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An example of a planar Anosov diffeomorphism without fixed points

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Abstract. We construct an example of a fixed point free Anosov diffeomorphism of the plane, which is not topologically conjugate to a translation.

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1. *Introduction*

In [**3**], White showed that a translation of the plane is an Anosov diffeomorphism in the sense of Definition 1.1. Mendes [**2**] studied properties of Anosov diffeomorphisms of the plane and conjectured that any planar fixed point free Anosov diffeomorphism is topologically conjugate to a translation. The purpose of this paper is to disprove this conjecture. First, let us recall the definition of Anosov diffeomorphisms of the plane.

Definition 1.1. A C^1 diffeomorphism F of the plane $\mathbb{R}^2$ is said to be an *Anosov diffeomorphism* if there is a continuous Riemannian metric m and there are two transversal continuous foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ by C^1-leaves with the following properties.

(1) The metric m is complete.

(2) The diffeomorphism F preserves the two foliations $\mathcal{F}^\sigma$, $\sigma = u, s$: i.e, maps each leaf of $\mathcal{F}^\sigma$ to a leaf of $\mathcal{F}^\sigma$.

(3) There are constants $C > 0$ and $\lambda > 0$ such that

$$\|DF^n(v)\|_m \geq C^{-1}e^{\lambda n}\|v\|_m \quad \text{for all } v \in T\mathcal{F}^u \text{ for all } n \in \mathbb{N} \tag{1.1}$$

and

$$\|DF^n(v)\|_m \leq Ce^{-\lambda n}\|v\|_m \quad \text{for all } v \in T\mathcal{F}^s \text{ for all } n \in \mathbb{N}. \tag{1.2}$$

Condition (1) is necessary in order to exclude trivial examples. Consider a linear diffeomorphism A defined by $A(x, y) = (2x, \frac{1}{2}y)$ and consider an A-invariant strip

$$C = \{(x, y) \in \mathbb{R}^2 \mid x > 0, 1 < xy < 2\}.$$

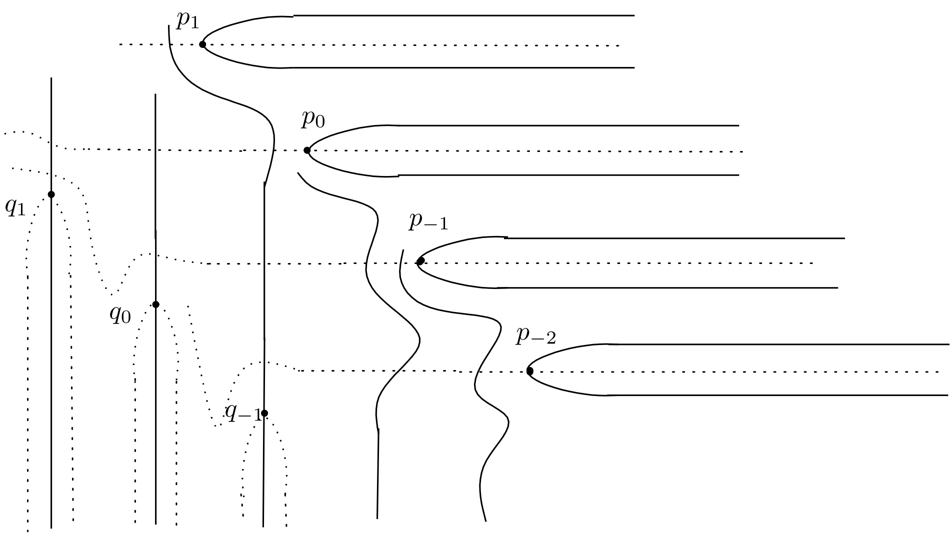

FIGURE 1. A schematic of the diffeomorphism of the Main Theorem.

Then $A|_C : C \to C$ satisfies conditions (2) and (3) with respect to the vertical and horizontal foliations and the metric m, which is the restriction of the Euclidean metric to C. However, m is not complete. The example in [**3**] is more involved.

MAIN THEOREM. *There is a fixed point free Anosov diffeomorphism which is not topologically conjugate to a translation.*

Groisman and Nitecki [**1**] proved the Mendes conjecture for a certain class of diffeomorphisms, i.e., the time-one maps of C^1-flows. In fact, they showed the following theorem.

THEOREM 1.2. *Let F be the time-one map of a fixed point free C^1 flow which is not topological conjugate to a translation. Assume that F preserves a continuous foliation $\mathcal{F}$ by C^1 leaves. Then some leaf L of $\mathcal{F}$ is left invariant by F.*

This quickly leads to the solution of the Mendes conjecture for this class of diffeomorphisms since, if $\mathcal{F} = \mathcal{F}^u$, F must have a fixed point in L by virtue of (1.1).

