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CONSERVATIONS OF FIRST-ORDER REFLECTIONS

TOSHIYASU ARAI

Abstract. The set theory $\mathsf{KP}\Pi_{N+1}$ for Π_{N+1}-reflecting universes is shown to be Π_{N+1}-conservative over iterations of Π_N-recursively Mahlo operations for each $N \geq 2$.

§1. Introduction. It is well known that the set of weakly Mahlo cardinals below a weakly compact cardinal is stationary. Furthermore any weakly compact cardinal κ is in the diagonal intersection $\kappa \in M^{\triangle} = \bigcap\{M(M^{\alpha}) : \alpha < \kappa\}$ for the α-th iterate M^{α} of the Mahlo operation M, where $\kappa \in M(X)$ iff $X \cap \kappa$ is stationary in κ.

The same holds for the recursive analogues of the indescribable cardinals, *reflecting ordinals* introduced by Richter and Aczel [12]. First, let us recall the ordinals briefly. For a full account of admissible set theory, see [8].

Δ_0 denotes the set of bounded formulae in the language $\{\in, =\}$ of set theories. Then the classes Σ_{i+1}, Π_{i+1} are defined recursively as usual. For set-theoretic formulae φ, let $P \models \varphi :\Leftrightarrow (P, \in) \models \varphi$.

The axioms of Kripke-Platek set theory, denoted KP are Extensionality, Foundation schema, Pair, Union, Δ_0-Separation, and Δ_0-Collection. BS denotes a weak subtheory of KP introduced in [4] and defined below, Definition 2.1, in which we can manipulate finite sequences, partially define truths, and show the existence of a universal Π_i-formula for each $i > 0$. BS is finitely axiomatized over Foundation schema by a Π_2-sentence bs, and KP is equivalent to BS plus Δ_0-Collection. $\mathsf{KP}\omega$ denotes the extension of KP by the axiom of Infinity, and $\mathsf{KP}\ell$ denotes the set theory for limits of admissible sets, which is obtained from KP minus Δ_0-Collection, or equivalently BS by adding the Π_2-axiom $lim :\Leftrightarrow \forall x \exists y[x \in y \wedge ad^{y}]$, where ad denotes a Π_3-sentence such that $P \models ad$ iff P is a transitive model of $\mathsf{KP}\omega$, and φ^{c} denotes the result of restricting any unbounded quantifiers $\exists x, \forall x$ in φ to $\exists x \in c, \forall x \in c$, resp. Again, $\mathsf{KP}\ell$ is finitely axiomatized over Foundation schema by Π_2-sentences bs and lim.

In what follows, V denotes a transitive and wellfounded model of $\mathsf{KP}\ell$, which is the universe of discourse. $P, Q, \ldots$ denote nonempty transitive sets in $V \cup \{V\}$.

A Π_i-recursively Mahlo operation for $2 \leq i < \omega$, is defined through a universal Π_i-formula $\Pi_i(a)$:

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$$P \in RM_i(\mathcal{X}) :\Leftrightarrow \forall b \in P[P \models \Pi_i(b) \to \exists Q \in \mathcal{X} \cap P(b \in Q \models \Pi_i(b))]$$

(read: P is Π_i-reflecting on $\mathcal{X}$.)

For the universe V, $V \in RM_i(\mathcal{X})$ denotes $\forall b[\Pi_i(b) \to \exists Q \in \mathcal{X}(b \in Q \models \Pi_i(b))]$. Suppose that there exists a first-order sentence φ, such that $P \in \mathcal{X} \Leftrightarrow P \models \varphi$ for any transitive $P \in V \cup \{V\}$. Then $RM_i(\mathcal{X})$ is Π_{i+1}, i.e., there exists a Π_{i+1}-sentence $rm_i(\mathcal{X})$, such that $P \in RM_i(\mathcal{X})$ iff $P \models rm_i(\mathcal{X})$ for any transitive set P.

The iteration of RM_i along a definable wellfounded relation $\prec$ is defined as follows.

$$P \in RM_i(a; \prec) :\Leftrightarrow a \in P \in \bigcap\{RM_i(RM_i(b; \prec)) : b \in P \models b \prec a\}.$$

Again $P \in RM_i(a; \prec)$ is a Π_{i+1}-relation.

Let Ord denote the class of ordinals in V. Let us write RM_i^α for $RM_i(\alpha; <)$ and ordinals $\alpha \in Ord$. A transitive set P is said to be Π_i-*reflecting* if $P \in RM_i = RM_i^1$.

$P \in RM_{i+1}$ is much stronger than $P \in RM_i$: assume $P \in RM_{i+1}$ and $P \models \Pi_i(b)$ for $b \in P$. Then $P \in RM_i$ and $P \models rm_i \wedge \Pi_i(b)$ for the Π_{i+1}-sentence rm_i, such that $P \in RM_i$ iff $P \models rm_i$. Hence there exists a $Q \in P$, such that $Q \models rm_i \wedge \Pi_i(b)$, i.e., $Q \in RM_i$ & $Q \models \Pi_i(b)$. This means $P \in RM_i^2 = RM_i(RM_i)$. Moreover P is in the diagonal intersection of RM_i, $P \in RM_i^\triangle$, i.e., $P \in \bigcap\{RM_i^\beta : \beta \in P \cap Ord\}$, and so on.

In particular, the set theory $\mathsf{KP}\Pi_{i+1}$ for universes in RM_{i+1} proves the consistency of a set theory for universes in $RM_i^\triangle$.

In this paper we address the problem: How far can we iterate lower recursive Mahlo operations in higher reflecting universes? In [1], we gave a sketchy proof of the following Theorem 1.1, which is implicit in ordinal analyses in [2, 7].

THEOREM 1.1. *For each $N \geq 2$ there exists a Σ_1-relation $\lhd_N$ on ω such that the set theory $\mathsf{KP}\ell$ for limits of admissibles proves the transfinite induction schema for $\lhd_N$ up to each $a \in \omega$, and $\mathsf{KP}\Pi_{N+1}$ is Π^1_1(on ω)-conservative over the theory*

$$\mathsf{KP}\ell + \{V \in RM_N(a; \lhd_N) : a \in \omega\}.$$

Theorem 1.1 suffices to approximate $\mathsf{KP}\Pi_{N+1}$ proof-theoretically in terms of iterations of Π_N-recursively Mahlo operations. However, $V \in RM_N(a; \prec)$ is a Π_{N+1}-formula for Σ_{N+1}-relation $\prec$, and the class Π^1_1 on ω is smaller than Π_{N+1}.

In this paper the set theory $\mathsf{KP}\Pi_{N+1}$ for Π_{N+1}-reflecting universes is shown to be Π_{N+1}-conservative over iterations of Π_N-recursively Mahlo operations RM_N for each $N \geq 2$ (Theorem 2.4). This result will be extended in [3, 5] to the indescribable cardinals over $\mathsf{ZF} + (V = L)$.

