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CENTRE MANIFOLD VARIETY FOR EIGHT-PARAMETER FAMILIES OF POLYNOMIAL VECTOR FIELDS OF ARBITRARY DEGREE

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Abstract We characterize the centre variety of the eight-parameter families of real planar polynomial vector fields given, in complex notation, by $\dot{z} = \mathrm{i}z + Az^{n_1}\bar{z}^{j_1} + Bz^{n_2}\bar{z}^{j_2} + Cz^{n_3}\bar{z}^{j_3} + Dz^{n_4}\bar{z}^{j_4}$, where $A, B, C, D \in \mathbb{C} \setminus \{0\}$, $(n_1, j_1) \neq (n_2, j_2) \neq (n_3, j_3) \neq (n_4, j_4)$, $n_k + j_k > 1$ for $k = 1, 2, 3, 4$, $n_1 + j_1 = n_2 + j_2 = n_3 + j_3 = n_4 + j_4$, $|1 - n_3 + j_3| = |1 - n_2 + j_2| \neq |1 - n_1 + j_1|$ and $j_4 = n_4 - 1$.

Keywords: centres; arbitrary degree; polynomial vector fields; eight-parameter families

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1. Introduction and statement of the main results

We consider planar real analytic differential systems that can be written in complex notation $z = x + \mathrm{i}y$, as

$$\dot{z} = \mathrm{i}z + F(z, \bar{z}). \tag{1.1}$$

We are concerned mainly with conditions under which the origin of (1.1) is a centre. This problem has attracted much attention over the years since the pioneering work of Poincaré in [**17**].

In its real formulation, systems like (1.1) are used to model various phenomena such as crystallization of agates (see [**20**, **25**]).

The usual method for looking for non-degenerate centres (i.e. those having purely imaginary eigenvalues) of planar polynomial vector fields such as (1.1) is to calculate the successive coefficients v_i of the return map of the vector field around the origin. That is, we choose a segment $(0, x_0]$ on the positive x-half-axis transversal to the flow of the vector field with parameter x, and represent the analytical return map by the series expansion $h(x) = x + \sum_{i\geqslant 2} v_i x^i$. It is known that the constants v_i are polynomial functions in the coefficients of the polynomial vector field. The terms v_k are functions of the previous v_i, for $i = 2, \ldots, k-1$, and it is well known (see, for example, [**1**, Chapter IX, Lemma 5, p. 243]) that when k is even v_k is zero. Therefore, the only interesting functions are those of the form v_{2i+1}. The functions v_{2i+1} are called the *Lyapunov constants* of the polynomial vector field.

When all the v_{2i+1} vanish, the origin is a *centre*. The set of coefficients for which all the v_i vanish is called the *centre variety* of the family of polynomial vector fields, which, by the Hilbert Basis Theorem, is an algebraic set.

In general is very difficult to study centres, requiring a good knowledge not only of common zeros of the polynomials v_i but also of the finitely generated ideal that they generate in the ring of polynomials with variables that are the coefficients of the polynomial vector field. Furthermore, in general, the calculation of the Lyapunov constants is not easy, and the computational complexity of finding their common zeros grows very quickly. A number of algorithms have been developed to compute them automatically (see [**3**, **6**, **7**, **13**–**16**] and the references therein).

The classification of centres in polynomial vector fields started with the quadratic ones in the works of Dulac, Kapteyn, Bautin, Żołądek and others (see [**19**] for references). It continued with symmetric cubic systems (those without quadratic terms) and projective quadratic systems [**12**, **21**, **26**]. Lyapunov constants are also well known for Liénard systems [**2**, **27**].

Our goal is to study the centres of *general* eight-parameter families of polynomial vector fields with arbitrary homogeneous nonlinearities. Despite the fact that nonlinear systems with quadratic and cubic nonlinearities are well understood [**15**, **18**, **26**], very few families of centres of arbitrary degree are known. See, for example, [**5**, **11**] for the case of four-parameter families of polynomial vector fields of arbitrary degree and [**10**, **23**] for the case of six-parameter families of polynomial vector fields. We emphasize that in the present paper we are dealing with *general* eight-parameter families of polynomial vector fields. It turns out that the two extra parameters make computations substantially harder and much more involved. We note that some eight-parameter families were considered earlier, in [**24**]; in fact, that paper deals with systems with an arbitrary number of parameters greater than or equal to six, with nonlinearities of arbitrary degree, but having very particular structure.

In this paper we shall use complex notation $z = x + \mathrm{i}y$. Then, any real polynomial differential system having the linear part at the origin of the form $\dot{x} = -y$, $\dot{y} = x$ can be written as

$$\dot{z} = \mathrm{i}z + \sum_{i+j\geqslant 2} A_{ij} z^i \bar{z}^j,$$

with $A_{ij} \in \mathbb{C}$. The dot denotes derivative with respect to the independent variable t. The complex notation is especially convenient for theoretical work. For instance, the conditions for which a polynomial vector field is symmetric with respect to a line can be easily obtained in this notation; later on we shall use this fact.

We consider the family of real polynomial differential equations in $(x, y) \in \mathbb{R}^2$ that in complex notation can be written

$$\dot{z} = \mathrm{i}z + Az^{n_1}\bar{z}^{j_1} + Bz^{n_2}\bar{z}^{j_2} + Cz^{n_3}\bar{z}^{j_3} + Dz^{n_4}\bar{z}^{j_4}, \tag{1.2}$$

with $A, B, C, D \in \mathbb{C} \setminus \{0\}$, and where $n_1, j_1, n_2, j_2, n_3, j_3, n_4$ and j_4 are non-negative integers such that $(n_1, j_1) \neq (n_2, j_2) \neq (n_3, j_3) \neq (n_4, j_4)$, $n_k + j_k > 1$ for $k = 1, 2, 3, 4$,

$n_1 + j_1 = n_2 + j_2 = n_3 + j_3 = n_4 + j_4$, $|1 - n_3 + j_3| = |1 - n_2 + j_2| \neq |1 - n_1 + j_1|$ and $j_4 = n_4 - 1$.

Note that since $j_4 = n_4 - 1$, $n_4 + j_4 = 2n_4 - 1$ is odd and thus system (1.2) is a nonlinear system with homogeneous nonlinearities of odd degree $2n_4 - 1$. Furthermore, from the facts that $(n_1, j_1) \neq (n_2, j_2) \neq (n_3, j_3) \neq (n_4, j_4)$, $j_4 = n_4 - 1$ and the nonlinear part of (1.2) is homogeneous, we get that $j_k \neq n_k - 1$ for $k = 1, 2, 3$ and

$$1 - n_3 + j_3 = -(1 - n_2 + j_2) = n_2 + j_2 - 1. \tag{1.3}$$

Hence, by (1.3) we can always assume that $(1 - n_1 + j_1)(1 - n_2 + j_2) > 0$.

We consider the case in which the parameters A, B, C, D are all non-zero since the case when at least two of the parameters are zero is studied in [**11**] in the more general case of the nonlinearity of the vector field not necessarily being homogeneous. Furthermore, the case in which $D = 0$ and $A, B, C \neq 0$ is studied in [**23**] and the case in which $D \neq 0$ and either $A = 0$ or $B = 0$ or $C = 0$ is studied in [**10**] also in the more general case of the not necessarily homogeneous nonlinearity.

For completeness of the present paper, the results in [**10**, **11**, **23**] are given in the appendix. The main results are the following.

