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Properties of the extremal solution for a fourth-order elliptic problem

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Let $\lambda^* > 0$ denote the largest possible value of λ such that the system

$$\Delta^2 u = \frac{\lambda}{(1-u)^p} \text{ in } \mathbb{B}, \qquad 0 < u \leqslant 1 \text{ in } \mathbb{B}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\mathbb{B},$$

has a solution, where $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$ centred at the origin, $p > 1$ and n is the exterior unit normal vector. We show that for $\lambda = \lambda^*$ this problem possesses a unique weak solution u^*, called the extremal solution. We prove that u^* is singular when $n \geqslant 13$ for p large enough and actually solve part of the open problem which Dávila *et al*. left unsolved.

1. Introduction and result

The main aim of this paper is to investigate regularity of the extremal solution for a class of fourth-order problem

$$\left.\begin{aligned} \Delta^2 u &= \frac{\lambda}{(1-u)^p} && \text{in } \mathbb{B}, \\ 0 < u &\leqslant 1 && \text{in } \mathbb{B}, \\ u = \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\mathbb{B}. \end{aligned}\right\} \tag{1.1$_\lambda$}$$

Here $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$, $n \geqslant 2$, centred at the origin, $\lambda > 0$, $p > 1$ and $\partial/\partial n$ denotes the differentiation with the respect to the exterior unit normal, i.e. in the radial direction. We consider only radial solutions, since all positive smooth solutions of $(1.1)_\lambda$ are radial [2].

The motivation for studying $(1.1)_\lambda$ stems from a model for the steady states of a simple microelectromechanical system (MEMS) which has the general form (see,

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for example, [11, 13])

$$\left.\begin{aligned} \alpha\Delta^2 u &= \left(\beta\int_\Omega |\nabla u|^2\,\mathrm{d}x + \gamma\right)\Delta u + \frac{\lambda f(x)}{(1-u)^2(1+\chi\int_\Omega \mathrm{d}x/(1-u)^2)} && \text{in } \Omega,\\ 0 &< u < 1 && \text{in } \Omega,\\ u &= \alpha\frac{\partial u}{\partial n} = 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where $\alpha, \beta, \gamma, \chi \geqslant 0$ are fixed, $f \geqslant 0$ represents the permittivity profile, Ω is a bounded domain in $\mathbb{R}^n$ and $\lambda > 0$ is a constant which is increasing with respect to the applied voltage.

Recently, (1.2), posed in $\Omega = \mathbb{B}$ with $\beta = \gamma = \chi = 0, \alpha = 1$ and $f(x) \equiv 1$, which is reduced to

$$\left.\begin{aligned} \Delta^2 u &= \frac{\lambda}{(1-u)^2} && \text{in } \mathbb{B},\\ 0 &< u < 1 && \text{in } \mathbb{B},\\ u &= \frac{\partial u}{\partial n} = 0 && \text{on } \partial\mathbb{B}, \end{aligned}\right\} \tag{1.3}$$

has been studied extensively in [6]. For convenience, we now give the following notion of the solution.

Definition 1.1. If u_λ is a solution of $(1.1)_\lambda$ such that for any other solution v_λ of $(1.1)_\lambda$ one has

$$u_\lambda \leqslant v_\lambda \quad \text{for almost every } x \in \mathbb{B},$$

we say that u_λ is a minimal solution of $(1.1)_\lambda$.

It was shown that there exists a critical value $\lambda^* > 0$ (pull-in voltage) such that if $\lambda \in (0, \lambda^*)$ (1.3) has a smooth minimal solution, while for $\lambda > \lambda^*$ (1.3) has no solution, even in a weak sense (see [5, 6]). Moreover, the branch $\lambda \to u_\lambda(x)$ is increasing for each $x \in \mathbb{B}$, and therefore the function $u^*(x) := \lim_{\lambda\to\lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution (which, by elliptic regularity theory, is equivalent to whether $\sup_{\mathbb{B}} u^* < 1$) is an important question for many reasons. For example, it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*) (see Figure 1). This issue turned out to depend closely on the dimension. Indeed, by the key uniform estimate of $\|(1-u)^{-3}\|_{L^1}$, Guo and Wei [10] obtained the regularity of the extremal solution for small dimensions and proved that for dimensions $n = 2, 3$, u^* is smooth. But, from their result, the regularity of the extremal solution of (1.3) is unknown for $n \geqslant 4$. Recently, using certain improved Hardy–Rellich inequalities, Cowan *et al.* [6] improved the above result and obtained that u^* is regular in dimensions $1 \leqslant n \leqslant 8$, while it is singular for $n \geqslant 9$, i.e. the critical dimension is 9. So the issue of the regularity of the extremal solution of $(1.1)_\lambda$ for power $p = 2$ is completely solved, but the critical dimension for general powers is *unknown*.

The multiplicity phenomenon for radial solutions of $(1.1)_\lambda$ and the regularity of the extremal solution of $(1.1)_\lambda$ for a large range of powers have recently been

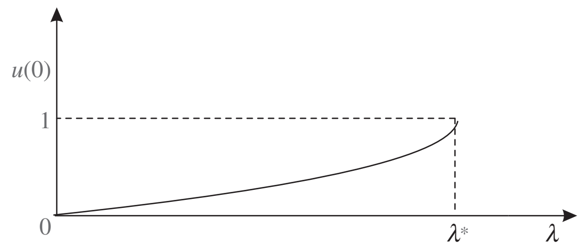

Figure 1. The bifurcation diagram in the case for the extremal solution is singular.

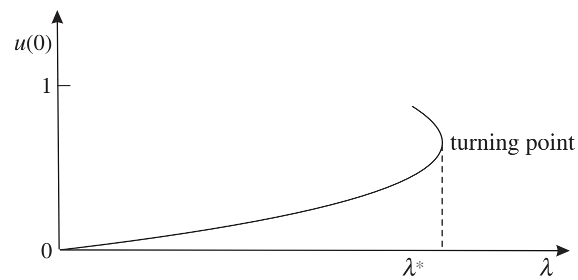

Figure 2. The local bifurcation diagram in the case for the extremal solution is regular.

studied extensively by Dávila *et al*. [7]. For convenience, we now define

$$p_c = \frac{n+2-\sqrt{4+n^2-4\sqrt{n^2+H_n}}}{n-6-\sqrt{4+n^2-4\sqrt{n^2+H_n}}} \quad \text{for } n \geqslant 3,$$

$$p_c^+ = \frac{n+2+\sqrt{4+n^2-4\sqrt{n^2+H_n}}}{n-6+\sqrt{4+n^2-4\sqrt{n^2+H_n}}} \quad \text{for } n \geqslant 3,\ n \neq 4,$$

with $H_n = (n(n-4)/4)^2$ and the numbers p_c and p_c^+ are such that when $-p = p_c$ or $-p = p_c^+$ we have

$$\left(\frac{4}{-p-1}+4\right)\left(\frac{4}{-p-1}+2\right)\left(n-2-\frac{4}{-p-1}\right)\left(n-4-\frac{4}{-p-1}\right) = H_n.$$

To explain our motivations, we now recall some corresponding results from [7].