§2. Conservation.

2.1. A weak base theory BS. A weak base theory BS is introduced in [4]. Consider the following functions $\mathcal{F}_i$ $(i < 9)$, $\mathcal{F}_0(x, y) = \{x, y\}$, $\mathcal{F}_1(x, y) = \cup x$, $\mathcal{F}_2(x, y) = x \setminus y$, $\mathcal{F}_3(x, y) = \{u \cup \{v\} : u \in x, v \in y\}$, $\mathcal{F}_4(x, y) = dom(x) = \{u \in \cup\cup x : \exists v \in \cup\cup x(\langle u, v\rangle \in x)\}$, $\mathcal{F}_5(x, y) = rng(x) = \{v \in \cup\cup x : \exists u \in \cup\cup x(\langle u, v\rangle \in x)\}$, $\mathcal{F}_6(x, y) = \{\langle v, u\rangle \in y \times x : v \in u\}$, $\mathcal{F}_7(x, y) = \{\langle u, v, w\rangle : \langle u, v\rangle \in x, w \in y\}$, and $\mathcal{F}_8(x, y) = \{\langle u, w, v\rangle : \langle u, v\rangle \in x, w \in y\}$, where $\langle v, u\rangle = \{v, \{v, u\}\}$ and $\langle u, v, w\rangle = \langle u, \langle v, w\rangle\rangle$.

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For each i, $\mathcal{F}_i(x,y,z)$ denotes a Δ_0-formula stating $\mathcal{F}_i(x,y)=z$.

Definition 2.1. BS is the set theory in the language $\{\in,=\}$. Its axioms are Extensionality, Foundation schema, and $\{\forall x, y \exists z\, \mathcal{F}_i(x,y,z) : i<9\}$.

bs denotes a Π_2-sentence, which is equivalent to the conjunction of Extensionality and $\{\forall x, y \exists z\, \mathcal{F}_i(x,y,z) : i<9\}$.

A set-theoretic function $f : V^n \to V$ is Σ_1^{BS}-*definable* if there exists a Σ_1-formula $\varphi(x_1,\ldots,x_n,y)$ for which $\mathsf{BS} \vdash \forall x_1,\ldots,x_n \exists y!\, \varphi(x_1,\ldots,x_n,y)$, and $f(x_1,\ldots,x_n)=y$ iff $V \models \varphi(x_1,\ldots,x_n,y)$.

A relation $R \subset V^n$ is Δ_1^{BS} if there exist Σ_1-formulae φ,ψ such that $\mathsf{BS} \vdash \forall x_1,\ldots,x_n[\varphi(x_1,\ldots,x_n) \leftrightarrow \neg\psi(x_1,\ldots,x_n)]$, and $(x_1,\ldots,x_n) \in R$ iff $V \models \varphi(x_1,\ldots,x_n)$.

Under a suitable encoding of the syntax, we can assume that the set $\ulcorner Fml \urcorner$ of codes $\ulcorner \varphi \urcorner$ of formulae φ as well as the set $\ulcorner Fml_{\Sigma_i} \urcorner$ of codes of Σ_i-formulae is Δ_1^{BS}. The set $\{n \in \omega : v_n$ occurs freely in the formula coded by $x\}$ is denoted by $var(x)$ for $x \in \ulcorner Fml \urcorner$, and $ass(x,y)$ the set of function $a : var(x) \to y$. Both $x \mapsto var(x)$ and $(x,y) \mapsto ass(x,y)$ are Σ_1^{BS}-functions. Let $\models \ulcorner \varphi \urcorner[a]$ denote the satisfaction relation for formulae φ and $a \in ass(\ulcorner \varphi \urcorner, y)$ for a y.

Lemma 2.2. *For each $i>0$, the satisfaction relation $\{(x,a) : x \in \ulcorner Fml_{\Sigma_i} \urcorner, a \in ass(x), \models x[a]\}$ for Σ_i-formulae φ is Σ_i-definable in* BS *in such a way that* BS *proves that $\varphi(v_0,\ldots,v_{m-1}) \leftrightarrow \models \ulcorner \varphi(v_0,\ldots,v_{m-1}) \urcorner[a]$ for $a(i)=v_i$, $\models \ulcorner \exists v_m\, \varphi \urcorner[a] \leftrightarrow \exists b[\models \ulcorner \varphi \urcorner[a \cup \{\langle m,b\rangle\}]$ for Σ_i-formula $\exists v_m\, \varphi$, and similarly for $\vee, \wedge, \forall$.*

Proof. It suffices to Δ_1^{BS}-define the satisfaction relation for Δ_0-formulae. This is seen as in [13, p. 613]. Note that we don't need the existence of transitive closures to bound range y of the assignments $a : var(x) \to y$ since there are only finitely many subformulae of a formula: Let x be a code of a Δ_0-formula, and n be the number of subformulae of the formula coded by x. Also let a be a function on $var(x)$ with its range $b = rng(a)$, and $var(\bar{x})$ the union of $var(y)$ for codes y of subformulae of the formula coded by x. Then in order to define $\models x[a]$ it suffices to consider assignments in $ass(var(\bar{x}), \cup^{(n)} b)$, where $\cup^{(0)} b = b$ and $\cup^{(n+1)} b = \cup(\cup^{(n)} b)$. Thus, the existence of the set $\cup^{(n)} b$ suffices for natural numbers n and sets b. Indeed, $(n,b) \mapsto \cup^{(n)} b$ is a Σ_1^{BS}-function as shown in [4]. $\dashv$

2.2. Codes of ordinals up to the next epsilon number. Next let us consider a well ordering $<^{\varepsilon}$ of type $\varepsilon_{\Omega+1}$, the next epsilon number to the order type Ω of the class Ord of all ordinals in the universe V. Here it is safe for us to work in a theory slightly stronger than BS, in which, additions and exponentiations on Ord are provably total. Let us work in Kripke-Platek set theory with the axiom of Infinity, denoted $\mathsf{KP}\omega$.