Theorem 1.1. *We consider the family of polynomial differential systems (1.2). Then the following statements hold:*

(a) $v_k = 0$ *for any* k *such that* $k/(n_4 - 1) \notin \mathbb{N}$*;*

(b) $\mathrm{Re}(D) = 0$ *is a necessary condition for system (1.2) to have a centre at the origin;*

(c) $\mathrm{Im}(BC) = 0$ *is a necessary condition for system (1.2) restricted to* $\mathrm{Re}(D) = 0$ *to have a centre at the origin.*

Set $N_1 = |1 - n_1 + j_1|$, $K_1 = |1 - n_2 + j_2|$, $M_1 = \gcd\{N_1, K_1\}$, $N_1 = M_1 N_2$ and $K_1 = M_1 K_2$. Note that, since $N_2, K_2 \geqslant 1$ and $\gcd\{N_2, K_2\} = 1$, $N_2 + K_2 \geqslant 3$. Moreover, as pointed out before, $n_1 + j_1 - 1 = 2(n_4 - 1)$ is even, $N_1, K_1 \geqslant 1$, $N_1 \neq K_1$ and $j_k \neq n_k - 1$ for $k = 1, 2, 3$.

Theorem 1.2. *We consider the family of polynomial differential systems (1.2) with* $\mathrm{Re}(D) = \mathrm{Im}(BC) = 0$*. Then* $v_{2k+1} = 0$ *for* $k = 2, 3, \dots, m$ *with* $m < (n_4 - 1)(K_2 + N_2)$*.*

We set

$$\mathbb{L}_{A,C}^{K_2,N_2} = \bar{A}^{K_2} \bar{C}^{N_2} - (-1)^{K_2+N_2} A^{K_2} C^{N_2}. \tag{1.4}$$

Theorem 1.3. *We consider the family of polynomial differential systems (1.2) with* $\mathrm{Re}(D) = \mathrm{Im}(BC) = 0$*. Then there exist real constants* L_k*,* $k = 0, \dots, N_2$*, such that, for* $m = (n_4 - 1)(K_2 + N_2)$*,*

$$v_{2m+1} = \mathbb{L}_{A,C}^{K_2,N_2} \sum_{k=0}^{N_2} L_k \left(\frac{B}{\bar{C}}\right)^k. \tag{1.5}$$

Furthermore, if $n_1 = 0$*, then*

$$v_{2m+1} = \frac{1}{n_2 \bar{C}} \mathbb{L}_{A,C}^{K_2,N_2} (n_2 B + n_3 \bar{C}) \sum_{k=1}^{N_2} L_k \sum_{j=0}^{k-1} \left(\frac{B}{\bar{C}}\right)^{k-1-j} \left(\frac{-n_3}{n_2}\right)^j. \tag{1.6}$$

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Theorem 1.4. *We consider the family of polynomial differential systems (1.2) with* $\mathrm{Re}(D) = \mathrm{Im}(BC) = 0$*. Then there exist real constants* M_k*,* $k = 0, \dots, N_2$ *such that, for* $m = (n_4 - 1)(K_2 + N_2 + 1)$*,*

$$v_{2m+1} = D\mathbb{L}_{A,C}^{K_2,N_2} \sum_{k=0}^{N_2} M_k \left(\frac{B}{\bar{C}}\right)^k. \tag{1.7}$$

Furthermore, if $n_1 = 0$*, then*

$$v_{2m+1} = \frac{D}{n_2\bar{C}} \mathbb{L}_{A,C}^{K_2,N_2} (n_2 B + n_3 \bar{C}) \sum_{k=1}^{N_2} M_k \sum_{j=0}^{k-1} \left(\frac{B}{\bar{C}}\right)^{k-1-j} \left(\frac{-n_3}{n_2}\right)^j. \tag{1.8}$$

Theorem 1.5. *We consider the family of polynomial differential systems (1.2) with* $\mathrm{Re}(D) = \mathrm{Im}(BC) = 0$*. Then the following hold.*

(a) *If* $n_1 \neq 0$ *and the constants* L_k *and* M_k *in Theorems 1.3 and 1.4 are such that the unique solution of*

$$\sum_{k=0}^{N_2} L_k \left(\frac{B}{\bar{C}}\right)^k = \sum_{k=0}^{N_2} M_k \left(\frac{B}{\bar{C}}\right)^k = 0$$

is $B = 0$*, the centre manifold variety of system (1.2) is*

$$\{(A, B, C, D) \in (\mathbb{C} \setminus \{0\})^4 : \mathrm{Re}(D) = \mathrm{Im}(BC) = \mathbb{L}_{A,C}^{K_2,N_2} = 0\}.$$

(b) *If* $n_1 = 0$*, and the constants* L_k *and* M_k *in Theorems 1.3 and 1.4 are such that the unique solution of*

$$\sum_{k=0}^{N_2} L_k \sum_{j=0}^{k-1} \left(\frac{B}{\bar{C}}\right)^{k-1-j} \left(\frac{-n_3}{n_2}\right)^j = \sum_{k=1}^{N_2} M_k \sum_{j=0}^{k-1} \left(\frac{B}{\bar{C}}\right)^{k-1-j} \left(\frac{-n_3}{n_2}\right)^j = 0$$

is $B = 0$*, the centre manifold variety of system (1.2) is*

$$\{(A, B, C, D) \in (\mathbb{C} \setminus \{0\})^4 : \mathrm{Re}(D) = 0,\ n_2 B + n_3 \bar{C} = 0\} \cup \{(A, B, C, D) \in (\mathbb{C} \setminus \{0\})^4 : \mathrm{Re}(D) = \mathrm{Im}(BC) = \mathbb{L}_{A,C}^{K_2,N_2} = 0\}.$$

We emphasize that Theorem 1.5 characterizes the centre manifold variety of system (1.2).

The proofs of Theorems 1.2, 1.3 and 1.4 are strongly based on algebraic properties of the Lyapunov constants of the polynomial vector fields. More concretely, to prove that the Lyapunov constants are zero, we will first see which are the admissible monomials that appear in v_{2m+1} for each m. After this, we will compute the coefficients of the polynomial v_{2m+1}. To do that we will use our knowledge of several types of centres for the systems (1.2): reversible centres and Hamiltonian centres (see § 3 for their definition).

The paper is organized as follows. In § 2 we illustrate our results with a family of nonlinear systems with homogeneous nonlinearities of odd degree n. In § 3 we present some basic results. In § 4 we prove Theorem 1.1 and in § 5 we provide the proof of Theorem 1.2. Theorems 1.3 and 1.4 are proved in § 6, while Theorem 1.5 is proved in § 7. For completeness, the case when at least two of the parameters A, B, C, D are zero (studied in [**11**]), the case in which $D \neq 0$ and either $A = 0$ or $B = 0$ or $C = 0$ (studied in [**10**]) and the case in which $D = 0$ and $A, B, C \neq 0$ (studied in [**23**]) are given in the appendix.