THEOREM 1.2 (Dávila *et al*. [7, theorem A]). *Assume*

$$n = 3 \text{ and } -p_c < p < -p_c^+, \quad \text{or} \quad 4 \leqslant n \leqslant 12 \text{ and } -p_c < p < \infty. \tag{1.4}$$

Then there exists a unique λ_s *such that* $(1.1)_\lambda$ *with* $\lambda = \lambda_s$ *has infinitely many radial smooth solutions. For* $\lambda \neq \lambda_s$ *there are finitely many radial smooth solutions and their number goes to infinity as* $\lambda \to \lambda_s$*. Moreover,* $\lambda_s < \lambda^*$ *and* u^* *is regular.*

From this theorem, we know that the extremal solution of $(1.1)_\lambda$ is regular for a certain range of p and n. At the same time, Dávila *et al*. left a open problem: if

$$n = 3 \quad \text{and} \quad p \in (1, -p_c] \cup [-p_c^+, 3]$$

or

$$5 \leqslant n \leqslant 12 \quad \text{and} \quad 1 < p < -p_{\mathrm{c}},$$

or

$$n \geqslant 13 \quad \text{and} \quad p > 1,\ 0 < p \leqslant 1,$$

is u^* singular?

In this paper, by constructing a semi-stable singular $H_0^2(\mathbb{B})$-weak subsolution of $(1.1)_\lambda$, we prove that if p is large enough, the extremal solution is singular for dimensions $n \geqslant 13$ and complete part of the above open problem. Our result is stated as follows.

Theorem 1.3.

(i) *For any $p > 1$, the unique extremal solution of $(1.1)_{\lambda^*}$ is regular for dimensions $n \leqslant 4$.*

(ii) *There exists $p_0 > 1$ large enough such that for $p \geqslant p_0$ the unique extremal solution of $(1.1)_{\lambda^*}$ is singular for dimensions $n \geqslant 13$.*

From the technical point of view, one of the basic tools in the analysis of nonlinear second-order elliptic problems in bounded and unbounded domains of $\mathbb{R}^n$, $n \geqslant 2$, is the maximum principle. However, for high-order problems, such a principle does not normally hold for general domains (at least for the clamped boundary conditions $u = \partial u/\partial n = 0$ on $\partial\Omega$), which causes several technical difficulties. One of reasons for studying $(1.1)_\lambda$ in a ball is that the maximum principle holds in this situation [1, 3]. The second obstacle is the well-known difficulty of extracting energy estimates for solutions of fourth-order problems from their stability properties. In addition, for the corresponding second-order problem, the starting point was an explicit singular solution for a suitable eigenvalue parameter, λ, which turned out to play a fundamental role in the shape of the corresponding bifurcation diagram [4]. When turning to the biharmonic problem $(1.1)_\lambda$, the second boundary condition $\partial u/\partial n = 0$ prevents us from finding an explicit singular solution. This means that the method used to analyse the regularity of the extremal solution for the second-order problem could not carry over to the corresponding problem for $(1.1)_\lambda$. In this paper, in order to overcome the third obstacle, we use the improved and non-standard Hardy–Rellich inequalities recently established in [8] to construct a semi-stable singular $H^2(\mathbb{B})$-weak subsolution of $(1.1)_\lambda$.

This paper is organized as follows. In the next section, some preliminaries are reviewed. In § 3, we give the uniform estimate of $\|(1-u)^{-(p+1)}\|_{L^1}$ according to the stability of the minimal solutions. We study the regularity of the extremal solution of $(1.1)_\lambda$ and theorem 1.3(i) is established in § 4. Finally, we shall show that the extremal solution u^* in dimensions $n \geqslant 13$ is singular by constructing a semi-stable singular $H^2(\mathbb{B})$-weak subsolution of $(1.1)_\lambda$.

2. Preliminaries

First we give some comparison principles which will be used throughout the paper.

LEMMA 2.1 (Boggio's principle [3]). *If $u \in C^4(\bar{\mathbb{B}}_R)$ satisfies*

$$\Delta^2 u \geqslant 0 \quad \text{in } \mathbb{B}_R,$$
$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\mathbb{B}_R,$$

then $u \geqslant 0$ in $\mathbb{B}_R$.

LEMMA 2.2. *Let $u \in L^1(\mathbb{B}_R)$ and suppose that*

$$\int_{\mathbb{B}_R} u\Delta^2\varphi \geqslant 0$$

for all $\varphi \in C^4(\bar{\mathbb{B}}_R)$ such that $\varphi \geqslant 0$ in $\mathbb{B}_R$,

$$\varphi|_{\partial\mathbb{B}_R} = \frac{\partial\varphi}{\partial n}\bigg|_{\partial\mathbb{B}_R} = 0.$$

Then $u \geqslant 0$ in $\mathbb{B}_R$. Moreover, $u \equiv 0$ or $u > 0$ almost everywhere (a.e.) in $\mathbb{B}_R$.

For a proof see [1, lemma 17].

LEMMA 2.3. *If $u \in H^2(\mathbb{B}_R)$ is radial, $\Delta^2 u \geqslant 0$ in $\mathbb{B}_R$ in the weak sense, i.e.*

$$\int_{\mathbb{B}_R} \Delta u \Delta\varphi \geqslant 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{B}_R), \varphi \geqslant 0$$

and

$$u|_{\partial\mathbb{B}_R} \geqslant 0, \qquad \frac{\partial u}{\partial n}\bigg|_{\partial\mathbb{B}_R} \leqslant 0$$

then $u \geqslant 0$ in $\mathbb{B}_R$.

Proof. We only deal with the case $R = 1$ for simplicity. Solve

$$\Delta^2 u_1 = \Delta^2 u \quad \text{in } \mathbb{B},$$
$$u_1 = \frac{\partial u_1}{\partial n} = 0 \qquad \text{on } \partial\mathbb{B},$$

in the sense that $u_1 \in H_0^2(\mathbb{B})$ and

$$\int_{\mathbb{B}} \Delta u_1 \Delta\varphi = \int_{\mathbb{B}} \Delta u \Delta\varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{B}).$$

Then $u_1 \geqslant 0$ in $\mathbb{B}$ by lemma 2.2.

Let $u_2 = u - u_1$ so that $\Delta^2 u_2 = 0$ in $\mathbb{B}$. Define $f = \Delta u_2$. Then $\Delta f = 0$ in $\mathbb{B}$ and since f is radial we find that f is a constant. It follows that $u_2 = ar^2 + b$. Using the boundary conditions, we deduce that $a + b \geqslant 0$ and $a \leqslant 0$, which imply $u_2 \geqslant 0$. □

As in [6], we are now led here to examine $(1.1)_\lambda$ with non-homogeneous boundary conditions such as

$$\left.\begin{aligned} \Delta^2 u &= \frac{\lambda}{(1-u)^p} && \text{in } \mathbb{B}, \\ \alpha < u &\leqslant 1 && \text{in } \mathbb{B}, \\ u = \alpha, \quad \frac{\partial u}{\partial n} &= \gamma && \text{on } \partial\mathbb{B}, \end{aligned}\right\} \tag{$(2.1)_{\lambda,\alpha,\gamma}$}$$

where α, γ are given.

Let Φ denote the unique solution of

$$\left.\begin{aligned} \Delta^2 \Phi &= 0 && \text{in } \mathbb{B}, \\ \Phi = \alpha, \quad \frac{\partial \Phi}{\partial n} &= \gamma && \text{on } \partial\mathbb{B}. \end{aligned}\right\} \tag{2.2}$$

We shall say that the pair (α, γ) is admissible if $\gamma \leqslant 0$, and $\alpha - \frac{1}{2}\gamma < 1$. We now introduce the notion of the weak solution.

Definition 2.4. We say that u is a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ if $\alpha \leqslant u \leqslant 1$ a.e. in Ω, $1/(1-u)^p \in L^1(\Omega)$ and if

$$\int_{\mathbb{B}} (u - \Phi)\Delta^2\varphi = \lambda \int_{\mathbb{B}} \frac{\varphi}{(1-u)^p} \quad \text{for all } \varphi \in C^4(\bar{\mathbb{B}}) \cap H_0^2(\mathbb{B}),$$

where Φ is given in (2.2). We say u is a weak supersolution (respectively, weak subsolution) of $(2.1)_{\lambda,\alpha,\gamma}$, if the equality is replaced with $\geqslant$ (respectively, $\leqslant$) for $\varphi \geqslant 0$.

