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Characterizations of bounded mean oscillation through commutators of bilinear singular integral operators

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We characterize bounded mean oscillation in terms of the boundedness of commutators of various bilinear singular integral operators with pointwise multiplication. In particular, we study commutators of a wide class of bilinear operators of convolution type, including bilinear Calderón–Zygmund operators and the bilinear fractional integral operators.

Keywords: bilinear operators; singular integrals; Calderón–Zygmund theory; commutators; characterization of BMO

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1. Introduction and statements of main results

Recall that the space of functions with bounded mean oscillation (BMO) consists of all locally integrable functions, b, such that

$$\|b\|_* := \sup_Q \fint_Q |b(x) - b_Q| \, \mathrm{d}x < \infty,$$

where Q is a cube with sides parallel to the axes, and b_Q is the average of b over Q.

In the linear setting, we define the commutator of a function, b, with an operator, T, acting on a function f as

$$[b, T](f)(x) := b(x)T(f)(x) - T(bf)(x).$$

In [3], Coifman *et al*. showed that when T is the Hilbert transform the linear commutator is bounded if and only if $b \in \mathrm{BMO}$. Note that for $f \in L^p$ and $g \in L^{p'}$ we have

$$\langle [b, T](f), g \rangle = \langle T(f)g - fT^*(g), b \rangle,$$

where T^* denotes the transpose of T. In this light, we see that the characterization of the boundedness of the commutator with BMO functions means $T(f)g - fT^*(g)$, which is clearly in L^1, is in fact in the Hardy space H^1, the pre-dual of BMO. This allowed Coifman *et al*. to achieve a factorization of H^1 in a higher-dimensional setting than had previously been done. Janson [5] and Uchiyama [10] each extended this characterization of BMO to commutators of Calderón–Zygmund operators of convolution type with smooth homogeneous kernels, and Chanillo [1] did the same for commutators of the fractional integral operator with the restriction that $n - \alpha$ be an even integer. The boundedness of commutators in the multilinear setting has been extensively studied (see, for example, [2,6–9]). However, until now it has been an open question as to whether they can be used to characterize BMO. In this paper we shall show that the characterizations of BMO can indeed be extended to a multilinear setting. For clarity we shall state and prove our results only for the bilinear cases.

The bilinear commutators we shall examine will be of the following form:

$$[b, T]_1(f, g)(x) := bT(f, g)(x) - T(bf, g)(x)$$

and

$$[b, T]_2(f, g)(x) := bT(f, g)(x) - T(f, bg)(x),$$

where b is a locally integral function and T is a bilinear singular integral operator.

Though they are not required for our main theorem, we wish to first define bilinear Calderón–Zygmund operators; they are important to the background work of this paper and will arise in corollary 3.2, which is itself a main result of this paper. In order to define bilinear Calderón–Zygmund operators, we first define the class of Calderón–Zygmund kernels. Let $K(x, y, z)$ be a locally integrable function defined away from the diagonal $x = y = z$. If for some parameters A and ε, both positive, we have

$$|K(y_0, y_1, y_2)| \leqslant \frac{A}{(\sum_{k,l=0}^{2} |y_k - y_l|)^{2n}}$$

and

$$|K(y_0, y_1, y_2) - K(y_0', y_1, y_2)| \leqslant \frac{A|y_0 - y_0'|^{\varepsilon}}{(\sum_{k,l=0}^{2} |y_k - y_l|)^{2n+\varepsilon}},$$

whenever $|y_0 - y_0'| \leqslant \frac{1}{2}\max_{0\leqslant k\leqslant 2} |y_0 - y_k|$, with similar inequalities for y_1 and y_2, then we say that K is a bilinear Calderón–Zygmund kernel. Suppose, for some bilinear operator, T, defined on $L^{p_1} \times L^{p_2}$, we have

$$T(f_1, f_2)(x) = \iint K(x, y, z) f_1(y) f_2(z) \,\mathrm{d}y \,\mathrm{d}z$$

for all $x \notin \operatorname{supp}(f_1) \cap \operatorname{supp}(f_2)$, where K is a Calderón–Zygmund kernel. Then if

$$T\colon L^{p_1} \times L^{p_2} \to L^{p},$$

for some $p_1, p_2 > 1$ satisfying $1/p = 1/p_1 + 1/p_2$, we say T is a bilinear Calderón–Zygmund operator. Many basic properties of these operators were thoroughly studied by Grafakos and Torres in [4].

Lastly, we say that a kernel is 'homogeneous of degree k' if, for any $\lambda > 0$, we have

$$K(\lambda x, \lambda y, \lambda z) = \lambda^k K(x, y, z),$$

and we say an operator is of 'convolution type' if the kernel $K(x,y,z)$ is actually of the form $K(x-y,x-z)$. Our first theorem can now be stated as follows.

