problem. The concept of the method we propose is easier to comprehend, and its calculation process is concise.

Table 2. Results of calculating astronomical vessel position by using the formulae of the spherical quadrilateral.

Equation	Input	Output	Solution
20	$d_A = 45 \cdot 973°$, $d_B = 49 \cdot 423°$, $H_A = 15 \cdot 322°$, $H_B = 77 \cdot 582°$, $HA = 128 \cdot 177°$	$AU = 87 \cdot 592°$	$P_1 \begin{pmatrix} 55°24 \cdot 1'\text{N} \\ 14°42 \cdot 5'\text{E} \end{pmatrix}$
21	$HA = 128 \cdot 177°$, $H_A = 15 \cdot 322°$, $H_B = 77 \cdot 582°$, $AU = 87 \cdot 592°$, $d_A = 45 \cdot 973°$, $d_B = 49 \cdot 428°$	$m = 19 \cdot 158°$	
22	$d_A = 45 \cdot 973°$, $H_A = 15 \cdot 322°$, $m = 19 \cdot 158°$	$L_1 = 55 \cdot 402°$	
23	$m = 19 \cdot 158°$, $d_A = 45 \cdot 973°$, $H_A = 15 \cdot 322°$	$t_1 = -33 \cdot 878° = 146 \cdot 122°\text{W}^*$	
20	$d_A = 45 \cdot 973°$, $d_B = 49 \cdot 428°$, $H_A = 15 \cdot 322°$, $H_B = 77 \cdot 582°$, $HA = 128 \cdot 177°$	$AU = 87 \cdot 592°$	$P_2 \begin{pmatrix} 41°39 \cdot 1'\text{N} \\ 17°07 \cdot 3'\text{W} \end{pmatrix}$
24	$HA = 128 \cdot 177°$, $H_A = 15 \cdot 322°$, $H_B = 77 \cdot 582°$, $AU = 87 \cdot 592°$, $d_A = 45 \cdot 973°$, $d_B = 49 \cdot 428°$	$M = 44 \cdot 922°$	
25	$d_A = 45 \cdot 973°$, $H_A = 15 \cdot 322°$, $M = 44 \cdot 922°$	$L_2 = 41 \cdot 652°$	
26	$M = 44 \cdot 922°$, $d_A = 45 \cdot 973°$, $H_A = 15 \cdot 322°$	$t_2 = -65 \cdot 709° = 114 \cdot 291°\text{W}^*$	

$* \tan(-\theta) = \tan(180° - \theta)$

5. CONCLUSIONS. This paper proposes two types of formulae for combined spherical triangles, which are the formulae of the divided spherical triangle and the formulae of the spherical quadrilateral. By using the divided spherical triangle formulae, we can directly obtain the positions of the waypoints without the extra calculation of the initial great circle course angle. By using the spherical quadrilateral formulae to solve astronomical vessel position problems, we can easier comprehend the calculation process concept than with some other methods.

REFERENCES

Bowditch, N. (2017). *The American Practical Navigator*, 2017 Edition, Volume I. National Geospatial-Intelligence Agency.

Chen, C.L. (2016). A Systematic Approach for Solving the Great Circle Track Problems Based on Vector Algebra. *Polish Maritime Research*, **23**, 3–13.

Chen, C.L., Hsieh, T.H. and Hsu, T.P. (2015). A Novel Approach to Solve the Great Circle Track Based on Rotation Transformation. *Journal of Marine Science and Technology*, **23**, 13–20.

Chen, C.L., Liu, P.F. and Gong, W.T. (2014). A Simple Approach to Great Circle Sailing: the COFI Method. *The Journal of Navigation*, **67**, 403–418.

Chiesa, A. and Chiesa, R. (1990). A Mathematical Method of Obtaining an Astronomical Vessel Position. *The Journal of Navigation*, **43**, 125–129.

Clough-Smith, J.H. (1978). *An Introduction to Spherical Trigonometry*. Brown, Son & Ferguson, Ltd.

Cutler, T.J. (2004). *Dutton's Nautical Navigation*, Fifteenth Edition. Naval Institute Press.

Gery, S.W. (1997). The Direct Fix of Latitude and Longitude from Two Observed Altitudes. *NAVIGATION: Journal of The Institute of Navigation*, **44**, 15–23.

Green, R.M. (1985). *Spherical Astronomy*. Cambridge University Press.

Holm, R.J. (1972). Great Circle Waypoints for Inertial Equipped Aircraft. *NAVIGATION: Journal of The Institute of Navigation*, **19**(2), 191–194.

Hsu, T.P., Chen, C.L. and Chang, J.R. (2005). New Computational Methods for Solving Problems of the Astronomical Vessel Position. *The Journal of Navigation*, **58**, 315–335.

Karl J.H. (2007). *Celestial Navigation in the GPS Age*, Paradise Cay Publications.

MathWorld. (1999). *Spherical Trigonometry*. http://mathworld.wolfram.com/. Accessed 19 August 2018.

Miller, A.R., Moskowitz, I.S. and Simmen, J. (1991). Traveling on the Curved Earth. *Journal of the Institute of Navigation*, **38**, 71–78.

Murray, D.A. (1908). *Spherical Trigonometry: for Colleges and Secondary Schools*. Longmans, Green and Co.

Nastro, V. and Tancredi, U. (2010). Great Circle Navigation with Vectorial Methods. *The Journal of Navigation*, **63**, 557–563.

Pepperday, M. (1992). The 'Two-body Problem' at Sea. *The Journal of Navigation*, **45**, 138–142.

Pierros, F. (2018). Stand-alone Celestial Navigation Positioning Method. *The Journal of Navigation*, doi:10.1017/S0373463318000401.

Royal Navy. (2008). *The Admiralty Manual of Navigation: The Principles of Navigation*, Volume 1, Tenth Edition. Nautical Institute.

Smart, W.M. (1977). *Text-Book on Spherical Astronomy*. Cambridge University Press.

Stuart, R.G. (2009). Applications of Complex Analysis to Celestial Navigation. *NAVIGATION: Journal of The Institute of Navigation*, **56**, 221–227.

Tseng, W.K. and Chang, W.J. (2014). Analogues between 2D Linear Equations and Great Circle Sailing. *The Journal of Navigation*, **67**, 101–112.

Todhunter, I. (1986). *Spherical Trigonometry: for the Use of Colleges and Schools*, Fifth Edition. Macmillan and Company.

Van Allen, J.A. (1981). An Analytical Solution of the Two Star Sight Problem of Celestial Navigation. *NAVIGATION: Journal of the Institute of Navigation*, **28**, 40–43.

Wentworth, G. and Smith, D.E. (1915). *Plane and Spherical Trigonometry*. Ginn and Company.

Wikipedia. (2001). *Spherical Trigonometry*. https://en.wikipedia.org/. Accessed 19 August 2018.