Definition 2.5. We say a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ is regular (respectively, singular) if $\|u\|_\infty < 1$ (respectively, $\|u\| = 1$) and stable (respectively, semi-stable) if

$$\mu_1(u) = \inf\left\{ \int_{\mathbb{B}} (\Delta\varphi)^2 - p\lambda \int_{\mathbb{B}} \frac{\varphi^2}{(1-u)^{p+1}} : \phi \in H_0^2(\mathbb{B}), \|\phi\|_{L^2} = 1 \right\}$$

is positive (respectively, non-negative).

We now define

$$\lambda^*(\alpha, \gamma) := \sup\{\lambda > 0 \colon (2.1)_{\lambda,\alpha,\gamma} \text{ has a classical solution}\}$$

and

$$\lambda_*(\alpha, \gamma) := \sup\{\lambda > 0 \colon (2.1)_{\lambda,\alpha,\gamma} \text{ has a weak solution}\}.$$

Observe that by the implicit function theorem, we can classically solve $(2.1)_{\lambda,\alpha,\gamma}$ for small λ. Therefore, $\lambda^*(\alpha, \gamma)$ and $\lambda_*(\alpha, \gamma)$ are well defined for any admissible pair (α, γ). For brevity of notation we shall not always indicate α and γ.

We now give the following standard existence result.

Theorem 2.6. *For every* $0 \leqslant f \in L^1(\Omega)$ *there exists a unique* $0 \leqslant u \in L^1(\mathbb{B})$ *which satisfies*

$$\int_{\mathbb{B}} u\Delta^2\varphi \, \mathrm{d}x = \int_{\mathbb{B}} f\varphi \, \mathrm{d}x$$

for all $\varphi \in C^4(\bar{\mathbb{B}}) \cap H_0^2(\mathbb{B})$.

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The proof is standard [9] so we omit it. From this theorem, we immediately have the following result.

Proposition 2.7. *Assume the existence of a weak supersolution U of (2.1)$_{\lambda,\alpha,\gamma}$. Then there exists a weak solution u of (2.1)$_{\lambda,\alpha,\gamma}$ so that $\alpha \leqslant u \leqslant U$ a.e in $\mathbb{B}$.*

For the sake of completeness, we include a brief proof here, which is called the 'weak' iterative scheme: let $u_0 = U$ and (inductively) let u_n, $n \geqslant 1$, be the solution of

$$\int_{\mathbb{B}} (u_n - \Phi)\Delta^2 \varphi = \lambda \int_{\mathbb{B}} \frac{\varphi}{(1-u_{n-1})^p} \quad \text{for all } \varphi \in C^4(\bar{\mathbb{B}}) \cap H_0^2(\mathbb{B}),$$

given by theorem 2.6. Since α is a subsolution of $(2.1)_{\lambda,\alpha,\gamma}$, inductively it is easily shown by lemma 2.2 that $\alpha \leqslant u_{n+1} \leqslant u_n \leqslant U$ for every $n \geqslant 0$. Since

$$(1-u_n)^{-p} \leqslant (1-U)^{-p} \in L^1(\mathbb{B}),$$

by the Lebesgue theorem the function $u = \lim_{n\to\infty} u_n$ is a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$ so that $\alpha \leqslant u \leqslant U$.

In particular, for every $\lambda \in (0, \lambda_*)$, we can find a weak solution of $(2.1)_{\lambda,\alpha,\gamma}$. In the same range of λ, this is still true for regular weak solutions, as shown in the following lemma.

Lemma 2.8. *Let (α, γ) be an admissible pair and let u be a weak solution of system (2.1)$_{\lambda,\alpha,\gamma}$. Then, there exists a regular solution for every $0 < \mu < \lambda$.*

Proof. Let $\epsilon \in (0, 1)$ be given and let $\bar{u} = (1-\epsilon)u + \epsilon\Phi$, where Φ is given in (2.2). By lemma 2.2 $\sup_{\mathbb{B}} \Phi < \sup_{\mathbb{B}} u \leqslant 1$. Hence,

$$\sup_{\mathbb{B}} \bar{u} \leqslant (1-\epsilon) + \epsilon \sup_{\mathbb{B}} \Phi < 1, \quad \inf_{\mathbb{B}} \bar{u} \geqslant (1-\epsilon)\alpha + \epsilon \inf_{\mathbb{B}} \Phi = \alpha$$

and

$$\begin{aligned}
\int_{\mathbb{B}} (\bar{u} - \Phi)\Delta^2\varphi &= (1-\epsilon) \int_{\mathbb{B}} (u - \Phi)\Delta^2\varphi \\
&= (1-\epsilon)\lambda \int_{\mathbb{B}} \frac{\varphi}{(1-u)^p} \\
&= (1-\epsilon)^{p+1}\lambda \int_{\mathbb{B}} \frac{\varphi}{(1-\bar{u}+\epsilon(\Phi-1))^p} \\
&\geqslant (1-\epsilon)^{p+1}\lambda \int_{\mathbb{B}} \frac{\varphi}{(1-\bar{u})^p}.
\end{aligned}$$

Note that $0 \leqslant (1-\epsilon)(1-u) = 1-\bar{u}+\epsilon(\Phi-1) < 1-\bar{u}$. So $\bar{u}$ is a weak supersolution of $(2.1)_{(1-\epsilon)^{p+1}\lambda,\alpha,\gamma}$ such that $\sup_{\mathbb{B}} \bar{u} < 1$. By lemma 2.2 we get the existence of a weak solution of $(2.1)_{(1-\epsilon)^{p+1}\lambda,\alpha,\gamma}$ so that $\alpha \leqslant \omega \leqslant \bar{u}$. In particular, $\sup_{\mathbb{B}} \bar{u} < 1$ and ω is a regular weak solution. Since $\epsilon \in (0, 1)$ is arbitrarily chosen, the proof is complete. □

Now we recall some basic facts about the minimal branch.

Theorem 2.9. *$\lambda^* \in (0, +\infty)$ and the following hold.*

(i) *For each $0 < \lambda < \lambda^*$ there exists a regular and minimal solution u_λ of (2.1)$_{\lambda,\alpha,\gamma}$.*

(ii) *For each $x \in \mathbb{B}$ the map $\lambda \to u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*

(iii) *For $\lambda > \lambda^*$ there are no weak solutions of (2.1)$_{\lambda,\alpha,\gamma}$.*

The proof is standard [6], so we omit it.

3. Stability of the minimal solutions

In this section we shall show that the extremal solution is regular in small dimensions. Let us begin with the following *a priori* estimates along the minimal branch u_λ. In order to achieve this, we shall need yet another notion of $H^2(\mathbb{B})$-weak solutions, which is an intermediate class between classical and weak solutions.

Definition 3.1. We say that u is an $H^2(\mathbb{B})$-weak solution of $(2.1)_{\lambda,\alpha,\beta}$ if $u - \Phi \in H_0^2(\mathbb{B})$, $\alpha \leqslant u \leqslant 1 \in \mathbb{B}$, $1/(1-u)^p \in L^1(\mathbb{B})$ and if

$$\int_{\mathbb{B}} \Delta u \Delta \phi = \lambda \int_{\mathbb{B}} \frac{\phi}{(1-u)^p} \quad \text{for all } \phi \in C^4(\bar{\mathbb{B}}) \cap H_0^2(\mathbb{B}),$$

where Φ is given in (2.2). We say that u is an $H^2(\mathbb{B})$-weak supersolution (respectively, $H^2(\mathbb{B})$-weak subsolution) of $(2.1)_{\lambda,\alpha,\beta}$ if for $\phi \geqslant 0$ the equality is replaced with $\geqslant$ (respectively, $\leqslant$) and $u \geqslant \alpha$ (respectively, $\leqslant$), $\partial u/\partial n \leqslant \beta$ (respectively, $\geqslant$) on $\partial\mathbb{B}$.

