Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2015.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

99.A (Tom Moore)

The sequence (P_n) of pentagonal numbers begins 1, 5, 12, 22, 35, 51, … and is generated by the formula $P_n = \frac{1}{2}n(3n - 1)$ for $n \geqslant 1$. Prove that infinitely many square numbers can be written in the form $P_a + P_b P_c$ for suitable positive integers a, b, c.

99.B (Michael Fox)

Find the smallest triangle whose sides are consecutive integers and whose area is an integer greater than 10^9.

99.C (Abdilkadir Altintaş)

In the triangle ABC, the medians from A, B, C meet the sides BC, AC, AB at A_1, B_1, C_1. Also, the internal angle bisectors of angles A, B, C meet the sides BC, AC, AB at A_2, B_2, C_2. Show that the area of triangle $A_2B_2C_2$ is never greater than the area of triangle $A_1B_1C_1$.

99.D (John D. Mahony)

The triangle ABC (labelled anti-clockwise) has a right-angle at A and side-lengths $a\,(= BC)$, $b\,(= CA)$ and $c\,(= AB)$ where $b < c < a$. Initially, three insects are at rest, one at each vertex of ABC. At the same instant, they start to chase each other in an anti-clockwise direction around the sides of the triangle, each moving the same relative distance $\alpha\,(< 1)$ along their respective pursuit sides before pausing to review their situations. Thus the insect at C stops at point P on CA where $CP = \alpha b$; points Q on AB and R on BC are similarly defined.

(a) If triangle PQR is right-angled at Q show that it is, in fact, similar to triangle ABC.

The insects then start moving again, this time in a clockwise direction along the sides of the right-angled triangle PQR, each moving the same relative distance α along their respective pursuit sides before pausing. The chase continues forever in this manner, alternating between clockwise and anti-clockwise directions of pursuit.

(b) At what point of triangle ABC do the insects eventually meet and how far has each insect travelled to reach that point?

(c) What happens if the right-angles at A and Q are replaced by 'the largest angle A in a scalene triangle ABC with $b < c < a$'?

https://doi.org/10.1017/mag.2015.17 Published online by Cambridge University Press

Solutions and comments on **98.E**, **98.F**, **98.G**, **98.H** (July 2014).

98.E (Stan Dolan and Nick Lord)

Let ϕ denote the golden ratio, $\frac{1}{2}(1 + \sqrt{5})$. If, for some integers a, b, c, $\phi^a + \phi^b + \phi^c$ is an integer other than 1 or 5, then prove that $\phi^a + \phi^b + \phi^c$ is a perfect square plus or minus 2.

First my apologies for an oversight in the setting of this problem: we inadvertently omitted a set of solutions which means that the wording of the question either needs expansion to include them or restriction (say, to a, b, c all non-zero) to exclude them. As ever, respondents took this in their stride, supplying full and detailed analyses which are briefly summarised below.

It is convenient to work with the extended Fibonacci sequence (F_n), $n \in \mathbb{Z}$ and we will use without proof the following standard results:

(1) $F_{-n} = (-1)^{n-1} F_n, n \in \mathbb{Z}$

(2) $\phi^n = F_n \phi + F_{n-1}, n \in \mathbb{Z}$

(3) $F_{2n-1} + F_{2n+1} = L_{2n} = L_n^2 - 2(-1)^n$ for $n \geqslant 1$ where $L_n = \phi^n + (-1)^n \phi^{-n}$ is the nth Lucas number.

By (2), $\phi^a + \phi^b + \phi^c = (F_a + F_b + F_c)\phi + (F_{a-1} + F_{b-1} + F_{c-1})$. Since ϕ is irrational, this will be an integer if, and only if, $F_a + F_b + F_c = 0$. We solve this by considering cases.

(A) If (say) $F_b = 0$, then either $F_a = F_c = 0$ or $F_a = -F_c$ with solutions $\{F_a, F_c\} = \{-1, 1\}$ or, by (1), $\{F_{-2n}, F_{2n}\}$.