Let $Ord \subset V$ denote the class of ordinals, $Ord^{\varepsilon} \subset V$ and $<^{\varepsilon}$ be Δ-predicates such that for any transitive and wellfounded model V of $\mathsf{KP}\omega$, $<^{\varepsilon}$ is a well ordering of type $\varepsilon_{\Omega+1}$ on Ord^{ε} for the order type Ω of the class Ord in V. Specifically, let us encode 'ordinals' $\alpha < \varepsilon_{\Omega+1}$ by codes $\ulcorner \alpha \urcorner \in Ord^{\varepsilon}$ as follows. $\ulcorner \alpha \urcorner = \langle 0, \alpha \rangle$ for $\alpha \in Ord$, $\ulcorner \Omega \urcorner = \langle 1, 0 \rangle$, $\ulcorner \omega^{\alpha} \urcorner = \langle 2, \ulcorner \alpha \urcorner \rangle$ for $\alpha > \Omega$, and $\ulcorner \alpha \urcorner = \langle 3, \ulcorner \alpha_1 \urcorner, \ldots, \ulcorner \alpha_n \urcorner \rangle$ if $\alpha = \alpha_1 + \cdots + \alpha_n > \Omega$ with $\alpha_1 \geq \cdots \geq \alpha_n$, $n>1$ and $\exists \beta_i (\alpha_i = \omega^{\beta_i})$ for each α_i. Then $\ulcorner \omega_n(\Omega+1) \urcorner \in Ord^{\varepsilon}$ denotes the code of the 'ordinal' $\omega_n(\Omega+1)$.

$<^\varepsilon$ is assumed to be a canonical ordering such that $\mathsf{KP}\omega$ proves the fact that $<^\varepsilon$ is a linear ordering, and for any formula φ and each $n < \omega$,

$$\mathsf{KP}\omega \vdash \forall x(\forall y <^\varepsilon x\, \varphi(y) \to \varphi(x)) \to \forall x <^\varepsilon \ulcorner \omega_n(\Omega+1)\urcorner \varphi(x). \quad (1)$$

For a definition of Δ-predicates Ord^ε and $<^\varepsilon$, and a proof of (1), cf. [6].

PROPOSITION 2.3. *$\mathsf{KP}\omega$ proves that if $P \in RM_N(\beta; <^\varepsilon)$, then $\forall \alpha <^\varepsilon \beta(\alpha \in P \to P \in RM_N(\alpha; <^\varepsilon))$.*

PROOF. This is seen from the fact that $<^\varepsilon$ is transitive in $\mathsf{KP}\omega$. ⊣

THEOREM 2.4. *For each $N \geq 2$, $\mathsf{KP}\Pi_{N+1}$ is Π_{N+1}-conservative over the theory*

$$\mathsf{KP}\omega + \{V \in RM_N(\ulcorner \omega_n(\Omega+1)\urcorner; <^\varepsilon) : n \in \omega\}.$$

From (1) we see that $\mathsf{KP}\Pi_{N+1}$ proves $V \in RM_N(\ulcorner \omega_n(\Omega+1)\urcorner; <^\varepsilon)$ for each $n \in \omega$. The converse is proved in Section 3.

PROPOSITION 2.5. *For any class Γ of Π_{N+1}-sentences, there exists a Σ_{N+1}-sentence A, such that $\mathsf{KP}\Pi_{N+1} \vdash A$, and $\mathsf{KP}\omega + \Gamma \nvdash A$ unless $\mathsf{KP}\omega + \Gamma$ is inconsistent.*

PROOF. This follows from the essential unboundedness theorem due to Kreisel and Lévy [10]. In this proof let $\vdash A :\Leftrightarrow \mathsf{KP}\omega \vdash A$ and Pr denote a standard provability predicate for $\mathsf{KP}\omega$. Also $\mathrm{Tr}_{\Pi_{N+1}}$ denotes a partial truth definition of Π_{N+1}-sentences.

Then, let A be a Σ_{N+1}-sentence saying that 'I am not provable from any true Π_{N+1}-sentence', $\vdash A \leftrightarrow \forall x \in \omega[\mathrm{Tr}_{\Pi_{N+1}}(x) \to \neg \mathrm{Pr}(x \dot{\to} \ulcorner A\urcorner)]$, where $\dot{\to}$ denotes a recursive function, such that $\ulcorner A\urcorner \dot{\to} \ulcorner B\urcorner = \ulcorner A \to B\urcorner$ for codes $\ulcorner A\urcorner$ of formulae A.

Suppose $\mathsf{KP}\omega + \Gamma \vdash A$. Pick a $C \in \Gamma$ so that $\vdash C \to A$. Then $\mathsf{KP}\omega + \Gamma \vdash \mathrm{Tr}_{\Pi_{N+1}}(\ulcorner C\urcorner) \wedge \mathrm{Pr}(\ulcorner C \to A\urcorner)$. Hence $\mathsf{KP}\omega + \Gamma \vdash \neg A$.

In what follows argue in $\mathsf{KP}\Pi_{N+1}$. Suppose A is false, and let C be any true Π_{N+1}-sentence. Since the universe V is Π_{N+1}-reflecting, there exists a transitive model $P \in V$ of $\mathsf{KP}\omega + \{C, \neg A\}$, which shows that $\mathsf{KP}\omega + \{C, \neg A\}$ is consistent. In other words, $\neg \mathrm{Pr}(\ulcorner C \to A\urcorner)$. Therefore, $\mathsf{KP}\Pi_{N+1} \vdash \neg A \to A$. ⊣

Thus, Theorem 2.4 is optimal with respect to the class Π_{N+1} of formulae provided that $\mathsf{KP}\Pi_{N+1}$ is consistent.

COROLLARY 2.6. *For each $N \geq 3$, $\mathsf{KP}\Pi_{N+1}$ + (Power) + (Σ_{N-3}-Separation) + (Π_{N-3}-Collection) is Π_{N+1}-conservative over the theory $\mathsf{KP}\omega + \{V \in RM_N(\ulcorner \omega_n(\Omega+1)\urcorner; <^\varepsilon) : n \in \omega\}$ + (Power) + (Σ_{N-3}-Separation) + (Π_{N-3}-Collection).*

PROOF. This follows from Theorem 2.4 and the facts that the axiom Power is a Π_3-sentence $\forall a \exists b \forall x \subset a(x \in b)$, and Σ_i-Separation or Π_i-Collection are Π_{i+3}-formulae. ⊣

Let us announce an extension of Theorem 2.4 in [3, 5] to the indescribable cardinals over $\mathsf{ZF} + (V = L)$.

Let $<^\varepsilon$ be an ε-ordering as above. Let M_N denote the Π^1_N-Mahlo operation defined for sets S of ordinals and uncountable regular cardinals κ: $\kappa \in M_N(S)$ iff $S \cap \kappa$ is Π^1_N-indescribable in κ. The Π^1_{N+1}-indescribability is proof-theoretically reducible to iterations of an operation along initial segments of $<^\varepsilon$ over $\mathsf{ZF} + (V = L)$. The operation is a mixture of the Π^1_N-Mahlo operation M_N and Mostowski collapsings.

For $\alpha <^{\varepsilon} \varepsilon_{\mathcal{K}+1}$ and finite sets $\Theta \subset_{fin} (\mathcal{K}+1)$, Π_{n+1}-classes $Mh^{\alpha}_{n}[\Theta]$ are defined so that the following holds.