2. Example

We consider the following linear systems with homogeneous nonlinearities of odd degree n, $n \geqslant 5$:

$$\dot{z} = \mathrm{i}z + (z\bar{z})^{(n-3)/2}(A\bar{z}^3 + Bz\bar{z}^2 + Cz^3 + Dz^2\bar{z}), \tag{2.1}$$

with $n \geqslant 5$ odd and $A, B, C, D \in \mathbb{C} \setminus \{0\}$. We note that when $n = 3$ we are considering a well-known homogeneous cubic system (for more details see [**15, 18**]). We refer to this system since it serves as a test for our results for infinitely many examples. Computing numerically the Lyapunov constants (see [**15**] for a more detailed explanation of the numerical method to compute the Lyapunov constants when $n = 3$ and [**9**] for a more detailed explanation of the computation of the Lyapunov constants when $n \geqslant 5$ odd) and removing the non-zero multiplicative factors, we have that

$$\left.\begin{aligned} v_k &= 0 \quad \text{if } 1 \leqslant k \leqslant n-1 \text{ and } v_n = D + \bar{D}, \\ v_k &= 0 \quad \text{if } n+1 \leqslant k \leqslant 2n-2 \text{ and } v_{2n-1} = \bar{B}\bar{C} - BC, \\ v_k &= 0 \quad \text{if } 2n \leqslant k \leqslant 3n-3, \\ v_{3n-2} &= -\left(3 + \frac{B}{\bar{C}}\right)(\bar{A}\bar{C}^2 + AC^2)\left((5-n) - (3+n)\frac{B}{\bar{C}}\right), \\ v_k &= 0 \quad \text{if } 3n-1 \leqslant k \leqslant 4n-4, \\ v_{4n-3} &= -\left(3 + \frac{B}{\bar{C}}\right)D(\bar{A}\bar{C}^2 + AC^2)\left((n-3) + (n+1)\frac{B}{\bar{C}}\right). \end{aligned}\right\} \tag{2.2}$$

In the notation of (1.2), we have that $n_1 = (n-3)/2$, $j_1 = (n+3)/2$, $n_2 = (n-1)/2$, $j_2 = (n+1)/2$, $n_3 = (n+3)/2$, $j_3 = (n-3)/2$ and $n_4 = (n+1)/2$, $j_4 = (n-1)/2$. Furthermore, $N_1 = 4$, $K_1 = 2$ and thus $M_1 = 2$, $N_2 = 2$ and $K_2 = 1$.

Then, since $n_4 - 1 = (n-1)/2$, Theorems 1.1 and 1.2 imply that if $\mathrm{Re}(D) = 0$ and $\mathrm{Im}(BC) = 0$, then $v_k = 0$ for $k < 3n-2$ and also that $v_k = 0$ for $3n-1 \leqslant k \leqslant 4n-4$. This matches perfectly well with the computed values of v_k in (2.2). Furthermore, for the value v_{2m+1} with $m = (n_4 - 1)(K_2 + N_2)$, i.e. for v_{3n-2}, from Theorem 1.3 we have that since system (2.1) has $\mathbb{L}^{1,2}_{A,C} = \bar{A}\bar{C}^2 + AC^2$, and not necessarily $n_1 = 0$, using that $BC = \bar{B}\bar{C}$, $B \neq 0$, we obtain

$$v_{3n-2} = (\bar{A}\bar{C}^2 + AC^2)\left(L_0 + L_1\frac{B}{\bar{C}} + L_2\left(\frac{B}{\bar{C}}\right)^2\right)\Bigg)^2\Bigg).$$

This is the same as v_{3n-2} in (2.2) taking $L_0 = 3(n-5)$, $L_1 = 4(n+1)$ and $L_2 = 3 + n$.

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Furthermore, by Theorem 1.4 and using the fact that $\bar{D}+D=0$, $BC=\bar{B}\bar{C}$, $B\neq 0$, we obtain

$$v_{4n-3}=D(AC^2+\bar{A}\bar{C}^2)\biggl(M_0+M_1\frac{B}{\bar{C}}+M_2\biggl(\frac{B}{\bar{C}}\biggr)^2\biggr).$$

This matches with v_{4n-3} in (2.2) taking $M_0=3(3-n)$, $M_1=-4n$ and $M_2=-(n+1)$.

3. Preliminary results

Using polar coordinates $r^2=z\bar{z}$, $\theta=\arctan(\operatorname{Im}(z)/\operatorname{Re}(z))$, in a neighbourhood of the origin system (1.1) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{r^2P_2(\theta)+r^3P_3(\theta)+\cdots}{1+rQ_2(\theta)+r^2Q_3(\theta)+\cdots}, \tag{3.1}$$

where

$$P_k(\theta)=\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\theta}F_k(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})),\qquad Q_k(\theta)=\operatorname{Im}(\mathrm{e}^{-\mathrm{i}\theta}F_k(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})),$$

and

$$F(z,\bar{z})=\sum_{k\geqslant 2}F_k(z,\bar{z})=\sum_{k\geqslant 2}F_k(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{-\mathrm{i}\theta})r^k.$$

Here F_k denotes the homogeneous part of degree k of F in the variables z and $\bar{z}$.

Consider the solution $r(\theta,x)$ of (3.1) that takes the value x at $\theta=0$. Then,

$$r(\theta,x)=u_1(\theta)x+u_2(\theta)x^2+\cdots,$$

with $u_1(0)=1$ and $u_k(0)=0$ for $k\geqslant 2$. Then $h(x)=r(2\pi,x)$ is the return map. Assume that system (3.1) satisfies $u_k(2\pi)=0$ for $k=2,3,\dots,2m$ and $u_{2m+1}(2\pi)\neq 0$. Therefore, $v_{2m+1}=u_{2m+1}(2\pi)$ is the mth Lyapunov constant.

It is known that the Lyapunov constant v_{2m+1} is a polynomial in the coefficients of F_i, $i=2,3,\dots,2m+1$, and their conjugates. We set

$$F(z,\bar{z})=\sum_{k+l\geqslant 2}A_{kl}z^k\bar{z}^l$$

and we will use the notation

$$v_{2m+1}=v_{2m+1}(F_2,F_3,\dots,F_{2m+1})=v_{2m+1}(A_{kl},\bar{A}_{kl}).$$

Some algebraic properties of the Lyapunov constants are established in the following well-known result. For its proof and additional properties of the Lyapunov constants we refer the reader to [**4**, **8**, **22**, **26**] and the references therein.

Proposition 3.1. *Let v_{2m+1} be the mth Lyapunov constant of system (1.1). Then it satisfies the following properties:*

(a) $v_{2m+1}(\lambda^{1-k+l}A_{kl},\lambda^{-(1-k+l)}\bar{A}_{kl})=v_{2m+1}(A_{kl},\bar{A}_{kl})$;

(b) $v_{2m+1}(\lambda^{k+l-1}A_{kl},\lambda^{k+l-1}\bar{A}_{kl})=\lambda^{2m}v_{2m+1}(A_{kl},\bar{A}_{kl})$.

Now there is an easy way to list the monomials that appear in v_{2m+1}. For $K \in \mathbb{R}$ let

$$\hat{M} = K\biggl(\prod_{i=1}^{r} A_{k_i l_i}^{m_i}\biggr)\biggl(\prod_{i=r+1}^{r+s} \bar{A}_{k_i l_i}^{m_i}\biggr) \tag{3.2}$$

be a monomial of v_{2m+1}. Then, Proposition 3.1 (a) implies

$$\sum_{i=1}^{r}(1-k_i+l_i)m_i = \sum_{i=r+1}^{r+s}(1-k_i+l_i)m_i, \tag{3.3}$$

and Proposition 3.1 (b) implies

$$\sum_{i=1}^{r+s}(k_i+l_i-1)m_i = 2m. \tag{3.4}$$

These last two equalities will be very useful for effective computation of the Lyapunov constants of system (1.2).

We say that a system (1.1) is *reversible* if it is invariant under the change of variables $\bar{w} = \mathrm{e}^{\mathrm{i}\gamma}z$, $\tau = -t$. For system (1.2) we have the following result.