THEOREM 1.1. *Fix $n \in \mathbb{N}$, $\alpha \in [0,2n)$ and $p_1, p_2 \geqslant 1$ so that q, defined by*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n},$$

is greater than 1. Let T be a bilinear operator defined on $L^{p_1} \times L^{p_2}$ that can be represented as

$$T(f,g)(x) = \iint K(x-y,x-z)f(y)g(z)\,\mathrm{d}y\,\mathrm{d}z$$

for all $x \notin \operatorname{supp}(f) \cap \operatorname{supp}(g)$, where K is a homogeneous kernel of degree $-2n+\alpha$, and such that on some ball $B \subset \mathbb{R}^{2n}$ we have that the Fourier series of $1/K$ is absolutely convergent. Then if $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $j = 1$ or 2, we have that

$$[b,T]_j \colon L^{p_1} \times L^{p_2} \to L^q \implies b \in \mathrm{BMO}(\mathbb{R}^n).$$

It is worth noting that the condition on the Fourier coefficients of the kernel will, for example, be satisfied if K is smooth, and this is the assumption that similar arguments have used in the past. For $\alpha = 0$, this theorem includes the case where the operator is a bilinear Calderón–Zygmund operator, whereas if $0 < \alpha < 2n$, it includes the case where it is the bilinear fractional integral operator defined by

$$I_\alpha(f,g)(x) := \iint \frac{f(y)g(z)}{(|x-y|^2 + |x-z|^2)^{n-\alpha/2}}\,\mathrm{d}y\,\mathrm{d}z.$$

Our proof also works in the linear case, closing a gap in the literature, since in [1] the necessity that $b \in \mathrm{BMO}$ for the boundedness of the commutator was only shown when $n - \alpha$ was an even integer.

It should perhaps also be said that the restrictions on α, p_1 and p_2 are not an artefact of the proof, and indeed the proof works for $\alpha \in \mathbb{R}$ and $p_1, p_2 > 0$. However, it is not known whether there exist appropriate operators outside of these ranges for which the commutators are bounded. Thus, we have stated the theorem with values restricted to situations where it is known that application is possible.

2. Proofs of the theorems

The proof of theorem 1.1 uses techniques applied by Janson in [5], modified to suit the multilinear setting and extended for kernels with different homogeneities. We note that, by symmetry, it is enough to prove this for $[b,T]_1$.

Proof of theorem 1.1. Let $B = B((y_0,z_0), \delta\sqrt{2n}) \subset \mathbb{R}^{2n}$ be a ball for which we can express $1/K(y,z)$ as an absolutely convergent Fourier series of the form

$$\frac{1}{K(y,z)} = \sum_j a_j \mathrm{e}^{\mathrm{i}\nu_j \cdot (y,z)}.$$

The specific vectors ν_j will not play a role in this proof. Note that due to the homogeneity of K we can take (y_0,z_0) such that $|(y_0,z_0)| > 2\sqrt{n}$ and take $\delta < 1$

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small such that $\bar{B} \cap \{0\} = \emptyset$. We do not care about the specific vectors $\nu_j \in \mathbb{R}^{2n}$, but we shall at times express them as $\nu_j = (\nu_j^1, \nu_j^2) \in \mathbb{R}^n \times \mathbb{R}^n$.

Set $y_1 = \delta^{-1} y_0$ and $z_1 = \delta^{-1} z_0$, and note that

$$(|y - y_1|^2 + |z - z_1|^2)^{1/2} < \sqrt{2n} \implies (|\delta y - y_0|^2 + |\delta z - z_0|^2)^{1/2} < \delta\sqrt{2n},$$

and so for all (y, z) satisfying the inequality on the left we have

$$\frac{1}{K(y,z)} = \frac{\delta^{-2n+\alpha}}{K(\delta y, \delta z)} = \delta^{-2n+\alpha} \sum_j a_j \mathrm{e}^{\mathrm{i}\delta\nu_j \cdot (y,z)}.$$

Let $Q = Q(x_0, r)$ be an arbitrary cube in $\mathbb{R}^n$. Set $\tilde{y} = x_0 + r y_1$, $\tilde{z} = x_0 + r z_1$ and take $Q' = Q(\tilde{y}, r) \subset \mathbb{R}^n$ and $Q'' = Q(\tilde{z}, r) \subset \mathbb{R}^n$. Then, for any $x \in Q$ and $y \in Q'$, we have

$$\left|\frac{x-y}{r} - y_1\right| \leqslant \left|\frac{x - x_0}{r}\right| + \left|\frac{y - \tilde{y}}{r}\right| \leqslant \sqrt{n}.$$

The same estimate holds for $x \in Q$ and $z \in Q''$, and so we have

$$\left(\left|\frac{x-y}{r} - y_1\right|^2 + \left|\frac{x-z}{r} - z_1\right|^2\right)^{1/2} \leqslant \sqrt{2n}.$$