Therefore our first task for the proof of Main Theorem is to construct a C^1 diffeomorphism F and two mutually transverse foliations, say, $\mathcal{F}^u$ and $\mathcal{F}^s$, invariant by F but without invariant leaves. The schematic idea can be found in Figure 1. The solid lines indicate the foliation $\mathcal{F}^u$, while dotted lines indicate $\mathcal{F}^s$. The diffeomorphism F maps p_i to p_{i+1} and q_i to q_{i-1}. Detailed construction is described in §§2 and 3. It may be worth mentioning that there is no contradiction with the Brouwer plane fixed point theorem. Horizontal and vertical 'Reeb components' are displaced and, outside them, the diffeomorphism is conjugate to a translation of the plane. Thus all the points are wandering. Sections 4 and 5 are devoted to the definition of the metric.

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2. *Construction of the diffeomorphism*

Notation 2.1. Denote by $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ the translation by $(-1, 1)$. Let

$$\Delta = \{x + y = 0\} \subset \mathbb{R}^2.$$

Denote by σ the symmetry at $\Delta : \sigma(x, y) = (-y, -x)$.

Notice that $\sigma\tau = \tau\sigma$ and $\sigma^2 = \text{id}$. The C^1 diffeomorphism F that we are going to construct will satisfy the following two properties.

$$\tau F = F\tau, \tag{2.1}$$

$$F^{-1} = \sigma F \sigma. \tag{2.2}$$

Let

$$P = [-2, \infty) \times [0, 1] \quad \text{and} \quad P' = [0, \infty) \times [0, 1].$$

We shall define a surjective diffeomorphism $\phi : P \to P'$ of the form

$$\phi(x, y) = (h_y(x), g(y)), \tag{2.3}$$

where $g : [0, 1] \to [0, 1]$ is a diffeomorphism with the following properties. See Figure 2.

(A) g is the time-one map of a C^1 flow g^t of the interval $[0, 1]$.

(B) For any $t \neq 0$, $g^t(y) = y$ if and only if $y \in \{0, \frac{1}{4}\} \cup [\frac{1}{2}, 1]$.

(C) $g^t|_{[0, \frac{1}{2}]}$ is symmetric at $\frac{1}{4}$, that is,

$$g^t(\tfrac{1}{2} - y) = \tfrac{1}{2} - g^t(y) \quad \text{for all } y \in [0, \tfrac{1}{2}].$$

(D) $g(y) < y$ for $y \in (0, \frac{1}{4})$.

(E) g is C^1 tangent to the identity at $y = 0$.

(F) g is affine of slope e^λ on the interval $[\frac{1}{4} - \delta, \frac{1}{4} + \delta]$, where λ is some positive number and δ is some *small* positive number.

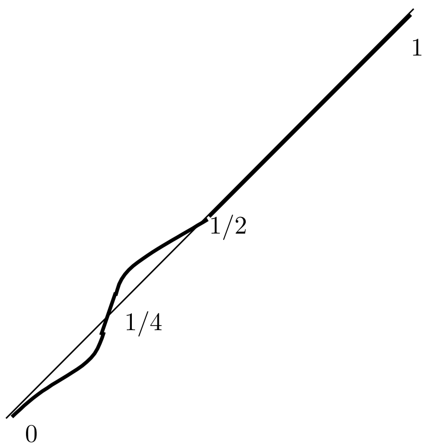

FIGURE 2. The graph of g.

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We also assume the following.

(G) ϕ sends the rectangle $[-2, -1] \times [0, 1]$ onto $[0, 1] \times [0, 1]$.

The positive number λ and a small positive number δ will appear in many places. One can show that there are such numbers which satisfy all the requirements we pose below.

Remark 2.2. Notice that the diffeomorphism ϕ of form (2.3) preserves the horizontal foliations, while it sends the vertical foliation to itself in the region where $h_y(x)$ does not depend on y.

Let $P_n = \tau^n(P)$ and $P'_n = \tau^n(P')$. (Thus $P = P_0$ and $P' = P'_0$.) We shall define a diffeomorphism $\phi_n : P_n \to P'_n$ by $\phi_n = \tau^n \circ \phi \circ \tau^{-n}$ and $\Phi : \bigcup_n P_n \to \bigcup_n P'_n$ as the union of the ϕ_n. In order that this defines a homeomorphism, we need the following condition.

$$h_1(x-1) = h_0(x) - 1 \quad \text{for all } x \geq 1. \tag{2.4}$$

Of course, for Φ to be a C^1 diffeomorphism, we need a bit more.

Define a map $F : \bigcup_n P_n \to \bigcup_n P'_n$ by

$$F = \tau \circ \Phi.$$

Consider a map $F' : \sigma(\bigcup_n P_n) \to \sigma(\bigcup_n P'_n)$ defined by the conjugation

$$F' = \sigma \circ F \circ \sigma.$$

The map F sends the rectangle $[-2, -1] \times [0, 1]$ to $[-1, 0] \times [1, 2]$, and, reciprocally, F' sends $[-1, 0] \times [1, 2]$ to $[-2, -1] \times [0, 1]$. Routine computation shows that the condition for F' to be the inverse of F on these rectangles is the following.

$$\phi(x, y) = (-g^{-1}(-x-1) + 1, g(y)) \quad \text{for all } (x, y) \in [-2, -1] \times [0, 1]. \tag{2.5}$$

With this condition, we can define a diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ by setting it to be equal to F on $\bigcup_n P_n$ and equal to $(F')^{-1}$ on $\bigcup_n \sigma(P'_n)$. Clearly,it satisfies (2.1) and (2.2).