Lemma 3.2. *System (1.2) is reversible if and only if* $A = -\bar{A}\mathrm{e}^{-\mathrm{i}(1-n_1+j_1)\gamma}$, $B = -\bar{B}\mathrm{e}^{-\mathrm{i}(1-n_2+j_2)\gamma}$, $C = -\bar{C}\mathrm{e}^{-\mathrm{i}(1-n_3+j_3)\gamma}$ *and* $D = -\bar{D}$ *for some* $\gamma \in \mathbb{R}$. *Furthermore, in this situation the origin of system (1.2) is a centre.*

Proof. The proof follows directly from the definition of reversibility. □

We say that a system (1.1) is *Hamiltonian* if it satisfies $\mathrm{Re}(\partial F/\partial z) = 0$. For system (1.2) we have the following result.

Lemma 3.3. *If* $n_1 \neq 0$, *system (1.2) is never Hamiltonian. Furthermore, if* $n_1 = 0$, *then system (1.2) is Hamiltonian if and only if* $\mathrm{Re}(D) = 0$ *and* $n_2B + n_3\bar{C} = 0$. *Furthermore, in this situation, the origin of system (1.2) is a centre.*

Proof. The proof follows directly from the definition of a system being Hamiltonian, taking into account the fact that $A \neq 0$. □

From (3.2) with $A_{n_1 j_1} = A$, $A_{n_2 j_2} = B$ and $A_{n_3 j_3} = C$ we have that the monomials which appear in v_{2m+1} are of the form

$$\hat{M} = KA^{m_1}B^{m_2}C^{m_3}D^{m_4}\bar{A}^{m_5}\bar{B}^{m_6}\bar{C}^{m_7}\bar{D}^{m_8},$$

with $K \in \mathbb{R}$ and m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8 being non-negative integers satisfying (3.3) and (3.4), that is,

$$(1-n_1+j_1)(m_1-m_5) + (1-n_2+j_2)(m_2-m_6) + (1-n_3-j_3)(m_3-m_7) = 0, \tag{3.5}$$

and since $n_1+j_1 = n_2+j_2 = n_3+j_3 = n_4+j_4 = 2n_4-1$ we get

$$(n_4-1)[m_1+m_5+m_2+m_6+m_3+m_7+m_4+m_8] = m. \tag{3.6}$$

From (1.3) and (3.5) we have that

$$\sigma_1 N_1(m_1 - m_5) + \sigma_2 K_1(m_2 - m_6 + m_7 - m_3) = 0, \tag{3.7}$$

where, for $k = 1, 2$,

$$\sigma_k = \begin{cases} 1 & \text{if } 1 - n_k + j_k > 0, \\ -1 & \text{if } 1 - n_k + j_k < 0. \end{cases} \tag{3.8}$$

Then, since $N_1 = M_1 N_2$, $K_1 = M_1 K_2$, after dividing by M_1, (3.7) becomes

$$\sigma_1 N_2(m_1 - m_5) + \sigma_2 K_2(m_2 - m_6 + m_7 - m_3) = 0. \tag{3.9}$$

Let

$$s_1 = m_2 + m_7, \qquad s_2 = m_3 + m_6. \tag{3.10}$$

Then from (3.9) we obtain

$$\sigma_1 N_2(m_1 - m_5) = \sigma_2 K_2(s_2 - s_1). \tag{3.11}$$

Now, since $\gcd\{N_2, K_2\} = 1$, from (3.11) we have that there exists $r \in \mathbb{Z} \cup \{0\}$ such that

$$m_1 = m_5 + \sigma_2 K_2 r \quad \text{and} \quad s_2 = s_1 + \sigma_1 N_2 r. \tag{3.12}$$

4. Proof of Theorem 1.1

Proof of Theorem 1.1 (a). By (3.6), $v_m = 0$ if m is not divisible by $n_4 - 1$. This finishes the proof of Theorem 1.1 (a). □

Proof of Theorem 1.1 (b). We can assume that $m = n_4 - 1$, since for $m < n_4 - 1$ we have that m is not divisible by $n_4 - 1$ and in view of Theorem 1.1 (a) we get $v_m = 0$. Therefore, we will compute $v_{2m+1} = v_{2n_4-1}$. Recall that $n_1 - j_1 - 1 = \sigma_1 N_1$, $n_2 - j_2 - 1 = \sigma_2 K_1$ and $n_3 - j_3 - 1 = -(n_2 - j_2 - 1) = -\sigma_2 K_1$, and thus if we introduce the notation

$$\begin{aligned} F_A(\theta) &= \mathrm{Re}(A)\cos(\sigma_1 N_1 \theta) - \mathrm{Im}(A)\sin(\sigma_1 N_1 \theta), \\ G_{B,C}(\theta) &= (\mathrm{Re}(B) + \mathrm{Re}(C))\cos(\sigma_2 N_2 \theta) - (\mathrm{Im}(B) - \mathrm{Im}(C))\sin(\sigma_2 N_2 \theta), \\ H_A(\theta) &= \mathrm{Re}(A)\sin(\sigma_1 N_1 \theta) + \mathrm{Im}(A)\cos(\sigma_1 N_1 \theta), \\ I_{B,C}(\theta) &= (\mathrm{Re}(B) - \mathrm{Re}(C))\sin(\sigma_2 N_2 \theta) + (\mathrm{Im}(B) + \mathrm{Im}(C))\cos(\sigma_2 N_2 \theta), \end{aligned}$$

system (1.2) in polar coordinates takes the form

$$\begin{aligned} \dot r &= r^{2n_4-1}(\mathrm{Re}(D) + F_A(\theta) + G_{B,C}(\theta)), \\ \dot\theta &= 1 + r^{2(n_4-1)}(\mathrm{Im}(D) + H_A(\theta) + I_{B,C}(\theta)), \end{aligned}$$

and thus,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r^{2n_4-1}(\mathrm{Re}(D) + F_A(\theta) + G_{B,C}(\theta))}{1 + r^{2(n_4-1)}(\mathrm{Im}(D) + H_A(\theta) + I_{B,C}(\theta))}.$$

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Therefore, in a neighbourhood of $r = 0$, $\mathrm{d}r/\mathrm{d}\theta$ can be written as

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} = r^{2n_4-1}(\mathrm{Re}(D) + F_A(\theta) + G_{B,C}(\theta)) \\ - r^{4n_4-3}(\mathrm{Re}(D) + F_A(\theta) + G_{B,C}(\theta)) \\ \times (\mathrm{Im}(D) + H_A(\theta) + I_{B,C}(\theta)) + O(r^{6n_4-5}).\end{aligned} \tag{4.1}$$

We denote by $r(\theta, x)$ the solution of (4.1) such that $r(0, x) = x$ and write it as

$$r(\theta, x) = x + u_{2n_4-1}(\theta)x^{2n_4-1} + \cdots .$$

Computing in (4.1) the terms with x^{2n_4-1}, we deduce that

$$\frac{\mathrm{d}u_{2n_4-1}(\theta)}{\mathrm{d}\theta} = \mathrm{Re}(D) + F_A(\theta) + G_{B,C}(\theta) \tag{4.2}$$

and thus, since $u_{2n_4-1}(0) = 0$, we have that

$$\begin{aligned}u_{2n_4-1}(\theta) = \mathrm{Re}(D)\theta + \frac{\mathrm{Re}(A)}{\sigma_1 N_1}\sin(\sigma_1 N_1\theta) + \frac{\mathrm{Im}(A)}{\sigma_1 N_1}\cos(\sigma_1 N_1\theta) \\ + \frac{\mathrm{Re}(B) + \mathrm{Re}(C)}{\sigma_2 N_2}\sin(\sigma_2 N_2\theta) + \frac{\mathrm{Im}(B) - \mathrm{Im}(C)}{\sigma_2 N_2}\cos(\sigma_2 N_2\theta) \\ - \frac{\mathrm{Im}(A)}{\sigma_1 N_1} - \frac{\mathrm{Im}(B) - \mathrm{Im}(C)}{\sigma_2 N_2}.\end{aligned} \tag{4.3}$$