Let $\sigma(x) = \operatorname{sgn}(b(x) - b_{Q'})$. We then have the following:

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}| \,\mathrm{d}x \\
&\quad= \int_Q (b(x) - b_{Q'})\sigma(x) \,\mathrm{d}x \\
&\quad= \frac{1}{|Q''|}\frac{1}{|Q'|}\int_Q \int_{Q'} \int_{Q''} (b(x) - b(y))\sigma(x) \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x \\
&\quad= r^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (b(x) - b(y)) \frac{r^{2n-\alpha} K(x-y, x-z)}{K((x-y)/r, (x-z)/r)} \\
&\qquad\qquad \times \sigma(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z) \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x \\
&\quad= \delta^{-2n+\alpha} r^{-\alpha} \iiint (b(x) - b(y)) K(x-y, x-z) \\
&\qquad\qquad \times \sum_j a_j \mathrm{e}^{\mathrm{i}(\delta/r)\nu_j \cdot (x-y, x-z)} \\
&\qquad\qquad \times \sigma(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z) \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}$$

Let

$$\begin{aligned}
f_j(y) &= \exp\left(-\mathrm{i}\frac{\delta}{r}\nu_j^1 \cdot y\right)\chi_{Q'}(y), \\
g_j(z) &= \exp\left(-\mathrm{i}\frac{\delta}{r}\nu_j^2 \cdot z\right)\chi_{Q''}(z), \\
h_j(x) &= \exp\left(\mathrm{i}\frac{\delta}{r}\nu_j \cdot (x, x)\right)\sigma(x)\chi_Q(x).
\end{aligned}$$

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Note that each of the above functions has an L^q norm of $|Q|^{1/q}$ for any $q \geqslant 1$. Since Q, Q' and Q'' all have side length r, we shall have that $Q \cap Q' \cap Q'' = \emptyset$ if either $|x_0 - \tilde{y}| > r\sqrt{n}$ or $|x_0 - \tilde{z}| > r\sqrt{n}$. By the size condition on (y_0, z_0) we have that either $|y_0| > \sqrt{n}$ or $|z_0| > \sqrt{n}$. If $|y_0| > \sqrt{n}$, we have

$$|x_0 - \tilde{y}| = \left| x_0 - x_0 + r\frac{y_0}{\delta} \right| \geqslant r|y_0| > r\sqrt{n},$$

with an identical calculation if $z_0 > \sqrt{n}$. Therefore, we have that $Q \cap Q' \cap Q'' = \emptyset$, since at least one of Q' and Q'' must be disjoint from Q, and, for all x, y and z in the supports of their respective characteristic functions, the point $(x - y, x - z)$ avoids the singularity of K. In particular, this means that the use of the kernel representation of $[b, T](f_j, g_j)$ is valid for all $x \in Q$. Continuing with the above calculations, we have

$$\begin{aligned}
\int_Q |b(x) - b_{Q'}| \, \mathrm{d}x
&= \delta^{-2n+\alpha} r^{-\alpha} \sum_j a_j \int h_j(x) \iint (b(x) - b(y)) \\
&\qquad\qquad \times K(x - y, x - z) f_j(y) g_j(z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \delta^{-2n+\alpha} |Q|^{-\alpha/n} \sum_j a_j \int h_j(x) [b, T]_1 (f_j, g_j)(x) \, \mathrm{d}x \\
&\leqslant \delta^{-2n+\alpha} |Q|^{-\alpha/n} \sum_j |a_j| \int |h_j(x)| |[b, T]_1 (f_j, g_j)(x)| \, \mathrm{d}x \\
&\leqslant \delta^{-2n+\alpha} |Q|^{-\alpha/n} \sum_j |a_j| \left(\int |h_j(x)|^{q'} \, \mathrm{d}x \right)^{1/q'} \left(\int |[b, T]_1 (f_j, g_j)(x)|^q \, \mathrm{d}x \right)^{1/q} \\
&\leqslant \delta^{-2n+\alpha} |Q|^{-\alpha/n} \sum_j |a_j| \|h_j\|_{L^{q'}} \|[b, T]_1\|_{L^{p_1} \times L^{p_2} \to L^p} \|f_j\|_{L^{p_1}} \|g_j\|_{L^{p_2}} \\
&= \delta^{-2n+\alpha} \|[b, T]_1\|_{L^{p_1} \times L^{p_2} \to L^p} \sum_j |a_j| |Q|^{1/q'} |Q|^{1/p_1} |Q|^{1/p_2} |Q|^{-\alpha/n} \\
&= \delta^{-2n+\alpha} |Q| \|[b, T]_1\|_{L^{p_1} \times L^{p_2} \to L^p} \sum_j |a_j|.
\end{aligned}$$