Otherwise none of F_a, F_b, F_c are zero.

(B) If $|F_a|, |F_b|, |F_c| > 1$ then, using (1) to make all subscripts positive we need $F_r + F_s = F_t$ to hold with r, s, t positive. For fixed r, $s = r \pm 1$ give solutions but there are no others for if $t > r + 2$ then $F_t - F_s \geqslant F_{r+3} - F_{r+2} = F_{r+1} > F_r$. Thus $\{F_r, F_s, F_t\}$ are consecutive Fibonacci numbers. For our signed Fibonacci numbers, this forces $F_{-2n} + F_{2n-2} + F_{2n-1} = 0$ or $F_{-2n} + F_{2n-2} + F_{-(2n-1)} = 0$.

(C) If (say) $|F_n| = 1$, we pick up the additional cases $\{F_a, F_b, F_c\} = \{1, 2, -3\}$ or $\{-1, -1, 2\}$.

The possible values of $\{a, b, c\}$ corresponding to the solution of $F_a + F_b + F_c = 0$ for each of the cases above are given in the table which also shows the corresponding value of $\phi^a + \phi^b + \phi^c = F_{a-1} + F_{b-1} + F_{c-1}$.

case	a	b	c	$\phi^a + \phi^b + \phi^c$
A	0	0	0	3
A	-2	0	± 1	2, 3
A	$-2n$	0	$2n$	$L_n^2 - 2(-1)^n + 1$
B	$-2n$	$2n - 2$	$2n - 1$	$L_n^2 - 2(-1)^n$
B	$-2n$	$2n - 2$	$-(2n - 1)$	$L_{n-1}^2 - 2(-1)^{n-1}$

C	−4	±1	±3	1, 2, 5, 6
C	−4	2	±3	3, 7
C	−2	−2	±3	1, 5

It was the infinite family in (A) that was omitted from the statement of **98.E**. The expressions involving Lucas numbers are easily verified; for example, $F_{-2n-1} + F_0 + F_{2n-1} = F_{2n-1} + F_{2n+1} + 1 = L_n^2 - 2(-1)^n + 1$, by (3), and $F_{-2n-1} + F_{2n-3} + F_{2n-2} = F_{2n-3} + F_{2n-2} + F_{2n+1} = F_{2n-1} + F_{2n+1} = L_n^2 - 2(-1)^n$, by (3).

Correct solutions were received from: N. Curwen, M. G. Elliott, M. Fox, GCHQ Problem Solving Group, A. P. Harrison, M. A. Hennings, G. Howlett, F. Hunt, P. F. Johnson, P. Kitchenside, J. A. Mundie, D. A. Quadling, B. N. Roth, C. Starr, G. Strickland, K. B. Subramaniam, G. B. Trustrum, L. Wimmer and the proposers S. Dolan and N. Lord.

98.F (Michael de Villiers)

Let $ABCD$ be a cyclic quadrilateral with E the point of intersection of its diagonals, F and G the respective circumcentre and incentre of triangle ABE and H and I the respective circumcentre and incentre of triangle DEC. Prove that $(DF^2 - CF^2) - (DG^2 - CG^2) = (AH^2 - BH^2) - (AI^2 - BI^2)$.

This striking geometrical problem clearly intrigued many readers. Direct computational derivations of the specific result in **98.F** are apt to be rather heavy but Stan Dolan, Michael Fox, Mark Hennings and Graham Howlett all noted that it was easier to prove a much more general result.

Stan Dolan's use of complex numbers below was notably succinct.

For any point P in the plane, represented by the complex number p with respect to some origin O, let P^* denote its image after reflection in the real axis followed by enlargement centre O, scale factor λ, so that $p^* = \lambda\bar{p}$. For any two points P and R, $|P^*R|^2 = (\lambda\bar{p} - r)(\lambda p - \bar{r}) = \lambda^2 p\bar{p} + r\bar{r} - \lambda(pr + \overline{pr})$ and $|R^*P|^2 = \lambda^2 r\bar{r} + p\bar{p} - \lambda(pr + \overline{pr})$ from which $|P^*R|^2 - |R^*P|^2 = (\lambda^2 - 1)(p\bar{p} - r\bar{r})$.