In Theorem 2.7, $\mathcal{K}$ is intended to denote the least Π^1_{N+1}-indescribable cardinal, and Ω the least weakly inaccessible cardinal above $\mathcal{K}$.

THEOREM 2.7. (*The case* $N = 0$ *in* [3], *and the general case in* [5].)

1. *For each* $n < \omega$,

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \textit{ is } \Pi^1_{N+1}\textit{-indescribable}) \vdash \mathcal{K} \in Mh_n^{\omega_n(\Omega+1)}[\emptyset].$$

2. *For any* Π^1_{N+1}*-sentences* φ, *if*

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \textit{ is } \Pi^1_{N+1}\textit{-indescribable}) \vdash \varphi^{L_{\mathcal{K}}},$$

then, we can find an $n < \omega$ *such that*

$$\mathsf{ZF} + (V = L) + (\mathcal{K} \in Mh_n^{\omega_n(\Omega+1)}[\emptyset]) \vdash \varphi^{L_{\mathcal{K}}}.$$

The classes $Mh^{\alpha}_{n}[\Theta]$ are defined from iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$, through which we described the limit of $\mathsf{ZF} + (V = L)$-provable countable ordinals in [6] as follows.

THEOREM 2.8. ([6])

$$|\mathsf{ZF} + (V=L)|_{\omega_1} := \inf\{\alpha \leq \omega_1 : \forall \varphi[\mathsf{ZF} + (V = L) \vdash \exists x \in L_{\omega_1}\varphi \Rightarrow \exists x \in L_{\alpha}\varphi]\}$$
$$= \Psi_{\omega_1}\varepsilon_{\Omega+1} := \sup\{\Psi_{\omega_1,n}\omega_n(\Omega+1) : n < \omega\}.$$

In Theorem 2.8, Ω *is intended to denote the least weakly inaccessible cardinal.*

§3. Proof of Theorem 2.4.

In this section, we prove Theorem 2.4. Our proof is extracted from M. Rathjen's ordinal analyses of Π_3-reflection in [11].

Let $N \geq 2$ denote a fixed integer. The axioms of the set theory $\mathsf{KP}\Pi_{N+1}$ for Π_{N+1}-reflecting universes are those of BS, and the axiom for Π_{N+1}-reflection: for Π_{N+1}-formulae φ, $\varphi(a) \to \exists c[ad^c \wedge a \in c \wedge \varphi^c(a)]$. Note that $\mathsf{KP}\Pi_{N+1}$ comprises $\mathsf{KP}\omega$, i.e., it proves Infinity and Δ_0-Collection for $N \geq 1$.

Throughout this section we work in an intuitionistic fixed point theory $\mathrm{FiX}^i(\mathsf{KP}\ell)$ over $\mathsf{KP}\ell$. The intuitionistic theory $\mathrm{FiX}^i(\mathsf{KP}\ell)$ is introduced in [4], and shown to be a conservative extension of $\mathsf{KP}\ell$. Let us reproduce definitions and results on $\mathrm{FiX}^i(\mathsf{KP}\ell)$ here.

Fix an X-strictly positive formula $\mathcal{Q}(X, x)$ in the language $\{\in, =, X\}$ with an extra unary predicate symbol X. In $\mathcal{Q}(X, x)$ the predicate symbol X occurs only strictly positive. This means that the predicate symbol X does not occur in the antecedent φ of implications $\varphi \to \psi$ nor in the scope of negations $\neg$ in $\mathcal{Q}(X, x)$. The language of $\mathrm{FiX}^i(\mathsf{KP}\ell)$ is $\{\in, =, Q\}$ with a fresh unary predicate symbol Q. The axioms in $\mathrm{FiX}^i(\mathsf{KP}\ell)$ consist of the following:

1. All provable sentences in $\mathsf{KP}\ell$ (in the language $\{\in, =\}$).
2. Induction schema for any formula φ in $\{\in, =, Q\}$:

$$\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x). \tag{2}$$

3. Fixed point axiom:

$$\forall x[Q(x) \leftrightarrow \mathcal{Q}(Q, x)].$$

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The underlying logic in $\mathrm{FiX}^i(\mathsf{KP}\ell)$ is defined to be the intuitionistic (first-order predicate) logic (with equality). Roughly, the exclude middle $\varphi \lor \neg\varphi$ is available in $\mathrm{FiX}^i(\mathsf{KP}\ell)$ only for set-theoretic formulae φ in the language $\{\in, =\}$, but not for formulae having the fixed point predicate Q.

(2) yields the following Lemma 3.1.

LEMMA 3.1. *Let $<^\varepsilon$ denote a Δ_1-predicate mentioned in the beginning of subsection* 2.2. *For each $n < \omega$ and each formula φ in $\{\in, =, Q\}$,*

$$\mathrm{FiX}^i(\mathsf{KP}\ell) \vdash \forall x(\forall y <^\varepsilon x\, \varphi(y) \to \varphi(x)) \to \forall x <^\varepsilon \lceil\omega_n(\Omega+1)\rceil\varphi(x).$$

In what follows, let us write $\alpha < \beta$ for $\alpha <^\varepsilon \beta$ for codes α, β of ordinals$< \varepsilon_{\Omega+1}$ when no confusion likely occurs.

The following Theorem 3.2 is shown in [4].

THEOREM 3.2. $\mathrm{FiX}^i(\mathsf{KP}\ell)$ *is a conservative extension of* $\mathsf{KP}\ell$.

In what follows, we work in $\mathrm{FiX}^i(\mathsf{KP}\ell)$.

Let V denote a transitive and wellfounded model of $\mathsf{KP}\ell$. Consider the language $\mathcal{L}_V = \{\in\} \cup \{c_a : a \in V\}$, where c_a denotes the name of the set $a \in V$. We identify the set a with its name c_a.

Our proof proceeds as follows. Assume that $\mathsf{KP}\Pi_{N+1} \vdash A$ for a Π_{N+1}-sentence A. $\mathsf{KP}\Pi_{N+1}$ is embedded to an infinitary system formulated in one-sided sequent calculus, and cut inferences are eliminated, which results in an infinitary derivation of height $\alpha < \varepsilon_{\Omega+1}$ with an inference rule (Ref_{N+1}) for Π_{N+1}-reflection. Then A is seen to be true in $P \in RM_N(\alpha; <^\varepsilon)$.

In one-sided sequent calculi, formulae are generated from atomic formulae and their negations $a \in b, a \notin b$ by propositional connectives $\lor, \land$ and quantifiers $\exists, \forall$. It is convenient here to have bounded quantifications $\exists x \in a, \forall x \in a$ besides unbounded ones $\exists x, \forall x$. The negation $\neg A$ of formulae A is defined recursively by de Morgan's law and elimination of double negations. Also $(A \to B) :\equiv (\neg A \lor B)$.