Now, writing $h(x) = r(2\pi, x)$ we have from (4.2) that

$$\begin{aligned}h(x) &:= x + v_{2n_4-1}x^{2n_4-1} + \cdots \\ &= u_1(2\pi)x + u_{2n_4-1}(2\pi)x^{2n_4-1} + \cdots \\ &= x + 2\pi\,\mathrm{Re}(D) + O(x^{2n_4}).\end{aligned}$$

Therefore, in order that $v_{2n_4-1} := u_{2n_4-1}(2\pi) \neq 0$, $\mathrm{Re}(D) = 0$ must hold, which yields that $\mathrm{Re}(D) = 0$ is a necessary condition for system (1.2) to have a centre at the origin. This finishes the proof of Theorem 1.1 (b). □

Proof of Theorem 1.1 (c). We restrict (4.1) and (4.3) to $\mathrm{Re}(D) = 0$ and compute $u_{4n_4-3}(\theta)$. From (4.1) and (4.3), computing the coefficients of x^{4n_4-3}, we obtain that

$$\begin{aligned}\frac{\mathrm{d}u_{4n_4-3}}{\mathrm{d}\theta} = (2n_4 - 1)u_{2n_4-1}(\theta)(F_A(\theta) + G_{B,C}(\theta)) \\ - (F_A(\theta) + G_{B,C}(\theta))(H_A(\theta) + I_{B,C}(\theta))\end{aligned}$$

and, since $u_{4n_4-3}(0) = 0$,

$$\begin{aligned}v_{4n_4-3} = u_{4n_4-3}(2\pi) &= \int_0^{2\pi} \frac{\mathrm{d}u_{4n_4-3}}{\mathrm{d}\theta}\,\mathrm{d}\theta \\ &= \mathrm{Re}(B)\,\mathrm{Im}(C) + \mathrm{Im}(B)\,\mathrm{Re}(C) \\ &= \mathrm{Im}(BC).\end{aligned}$$

Therefore, to have $v_{4n_4-3}=0$ we must have $\mathrm{Im}(BC)=0$, which means that $\mathrm{Im}(BC)=0$ is a necessary condition for system (1.2) restricted to $\mathrm{Re}(D)=0$ to have a centre at the origin. □

5. Proof of Theorem 1.2

By Theorem 1.1 (a), $m=(n_4-1)\hat{m}$, for some positive integer $\hat{m}$. Thus, (3.6) together with (3.12) imply that

$$2m_5+2s_1+m_4+m_8+(\sigma_2K_2+\sigma_1N_2)r=\hat{m}. \tag{5.1}$$

We consider two different cases.

Case 1 ($1+j_1>n_1$ and $1+j_2>n_2$). In this case, if $\hat{m}<K_2+N_2$, then the unique solution of (3.12) and (5.1) with $\sigma_1=\sigma_2=1$ is $r=0$. Indeed, since $\hat{m}<K_2+N_2$ and m_4, s_1, m_5, m_8 are positive integers, from (5.1) with $\sigma_1=\sigma_2=1$, we have that $r\leqslant 0$. Similarly, from (3.12) and (5.1) with $\sigma_1=\sigma_2=1$ we obtain that $2m_1+2s_2+m_4+m_8-(K_2+N_2)r=\hat{m}<K_2+N_2$. Now, using the fact that m_1, s_2, m_4, m_8 are positive integers, we deduce that $r\geqslant 0$. Thus, $r=0$ and from (3.12) and (5.1) we have that $m_1=m_5$, $s_2=s_1$ and $2(m_5+s_1)+m_4+m_8=\hat{m}$.

Let $s_3=m_4+m_8$, and introduce the set

$$S=\{(m_5,s_1,s_3)\in\mathbb{N}^3:2(m_5+s_1)+s_3=\hat{m}\}.$$

The expression for v_{2m+1} with $m=(n_4-1)\hat{m}$ becomes

$$v_{2m+1}=\sum_{(m_5,s_1,s_3)\in S}(A\bar{A})^{m_5}\sum_{m_4=0}^{s_3}D^{m_4}\bar{D}^{s_3-m_4} \times\sum_{m_2=0}^{s_1}\sum_{m_6=0}^{s_1}\beta_{m_2,m_6,s_3,m_4}B^{m_2}\bar{B}^{m_6}C^{s_1-m_6}\bar{C}^{s_1-m_2}, \tag{5.2}$$

with β_{m_2,m_6,s_3,m_4} real constants. Let $\varGamma_1=\mathrm{e}^{\mathrm{i}(1-n_1+j_1)\gamma}$ and $\varGamma_2=\mathrm{e}^{\mathrm{i}(1-n_2+j_2)\gamma}$. From Lemma 3.2, if $\bar{A}=-A\varGamma_1$, $\bar{B}=-B\varGamma_2$, $\bar{C}=-C\varGamma_2^{-1}$ and $\bar{D}=-D$, then $v_{2m+1}=0$, and thus

$$0=\sum_{(m_5,s_1,s_3)\in S}(-1)^{m_5+s_1+s_3}A^{2m_5}\varGamma_1^{m_5}\varGamma_2^{s_1}D^{s_3} \times\sum_{m_2=0}^{s_1}\sum_{m_6=0}^{s_1}(-1)^{m_6-m_2}\varGamma_2^{m_2+m_6}B^{m_2+m_6}C^{2s_1-(m_2+m_6)}\sum_{m_4=0}^{s_3}(-1)^{m_4}\beta_{m_2,m_6,s_3,m_4}. \tag{5.3}$$

Now, we note that (5.3) is valid for any values of the complex numbers A, B, C and D. Thus, computing the different powers of A, D and the degree of BC (which is $2s_1$), we have that, for every $(m_5,s_1,s_3)\in S$,

$$\sum_{m_2=0}^{s_1}\sum_{m_6=0}^{s_1}(-\varGamma_2)^{m_2+m_6}B^{m_2+m_6}C^{2s_1-(m_2+m_6)}\sum_{m_4=0}^{s_3}(-1)^{m_4}\beta_{m_2,m_6,s_3,m_4}=0. \tag{5.4}$$

Let $r = m_2 + m_6$. Then (5.4) becomes

$$\biggl(\sum_{r=0}^{s_1}\sum_{m_2=0}^{r} + \sum_{r=s_1+1}^{2s_1}\sum_{m_2=r-s_1}^{s_1}\biggr)(-\Gamma_2)^r B^r C^{2s_1-r}\sum_{m_4=0}^{s_3}(-1)^{m_4}\beta_{m_2,r-m_2,s_3,m_4} = 0,$$

where we have used the notation

$$\biggl(\sum_{r=0}^{s_1}\sum_{m_2=0}^{r} + \sum_{r=s_1+1}^{2s_1}\sum_{m_2=r-s_1}^{s_1}\biggr)a_{r,m_2} = \sum_{r=0}^{s_1}\sum_{m_2=0}^{r} a_{r,m_2} + \sum_{r=s_1+1}^{2s_1}\sum_{m_2=r-s_1}^{s_1} a_{r,m_2}.$$