In addition to (2.5), we assume further conditions on ϕ: on $[-1, 0] \times [0, \frac{1}{2}]$, it is the conjugate of $\phi|_{[-2,-1]\times[0,\frac{1}{2}]}$ by the translation by (1,0), that is,

$$\phi(x, y) = (-g^{-1}(-x) + 2, g(y)) \quad \text{on } [-1, 0] \times [0, \tfrac{1}{2}]. \tag{2.6}$$

This condition is helpful for making the assembled map Φ a C^1 diffeomorphism. Moreover, we assume the following.

$$\phi(x, y) = (x + 2, g(y)) \quad \text{on } C, \tag{2.7}$$

where C is the union of the subsets

$$[-1, \infty) \times [\tfrac{3}{4}, 1], \quad [0, \infty) \times [\tfrac{1}{2}, 1], \quad [0, \infty) \times \{0\} \quad \text{and} \quad [0, 1] \times [0, 1].$$

See Figure 3.

The map ϕ is already determined on the boundary of $D = [-1, 0] \times [\frac{1}{2}, \frac{1}{4}]$. On D, ϕ is to be any extension of it of the form (2.3). Notice that the map ϕ defined by (2.5), (2.6) and (2.7) satisfies condition (2.4).

The foliation $\mathcal{F}^u$ is defined to be the image by the iterates of F of the vertical foliation on $\bigcup_n \sigma(P_n)$. Conversely, $\mathcal{F}^s$ is to be the image by the iterates of F^{-1} of the horizontal foliation on $\bigcup_n P_n$. More concretely, on P, $\mathcal{F}^s$ is the horizontal foliation, while $\mathcal{F}^u$ is the image by the iterates of ϕ of the vertical foliation on $[-2, 0] \times [0, 1]$. Since the product map sends the vertical foliation to the vertical foliation, we have the following lemma.

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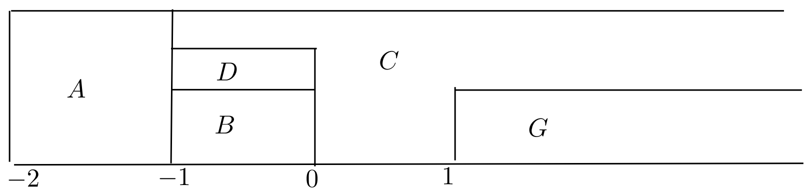

FIGURE 3. Rectangles A and B are mapped by ϕ onto the rectangles two units to the right to them by a product map (2.5) and (2.6). On the other hand, $\phi(x, y) = (x + 2, g(y))$ on C.

LEMMA 2.3. *The foliation $\mathcal{F}^u$ is vertical on $[-2, \infty) \times [\frac{3}{4}, 1]$ and also on $[-2, 3] \times [0, \frac{1}{2}]$.*

3. *More conditions on the map ϕ*

In this section, we shall define a map ϕ on the region G in Figure 3. To do this, we first define a Reeb component R of the foliation $\mathcal{F}^u$ in P, as in Figure 5. Let us define its boundary ∂R to be the graph of a function $\theta : [0, \frac{1}{2}] \to [4, \infty)$ symmetric at $y = \frac{1}{4}$. By the symmetry, we need to define θ only on $[0, \frac{1}{4}]$. Recall that the map $g : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ is the time-one map of the flow g^t. If we put $y(t) = g^t(1/8)$, it is monotone decreasing and satisfies $\lim_{t\to-\infty} y(t) = \frac{1}{4}$ and $\lim_{t\to\infty} y(t) = 0$. First, let us define a curve $x(t) \in [4, \infty)$, $t \in \mathbb{R}$, and then a function θ by $\theta(y(t)) = x(t)$. The conditions for $x(t)$ are the following.

(H) $x(t) = 4$ for $t < t_0$ for some $t_0 < 0$: equivalently, $\theta(y) = 4$ if y is δ-near to $\frac{1}{4}$, where $\delta = \frac{1}{4} - y(t_0) > 0$ is some small number.

(I) $x'(t) \in [0, 2)$ and $x'(t)$ is strictly monotone increasing for $t > t_0$ and

$$\lim_{t\to\infty} x'(t) = 2.$$

Thus $x(t)$ itself is monotone increasing. Moreover, we have $x(t + 1) < x(t) + 2$ and its difference tends to zero monotonically. We define the Reeb component R by

$$R = \{x \geq \theta(y), 0 < y < \tfrac{1}{2}\}.$$

We have

$$\partial R = \mathrm{Graph}(\theta).$$

Conditions (H) and (I) imply that ∂R is vertical on the region $|y - \frac{1}{4}| < \delta$ and is strictly convex leftward outside this region.