In **98.F**, take the origin at E and the real axis as the bisector of $\angle BEC$. Then, with λ the scale factor between the similar triangles ΔABE and ΔDCE, we have $D = A^*$, $C = B^*$, $H = F^*$, $I = G^*$. Notice that there is nothing special about the circumcentre and incentre here: any pair of corresponding points under the map $P \to P^*$ will suffice.

Then

$$AH^2 - DF^2 = (\lambda^2 - 1)(f\bar{f} - a\bar{a})$$

and

$$CF^2 - BH^2 = (\lambda^2 - 1)(b\bar{b} - f\bar{f})$$

https://doi.org/10.1017/mag.2015.17 Published online by Cambridge University Press

so that

$$\left(AH^2 - BH^2\right) - \left(DF^2 - CF^2\right) = \left(\lambda^2 - 1\right)\left(b\bar{b} - a\bar{a}\right) \qquad (*)$$

where the RHS is independent of F. Replacing F with G thus establishes that

$$\left(AH^2 - BH^2\right) - \left(DF^2 - CF^2\right) = \left(AI^2 - BI^2\right) - \left(DG^2 - CG^2\right)$$

and hence **98.F**.

Since $E = E^*$, the RHS of (*) is just $\left(AE^2 - BE^2\right) - \left(DE^2 - CE^2\right)$; James Mundie evaluated this constant in terms of the side-lengths of the quadrilateral. With $\alpha = AB, \beta = BC, \gamma = CD, \delta = DA,$

$$\left(AE^2 - BE^2\right) - \left(DE^2 - CE^2\right) = \frac{\left(\alpha^2 - \gamma^2\right)\left(\delta^2 - \beta^2\right)\left(\alpha\gamma + \beta\delta\right)}{\left(\alpha\beta + \gamma\delta\right)\left(\alpha\delta + \beta\gamma\right)}.$$

Along with many other geometrical treasures, this problem is on Michael de Villiers' website ⟨http://dynamicmathematicslearning.com/homepage4.html⟩ where he will post three other solutions that he has received from Dirk Basson, Michael Fox and Waldemar Pompe.

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, M. Fox, GCHQ Problem Solving Group, A. P. Harrison, M. A. Hennings, G. Howlett, J. A. Mundie and the proposer M. de Villiers.

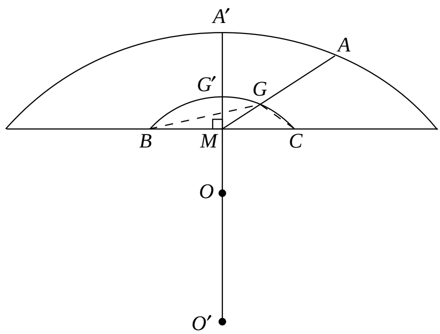
98.G (M. N. Deshpande)

The triangle ABC has centroid G. If $\angle BGC = 120°$, between what limits does $\angle BAC$ lie?

Answer: $0° < \angle BAC \leqslant 60°$.

Solvers employed a range of geometrical techniques to tackle this problem including coordinates, trigonometry and vectors but my eye was caught by several similar solutions which argued as follows and covered the general case where $\angle BGC = \theta$.

For fixed B and C, if $\angle BGC = \theta$, then G lies on a circular arc through B and C with centre O as shown in the Figure.