Theorem 3.2. *Suppose that (α, γ) is an admissible pair.*

(i) *The minimal solution u_λ is stable and is the unique semi-stable $H^2(\mathbb{B})$-weak solution of (2.1)$_{\lambda,\alpha,\gamma}$.*

(ii) *The function $u^* := \lim_{\lambda\to\lambda^*} u_\lambda$ is a well-defined semi-stable $H^2(\mathbb{B})$-weak solution of* $(2.1)_{\lambda^*,\alpha,\gamma}$.

(iii) *u^* is the unique $H^2(\mathbb{B})$-weak solution of* $(2.1)_{\lambda^*,\alpha,\gamma}$, *and, when u^* is classical solution, $\mu_1(u^*) = 0$,*

(iv) *If v is a singular, semi-stable $H^2(\mathbb{B})$-weak solution of (2.1)$_{\lambda,\alpha,\gamma}$, then $v = u^*$ and $\lambda = \lambda^*$.*

The main tool which we use to prove theorem 3.2 is the following comparison lemma, which is valid exactly in the class $H^2(\mathbb{B})$.

Lemma 3.3. *Let (α, γ) be an admissible pair and let u be a semi-stable $H^2(\mathbb{B})$-weak solution of (2.1)$_{\lambda,\alpha,\gamma}$. Assume U is an $H^2(\mathbb{B})$-weak supersolution of (2.1)$_{\lambda,\alpha,\gamma}$. Then*

(i) *$u \leqslant U$ a.e. in $\mathbb{B}$,*

(ii) *if u is a classical solution and $\mu_1(u) = 0$, then $U = u$.*

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A more general version of lemma 3.3 is available, as follows.

LEMMA 3.4. *Let* (α, γ) *be an admissible pair and let* $\gamma' \leqslant 0$. *Let* u *be a semi-stable* $H^2(\mathbb{B})$*-weak subsolution of* $(2.1)_{\lambda,\alpha,\gamma}$ *with* $u = \alpha' \leqslant \alpha$, $\partial u/\partial n = \gamma' \geqslant \gamma$ *on* $\partial\mathbb{B}$. *Assume that* U *is an* $H^2(\mathbb{B})$*-weak supersolution of* $(2.1)_{\lambda,\alpha,\gamma}$ *with* $U = \alpha$, $\partial U/\partial n = \gamma$ *on* $\partial\mathbb{B}$. *Then* $U \geqslant u$ *a.e. in* $\mathbb{B}$.

The proofs of lemmas 3.3 and 3.4 are the same as in [6, 12], so we omit them here. Also, we need some *a priori* estimates along the minimal branch u_λ.

LEMMA 3.5. *Let* (α, γ) *be an admissible pair. Then for every* $\lambda \in (0, \lambda^*)$ *we have*

$$p \int_{\mathbb{B}} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} \leqslant \int_{\mathbb{B}} \frac{u_\lambda - \Phi}{(1-u_\lambda)^p},$$

where Φ *is given by (2.2). In particular, there is a constant* C *independent of* λ *so that*

$$\int_{\mathbb{B}} |\Delta u_\lambda|^2 \, dx + \int_{\mathbb{B}} \frac{1}{(1-u_\lambda)^{p+1}} \leqslant C. \tag{3.1}$$

Proof. Testing $(2.1)_{\lambda,\alpha,\gamma}$ on $u_\lambda - \Phi \in W^{4,2}(\mathbb{B}) \cap H_0^2(\mathbb{B})$, we see that

$$\lambda \int_{\mathbb{B}} \frac{u_\lambda - \Phi}{(1-u_\lambda)^p} = \int_{\mathbb{B}} (\Delta(u_\lambda - \Phi))^2 \, dx \geqslant p\lambda \int_{\mathbb{B}} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} \, dx$$

in the view of $\Delta^2 \Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\begin{aligned} \int_{|u_\lambda - \Phi| \geqslant \delta} \frac{1}{(1-u_\lambda)^{p+1}} &\leqslant \frac{1}{\delta^2} \int_{|u_\lambda - \Phi| \geqslant \delta} \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^{p+1}} \\ &\leqslant \frac{1}{\delta^2} \int_{\mathbb{B}} \frac{1}{(1-u_\lambda)^p} \\ &\leqslant \delta^{p-1} \int_{\mathbb{B}} \frac{1}{(1-u_\lambda)^{p+1}} + C_\delta \end{aligned}$$

by means of Young's inequality. Since for δ small

$$\int_{|u_\lambda - \Phi| \leqslant \delta} \frac{1}{(1-u_\lambda)^{p+1}} \leqslant C$$

for some $C > 0$, we get that

$$\int_{\mathbb{B}} \frac{1}{(1-u_\lambda)^{p+1}} \leqslant C$$

for some $C > 0$ and for every $\lambda \in (0, \lambda^*)$. By Young's and Hölder's inequalities, we have

$$\begin{aligned} \int_{\mathbb{B}} |\Delta u_\lambda|^2 \, dx &= \int_{\mathbb{B}} \Delta u_\lambda \Delta \Phi \, dx + \lambda \int_{\mathbb{B}} \frac{u_\lambda - \Phi}{(1-u_\lambda)^p} \, dx \\ &\leqslant \delta \int_{\mathbb{B}} |\Delta u_\lambda|^2 \, dx + C_\delta + C \left(\int_{\mathbb{B}} \frac{dx}{(1-u_\lambda)^{p+1}} \right)^{p/p+1}. \end{aligned}$$

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So we have

$$\int_{\mathbb{B}} |\Delta u_\lambda|^2 \, dx + \int_{\mathbb{B}} \frac{dx}{(1-u_\lambda)^{p+1}} \leqslant C,$$

where C is absolute constant. □

Proof of theorem 3.2.
(i) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. Since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in \mathbb{B}$, it is easily seen that $\lambda \to \mu_1(u_\lambda)$ is a decreasing and continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* \colon \mu_1(u_\lambda) > 0\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have $\mu_1(u_{\lambda_{**}}) = 0$, and, for every $\mu \in (\lambda_{**}, \lambda^*)$, u_μ would be a classical supersolution of $(2.1)_{\lambda_{**},\alpha,\gamma}$. A contradiction arises since lemma 3.3 implies $u_\mu = u_{\lambda_{**}}$. Finally, lemma 3.3 guarantees the uniqueness in the class of semi-stable $H^2(\mathbb{B})$-weak solutions.

(ii) It follows from (3.1) that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(\mathbb{B})$, and $1/(1-u^*) \in L^{p+1}$. In particular, u^* is an $H^2(\mathbb{B})$-weak solution of $(2.1)_{\lambda_{**},\alpha,\gamma}$ that is also semi-stable as the limiting function of the semi-stable solutions $\{u_\lambda\}$.

(iii) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the implicit function theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By lemma 3.3, u^* is then the unique $H^2(\mathbb{B})$-weak solution of $(2.1)_{\lambda_*,\alpha,\gamma}$..