Next, we shall define the diffeomorphism

$$\phi : ([0, \infty) \times [0, \tfrac{1}{2}]) \setminus R \to ([2, \infty) \times [0, \tfrac{1}{2}]) \setminus R.$$

Again, ϕ is to be symmetric with respect to the line $y = \frac{1}{4}$, and we shall define it only on $([0, \infty) \times [0, \frac{1}{4}]) \setminus R$.

- On $\partial R \cap ([0, \infty) \times [0, \frac{1}{4}])$, ϕ is defined by $\phi(x(t), y(t)) = (x(t + 1), y(t + 1))$.
- On $([0, \infty) \times [0, \frac{1}{4}]) \setminus \mathrm{Int}\,(R)$, ϕ maps the interval $[0, x(t)] \times \{y(t)\}$ to the interval $[2, x(t + 1)] \times \{y(t + 1)\}$ by the formula

$$\phi(x, y(t)) = (h_{y(t)}(x), y(t + 1)),$$

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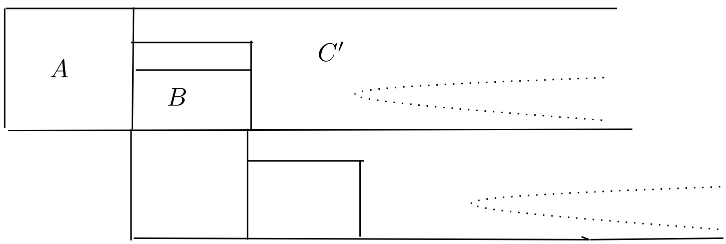

FIGURE 4. The dotted curve indicates $\{x = \theta(y)/4\}$. $\phi(x, y) = (x + 2, g(y))$ on the region C', which is outside the dotted curve.

where the diffeomorphism

$$h_{y(t)} : [0, x(t)] \to [2, x(t + 1)]$$

satisfies

$$h_{y(t)}(x) = x + 2 \quad \text{if } 0 \le x \le x(t)/4, \tag{3.1}$$

$$\Lambda = \{(x, y(t)) \mid h'_{y(t)}(x) = e^{-\lambda}\} \text{ is a neighbourhood of } \partial R \text{ in } P \setminus \mathrm{Int}(R). \tag{3.2}$$

Recall that $\lambda > 0$ is a constant which appeared in condition (F) on g.

$$h'_{y(t)}(x) \le 1 \quad \text{for all } x \in [0, x(t)]. \tag{3.3}$$

$$h_{y(t)} \quad \text{does not depend on } y(t) \text{ if } \tfrac{1}{4} - \delta \le g(y(t)) = y(t + 1) \le \tfrac{1}{4}. \tag{3.4}$$

The following lemma is a restatement of (3.1). See Figure 4.

LEMMA 3.1. *On the region* $\{0 < x < \theta(y)/4, 0 < y < \frac{1}{2}\}$, *we have*

$$\phi(x, y) = (x + 2, g(y)). \qquad \square$$

This lemma, together with the fact that $g'(0) = g'(1) = 1$, shows that the assembled map $\Phi : \bigcup_n P_n \to \bigcup_n P'_n$ is actually a C^1 diffeomorphism. Denote the Euclidean norm on $\mathbb{R}^2$ by $|\cdot|$.

COROLLARY 3.2. *The tangent bundle* $T\mathcal{F}^u$ *of the foliation* $\mathcal{F}^u$ *is vertical in a neighbourhood of* $[0, \infty) \times \{0, 1\}$, *and if* $v \in T\mathcal{F}^u_p$, $p \in [0, \infty) \times \{0, 1\}$, *then*

$$|D\phi(v)| = |v|.$$

Proof. The first assertion follows from Lemmas 2.3 and 3.1, while the last follows from $g'(0) = g'(1) = 1$. $\square$

So far, we have defined the diffeomorphism ϕ, and hence the foliation $\mathcal{F}^u$, everywhere apart from the interior of the Reeb component R. On $R = \{x \ge \theta(y), 0 < y < \frac{1}{2}\}$, define the foliation $\mathcal{F}^u$ by the horizontal translation of the boundary curve ∂R (see Figure 5). Let $L = [4, \infty) \times \{\frac{1}{4}\}$ be the centre ray of R. The two transverse foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ define a product structure on R: that is,

$$R \approx \partial R \times L.$$

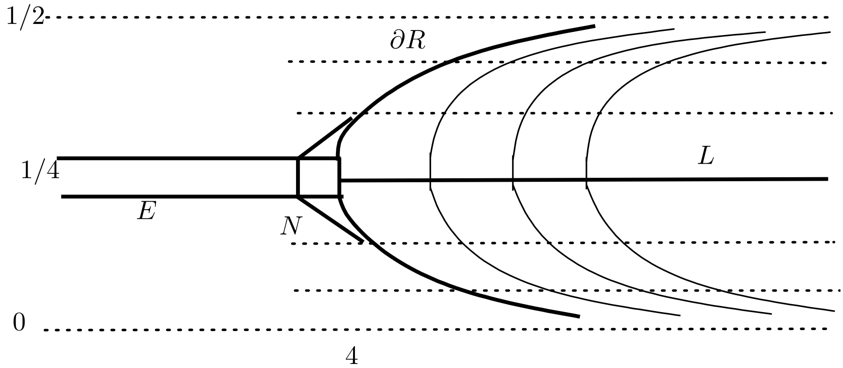

FIGURE 5. 1/2, 1/4 and 0 denote the y-coordinate, while 4 is the x-coordinate.