Then, for each degree $2s_1$, computing the different powers of B and C, we have that, for $r = 0, \dots, s_1$,

$$\sum_{m_2=0}^{r}\sum_{m_4=0}^{s_3}(-1)^{m_4}\beta_{m_2,r-m_2,s_3,m_4} = 0$$

and thus,

$$\beta_{0,r,s_3,0} = -\sum_{m_2=1}^{r}\beta_{m_2,r-m_2,s_3,0} - \sum_{m_2=0}^{r}\sum_{m_4=1}^{s_3}(-1)^{m_4}\beta_{m_2,r-m_2,s_3,m_4}. \tag{5.5}$$

Furthermore, for $r = s_1 + 1, \dots, 2s_1$ we deduce that

$$\sum_{m_2=r-s_1}^{s_1}\sum_{m_4=0}^{s_3}(-1)^{m_4}\beta_{m_2,r-m_2,s_3,m_4} = 0,$$

and thus

$$\beta_{0,r,s_3,0} = -\sum_{m_2=r-s_1+1}^{s_1}\beta_{m_2,r-m_2,s_3,0} - \sum_{m_2=r-s_1}^{s_1}\sum_{m_4=1}^{s_3}(-1)^{m_4}\beta_{m_2,r-m_2,s_3,m_4}. \tag{5.6}$$

Now, we write v_{2m+1} in (5.2) as

$$\begin{aligned} v_{2m+1} = \sum_{(m_5,s_1,s_3)\in S}(A\bar{A})^{m_5}\sum_{m_4=0}^{s_3}D^{m_4}\bar{D}^{s_3-m_4} \\ \times\biggl(\sum_{r=0}^{s_1}\sum_{m_2=0}^{r} + \sum_{r=s_1+1}^{2s_1}\sum_{m_2=s_1-r}^{s_1}\biggr)\beta_{m_2,r-m_2,s_3,m_4}B^{m_2}\bar{B}^{r-m_2} \\ \times C^{s_1-r+m_2}\bar{C}^{s_1-m_2}. \end{aligned} \tag{5.7}$$

Then, (5.7) together with (5.5) and (5.6) imply

$$\begin{aligned} v_{2m+1} = \sum_{(m_5,s_1,s_3)\in S}(A\bar{A})^{m_5}\bar{D}^{s_3} \\ \times\biggl(\sum_{r=0}^{s_1}\sum_{m_2=1}^{r} + \sum_{r=s_1+1}^{2s_1}\sum_{m_2=s_1-r}^{s_1}\biggr)\beta_{m_2,r-m_2,s_3,0} \\ \times[-\bar{B}^r C^{s_1-r}\bar{C}^{s_1} + B^{m_2}\bar{B}^{r-m_2}C^{s_1-r+m_2}\bar{C}^{s_1-m_2}] \end{aligned}$$

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$$+\sum_{(m_5,s_1,s_3)\in S}(A\bar{A})^{m_5}\biggl(\sum_{r=0}^{s_1}\sum_{m_2=1}^{r}+\sum_{r=s_1+1}^{2s_1}\sum_{m_2=s_1-r}^{s_1}\biggr)$$
$$\times\sum_{m_4=1}^{s_3}\beta_{m_5,s_1,m_2,r-m_2,s_3,m_4}$$
$$\times[(-1)^{m_4+1}\bar{B}^{r}C^{s_1-r}\bar{C}^{s_1}\bar{D}^{s_3}+B^{m_2}\bar{B}^{r-m_2}C^{s_1-r+m_2}\bar{C}^{s_1-m_2}D^{m_4}\bar{D}^{s_3-m_4}]. \tag{5.8}$$

Clearly, we can rewrite $v_{2\hat{m}+1}$ in (5.8) as

$$v_{2m+1}=\sum_{(m_5,s_1,s_3)\in S}(A\bar{A})^{m_5}\bar{D}^{s_3}$$
$$\times\biggl(\sum_{r=0}^{s_1}\sum_{m_2=1}^{r}+\sum_{r=s_1+1}^{2s_1}\sum_{m_2=s_1-r}^{s_1}\biggr)\beta_{m_5,s_1,m_2,r-m_2,s_3,0}\bar{B}^{r-m_2}C^{s_1-r}\bar{C}^{s_1-m_2}$$
$$\times[-B^{m_2}C^{m_2}+\bar{B}^{m_2}\bar{C}^{m_2}]$$
$$+\sum_{(m_5,s_1,s_3)\in S}(A\bar{A})^{m_5}\biggl(\sum_{r=0}^{s_1}\sum_{m_2=1}^{r}+\sum_{r=s_1+1}^{2s_1}\sum_{m_2=s_1-r}^{s_1}\biggr)$$
$$\times\sum_{m_4=1}^{s_3}\beta_{m_5,s_1,m_2,r-m_2,s_3,m_4}\bar{B}^{r-m_2}C^{s_1-r}\bar{C}^{s_1-m_2}\bar{D}^{s_3-m_4}$$
$$\times[(-1)^{m_4+1}B^{m_2}C^{m_2}\bar{D}^{m_4}+\bar{B}^{m_2}\bar{C}^{m_2}D^{m_4}]. \tag{5.9}$$

Since by hypothesis $\mathrm{Re}(D)=\mathrm{Im}(BC)=0$, i.e. $D=-\bar{D}$ and $BC=\bar{B}\bar{C}$, from (5.9) we conclude that $v_{2m+1}=0$ for any $2m=(n_1+j_1-1)\hat{m}$ with $\hat{m}<N_2+K_2$. This finishes the proof of the theorem.

Case 2 ($1+j_1<n_1$ and $1+j_2<n_2$). In this case $1-n_1+j_1=-N_1$ and $1-n_2+j_2=-K_1$. The same arguments as were used in Case 1, but working with (3.12) and (5.1) with $\sigma_1=-1$, $\sigma_2=-1$, imply the theorem in this case.

6. Proof of Theorems 1.3 and 1.4

Let $m=(n_4-1)(K_2+N_2+\lambda)$, $\lambda\in\{0,1\}$. We note that when $\lambda=0$ we are proving Theorem 1.3 and when $\lambda=1$, we are proving Theorem 1.4. We fix $\lambda\in\{0,1\}$. We consider two different cases.

Case 1 ($1+j_1>n_1$ and $1+j_2>n_2$). In this case, we first note that $1-n_1+j_1=N_1$, $1-n_2+j_2=K_1$ and if $\hat{m}=K_2+N_2+\lambda$, then the unique solution of (3.12), and (5.1) with $\sigma_1=\sigma_2=1$, is $r\in\{-1,0,1\}$. Indeed, from (5.1) we get that $2m_5+2s_1+m_4+m_8+(K_2+N_2)r=K_2+N_2+\lambda$. Thus, since m_4, s_1, m_5 and m_8 are positive integers, $K_2+N_2\geqslant 3$ and $\lambda\in\{0,1\}$, we have that $r\leqslant 1$. Furthermore, (3.12) and (5.1) imply that $2m_1+2s_2+m_4+m_8-(K_2+N_2)r=K_2+N_2+\lambda$. Thus, since m_1, s_2, m_4, m_8 are positive integers, $K_2+N_2\geqslant 3$ and $\lambda\in\{0,1\}$, we deduce that $r\geqslant -1$. Thus, $r\in\{-1,0,1\}$.