(iv) If $\lambda < \lambda^*$, we get by uniqueness that $v = u_\lambda$. So v is not singular and a contradiction arises. Since v is a semi-stable $H^2(\mathbb{B})$-weak solution of $(2.1)_{\lambda_*,\alpha,\gamma}$ and u^* is an $H^2(\mathbb{B})$-weak supersolution of $(2.1)_{\lambda_*,\alpha,\gamma}$, we can apply lemma 3.3 to get $v \leqslant u^*$ a.e. in Ω. Since u^* is also a semi-stable solution, we can reverse the roles of v and u^* in lemma 3.3 to see that $v \geqslant u^*$ a.e. in $\mathbb{B}$. So equality $v = u^*$ holds and the proof is complete. □

4. Regularity of the extremal solution and the proof of theorem 1.3(i)

In this section we first show that the extremal solution is regular in small dimensions by the uniformly bounded of u_λ in $H_0^2(\mathbb{B})$. Now we give the proof of theorem 1.3(i).

Proof of theorem 1.3(i). As already observed, estimate (3.1) implies that $f(u^*) = (1-u^*)^{-p} \in L^{(p+1)/p}(\mathbb{B})$. Since u^* is radial and radially decreasing. We need to show that $u^*(0) < 1$ to get the regularity of u^*. In fact, on the contrary, suppose that $u^*(0) = 1$. By the standard elliptic regularity theory shows that $u^* \in W^{4,(p+1)/p}$. By the Sobolev imbedding theorem, i.e.

$$W^{4,(p+1)/p} \hookrightarrow C^m\left(0 < m \leqslant 4 - \frac{pn}{p+1},\ 1 \leqslant n \leqslant 4\right).$$

We have that u^* is a Lipschitz function in $\mathbb{B}$ for $1 \leqslant n \leqslant 3$.

Now suppose $u^*(0) = 1$ and $1 \leqslant n \leqslant 2$. Since

$$\frac{1}{1-u^*} \geqslant \frac{C}{|x|} \quad \text{in } \mathbb{B}$$

for some $C > 0$. We see that

$$+\infty = C^{p+1} \int_{\mathbb{B}} \frac{1}{|x|^{p+1}} \leqslant \int_{\mathbb{B}} \frac{1}{(1-u^*)^{p+1}} < +\infty.$$

A contradiction arises and hence u^* is regular for $1 \leqslant n \leqslant 2$.

For $n = 3$, by the Sobolev imbedding theorem, we have $u^* \in C^{(p+4)/(p+1)}(\bar{B})$. If $(p+4)/(p+1) \geqslant 2$, then $u^*(0) = 1$, $Du^*(0) = 0$ and

$$|Du^*(\varepsilon) - Du^*(0)| \leqslant M|\varepsilon| \leqslant M|x|,$$

where $0 < |\varepsilon| < |x|$. Thus,

$$|u(x) - u(0)| \leqslant |Du(\varepsilon)||x| \leqslant M|x|^2.$$

This inequality shows that

$$+\infty = C^{p+1} \int_{\mathbb{B}} \frac{1}{|x|^{2(p+1)}} \leqslant \int_{\mathbb{B}} \frac{1}{(1-u^*)^{p+1}} < +\infty.$$

A contradiction arises and hence u^* is regular for $n = 3$; if $(p+4)/(p+1) < 2$, then

$$|Du(\varepsilon) - Du(0)| \leqslant M|\varepsilon|^{4/(p-1)-1} \leqslant M|x|^{3/(p+1)},$$

where $0 < |\varepsilon| < |x|$. Thus,

$$|u(x) - u(0)| \leqslant |Du(\varepsilon)||x| \leqslant M|x|^{(4+p)/(p+1)},$$

and a contradiction is obtained, as above.

For $n = 4$, by the Sobolev imbedding theorem, we have $u^* \in C^{4/(p+1)}(\bar{\mathbb{B}})$. If $1 < 4/(p+1) < 2$, then $u^*(0) = 1$, $Du^*(0) = 0$ and

$$|Du(\varepsilon) - Du(0)| \leqslant M|\varepsilon|^{4/(p+1)-1} \leqslant M|x|^{4/(p+1)-1},$$

where $0 < |\varepsilon| < |x|$. Thus,

$$|u(x) - u(0)| \leqslant |Du(\varepsilon)||x| \leqslant M|x|^{4/(p+1)}.$$

If $4/(p+1) \leqslant 1$, then u^* is Hölder continuous and

$$1 - u^*(x) \leqslant M|x|^{4/(p+1)},$$

and we obtain a contradiction as above. □

Now we give the point estimate of singular extremal solution for dimensions $n \geqslant 5$.

Theorem 4.1. *Let $n \geqslant 5$ and (u^*, λ^*) be the extremal pair of (1.1)$_\lambda$, when u^* is singular. Then*

$$1 - u^* \leqslant C_0|x|^{4/(p+1)},$$

where $C_0 := (\lambda^*/K_0)^{1/(p+1)}$ and

$$K_0 := \frac{8(p-1)}{(p+1)^2}\left[n - \frac{2(p-1)}{p+1}\right]\left[n - \frac{4p}{p+1}\right].$$

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In order to prove theorem 4.1, we need the lower bounds of λ^*, which we state as follows.

Lemma 4.2. *λ^* satisfies the following lower bounds for $n \geqslant 4$:*

$$\lambda^* \geqslant K_0,$$

where

$$K_0 := \frac{8(p-1)}{(p+1)^2}\left[n - \frac{2(p-1)}{p+1}\right]\left[n - \frac{4p}{p+1}\right].$$

Proof. The proof is standard, but we include it for the sake of completeness. Note that for $n \geqslant 4$ the function $\bar{u} = 1 - |x|^{4/(p+1)}$ satisfies

$$\frac{1}{(1-\bar{u})^p} \in L^1(\mathbb{B})$$

and $\bar{u}$ is a weak solution of

$$\Delta^2 \bar{u} = \frac{K_0}{(1-\bar{u})^p},$$

and

$$\bar{u}(1) = 0 = u_\lambda(1); \qquad \frac{\partial u_\lambda}{\partial n}(1) \geqslant \frac{\partial \bar{u}_\lambda}{\partial n}(1).$$

Therefore, $\bar{u}$ turns out to be a weak supersolution of $(1.1)_\lambda$ provided $\lambda \leqslant K_0$. Thus, necessarily, we have

$$\lambda^* = \lambda_* \geqslant K_0.$$

The proof is complete. □

Proof of theorem 4.1. First note that lemma 4.2 gives the lower bound

$$\lambda^* \geqslant K_0.$$

For $\delta > 0$, we define $u_\delta(x) := 1 - C_\delta |x|^{4/(p+1)}$ with

$$C_\delta := \left(\frac{\lambda^*}{K_0} + \delta\right)^{1/(p+1)} > 1.$$

Since $n \geqslant 5$, we have that

$$u_\delta \in H^2_{\mathrm{loc}}(\mathbb{R}^n), \qquad \frac{1}{1-u_\delta} \in L^3_{\mathrm{loc}}(\mathbb{R}^n)$$

and u_δ is an H^2-weak solution of

$$\Delta^2 u_\delta = \frac{\lambda^* + \delta K_0}{(1-u_\delta)^p} \quad \text{in } \mathbb{R}^n.$$

We claim that $u_\delta \leqslant u^*$ in $\mathbb{B}$, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set

$$\Gamma := \{r \in (0,1) : u_\delta(r) > u^*(r)\}$$

is non-empty, and let $r_1 = \sup \Gamma$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$ and we can infer that

$$\alpha := u^*(r_1) = u_\delta(r_1), \qquad \beta = (u^*)'(r_1) \geqslant u'_\delta(r_1).$$

Setting $u_{\delta,r_1}(r) = r_1^{-4/(p+1)}(u_\delta(r_1 r) - 1) + 1$, we easily see that the function $u_{\delta,r_1}(r)$ is an $H^2(\mathbb{B})$-weak supersolution of $(2.1)_{\lambda^*+\delta K_0,\alpha',\beta'}$, where

$$\alpha' := r_1^{-4/(p+1)}(u^*(r_1 r) - 1) + 1, \qquad \beta' := r_1^{(p-3)/(p+1)}\beta.$$

Similarly, defining $u^*_{r_1} = r_1^{-4/(p+1)}(u^*(r_1 r) - 1) + 1$, we have that $u^*_{r_1}$ is singular semi-stable $H^2(\mathbb{B})$-weak solution of $(2.1)_{\lambda^*,\alpha',\beta'}$.