We have already defined the map ϕ on ∂R. Let us define it on L to be the contraction of ratio $e^{-\lambda}$ centred at $(\frac{1}{4}, 4)$. Finally, define the map $\phi : R \to R$ as the product of these two maps. By virtue of (3.2), $\phi : P \to P'$ is a C^1 diffeomorphism. Recall that it has the form $\phi(x, y) = (h_y(x), g(y))$.

LEMMA 3.3.
(1) $h'_y(x) \in (0, 1]$ *for any* $(x, y) \in P'$.
(2) *There is a neighbourhood* N *of the point* $(4, \frac{1}{4})$ *such that if* $(x, y) \in N \cup R$, *then* $h'_y(x) = e^{-\lambda}$.
(3) *Moreover, one can choose* N *of (2) large enough so that if* $(x, y) \in P \setminus (N \cup R)$, *then* $h_y(x) - x > \alpha$ *for some fixed* $\alpha > 0$.

Proof. (1) follows from (3.3) and the construction on R. For (2), one can choose N to be any neighbourhood of $(4, \frac{1}{4})$ in $\Lambda \cup R$, where Λ is a set given by (3.2). Let us show (3). Conditions (H) and (I) imply that $x(t)$ is strictly monotone increasing if $x(t) > 4$. This, together with (3.1) and (3.3), shows that the set

$$K = \{(x, y) \in P \setminus \mathrm{Int}\,(R) \mid h_y(x) \le x\}$$

coincides with a compact interval

$$I = \{(x(t), y(t)) \mid x(t+1) = x(t) = 4\}$$

of ∂R. One can choose a neighbourhood N of I contained in the set Λ and set

$$\alpha = \min\{h_y(x) - x \mid (x, y) \in P \setminus (N \cup R)\}. \qquad \square$$

To restate Lemma 3.3, we get the following corollary.

COROLLARY 3.4. *The diffeomorphism* ϕ *is* 1*-contracting along* $T\mathcal{F}^s$ *on* P': *that is, if* $v \in T_p\mathcal{F}^s$, $p \in P'$, *then* $|D\phi(v)| \le |v|$. *If, furthermore,* $p \in N \cup R$, *then* $|D\phi(v)| = e^{-\lambda}|v|$.

The strip

$$E' = \{0 \le x \le 4, |y - \tfrac{1}{4}| < e^{-\lambda}\delta\}$$

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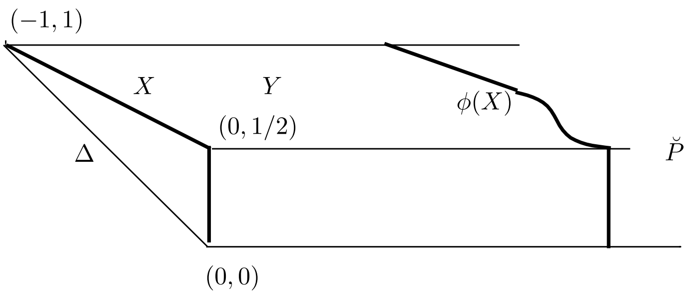

FIGURE 6. The graph of $\check{P}$.

is mapped to the strip

$$E = \{2 \leq x \leq 4,\ |y - \tfrac{1}{4}| < \delta\}$$

by a product map by virtue of (3.4). Together with Lemma 2.3, we have the following lemma.

LEMMA 3.5. *The foliation $\mathcal{F}^u$ is vertical on the strip E.*

We also have the following lemma by virtue of condition (F).

LEMMA 3.6. *If v is a vertical vector at a point on E', then $|D\phi(v)| = e^{\lambda}|v|$.*

4. *Expanding norm on $T\mathcal{F}^u$*

Let

$$\check{P} = ([0, \infty) \times [0, \tfrac{1}{2}]) \cup \{x \geq 1 - 2y,\ \tfrac{1}{2} \leq y \leq 1\}.$$

See Figure 6. We shall define a metric $\|\cdot\|$ of $T\mathcal{F}^u|_{\check{P}}$ which is e^{λ}-expanding by ϕ in the sense that $\|D\phi(v)\| \geq e^{\lambda}\|v\|$ for all $v \in T\mathcal{F}^u|_{\check{P}}$. The overall strategy is as follows. Suppose $\|\cdot\|_{\phi^{-1}(p)}$ is given. For any $v \in T\mathcal{F}^u_p$, we shall define $\|v\|_p$ by

$$\|v\|_p = \begin{cases} e^{\lambda}\|D\phi^{-1}(v)\|_{\phi^{-1}(p)} & \text{if } \dfrac{|v|}{|D\phi^{-1}(v)|} \leq e^{\lambda}, \\ \frac{|v|}{|D\phi^{-1}(v)|} \cdot \|D\phi^{-1}(v)\|_{\phi^{-1}(p)} & \text{if } \dfrac{|v|}{|D\phi^{-1}(v)|} \geq e^{\lambda}, \end{cases} \tag{4.1}$$

where $|\cdot|$ denotes the Euclidean norm. Let

$$X = (\{0\} \times [0, \tfrac{1}{2}]) \cup \{x = 1 - 2y,\ \tfrac{1}{2} \leq y \leq 1\}.$$