$\Gamma, \Delta, \dots$ denote finite sets of sentences, called *sequents* in the language $\mathcal{L}_V$. Γ, Δ denotes the union $\Gamma \cup \Delta$, and Γ, A the union $\Gamma \cup \{A\}$. A finite set Γ of sentences is intended to denote the disjunction $\bigvee\Gamma := \bigvee\{A : A \in \Gamma\}$. Γ is *true* in $P \in V \cup \{V\}$ iff $\bigvee\Gamma$ is true in P iff $\bigvee\Gamma^P$ is true.

Classes $\Delta_0, \Sigma_{i+1}, \Pi_{i+1}$ of sentences in $\mathcal{L}_V$ are defined as usual.

We assign disjunctions or conjunctions to sentences as follows. When a disjunction $\bigvee(A_\iota)_{\iota\in J}$ [a conjunction $\bigwedge(A_\iota)_{\iota\in J}$] is assigned to A, we denote $A \simeq \bigvee(A_\iota)_{\iota\in J}$ [$A \simeq \bigwedge(A_\iota)_{\iota\in J}$], resp.

DEFINITION 3.3. 1. For a Δ_0-sentence M

$$M :\simeq \begin{cases} \bigvee(A_\iota)_{\iota\in J} & \text{if } M \text{ is false in } V \\ \bigwedge(A_\iota)_{\iota\in J} & \text{if } M \text{ is true in } V \end{cases} \text{ with } J := \emptyset.$$

In what follows, we consider the unbounded sentences.

2. $(A_0 \lor A_1) :\simeq \bigvee(A_\iota)_{\iota\in J}$ and $(A_0 \land A_1) :\simeq \bigwedge(A_\iota)_{\iota\in J}$ with $J := 2$.
3. $\exists x \in a\, A(x) :\simeq \bigvee(A(b))_{b\in J}$ and $\forall x \in a\, A(x) :\simeq \bigwedge(A(b))_{b\in J}$ with $J := a$.
4. $\exists x\, A(x) :\simeq \bigvee(A(b))_{b\in J}$ and $\forall x\, A(x) :\simeq \bigwedge(A(b))_{b\in J}$ with $J := V$.

DEFINITION 3.4. The *depth* $\mathrm{dp}(A) < \omega$ of $\mathcal{L}_V$-sentences A is defined recursively as follows.

1. $\mathrm{dp}(A) = 0$ if $A \in \Delta_0$.
   In what follows we consider unbounded sentences A.
2. $\mathrm{dp}(A) = \max\{\mathrm{dp}(A_i) : i < 2\} + 1$ if $A \equiv (A_0 \circ A_1)$ for $\circ \in \{\vee, \wedge\}$.
3. $\mathrm{dp}(A) = \mathrm{dp}(B(\emptyset)) + 1$ if $A \in \{(Qx\, B(x)), (Qx \in a\, B(x)) : a \in V\}$ for $Q \in \{\exists, \forall\}$.

DEFINITION 3.5. 1. For $\mathcal{L}_V$-sentences A, $\mathsf{k}(A) := \{a \in V : c_a \text{ occurs in } A\}$.
2. For sets Γ of sentences, $\mathsf{k}(\Gamma) := \bigcup\{\mathsf{k}(A) : A \in \Gamma\}$.
3. For $\imath \in V$ and a transitive model $P \in V$ of $\mathsf{KP}\omega$, $P(\imath) \in V \cup \{V\}$ denotes the smallest transitive model of $\mathsf{KP}\omega$ such that $P \cup \{\imath\} \subset P(\imath)$. Note that V is assumed to be a model of $\mathsf{KP}\ell$.
   For finite lists $\vec{a} = (a_1, \ldots, a_n)$, $P(\vec{a}) := (\cdots P(a_1)\cdots)(a_n)$.

Inspired by operator controlled derivations due to W. Buchholz [9], let us define a relation $P \vdash^\alpha_m \Gamma$ for transitive models $P \in V \cup \{V\}$ of $\mathsf{KP}\omega$. The relation $P \vdash^\alpha_m \Gamma$ is defined as a fixed point of a strictly positive formula H

$$H(P, \alpha, m, \Gamma) \Leftrightarrow P \vdash^\alpha_m \Gamma$$

in $\mathrm{FiX}^i(\mathsf{KP}\ell)$.

Note that P contains the code $\langle 1, 0\rangle = \ulcorner\Omega\urcorner$, and is closed under ordinal addition $(\alpha, \beta) \mapsto \alpha + \beta$, exponentiation $\alpha \mapsto \omega^\alpha$ for $\alpha, \beta \in Ord^\varepsilon$ and $a \mapsto rank(a)$ for $rank(a) = \sup\{rank(b) + 1 : b \in a\}$.

DEFINITION 3.6. Let $P \in V \cup \{V\}$ be a transitive model of $\mathsf{KP}\omega$, codes $\alpha < \varepsilon_{\Omega+1}$ and $m < \omega$.

$P \vdash^\alpha_m \Gamma$ holds if

$$\mathsf{k}(\Gamma) \cup \{\alpha\} \subset P \tag{3}$$

and one of the following cases holds:

$(\bigvee)$: there is an $A \in \Gamma$, such that $A \simeq \bigvee(A_\imath)_{\imath\in J}$, and for an $\imath \in J$ and an $\alpha(\imath) < \alpha$, $P \vdash^{\alpha(\imath)}_m \Gamma, A_\imath$.

$$\frac{P \vdash^{\alpha(\imath)}_m \Gamma, A_\imath}{P \vdash^\alpha_m \Gamma}\ (\bigvee)$$

$(\bigwedge)$: there is an $A \in \Gamma$, such that $A \simeq \bigwedge(A_\imath)_{\imath\in J}$, and for any $\imath \in J$, there is an $\alpha(\imath)$, such that $\alpha(\imath) < \alpha$ and $P(\imath) \vdash^{\alpha(\imath)}_m \Gamma, A_\imath$.

$$\frac{\{P(\imath) \vdash^{\alpha(\imath)}_m \Gamma, A_\imath : \imath \in J\}}{P \vdash^\alpha_m \Gamma}\ (\bigwedge)$$

(cut): there are C and α_0, α_1, such that $\mathrm{dp}(C) < m$, $\alpha_0, \alpha_1 < \alpha$, and $P \vdash^{\alpha_0}_m \Gamma, \neg C$ and $P \vdash^{\alpha_1}_m C, \Gamma$.

$$\frac{P \vdash^{\alpha_0}_m \Gamma, \neg C \qquad P \vdash^{\alpha_1}_m C, \Gamma}{P \vdash^\alpha_m \Gamma}\ (cut)$$

(Ref_{N+1}): there are $A(c) \in \Pi_{N+1}$ and $\alpha_0, \alpha_1 < \alpha$, such that $P \vdash^{\alpha_0}_m \Gamma, A(c)$ and $P \vdash^{\alpha_1}_m \forall z[ad^z \to c \in z \to \neg A^z(c)], \Gamma$.