Now we claim that (α', β') is an admissible pair. Since u^* is radially decreasing, we have that $\beta' \leqslant 0$. Define the function

$$\omega(r) := (\alpha' - \tfrac{1}{2}\beta') + \tfrac{1}{2}\beta'|x|^2 + \gamma(x),$$

where $\gamma(x)$ is a solution of $\Delta^2\gamma = \lambda^*$ in $\mathbb{B}$ with $\gamma = \partial_v\gamma = 0$ on $\partial\mathbb{B}$. Then ω is a classical solution of

$$\begin{aligned} \Delta^2\omega &= \lambda^* \quad \text{in } \mathbb{B}, \\ \omega = \alpha', \quad \partial_v\omega &= \beta' \quad \text{on } \partial\mathbb{B}. \end{aligned}$$

Since

$$\frac{\lambda^*}{(1 - u^*_{r_1})^p} \geqslant \lambda^*,$$

by lemma 2.1 we have

$$u^*_{r_1} \geqslant \omega \quad \text{a.e. in } \mathbb{B}.$$

Since $\omega(0) = \alpha' - \frac{1}{2}\beta' + \gamma(0)$ and $\gamma(0) > 0$, we have

$$\alpha' - \tfrac{1}{2}\beta' < 1.$$

So (α', β') is an admissible pair and by theorem 3.2(iv) we get that $(u^*_{r_1}, \lambda^*)$ coincides with the extremal pair of $(2.1)_{\lambda,\alpha',\beta'}$ in $\mathbb{B}$.

Since (α', β') is an admissible pair and u_{δ,r_1} is an $H^2(\mathbb{B})$-weak supersolution of $(2.1)_{\lambda^*+\delta K_0,\alpha',\beta'}$. We obtain from proposition 2.7 the existence of a weak solution of $(2.1)_{\lambda^*+\delta K_0,\alpha',\beta'}$. Since

$$\lambda^* + \delta K_0 > \lambda^*,$$

we contradict the fact that λ^* is the extremal parameter of $(2.1)_{\lambda,\alpha',\beta'}$. □

Due to the lower estimate on u^*, we obtain the following result.

Corollary 4.3. *In any dimension $n \geqslant 1$, we have*

$$\lambda^* > K_0 = \frac{8(p-1)}{(p+1)^2}\left[n - \frac{2(p-1)}{p+1}\right]\left[n - \frac{4p}{p+1}\right].$$

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Proof. $\bar{u} := 1 - |x|^{4/(p+1)}$ is an $H^2(\mathbb{B})$-weak solution of $(2.1)_{K_0,0,-4/(p+1)}$. If by contradiction $\lambda^* = K_0$, then $\bar{u}$ is an $H^2(\mathbb{B})$-weak supersolution of $(1.1)_\lambda$ for every $\lambda \in (0, \lambda^*)$. By lemma 3.3 we obtain that $u_\lambda \leqslant \bar{u}$ for all $\lambda < \lambda^*$, and then $u^* \leqslant \bar{u}$ a.e. in $\mathbb{B}$.

If $1 \leqslant n \leqslant 4$, u^* is then regular by theorem 1.3(i). By theorem 3.2(iii) it holds that $\mu_1(u^*) = 0$. Lemma 3.4 then yields that $u^* = \bar{u}$, which is a contradiction, since then u^* will not satisfy the boundary conditions.

If now $n \geqslant 5$ and $\lambda^* = K_0$, then $C_0 = 1$ in theorem 4.1, and we have $u^* \geqslant \bar{u}$. This again means that $u^* = \bar{u}$: a contradiction that completes the proof. □

In what follows, we shall show that the extremal solution u^* of $(1.1)_\lambda$ in dimensions $n \geqslant 13$ is singular.

5. The extremal solution is singular for $n \geqslant 13$

We prove in this section that the extremal solution is singular for $n \geqslant 13$ and p large enough. For this we shall need a suitable Hardy–Rellich-type inequality such as that established in [8]. As in the previous section, (u^*, λ^*) denotes the extremal pair of $(2.1)_{\lambda^*,0,0}$.

Lemma 5.1. *Let $n \geqslant 5$ and let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$. Then there exists $C > 0$ such that the following improved Hardy–Rellich inequality holds for all $\varphi \in H_0^2(\mathbb{B})$:*

$$\int_{\mathbb{B}} (\Delta\varphi)^2 \, dx \geqslant \frac{n^2(n-4)^2}{16} \int_{\mathbb{B}} \frac{\varphi^2}{|x|^4} \, dx + C \int_{\mathbb{B}} \varphi^2 \, dx.$$

Lemma 5.2. *Let $n \geqslant 5$ and let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$. Then the following improved Hardy–Rellich inequality holds for all $\varphi \in H_0^2(\mathbb{B})$:*

$$\int_{\mathbb{B}} (\Delta\varphi)^2 \, dx \geqslant \frac{(n-2)^2(n-4)^2}{16} \int_{\mathbb{B}} \frac{\varphi^2 \, dx}{(|x|^2 - 0.9|x|^{n/2+1})(|x|^2 - |x|^{n/2})} + \frac{(n-1)(n-4)^2}{4} \int_{\mathbb{B}} \frac{\varphi^2 \, dx}{|x|^2(|x|^2 - |x|^{n/2})}. \tag{5.1}$$

As a consequence, the following improvement of the classical Hardy–Rellich inequality holds:

$$\int_{\mathbb{B}} (\Delta\varphi)^2 \, dx \geqslant \frac{n^2(n-4)^2}{16} \int_{\mathbb{B}} \frac{\varphi^2}{|x|^2(|x|^2 - |x|^{n/2})}.$$

Lemma 5.3. *If $n \geqslant 13$, then $u^* \leqslant 1 - |x|^{4/(p+1)}$.*

Proof. Recall from corollary 4.3 that $K_0 < \lambda^*$. Let $\bar{u} = 1 - |x|^{4/(p+1)}$. We now claim that $u_\lambda \leqslant \bar{u}$ for all $\lambda \in (K_0, \lambda^*)$. Indeed, fix such a λ and assume by contradiction that

$$R_1 := \inf\{0 \leqslant R \leqslant 1 : u_\lambda < \bar{u} \text{ in the interval } (R, 1)\} > 0.$$

From the boundary condition, one has that $u_\lambda < \bar{u}(r)$ as $r \to 1^-$. Hence, $0 < R_1 < 1, \alpha := u_\lambda(R_1) = \bar{u}(R_1)$ and $\beta := u'_\lambda(R_1) \leqslant \bar{u}'(R_1)$. As in the proof of theorem 4.1, we have $u_\lambda \geqslant \bar{u}$ in $\mathbb{B}_{R_1}$ and a contradiction arises in view of the fact that $\lim_{x \to 0} \bar{u}(x) = 1$ and $\|u_\lambda\|_\infty < 1$. It follows that $u_\lambda \leqslant \bar{u}$ in $\mathbb{B}$ for every $\lambda \in (K_0, \lambda^*)$ and in particular $u^* \leqslant \bar{u}$ in $\mathbb{B}$. □