We put the Euclidean norm on X and apply the above strategy to get a norm $\|\cdot\|$ on $\phi(X)$. We then interpolate in the region Y bounded by X and $\phi(X)$ the two norms monotonically along the $\mathcal{F}^s$-leaves. We apply the same strategy to $\phi(Y)$, and then to $\phi^2(Y)$ and so on. Thus we obtain a norm on $\check{P} \setminus R$. But, in fact, we can get a bit more. As we remarked in Lemma 3.6, the map $\phi : E' \to E$ is already e^{λ}-expanding along $T\mathcal{F}^u$ with respect to the Euclidean norm. Therefore the norm we obtained on E is nothing but the Euclidean norm. Thus it extends continuously to $\partial R \cap E$, and one can apply the same strategy as in (4.1) including this set. This way, we obtain a continuous norm on the closed set $\check{P} \setminus \mathrm{Int}(R)$

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which is e^{λ}-expanding. Next, we shall extend the norm to R. Recall that the $\mathcal{F}^u$-leaves in R are the horizontal translates of ∂R. Define the norm on each leaf simply as the translate of the norm $\|\cdot\|$ on ∂R. By the product structure of R,

$$R \approx \partial R \times L,$$

given by $\mathcal{F}^u$ and $\mathcal{F}^s$, this norm on R is also e^{λ}-expanding by ϕ.

By Corollary 3.2, the norm we obtained on the upper boundary of $\check{P}$ is the image by τ of the norm on the lower boundary, as long as the interpolation in Y is chosen to be τ-invariant on the horizontal boundaries. Therefore, by distributing the norm by the iterates of τ, we get a continuous norm on $\bigcup_n \tau^n(\check{P})$ which is e^{λ}-expanding by Φ and therefore by F. Let

$$\mathbb{P} = \{x + y \geq 0\}.$$

Extend the norm $\|\cdot\|$ of $T\mathcal{F}^u$ from $\bigcup_n \tau^n(\check{P})$ to $\mathbb{P}$, just setting it to be the Euclidean norm on the difference set. Summarizing the content of this section, we get the following lemma.

LEMMA 4.1. *There is a continuous norm $\|\cdot\|$ on $T_{\mathbb{P}}\mathcal{F}^u$ with the following properties.*
- $\|v\| \geq |v|$ *for any* $v \in T_{\mathbb{P}}\mathcal{F}^u$.
- $\|v\| = |v|$ *for any* $v \in T_{\Delta}\mathcal{F}^u$.
- $\|DF(v)\| \geq e^{\lambda}\|v\|$ *for any* $v \in T_p\mathcal{F}^u$, $p \in \bigcup_n \tau^n(\check{P})$.

5. *Final step*

We shall construct norms $|||\cdot|||$ along $T\mathcal{F}^s$ and $T\mathcal{F}^u$ on $\mathbb{P}$ for which F is hyperbolic, i.e., conditions (1.1) and (1.2) are satisfied. Recall that $T_p\mathcal{F}^s$ is a horizontal line and $T_p\mathcal{F}^u$ is a vertical line for $p \in \Delta$ and that the differential $D\sigma$ of the involution σ maps $T_p\mathcal{F}^s$ onto $T_p\mathcal{F}^u$. We shall construct $|||\cdot|||$ in such a way that

$$|||D\sigma(v)||| = |||v||| \quad \text{for all } v \in T_p\mathcal{F}^s \text{ for all } p \in \Delta. \tag{5.1}$$

Recall that $F^{-1} = \sigma F \sigma$, and we have $\sigma\mathcal{F}^u = \mathcal{F}^s$ and $\sigma\mathcal{F}^s = \mathcal{F}^u$. After we have constructed the norms on $\mathbb{P}$, norms on $\sigma(\mathbb{P})$ will be given as the $D\sigma$-images: that is,

$$|||v|||_p = |||D\sigma(v)|||_{\sigma(p)} \quad \text{for all } v \in T_p\mathcal{F}^s \cup T_p\mathcal{F}^u,\ p \in \sigma(\mathbb{P}).$$

Let $U = \{|x + y| < 1\}$, a partial fundamental domain of F. We shall estimate the ratio $|||DF(v)|||/|||v|||$, $v \in T_p\mathcal{F}^u \cup T_p\mathcal{F}^s$, only when both p and $F(p)$ are above U or below U. By the construction of $|||\cdot|||$ which follows, this ratio is bounded when one of p or $F(p)$ is contained in U. To get the hyperbolicity, it is not a problem to skip one or two steps: conditions (1.1) and (1.2) are asymptotic in nature. Also, hyperbolicity for the region below U follows from the hyperbolicity above U by the symmetry.