$$\frac{P \vdash^{\alpha_0}_{m} \Gamma, A(c) \qquad P \vdash^{\alpha_1}_{m} \forall z[ad^{z} \to c \in z \to \neg A^{z}(c)], \Gamma}{P \vdash^{\alpha}_{m} \Gamma} \ (Ref_{N+1})$$

In what follows, let us fix an integer n_0 and restrict (codes of) ordinals to $\alpha <^{\varepsilon} \ulcorner \omega_{n_0}(\Omega + 1) \urcorner$. n_0 is chosen from the given finite proof of a Π_{N+1}-sentence A in $\mathsf{KP}\Pi_{N+1}$, cf. Corollary 3.9 (Embedding). Since n_0 is a constant, we see from Lemma 3.1 that $\mathrm{FiX}^{i}(\mathsf{KP}\ell)$ proves transfinite induction schema up to $\ulcorner \omega_{n_0}(\Omega+1) \urcorner$ for any formula in which the derivability relation $P \vdash^{\alpha}_{m} \Gamma$ may occur.

Proposition 3.7. *Let $P' \supset P$ be transitive models of* $\mathsf{KP}\omega$, $\alpha \leq \alpha'$, $m \leq m' < \omega$ *and* $\mathsf{k}(\Delta) \cup \{\alpha'\} \subset P'$. *If* $P \vdash^{\alpha}_{m} \Gamma$, *then* $P' \vdash^{\alpha'}_{m'} \Gamma, \Delta$.

In embedding $\mathsf{KP}\Pi_{N+1}$ in the infinitary calculus, it is convenient to formulate $\mathsf{KP}\Pi_{N+1}$ in (finitary) one-sided sequent calculus of the language $\{\in, 0\}$ with the individual constant 0 for the empty set. Axioms are logical ones $\Gamma, \neg A, A$ for any formulae A, and axioms in the theory $\mathsf{KP}\Pi_{N+1}$. Inference rules are $(\vee)$, $(\wedge)$ for propositional connectives, $(b\exists)$, $(b\forall)$ for bounded quantifications, $(\exists)$, $(\forall)$ for unbounded quantifications, and (cut). For details, see the proof of the following Lemma 3.8.

Though the following Lemmata 3.8, 3.10, and 3.11 are seen as in [9], we give proofs of them for readers' convenience.

Let $(m, \vec{a}) := \Omega \cdot m + 3rank(a_1)\# \cdots \# 3rank(a_n)$ for $\vec{a} = (a_1, \ldots, a_n)$ and the natural (commutative) sum $\alpha\#\beta$ of ordinals α, β.

Lemma 3.8. *Suppose* $\mathsf{KP}\Pi_{N+1} \vdash \Gamma(\vec{x})$, *where the free variables occurring in the sequent are among the list $\vec{x}$. Then there is an $m < \omega$ such that for any $\vec{a} \subset V$ and any transitive model $P \in V \cup \{V\}$ of* $\mathsf{KP}\omega$, $P(\vec{a}) \vdash^{(m,\vec{a})}_{m} \Gamma(\vec{a})$.

Proof. First consider the logical axiom $\Gamma(\vec{x}), \neg A(\vec{x}), A(\vec{x})$. We see that for any $\vec{a}$

$$P(\vec{a}) \vdash^{2d}_{0} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a}) \tag{4}$$

by induction on $d = \mathrm{dp}(A)$.

Then by Proposition 3.7, we have, $P(\vec{a}) \vdash^{(2d,\vec{a})}_{2d} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a})$.

If $d = 0$, then $A \in \Delta_0$ and one of $\neg A(\vec{a})$ and $A(\vec{a})$ is true. Hence by $(\bigwedge)$ we have, $P(\vec{a}) \vdash^{0}_{0} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a})$.

Next consider the case when $A \equiv (\exists y\, B(\vec{x}, y)) \notin \Delta_0$ with $\mathrm{dp}(B(\vec{x}, y)) = d - 1$. By IH(=Induction Hypothesis) we have for any $\vec{a} \subset V$ and any $b \in V$, $P(\vec{a} * (b)) \vdash^{2d-2}_{0} \Gamma(\vec{a}), \neg B(\vec{a}, b), B(\vec{a}, b)$, where $(a_1, \ldots, a_n) * (b) = (a_1, \ldots, a_n, b)$. $(\bigvee)$ yields $P(\vec{a} * (b)) \vdash^{2d-1}_{0} \Gamma(\vec{a}), \neg B(\vec{a}, b), \exists y\, B(\vec{a}, y)$. Hence $(\bigwedge)$ with $P(\vec{a} * (b)) = P(\vec{a})(b)$ yields $P(\vec{a}) \vdash^{2d}_{0} \Gamma(\vec{a}), \neg \exists y\, B(\vec{a}, y), \exists y\, B(\vec{a}, y)$.

The cases $A \equiv (\exists y \in a\, B(\vec{x}, y)) \notin \Delta_0$ and $A \equiv (B_0 \vee B_1) \notin \Delta_0$ are similar. Thus (4) is shown.

Second consider the inference rule $(\exists)$ with $\exists y\, A(\vec{x}, y) \in \Gamma(\vec{x})$

$$\frac{\Gamma(\vec{x}), A(\vec{x}, t)}{\Gamma(\vec{x})} \ (\exists)$$

When t is a variable y, we can assume that y is an x_i in the list $\vec{x}$, for otherwise substitute 0 for y. By IH there is an m such that $P(\vec{a}) \vdash^{(m,\vec{a})}_{m} \Gamma(\vec{a}), A(\vec{a}, t')$, where $t' \equiv a_i$ if $t \equiv x_i$, and $t' \equiv 0$, otherwise. Thus $P(\vec{a}) \vdash^{(m+1,\vec{a})}_{m+1} \Gamma(\vec{a})$.

Third consider the inference rule $(\forall)$ with $\forall y\, A(\vec{x}, y) \in \Gamma(\vec{x})$

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$$\frac{\Gamma(\vec{x}), A(\vec{x}, y)}{\Gamma(\vec{x})}\ (\forall),$$

where the variable y does not occur in $\Gamma(\vec{x})$. IH yields for an m, $P(\vec{a} * (b)) \vdash_m^{(m, \vec{a}*(b))} \Gamma(\vec{a}), A(\vec{a}, b)$. $(\bigwedge)$ with $(m+1, \vec{a}) > (m, \vec{a} * (b))$ yields $P(\vec{a}) \vdash_{m+1}^{(m+1, \vec{a})} \Gamma(\vec{a})$.