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LEMMA 5.4. *Let $n \geqslant 13$. Suppose there exist $\lambda' > 0$ and a singular radial function $\omega(r) \in H^2(\mathbb{B})$ with $1/(1-\omega) \in L^\infty_{\mathrm{loc}}(\bar{\mathbb{B}} \setminus \{0\})$ such that*

$$\left.\begin{aligned} \Delta^2\omega &\leqslant \frac{\lambda'}{(1-\omega)^p} \quad \text{for } 0 < r < 1, \\ \omega(1) &= 0, \qquad \omega'(1) = 0, \end{aligned}\right\} \tag{5.2}$$

and

$$p\beta \int_{\mathbb{B}} \frac{\varphi^2}{(1-\omega)^{p+1}} \leqslant \int_{\mathbb{B}} (\Delta\varphi)^2 \quad \text{for all } \varphi \in H^2_0(\mathbb{B}). \tag{5.3}$$

(i) *If $\beta \geqslant \lambda'$, then $\lambda^* \leqslant \lambda'$.*

(ii) *If either $\beta > \lambda'$ or $\beta = \lambda' = H_n/p$, then the extremal solution u^* is necessarily singular.*

Proof.
(i) First, note that (5.3) and $1/(1-\omega) \in L^\infty_{\mathrm{loc}}(\bar{\mathbb{B}} \setminus \{0\})$ yield $1/(1-\omega) \in L^1(\mathbb{B})$.

At the same time, (5.2) implies that $\omega(r)$ is an $H^2(\mathbb{B})$-weak stable subsolution of $(1.1)_{\lambda'}$. If now $\lambda' < \lambda^*$, then by lemma 3.4 we have

$$\omega(r) < u_{\lambda'},$$

which ia a contradiction since ω is singular, while $u_{\lambda'}$ is regular.

(ii) Suppose first that $\beta = \lambda' = H_n/p$ and that $n \geqslant 13$. Since by (i) we have $\lambda^* \leqslant H_n/p$, we obtain from lemma 5.3 and the improved Hardy–Rellich inequality that there exists $C > 0$ such that, for all $\phi \in H^2_0(\mathbb{B})$,

$$\int_{\mathbb{B}} (\Delta\phi)^2 - p\lambda^* \int_{\mathbb{B}} \frac{\phi^2}{(1-u^*)^{p+1}} \geqslant \int_{\mathbb{B}} (\Delta\phi)^2 - H_n \int_{\mathbb{B}} \frac{\phi^2}{|x|^4} \geqslant C \int_{\mathbb{B}} \phi^2.$$

It follows that $\mu_1(u^*) > 0$ and u^* must therefore be singular, since otherwise one could use the implicit function theorem to continue the minimal branch beyond λ^*.

Suppose now that $\beta > \lambda'$, and let $\lambda'/\beta < \gamma < 1$ in such a way that

$$\alpha := \left(\frac{\gamma\lambda^*}{\lambda'}\right)^{1/(p+1)} < 1.$$

Setting $\bar{\omega} := 1 - \alpha(1-\omega)$, we claim that

$$u^* \leqslant \bar{\omega} \quad \text{in } \mathbb{B}. \tag{5.4}$$

Note that by the choice of α we have $\alpha^{p+1}\lambda' < \lambda^*$, and therefore to prove (5.4) it suffices to show that for $\alpha^{p+1}\lambda' \leqslant \lambda < \lambda^*$ we have $u_\lambda \leqslant \bar{\omega}$ in $\mathbb{B}$. Indeed, fix such a λ and note that

$$\Delta^2\bar{\omega} = \alpha\Delta^2\omega \leqslant \frac{\alpha\lambda'}{(1-\omega)^p} = \frac{\alpha^{p+1}\lambda'}{(1-\bar{\omega})^p} \leqslant \frac{\lambda}{(1-\bar{\omega})^p}.$$

Assume that $u_\lambda \leqslant \bar{\omega}$ dose not hold in $\mathbb{B}$, and consider

$$R_1 := \sup\{0 \leqslant R \leqslant 1 \mid u_\lambda(R) > \bar{\omega}(R)\} > 0$$

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Since $\bar{\omega}(1) = 1 - \alpha > 0 = u_\lambda(1)$, we then have

$$R_1 < 1, \quad u_\lambda(R_1) = \bar{\omega}(R_1) \quad \text{and} \quad u'_\lambda(R_1) \leqslant \bar{\omega}'(R_1).$$

Introduce, as in the proof of theorem 4.1, the functions u_{λ,R_1} and $\bar{\omega}_{R_1}$. We have that u_{λ,R_1} is a classical solution of $(2.1)_{\lambda,\alpha',\beta'}$, where

$$\alpha' := R_1^{-4/(p+1)}(u_\lambda(R_1) - 1) + 1, \qquad \beta' := R_1^{(p-3)/(p+1)} u'_\lambda(R_1).$$

Since $\lambda < \lambda^*$ and then

$$\frac{p\lambda}{(1-\bar{\omega})^{p+1}} \leqslant \frac{p\lambda^*}{\alpha^{p+1}(1-\omega)^{p+1}} < \frac{p\beta}{(1-\omega)^{p+1}},$$

by (5.3) $\bar{\omega}_{R_1}$ is a stable $H^2(\mathbb{B})$-weak subsolution of $(2.1)_{\lambda,\alpha',\beta'}$. By lemma 3.4, we deduce that $u_\lambda \geqslant \bar{\omega}$ in $\mathbb{B}_{R_1}$, which is impossible, since $\bar{\omega}$ is singular, while u_λ is regular. This establishes claim (5.4) which, combined with the above inequality, yields

$$\frac{p\lambda^*}{(1-u^*)^{p+1}} \leqslant \frac{p\lambda^*}{\alpha^{p+1}(1-\omega)^{p+1}} < \frac{p\beta}{(1-\omega)^{p}},$$

and thus

$$\inf_{\varphi \in C_0^\infty(\mathbb{B})} \int_{\mathbb{B}} \left[(\Delta\varphi)^2 - \frac{p\lambda^*\varphi^2}{(1-u^*)^{p+1}}\right] \mathrm{d}x \left(\int_{\mathbb{B}} \varphi^2 \,\mathrm{d}x\right)^{-1} > 0.$$

This is not possible if u^* is a smooth function, since otherwise one could use the implicit function theorem to continue the minimal branch beyond λ^*. □

Proof theorem 1.3(ii). Now we prove that u^* is a singular solution of $(1.1)_{\lambda^*}$ for $n \geqslant 13$. In order to achieve this, we shall find a singular $H^2(\mathbb{B})$-weak subsolution of $(1.1)_{\lambda'}$, denoted by $\omega_m(r)$, which is stable, according to lemma 5.4.