Construction of $|||\cdot|||$ for $\mathbb{P} \cap \{y < 0\}$ is given in (I), and for $\mathbb{P} \cap \{y > 0\}$ it is in (II). In (III), we shall show that the norms constructed yield a compete Riemannian metric. Let ϵ be a positive number that is small compared with λ.

(I) *Construction for* $\mathbb{P}_- = \mathbb{P} \cap \{y < 0\}$. For $v \in T_{(x,y)}\mathcal{F}^s$, we let $|||v||| = e^{-\epsilon y}|v|$. By Corollary 3.4, Φ is 1-contracting along $T\mathcal{F}^s$ with respect to the Euclidean metric and τ is $e^{-\epsilon}$-contracting on $F^{-1}(\mathbb{P}_-) \setminus U$ with respect to $|||\cdot|||$. Now it follows that F is

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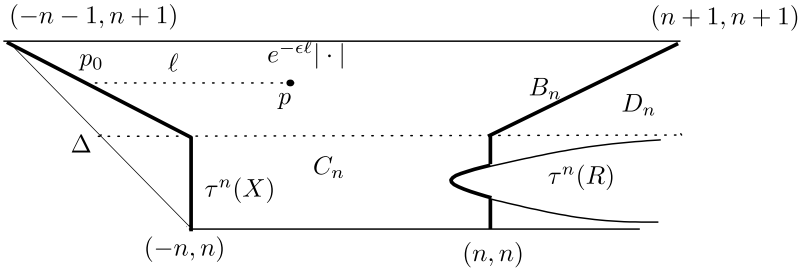

FIGURE 7. B_n is a curve composed of line segments and a boundary curve of $\tau^n(R)$. When $n = 0$, 1, 2, B_n consists of just two line segments. D_n is the region to the right to B_n, and thus $\tau^n(R) \subset D_n$.

$e^{-\epsilon}$-contracting along $T\mathcal{F}^s$ on this set. For $v \in T\mathcal{F}^u$, define $|||v||| = e^{-\epsilon y}\|v\|$. Then F is clearly $e^{\lambda-\epsilon}$-expanding along $T\mathcal{F}^u$ on $F^{-1}(\mathbb{P}) \setminus U$.

Notice that the symmetry (5.1) is satisfied. We do not estimate the contraction/expansion ratio on $\mathbb{P} \cap \{|y| < 1\}$ for the same reason as explained before. This is enough for robust asymptotic estimates as in (1.1). Notice that $|||v||| \geq |v|$ for $v \in T_p\mathcal{F}^u \cup T_p\mathcal{F}^s$, $p \in \mathbb{P}_-$.

(II) *Construction for* $\mathbb{P}_+ = \mathbb{P} \cap \{y > 0\}$. If we did the same construction as in (I) for the whole $\mathbb{P}$, an upward $\mathcal{F}^u$-ray would have finite length, contrary to the completeness of the metric. So we need a different construction for $\mathbb{P}_+$.

As for $T\mathcal{F}^u$, we just put $|||\cdot||| = \|\cdot\|$. Then F is e^λ-expanding along $T\mathcal{F}^u$ on $\mathbb{P}_+ \setminus U$.

To define $|||\cdot|||$ on $T\mathcal{F}^s$, consider an arbitrary point p from the region $C_n \subset \tau^n(\check{P})$ in Figure 7. The point p lies on a horizontal line segment which starts at a point $p_0 \in \tau^n(X)$. Let ℓ be the distance between p and p_0. Define

$$|||v||| = e^{-\epsilon\ell}|v| \quad \text{for all } v \in T_p\mathcal{F}^s.$$

Next, for a point q from the region D_n, let $q_0 \in B_n$ be the point on the horizontal line passing through q. Define $|||\cdot|||$ on $T_q\mathcal{F}^s$ to be equal to that on $T_{q_0}\mathcal{F}^s$. Here we make a natural identification of the horizontal line field: $T\mathcal{F}^s_q \approx T\mathcal{F}^s_{q_0}$. Finally, on the subset $\mathbb{P}_+ \setminus \bigcup_{n\geq 0}\tau^n(\check{P})$ (consisting of small triangles), put $|||\cdot||| = |\cdot|$. Again, the symmetry (5.1) is satisfied.

Now let us show that F is $e^{-\alpha\epsilon}$-contracting along $T\mathcal{F}^s$ on

$$P''_n := [-n, \infty) \times [n, n+1],$$

where the constant $\alpha > 0$ is from Lemma 3.3. We assume that $\alpha < 1$. First, consider the case where p lies on the upper half of P''_n: $p \in [-n, \infty) \times [n + \frac{1}{2}, n+1]$. There Φ is the (2, 0)-translation and thus $F = \tau \circ \Phi$ is the (1, 1)-translation. (The upper half of P''_0 is contained in the region C of Figure 3.) If $p \in C_n$ (respectively, $p \in D_n$), $f(p) \in C_{n+1}$

(respectively, $f(p) \in D_{n+1}$). In both cases, we have

$$|||DF(v)||| = e^{-2\epsilon}|||v||| \quad \text{for all } v \in T_p\mathcal{F}^s,$$

as desired.