If $r=1$, then (5.1) implies that $2m_5+2s_1+m_4+m_8=\lambda$, which yields, since $\lambda\in\{0,1\}$, $m_5=s_1=0$ and $m_4+m_8=\lambda$. Therefore, (3.12) with $\sigma_1=\sigma_2=1$ and (3.10) yield $m_2=m_7=m_5=0$, $m_4+m_8=\lambda$, and $m_1=K_2$, $s_2=m_3+m_6=N_2$.

If $r=0$, proceeding as in the proof of Theorem 1.1, we have that $m_1=m_5$, $s_2=s_1$ and $2m_5+2s_1+m_4+m_8=N_2+K_2+\lambda$.

Finally, if $r=-1$, then (5.1) implies that $2m_1+2s_2+m_4+m_8=\lambda$ which, again since $\lambda\in\{0,1\}$, yields $m_1=s_2=0$, $m_4+m_8=\lambda$. Therefore, (3.12) with $\sigma_1=\sigma_2=-1$ and (3.10) yield $m_1=m_3=m_6=0$, $m_4+m_8=\lambda$ and $m_5=K_2$, $s_1=m_2+m_7=N_2$.

Let $s_3=m_4+m_8$ and set

$$S_0=\{(m_5,s_1,s_3)\in\mathbb{N}^3:2(m_5+s_1)+s_3=N_2+K_2+\lambda\}.$$

Then v_{2m+1} becomes

$$\begin{aligned}v_{2m+1}&=A^{K_2}\sum_{k=0}^{N_2}\bar{B}^kC^{N_2-k}\sum_{m_4=0}^{\lambda}\hat{\alpha}_{k,\lambda}D^{m_4}\bar{D}^{\lambda-m_4}\\&\quad+\bar{A}^{K_2}\sum_{k=0}^{N_2}B^k\bar{C}^{N_2-k}\sum_{m_4=0}^{\lambda}\hat{\gamma}_{k,\lambda}D^{m_4}\bar{D}^{\lambda-m_4}\\&\quad+\sum_{(m_5,s_1,s_3)\in S_0}(A\bar{A})^{m_5}\sum_{m_2=0}^{s_1}\sum_{m_6=0}^{s_1}\sum_{m_4=0}^{s_3}\beta_{m_2,m_6,s_3,m_4,\lambda}\\&\qquad\times B^{m_2}\bar{B}^{m_6}C^{s_1-m_6}\bar{C}^{s_1-m_2}D^{m_4}\bar{D}^{s_3-m_4},\end{aligned}\tag{6.1}$$

where $\hat{\alpha}_{k,\lambda}$, $\hat{\gamma}_{k,\lambda}$ and $\beta_{m_2,m_6,s_3,m_4,\lambda}$ are real constants. Since $\mathrm{Re}(D)=0$, i.e. $\bar{D}=-D$, from (6.1) we have that

$$\begin{aligned}v_{2m+1}&=D^{\lambda}\sum_{k=0}^{N_2}[A^{K_2}\alpha_{k,\lambda}\bar{B}^kC^{N_2-k}+\bar{A}^{K_2}\gamma_{k,\lambda}B^k\bar{C}^{N_2-k}]\\&\quad+\sum_{(m_5,s_1,s_3)\in S_0}(A\bar{A})^{m_5}\sum_{m_2=0}^{s_1}\sum_{m_6=0}^{s_1}\sum_{m_4=0}^{s_3}\beta_{m_2,m_6,s_3,m_4,\lambda}\\&\qquad\times B^{m_2}\bar{B}^{m_6}C^{s_1-m_6}\bar{C}^{s_1-m_2}D^{m_4}\bar{D}^{s_3-m_4},\end{aligned}\tag{6.2}$$

where

$$\alpha_{k,\lambda}=\sum_{m_4=0}^{\lambda}(-1)^{\lambda-m_4}\hat{\alpha}_{k,\lambda}\quad\text{and}\quad\gamma_{k,\lambda}=\sum_{m_4=0}^{\lambda}(-1)^{\lambda-m_4}\hat{\gamma}_{k,\lambda}.$$

From Lemma 3.2, if $\bar{A}=-A\Gamma_1$, $\bar{B}=-B\Gamma_2$ $\bar{C}=-C\Gamma_2^{-1}$, $\bar{D}=-D$, then $v_{2m+1}=0$, and thus

$$\begin{aligned}0&=D^{\lambda}A^{K_2}\sum_{k=0}^{N_2}[\alpha_{k,\lambda}(-1)^k\Gamma_2^k+\gamma_{k,\lambda}(-1)^{K_2+N_2-k}\Gamma_1^{K_2}\Gamma_2^{-N_2+k}]B^kC^{N_2-k}\\&\quad+\sum_{(m_5,s_1,s_3)\in S_0}(-1)^{m_5+s_1+s_3}A^{2m_5}\Gamma_1^{m_5}\Gamma_2^{-s_1}D^{s_3}\\&\quad\times\sum_{m_2=0}^{s_1}\sum_{m_6=0}^{s_1}(-\Gamma_2)^{m_2+m_6}B^{m_2+m_6}C^{2s_1-m_2-m_6}\sum_{m_4=0}^{s_3}(-1)^{m_4}\beta_{m_2,m_6,s_3,m_4,\lambda}.\end{aligned}\tag{6.3}$$

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Now we point out that if K_2+N_2 is even, since K_2 and N_2 are coprime, then K_2 and N_2 are both odd. Thus, for any $m_5 \in S_0$, $A^{2m_5} \neq A^{K_2}$. Furthermore, if K_2+N_2 is odd and $\lambda = 0$, then $(m_5, s_1) \in S_0$ satisfy that $2(m_5+s_1) \leqslant K_2+N_2-1$ and thus the degree of ABC in the first sum in (6.3) is K_2+N_2, while the rest of terms in (6.3) have degree in ABC equal to $2(m_5+s_1) \leqslant N_2+K_2-1$. On the other hand, if K_2+N_2 is odd and $\lambda = 1$, since $s_3 \in S_0$, we have that s_3 is even. Moreover, the first sum in (6.3) has degree 1 in the parameter D, while the rest of the terms in (6.3) have degree s_3 (even) in the parameter D. From this discussion, computing the different degrees of A, B, C and D in (6.3) and proceeding as in the proof of Theorem 1.2, we have that $\beta_{m_2,m_6,s_3,m_4,\lambda} = 0$ and

$$\alpha_{k,\lambda}(-1)^k \Gamma_2^k + \gamma_{k,\lambda}(-1)^{K_2+N_2-k}\Gamma_1^{K_2}\Gamma_2^{-N_2+k} = 0 \quad \text{for } k = 0, \dots, N_2.$$

Now, using the facts that $\Gamma_1^{K_2}\Gamma_2^{-N_2} = 1$ and $\Gamma_2^k \neq 0$, we obtain

$$\alpha_{k,\lambda} + (-1)^{K_2+N_2}\gamma_{k,\lambda} = 0 \quad \text{for } k = 0, \dots, N_2;$$

that is,

$$\alpha_{k,\lambda} = (-1)^{K_2+N_2+1}\gamma_{k,\lambda} \quad \text{for } k = 0, \dots, N_2. \tag{6.4}$$

Thus, from (6.2) and (6.4) we have that

$$v_{2m+1} = D^\lambda \sum_{k=0}^{N_2} \gamma_{k,\lambda}[\bar{A}^{K_2} B^k \bar{C}^{N_2-k} - (-1)^{K_2+N_2} A^{K_2} \bar{B}^k C^{N_2-k}].$$

Now, since $\operatorname{Im}(BC) = 0$, i.e. $\bar{B}\bar{C} = BC$ with the notation introduced in (1.4) we have that

$$v_{2m+1} = D^\lambda \mathbb{L}_{A,C}^{K_2,N_2} \sum_{k=0}^{N_2} \gamma_{k,\lambda} \left(\frac{B}{\bar{C}}\right)^k, \tag{6.5}$$

which finishes the proof of the first statement of the theorem in this case.