Choosing

$$\omega_m(r) = 1 - a_1 r^{4/(p+1)} + a_2 r^m, \quad K_0 = \frac{8(p-1)}{(p+1)^2}\left[n - \frac{2(p-1)}{p+1}\right]\left[n - \frac{4p}{p+1}\right],$$

since $\omega(1) = \omega'(1) = 0$, we have

$$a_1 = \frac{m}{m - 4/(p+1)}, \qquad a_2 = \frac{4/(p+1)}{m - 4/(p+1)}.$$

For any m fixed, when $p \to \infty$, we have

$$a_1 = 1 + \frac{4}{(p+1)m} + o(p^{-1}) \quad \text{and} \quad a_2 = a_1 - 1 = \frac{4}{(p+1)m} + o(p^{-1})$$

and

$$K_0 = \frac{8(n-2)(n-4)}{p} + o(p^{-1}).$$

Note that

$$
\begin{aligned}
&\frac{\lambda' K_0}{(1-\omega_m(r))^p} - \Delta^2\omega_m(r)\\
&\quad = \frac{\lambda' K_0}{(1-\omega_m(r))^p} - a_1K_0r^{-4p/(p+1)} - K_1r^{m-4}\\
&\quad = \frac{\lambda' K_0}{(a_1r^{4/(p+1)} - a_2r^m)^p} - a_1K_0r^{-4p/(p+1)} - a_2K_1r^{m-4}\\
&\quad = K_0r^{-4p/(p+1)}\biggl[\frac{\lambda'}{(a_1 - a_2r^{m-4/(p+1)})^p} - a_1 - a_2K_1K_0^{-1}r^{4p/(p+1)+m-4}\biggr]\\
&\quad = K_0r^{-4p/(p+1)}\biggl[\frac{\lambda'}{(a_1 - a_2r^{m-4/(p+1)})^p} - a_1 - a_2K_1K_0^{-1}r^{m-4/(p+1)}\biggr]\\
&\quad = \frac{K_0r^{-4p/(p+1)}}{(a_1 - a_2r^{m-4/(p+1)})^p}[\lambda' - H(r^{m-4/(p+1)})] \qquad (5.5)
\end{aligned}
$$

with

$$
\left.\begin{aligned}
H(x) &= (a_1 + a_2x)^p[a_1 + a_2K_1K_0^{-1}x],\\
K_1 &= m(m-2)(m+n-2)(m+n-4).
\end{aligned}\right\} \qquad (5.6)
$$

(i) Let $m = 2$, $n \geqslant 32$. Then we can prove that

$$\sup_{[0,1]} H(x) = H(0) = a_1^{p+1} \to \mathrm{e}^2 \quad \text{as } p \to +\infty.$$

So (5.5) $\geqslant 0$ is valid as long as

$$\lambda' = \mathrm{e}^2.$$

At the same time, we have (since $a_1 - a_2r^{2-4/(p+1)} \geqslant a_1 - a_2 \geqslant 1$ in $[0,1]$)

$$\frac{n^2(n-4)^2}{16}\frac{1}{r^4} - \frac{p\beta}{r^4(a_1 - a_2r^{2-4/(p+1)})^{p+1}} \geqslant r^{-4}\biggl[\frac{n^2(n-4)^2}{16} - p\beta\biggr]. \qquad (5.7)$$

Let $\beta = (\lambda' + \varepsilon)K_0$, where ε is arbitrarily sufficiently small. Finally, we need here

$$\frac{n^2(n-4)^2}{16} - p\beta = \frac{n^2(n-4)^2}{16} - p(\lambda' + \varepsilon)K_0 > 0.$$

To show this, it is sufficient to have, for $p \to +\infty$,

$$\frac{n^2(n-4)^2}{16} - 8(\mathrm{e}^2 + \varepsilon)(n-2)(n-4) + o\biggl(\frac{1}{p}\biggr) > 0.$$

So (5.7) $\geqslant 0$ holds only for $n \geqslant 32$ when $p \to +\infty$. Moreover, for p large enough

$$8\mathrm{e}^2(n-2)(n-4)\int_{\mathbb{B}} \frac{\varphi^2}{(1-\omega_2)^{p+1}} \leqslant H_n\int_{\mathbb{B}}\frac{\varphi^2}{|x|^4} \leqslant \int_{\mathbb{B}}|\Delta\varphi|^2.$$

Thus, it follows from lemma 5.4 that u^* is singular with $\lambda' = \mathrm{e}^2K_0$, $\beta = (\mathrm{e}^2K_0 + \varepsilon(n,p))$ and $\lambda^* \leqslant \mathrm{e}^2K_0$.

Table 1.

n	λ'	β
31	$3.15K_0$	$4.00K_0$
30–19	$4.00K_0$	$10.00K_0$
18	$3.19K_0$	$3.22K_0$
17	$3.15K_0$	$3.18K_0$
16	$3.13K_0$	$3.14K_0$
15	$2.76K_0$	$3.12K_0$
14	$2.34K_0$	$2.96K_0$
13	$2.03K_0$	$2.15K_0$

(ii) Assume $13 \leqslant n \leqslant 31$. We shall show that $u = \omega_{3.5}$ satisfies the assumptions of lemma 5.4 for each dimension $13 \leqslant n \leqslant 31$. Using Maple, for each dimension $13 \leqslant n \leqslant 31$ one can verify that inequality (5.5) $\geqslant 0$ holds for the λ' given in table 1. Then, by using Maple again, we show that there exists $\beta > \lambda'$ such that

$$\frac{(n-2)^2(n-4)^2}{16}\frac{1}{(|x|^2-0.9|x|^{n/2+1})(|x|^2-|x|^{n/2})} + \frac{(n-1)(n-4)^2}{4}\frac{1}{|x|^2(|x|^2-|x|^{n/2})} \geqslant \frac{p\beta}{(1-w_{3.5})^{p+1}}.$$

The above inequality and improved Hardy–Rellich inequality (5.1) guarantee that the stability condition (5.3) holds for $\beta > \lambda'$. Hence, by lemma 5.4 the extremal solution is singular for $13 \leqslant n \leqslant 31$ the value of λ' and β are shown in table 1. □

Remark 5.5. The values of λ' and β in table 1 are not optimal.

Remark 5.6. The improved Hardy–Rellich inequality (5.1) is crucial to prove that u^* is singular in dimensions $n \geqslant 13$. Indeed by the classical Hardy–Rellich inequality and $u := w_2$, lemma 5.4 only implies that u^* is singular for dimensions $n \geqslant 32$.

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References

1 G. Arioli, F. Gazzola, H.-C. Grunau and E. Mitidieri. A semilinear fourth-order elliptic problem with exponential nonlinearity. *SIAM J. Math. Analysis* **36** (2005), 1226–1258.

2 E. Berchio, F. Gazzola and T. Weth. Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems. *J. Reine Angew. Math.* **620** (2008), 165–183.

3 T. Boggio. Sulle funzioni di Green d'ordine m. *Rend. Circ. Mat. Palermo* **20** (1905), 97–135.

4 H. Brézis, J. L. Vázquez. Blow up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Compl. Madrid* **10** (1997), 443–469.

5 D. Cassani, J. do Ó and N. Ghoussoub. On a fourth-order elliptic problem with a singular nonlinearity. *Adv. Nonlin. Studies* **9** (2007), 177–197.

6 C. Cowan, P. Esposito, N. Ghoussoub and A. Moradifam. The critical dimension for a fourth-order elliptic problem with singular nonlinearity. *Arch. Ration. Mech. Analysis* **198** (2010), 763–787.

7 J. Dávila, I. Flores and I. Guerra. Multiplicity of solutions for a fourth-order equation with power-type nonlinearity. *Math. Annalen* **348** (2009), 143–193.

8 N. Ghoussoub and A. Moradifam. Bessel pairs and optimal Hardy and Hardy–Rellich inequalities. Preprint, arXiv:0709.1954.

9 Z. M. Guo and J. C. Wei. Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity. *Manuscr. Math.* **120** (2006), 193–209.

10 Z. Guo and J. Wei. On a fourth-order nonlinear elliptic equation with negative exponent. *SIAM J. Math. Analysis* **40** (2009), 2034–2054.

11 F. Lin and Y. Yang. Nonlinear non-local elliptic equation modelling electrostatic actuation. *Proc. R. Soc. Lond.* A **463** (2007), 1323–1337.

12 A. Moradifam. On the critical dimension of a fourth-order elliptic problem with negative exponent. *J. Diff. Eqns* **248** (2010), 594–616.

13 J. A. Pelesko and A. A. Bernstein. *Modeling MEMS and NEMS* (Boca Raton, FL: Chapman & Hall/CRC Press, 2002).

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