Recall that

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q| \, \mathrm{d}x \leqslant \frac{2}{|Q|} \int_Q |f(x) - C|$$

for any C, and so for any arbitrary $Q \subset \mathbb{R}^n$ we have

$$\frac{1}{|Q|} \int_Q |b(x) - b_Q| \leqslant \frac{2}{|Q|} \int_Q |b(x) - b_{Q'}| \, \mathrm{d}x \leqslant 2 \|[b, T]_1\|_{L^{p_1} \times L^{p_2} \to L^p} \sum_j |a_j|.$$

Therefore, $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

3. Corollaries and closing remarks

In [7, proposition 3.1], Pérez and Torres showed that $b \in \mathrm{BMO}$ was sufficient to show the boundedness of commutators with m-linear Calderón–Zygmund operators, which we state in a simpler bilinear format without proof.

PROPOSITION 3.1. *If T is a bilinear Calderón–Zygmund operator and $b \in \mathrm{BMO}$, then $[b,T]_j\colon L^{p_1} \times L^{p_2} \to L^p$, for $j = 1$ or 2 and $1/p = 1/p_1 + 1/p_2$ with $1 < p, p_1, p_2 < \infty$.*

This, combined with theorem 1.1, immediately gives us the following.

COROLLARY 3.2. *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let T be a bilinear Calderón–Zygmund operator of convolution type with kernel K, a homogeneous function of degree $-2n$. Suppose that on some ball, B, in $\mathbb{R}^{2n}$ we have that the Fourier series of $1/K$ is absolutely convergent. Then, for $1 > 1/p = 1/p_1 + 1/p_2$, and $j = 1$ or 2,*

$$[b,T]_j\colon L^{p_1} \times L^{p_2} \to L^p \iff b \in \mathrm{BMO}(\mathbb{R}^n).$$

For $T = I_\alpha$, the sufficiency of $b \in \mathrm{BMO}$ to conclude the boundedness of $[b, I_\alpha]_i$ was shown for a class of weights that includes the unweighted case of [2, theorem 2.7]. As before, we state without proof a particular case of this theorem that suits our needs.

PROPOSITION 3.3. *Let $0 < \alpha < 2n$ and $1 \leqslant p_1, p_2$, and let q be such that*

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q}.$$

Then

$$\|[b, I_\alpha]_j(f,g)\|_{L^q} \lesssim \|b\|_* \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

for $j = 1$ or 2.

The kernel of I_α has precisely the homogeneity required by theorem 1.1, and the reciprocal of the convolution kernel of I_α, $(|y|^2 + |z|^2)^{n-\alpha/2}$, is smooth away from the origin and so its Fourier series will indeed have regions in which it is absolutely convergent. These facts give us the following result.

COROLLARY 3.4. *For $b \in L^1_{\mathrm{loc}}$, $0 < \alpha < 2n$ and $1 < p_1, p_2$, and for q satisfying*

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{q} < 1$$

we have

$$\|[b, I_\alpha]_j\|_{p_1 \times p_2 \to q} \approx \|b\|_* \quad \text{for } j = 1 \text{ or } 2.$$

In particular, for $j = 1$ or 2,

$$[b, I_\alpha]_j\colon L^{p_1} \times L^{p_2} \to L^q \iff b \in \mathrm{BMO}.$$

With regards to our main theorem, two key things should be noted. First, the proof easily generalizes to commutators with the m-linear operators and homogeneous kernels of degree $-mn + \alpha$. The original statements in [7, proposition 3.1] and [2, proposition 3.3] are for m-linear commutators, so corollaries 3.2 and 3.4 hold in the m-linear setting as well. Second, we assume that only the commutator, not the underlying operator, is bounded and this could allow us to potentially investigate situations with $\alpha \notin [0, 2n)$, such as the hypersingular pseudo-differential operators corresponding to $\alpha < 0$. As was stated at the end of § 1, it is unclear whether there exist such an appropriately homogeneous unbounded T and non-trivial b so that $[b, T]_j$ is bounded. However, we know that should they exist, it must be the case that b is in BMO.

Finally, we observe that since our proof required the use of Hölder's inequality with q and q', the exponent in our target space must be larger than 1. We do not know if it is possible to characterize BMO in terms of the boundedness of commutators for $L^{p_1} \times L^{p_2} \to L^q$ for $\frac{1}{2} < q < 1$. This is of interest because bounds of this form have indeed been shown. In particular, in [6], Lerner *et al*. showed that commutators with m-linear Calderón–Zygmund operators are bounded from $\prod_{j=1}^{m} L^{p_j}$ to L^p, for any $1 < p_1, \dots, p_m$ such that

$$\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j},$$

provided that $b \in$ BMO. In [9], Tang obtained this result for commutators of vector-valued multilinear Calderón–Zygmund operators, again without the restriction that p be greater than 1.

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