If M is the midpoint of BC, then A is given by $\overrightarrow{MA} = 3\overrightarrow{MG}$ so that A lies on a circular arc with centre O' where $\overrightarrow{MO'} = 3\overrightarrow{MO}$. For a given value of $\angle BAC$, A lies on a circular arc through B and C; this circle is smallest (so $\angle BAC$ is biggest) when $A = A'$. From $\Delta BA'M$ and $\Delta BG'M$ we have $\tan\angle BA'M = \frac{1}{3}\tan\angle BG'M = \frac{1}{3}\tan\frac{1}{2}\theta$ so that $\angle BA'C = 2\tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}\theta\right)$. Also, as G approaches B or C, $\angle BAC$ approaches 0. So the general bounds are $0 < \angle BAC \leqslant 2\tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}\theta\right)$ which gives $0 < \angle BAC \leqslant 60°$ when $\theta = 120°$.

Other solutions produced other insights. For example, reconciling two apparently different expressions for $\angle BA'C$ gave the pretty trigonometrical identity $2\tan^{-1}\left(\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta\right) = \cos^{-1}\left(\frac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}\right)$. And the proposer's solution showed that when $\theta = 120°$ the extreme values of the ratio $b : c$ occur when $\angle BAC = \cos^{-1}\left(\frac{5\sqrt{7}}{14}\right) \approx 19.1°$.

Correct solutions were received from: M. Bataille, N. Curwen, S. Dolan, M. G. Elliott, M. Fox, GCHQ Problem Solving Group, A. P. Harrison, M. A. Hennings, G. Howlett, P. F. Johnson, J. A. Mundie, D. A. Quadling, G. Strickland, K. B. Subramaniam, G. B. Trustrum, L. Wimmer and the proposer M. N. Deshpande.

98.H (S. N. Maitra)

A composite solid body consists of a right circular cone of fixed height H sharing a base with a spherical cap of height h cut from a sphere of fixed radius R. Find the value of h which maximises the surface area of the body.

Answer: $h = R + \sqrt{R^2 - r^2}$ where r satisfies $2r^3 + H^2r - H^2R = 0$.

Let r be the radius of the base of the cone. For $0 < h \leqslant R$, it is clear that the surface area of the body, $S(r)$, increases as h (and r) increases, so we may suppose that $R \leqslant h < 2R$ where for $0 < r \leqslant R$ the surface area is given by

$$S(r) = 2\pi R\left(R + \sqrt{R^2 - r^2}\right) + \pi r\sqrt{H^2 + r^2}$$

for which

$$S'(r) = \pi\left[\frac{H^2 + 2r^2}{\sqrt{H^2 + r^2}} - \frac{2Rr}{\sqrt{R^2 - r^2}}\right].$$

Now $S'(0) > 0$ and $S'\left(R^-\right) < 0$ while $S'(r) = 0$ rearranges to give

$$4r^6 + 4H^2r^4 + H^4r^2 - H^4R^2 = 0$$

$$\text{or } \left(2r^3 + H^2r + H^2R\right)\left(2r^3 + H^2r - H^2R\right) = 0$$

$$\text{so that } f(r) := 2r^3 + H^2r - H^2R = 0.$$

For $0 \leqslant r \leqslant R$, $f(r)$ increases from $f(0) = -H^2R$ to $f(R) = 2R^3$ so there is a unique solution to $f(r) = 0$ in this interval which maximises $S(r)$. This may be found using Cardan's formula which yields

$$r = \sqrt[3]{\frac{H^2}{4}\left(R + \sqrt{R^2 + \frac{2H^2}{27}}\right)} + \sqrt[3]{\frac{H^2}{4}\left(R - \sqrt{R^2 + \frac{2H^2}{27}}\right)}$$

with $h = R + \sqrt{R^2 - r^2}$.

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, M. A. Hennings, G. Howlett, P. F. Johnson, J. A. Mundie, G. B. Trustrum, and the proposer S. N. Maitra.

Readers will be saddened by the news of the recent death of John Rigby. His encyclopaedic knowledge of all areas of geometry and his ready willingness to share his erudition patiently and enthusiastically will be greatly missed. He contributed many beautiful articles, problems and solutions to the *Gazette* over many years and he was a regular speaker at M. A. Conferences.

doi: 10.1017/mag.2015.17 N.J.L.