The following cases are similarly seen.

$$\frac{\Gamma, t \in s \qquad \Gamma, B(\vec{x}, t)}{\Gamma, \exists y \in s\, B(\vec{x}, y)}\ (b\exists) \quad \frac{\Gamma, y \notin s, B(\vec{x}, y)}{\Gamma, \forall y \in s\, B(\vec{x}, y)}\ (b\forall)$$

$$\frac{\Gamma, A_0, A_1}{\Gamma, A_0 \vee A_1}\ (\vee) \quad \frac{\Gamma, A_0 \qquad \Gamma, A_1}{\Gamma, A_0 \wedge A_1}\ (\wedge)$$

In a cut inference

$$\frac{\Gamma(\vec{x}), \neg A(\vec{x}) \qquad A(\vec{x}), \Gamma(\vec{x})}{\Gamma(\vec{x})}\ (cut)$$

if the cut formula $A(\vec{x})$ has free variables $\vec{y}$ other than $\vec{x}$, then substitute 0 for $\vec{y}$.

In what follows, let us suppress parameters.

Fourth, consider the axioms other than Foundation. First consider a Π_2-axiom $\forall x, y \exists z\, \mathcal{F}_i(x, y, z)$ in BS stating that $\mathcal{F}_i(x, y)$ exists for $i < 9$. Let $a, b \in V$. Since $P(a, b)$ is a transitive model of $\mathsf{KP}\omega$ and $a, b \in P(a, b)$, pick a $c \in P(a, b)$ such that the Δ_0-formula $\mathcal{F}_i(a, b, c)$ holds in $P(a, b)$, and in V. Since this is a true Δ_0-sentence, we have $P(a, b) \vdash_0^0 \mathcal{F}_i(a, b, c)$, and $P \vdash_0^3 \forall x, y \exists z\, \mathcal{F}_i(x, y, z)$.

Next consider the axiom $A(c) \to \exists z[ad^z \wedge c \in z \wedge A^z(c)]$ for $A \in \Pi_{N+1}$. We have by (4) for $d = \mathrm{dp}(A)$

$$\frac{P(c) \vdash_0^{2d} \neg A(c), A(c) \qquad P(c) \vdash_0^2 \forall z[ad^z \to c \in z \to \neg A^z(c)], \exists z[ad^z \wedge c \in z \wedge A^z(c)]}{P(c) \vdash_0^{2d+1} \neg A(c), \exists z[ad^z \wedge c \in z \wedge A^z(c)]}\ (Ref_{N+1})$$

In this way, we see that there are cut-free infinitary derivations of finite heights deducing axioms in $\mathsf{KP}\Pi_{N+1}$ other than Foundation.

Finally consider Foundation. Let $d = \mathrm{dp}(A)$ and $B \equiv (\neg \forall x (\forall y \in x\, A(y) \to A(x)))$. We show by induction on $rank(a)$ that

$$P(a) \vdash_0^{2d+3rank(a)} B, \forall x \in a\, A(x) \tag{5}$$

By IH we have for any $b \in a$, $P(b) \vdash_0^{2d+3rank(b)} B, \forall x \in b\, A(x)$. Thus we have by (4)

$$\cfrac{\cfrac{P(b) \vdash_0^{2d+3rank(b)} B, \forall x \in b\, A(x) \qquad P(b) \vdash_0^{2d} \neg A(b), A(b)}{P(b) \vdash_0^{2d+3rank(b)+1} B, \forall x \in b\, A(x) \wedge \neg A(b), A(b)}\ (\bigwedge)}{P(b) \vdash_0^{2d+3rank(b)+2} B, A(b)}\ (\bigvee)$$

Therefore (5) is shown.

$$\frac{\{P(a, b) \vdash_0^{2d+3rank(b)+2} B, A(b) : b \in a\}}{P(a) \vdash_0^{2d+3rank(a)} B, \forall x \in a\, A(x)}\ (\bigwedge)$$

⊣

Corollary 3.9 (Embedding). *If* $\mathsf{KP}\Pi_{N+1} \vdash A$ *for a sentence* A, *then there is an* $m < \omega$ *such that for any transitive model* $P \in V \cup \{V\}$ *of* $\mathsf{KP}\omega$, $P \vdash_m^{\Omega \cdot m} A$.

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LEMMA 3.10 (Reduction). *Let* $C \simeq \bigvee(C_\iota)_{\iota \in J}$. *Then*

$$(P \vdash^{\alpha}_{m} \Delta, \neg C) \,\&\, (P \vdash^{\beta}_{m} C, \Gamma) \,\&\, (\mathrm{dp}(C) \leq m) \Rightarrow P \vdash^{\alpha+\beta}_{m} \Delta, \Gamma.$$

PROOF. This is seen by induction on β.

Consider first the case when C is a Δ_0-sentence. Then C is false and $J = \emptyset$. From $P \vdash^{\beta}_{m} C, \Gamma$ we see that $P \vdash^{\beta}_{m} \Gamma$. $\beta \leq \alpha + \beta$ yields $P \vdash^{\alpha+\beta}_{m} \Delta, \Gamma$.

Next assume that the last inference rule in $P \vdash^{\beta}_{m} C, \Gamma$ is a $(\bigvee)$ with the main formula $C \notin \Delta_0$:

$$\frac{P \vdash^{\beta(\iota)}_{m} C, C_\iota, \Gamma}{P \vdash^{\beta}_{m} C, \Gamma}\,(\bigvee),$$

where $\iota \in J$ and $\beta(\iota) < \beta$. We can assume that ι occurs in C_ι. Otherwise, set $\iota = 0$. Thus, $\iota \in P$ by (3). On the other hand, we have $P(\iota) \vdash^{\alpha}_{m} \Delta, \neg C_\iota$ by inversion, and hence $P \vdash^{\alpha}_{m} \Delta, \neg C_\iota$ by $\iota \in P$.

IH yields $P \vdash^{\alpha+\beta(\iota)}_{m} C_\iota, \Delta, \Gamma$. A cut inference with $P \vdash^{\alpha}_{m} \Delta, \neg C_\iota$ and $\mathrm{dp}(C_\iota) < \mathrm{dp}(C) \leq m$ yields $P \vdash^{\alpha+\beta}_{m} \Delta, \Gamma$.

Other cases are easily seen from IH. ⊣

LEMMA 3.11 (Predicative Cut-elimination). $P \vdash^{\alpha}_{m+1} \Gamma \Rightarrow P \vdash^{\omega^\alpha}_{m} \Gamma$.

PROOF. This is seen by induction on α using Lemma 3.10 and the fact: $\beta < \alpha \Rightarrow \omega^\beta + \omega^\beta \leq \omega^\alpha$. ⊣

For $\alpha <^{\varepsilon} \lceil \omega_n(\Omega + 1) \rceil$, set $RM^{\alpha}_{N} := RM_N(\alpha; <^{\varepsilon})$.