Next, consider the case where p lies in the lower half of P''_n but not in the Reeb components $\tau^n(R)$. Notice that, in this part, the boundaries of C_n are vertical and the norm $|||\cdot|||$ depends only on the x-coordinate. Thus, in the computation of contraction ratio, we only need to consider the function $x \mapsto h_y(x)$: we do not have to care about the variation of y-coordinate $y \mapsto g(y)$.

If $F(p) \notin C_{n+1}$, then F is $e^{-2\epsilon}$-contracting at p. So consider the case $F(p) \in C_{n+1}$. If p does not lie in $\tau^n(N)$, then F is $e^{-\alpha\epsilon}$-contracting by virtue of Lemma 3.3. If $p \in \tau^n(N)$, then F is $e^{-\lambda}$-contracting by virtue of Corollary 3.4 and the fact that Φ does not decrease the x-coordinate in N. This holds true regardless of whether $p \in C_n$ or not.

For the Reeb component $\tau^n(R)$, consider a horizontal ray r contained in $\tau^n(R)$ with initial point on $\tau^n(\partial R)$. The norm of $T_r\mathcal{F}^s$ is determined by the x-coordinate of the initial point. Now the x-coordinate of the initial point of $\Phi(r)$ is not less than the x-coordinate of the initial point q of r. This shows that F is $e^{-\lambda}$-contracting on $\tau^n(R)$ by Corollary 3.4.

It is clear that $|||v||| \geq |v|$ for $v \in T_p\mathcal{F}^u$, $p \in \mathbb{P}_+$. Our construction $|||\cdot|||$ on $T_p\mathcal{F}^s$ satisfies the following property, which turns out to be useful in (III).

- *For any $n \in \mathbb{N}$, there exists $c > 0$ such that $|||v||| \geq c|v|$ for*

$$v \in T_p\mathcal{F}^s, \quad p \in \mathbb{P}_+ \cup \{y < n\}.$$

(III) We have defined the norm $|||\cdot|||$ on $\mathbb{P}$. As we said earlier, we define the norm $|||\cdot|||$ on $\sigma(\mathbb{P})$ by transforming the former by $D\sigma$. Define a Riemannian metric m on $\mathbb{R}^2$ by using these norms and setting the two subspaces $T_p\mathcal{F}^u$ and $T_p\mathcal{F}^s$ to be orthogonal. We shall denote by $\|\cdot\|_m$ the norm of m. We have already shown that F satisfies the hyperbolicity conditions (1.1) and (1.2). We now need to show that m is complete. Given arbitrarily large $R > 0$, we shall show that the set $B(R)$ of points which are R-near to $(0, 0)$ with respect to m is bounded. First, given $n \in \mathbb{Z}$, consider the set

$$Q_n = \{n - \tfrac{1}{4} < y < n\} \cap \mathbb{P}.$$

By Lemma 2.3, the foliation $\mathcal{F}^u$ is vertical on Q_n and $\|v\|_m \geq |v|$ for any vertical vector v of Q_n. This shows that any path in $\mathbb{P}$ which crosses the strip Q_n must have m-length $\geq \frac{1}{4}$. The same is true for the strip $\sigma(Q_n)$ in the region $\sigma(\mathbb{P})$. Thus the set $B(R)$ must be contained in the region Y bounded by $Q_{-n} \cup \sigma(Q_{-n})$ and $Q_n \cup \sigma(Q_n)$ for some $n > 0$. But, in Y, there is $c > 0$ such that $\|v\|_m \geq c|v|$ for any tangent vector v on Y. In fact, if

$$v = v_1 + v_2 \quad \text{for all } v_1 \in T\mathcal{F}^s,\ v_2 \in T\mathcal{F}^u,$$

we have $\|v\|_m \geq \|v_i\|_m$ for each i since $T\mathcal{F}^u$ and $T\mathcal{F}^s$ are orthogonal. On the other hand, there is $c > 0$ depending on Y such that $\|v_i\|_m \geq 2c|v_i|$ for each i. Now, by the triangle inequality, there is i such that $|v_i| \geq |v|/2$. Then

$$\|v\|_m \geq \|v_i\|_m \geq 2c|v_i| \geq c|v|,$$

as desired. Now the set $B(R)$ must be contained in the Euclidean R/c-ball centred at $(0, 0)$. The proof of the completeness is now complete.

Final remark. The diffeomorphism F is not topologically conjugate to a translation, since the quotient space $\mathbb{R}^2/\langle F\rangle$ is not Hausdorff. To show this, notice that any small piece of the $\mathcal{F}^s$-leaf passing through a point p from the boundary of the Reeb component ∂R and any small piece of the $\mathcal{F}^u$-leaf passing through the point $\sigma(p)$ contain a common orbit.

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