PROPOSITION 3.12. *Let* $\Gamma \subset \Pi_{N+1}$ $(N \geq 2)$ *and* $P \in RM^{\alpha}_{N}$ *be a transitive model of* $\mathsf{KP}\ell$. *Assume*

$$\exists \xi, x \in P(\xi <^{\varepsilon} \alpha \wedge \forall Q \in RM^{\xi}_{N} \cap P(x \in Q \models \mathsf{KP}\ell \to \Gamma \text{ is true in } Q)).$$

Then Γ *is true in* P.

PROOF. By $P \in RM^{\alpha}_{N}$ we have $P \in RM_N(RM^{\xi}_{N})$ for any $\xi \in P$, such that $\xi <^{\varepsilon} \alpha$.

Suppose contrarily that the Σ_{N+1}-sentence $\varphi := \bigwedge \neg \Gamma := \bigwedge\{\neg \theta : \theta \in \Gamma\}$ is true in P. Since $P \models \mathsf{KP}\ell$, the conjunction of Π_2-axioms of *bs* and *lim* (except the Foundation) holds in P. Then for any $\xi \in P$ with $\xi <^{\varepsilon} \alpha$ and $x \in P$ there exists a transitive model $Q \in RM^{\xi}_{N} \cap P$ of $\mathsf{KP}\ell$ such that $x \in Q$ and φ is true in Q. ⊣

LEMMA 3.13 (Elimination of $(Ref_{N+1}))$). *Let* $\Gamma \subset \Pi_{N+1}$. *Suppose* $P_0 \vdash^{\alpha}_{0} \Gamma$, $P_0 \in P$ *and* $P \in RM^{\alpha}_{N}$ *for a transitive model* P *of* $\mathsf{KP}\ell$. *Then,* Γ *is true in* P.

PROOF. This is seen by induction on α. Let $P_0 \vdash^{\alpha}_{0} \Gamma$, $P_0 \in P$, and $P \in RM^{\alpha}_{N}$ be a transitive model P of $\mathsf{KP}\ell$. Note that any sentence occurring in the witnessed derivation of $P_0 \vdash^{\alpha}_{0} \Gamma$ is Π_{N+1}.

CASE 1. When the last inference is a (Ref_{N+1}): By (3) we have $\{\alpha_\ell, \alpha_r\} \subset P_0 \subset P$, $\max\{\alpha_\ell, \alpha_r\} <^{\varepsilon} \alpha$, $A \in \Pi_{N+1}$.

$$\frac{P_0 \vdash^{\alpha_\ell}_{0} \Gamma, A(c) \qquad P_0 \vdash^{\alpha_r}_{0} \forall z[ad^z \to c \in z \to \neg A^z(c)], \Gamma}{P_0 \vdash^{\alpha}_{0} \Gamma}\,(Ref_{N+1})$$

We can assume that c occurs in $A(c)$, and hence $c \in P_0$.

By Proposition 2.3, we have $P \in RM^{\alpha_r}_{N}$. From IH we see that

$$\text{either } \forall z \in P[ad^z \to c \in z \to \neg A^z(c)] \text{ or } \bigvee \Gamma^P \text{ is true.} \tag{6}$$

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On the other hand, by IH, we have for any $Q \in RM_N^{\alpha_\ell} \cap P$ with $c \in P_0 \in Q \models \mathsf{KP}\ell$ that either $\bigvee \Gamma^Q$ is true or $A(c)^Q$ is true. By (6) for any $Q \in RM_N^{\alpha_\ell} \cap P$ with $P_0 \in Q \models \mathsf{KP}\ell$, $\bigvee \Gamma^Q \vee \bigvee \Gamma^P$ is true. From Proposition 3.12, we see that $\bigvee \Gamma^P$ is true.

Case 2. When the last inference is a $(\bigwedge)$: we have $A \simeq \bigwedge(A_\iota)_{\iota \in J}$, $A \in \Gamma$, and $\alpha(\iota) < \alpha$ for any $\iota \in J$

$$\frac{\{P_0(\iota) \vdash_0^{\alpha(\iota)} \Gamma, A_\iota : \iota \in J\}}{P_0 \vdash_0^{\alpha} \Gamma} (\bigwedge)$$

For any $\iota \in P$ we have $P_0(\iota) \in P$ since P is assumed to be a limit of transitive models of $\mathsf{KP}\omega$.

IH yields for any $\iota \in P$ that either $\bigvee \Gamma^P$ is true or A_ι^P is true. If $J = V$, then we are done. If $J = a \in V$, then $a \in P_0 \subset P$ by (3), and hence $a \subset P$.

Case 3. When the last inference is a $(\bigvee)$: we have $A \simeq \bigvee(A_\iota)_{\iota \in J}$, $A \in \Gamma$, and $\alpha(\iota) < \alpha$ for an $\iota \in J$

$$\frac{P_0 \vdash_0^{\alpha(\iota)} \Gamma, A_\iota}{P_0 \vdash_0^{\alpha} \Gamma} (\bigvee)$$

IH yields that either $\bigvee \Gamma^P$ is true or A_ι^P is true. Consider the case when $J = V$. We can assume that ι occurs in A_ι. Then $\iota \in P_0 \subset P$. Hence $\bigvee \Gamma^P$ is true. $\dashv$

Let us prove Theorem 2.4. Let $N \geq 2$, and A be a Π_{N+1}-sentence provable in $\mathsf{KP}\Pi_{N+1}$. Then $\mathsf{k}(A) = \emptyset$, and by Corollary 3.9 and Lemma 3.11, we have for an $n < \omega$ such that $P \vdash_0^{\omega_n(\Omega+1)} A$, for each transitive model $P \in V \cup \{V\}$ of $\mathsf{KP}\omega$. If $V \in RM_N^{\omega_n(\Omega+1)}$, then $L_{\omega_1^{CK}} \in V \models \mathsf{KP}\ell$, and A is true (in V) by Elimination of (Ref_{N+1}) 3.13.

By formalizing the above proof in $\mathrm{FiX}^i(\mathsf{KP}\ell)$ with Lemma 3.1 yields

$$\mathrm{FiX}^i(\mathsf{KP}\ell) \vdash V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^{\varepsilon}) \rightarrow A.$$

In the formalization note that, we have in $\mathrm{FiX}^i(\mathsf{KP}\ell)$, a partial truth definition of Π_{N+1}-sentences, cf. Lemma 2.2. Then by Theorem 3.2

$$\mathsf{KP}\ell \vdash V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^{\varepsilon}) \rightarrow A.$$

Finally noting that over $\mathsf{KP}\omega$, $V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^{\varepsilon})$ implies *lim*, the unboundedness of admissible sets, we conclude

$$\mathsf{KP}\omega \vdash V \in RM_N(\lceil \omega_n(\Omega+1) \rceil; <^{\varepsilon}) \rightarrow A.$$

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