Now assume that $n_1 = 0$. By Lemma 3.3 with $n_1 = 0$, if $n_2 B + n_3 \bar{C} = 0$, we have that $v_{2m+1} = 0$ and thus, from (6.5),

$$D^\lambda \mathbb{L}_{A,C}^{K_2,N_2} \sum_{k=0}^{N_2} \gamma_{k,\lambda} \left(\frac{-n_3}{n_2}\right)^k = 0, \quad \text{that is, } \gamma_{0,\lambda} = -\sum_{k=1}^{N_2} \gamma_{k,\lambda} \left(\frac{-n_3}{n_2}\right)^k. \tag{6.6}$$

Therefore, inserting (6.6) into (6.5), we obtain

$$\begin{aligned} v_{2m+1} &= D^\lambda \mathbb{L}_{A,C}^{K_2,N_2} \left[\gamma_{0,\lambda} + \sum_{k=1}^{N_2} \gamma_{k,\lambda} \left(\frac{B}{\bar{C}}\right)^k\right] \\ &= D^\lambda \mathbb{L}_{A,C}^{K_2,N_2} \sum_{k=1}^{N_2} \gamma_{k,\lambda} \left[\left(\frac{B}{\bar{C}}\right)^k - \left(\frac{-n_3}{n_2}\right)^k\right] \\ &= \frac{D^\lambda}{n_2 \bar{C}} \mathbb{L}_{A,C}^{K_2,N_2} (n_2 B + n_3 \bar{C}) \sum_{k=1}^{N_2} \gamma_{k,\lambda} \sum_{j=0}^{k-1} \left(\frac{B}{\bar{C}}\right)^{k-1-j} \left(\frac{-n_3}{n_2}\right)^j. \end{aligned}$$

This completes the proof of the theorem in this case.

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Case 2 ($\mathbf{1+j_1<n_1}$ and $\mathbf{1+j_2<n_2}$). In this case $1-n_1+j_1=-N_1$ and $1-n_2+j_2=-K_1$. The same arguments as were used in Case 1, but working with (3.12) and (5.1) with $\sigma_1=-1$ and $\sigma_2=-1$ (see (3.8)), imply the theorem in this case.

7. Proof of Theorem 1.5

We recall that $\mathrm{Re}(D)=\mathrm{Im}(BC)=0$. We consider two different cases.

Case 1 ($\boldsymbol{n_1\neq 0}$). In this case we will prove that $\mathbb{L}_{A,C}^{K_2,N_2}=0$ is a necessary condition for system (1.2) to have a centre at the origin. We proceed by contradiction. Assume that the origin of (1.2) is a centre and that $\mathbb{L}_{A,C}^{K_2,N_2}\neq 0$. Then, by hypothesis, together with the fact that $B,C\neq 0$ and making use of (1.5) and (1.7), we have that either

$$v_{(n_1+j_1)(K_2+N_2)+1}\neq 0 \quad \text{or} \quad v_{(n_1+j_1)(K_2+N_2+1)+1}\neq 0,$$

a contradiction to the fact that the origin of (1.2) is a centre.

Now, we shall prove that $\mathrm{Re}(D)=\mathrm{Im}(BC)=\mathbb{L}_{A,C}^{K_2,N_2}=0$ is also a sufficient condition for system (1.2) to have a centre at the origin. From Lemma 3.2 we know that system (1.2) has a reversible centre at the origin if and only if, for some α, we have

$$\begin{aligned} A &= -\bar{A}\mathrm{e}^{-\mathrm{i}(1-n_1+j_1)\alpha},\\ B &= -\bar{B}\mathrm{e}^{-\mathrm{i}(1-n_2+j_2)\alpha},\\ C &= -\bar{C}\mathrm{e}^{\mathrm{i}(1-n_2+j_2)\alpha},\\ D &= -\bar{D}. \end{aligned}$$

Thus, we just need to show that

$$A^{K_2}C^{N_2}=(-1)^{K_2+N_2}\bar{A}^{K_2}\bar{C}^{N_2}$$

and $\mathrm{Re}(D)=\mathrm{Im}(BC)=0$ is a sufficient condition for the centre variety. So, since $BC=\bar{B}\bar{C}$, we have that

$$\left(\frac{-\bar{A}}{A}\right)^{K_2}=\left(\frac{-C}{\bar{C}}\right)^{N_2}=\left(\frac{-\bar{B}}{B}\right)^{N_2}. \tag{7.1}$$

Now let θ_1 and θ_2 be such that $\mathrm{e}^{\mathrm{i}\theta_1}=-\bar{A}/A$ and $\mathrm{e}^{\mathrm{i}\theta_2}=-\bar{B}/B$. Then from (7.1), together with the facts that $N_2=N_1/M_1$ and $K_2=K_1/M_1$, we have that

$$K_1\theta_1=N_1\theta_2(\mathrm{mod}\,2\pi). \tag{7.2}$$

Now using the fact that $N_1=|1-n_1+j_1|$, that $K_1=|1-n_2+j_2|$ and that $\mathrm{sgn}(1-n_1+j_1)(1-n_2+j_2)>0$, (7.2) becomes

$$(1-n_1+j_1)\theta_1=(1-n_2+j_2)\theta_2(\mathrm{mod}\,2\pi). \tag{7.3}$$

Set $\alpha=\theta_1/(1-n_1+j_1)$. Then, using (7.3), we have that

$$\mathrm{e}^{\mathrm{i}(1-n_1+j_1)\alpha}=\mathrm{e}^{\mathrm{i}\theta_1}=-\frac{\bar{A}}{A} \quad \text{and} \quad \mathrm{e}^{\mathrm{i}(1-n_2+j_2)\alpha}=-\frac{\bar{B}}{B}. \tag{7.4}$$

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Consequently, the sufficiency of the condition follows from (7.4) and the fact that $\mathrm{Re}(D) = \mathrm{Im}(BC) = 0$. Thus, the theorem is proved in this case.

Case 2 ($n_1 = 0$). In this case we will show that $(n_2 B + n_3)\mathbb{L}_{A,C}^{K_2,N_2} = 0$ is a necessary condition in order that system (1.2) has a centre at the origin. We proceed by contradiction. Assume the origin of (1.2) is a centre and $(n_2 B + n_3 \bar{C})\mathbb{L}_{A,C}^{K_2,N_2} \neq 0$. Then, by hypothesis, together with the fact that $B, C \neq 0$, and making use of (1.6) and (1.8), we get that either

$$v_{(n_1+j_1)(K_2+N_2)+1} \neq 0 \quad \text{or} \quad v_{(n_1+j_1)(K_2+N_2+1)+1} \neq 0,$$

a contradiction to the fact that the origin of (1.2) is a centre. Furthermore, Lemma 3.3 states that $\mathrm{Re}(D) = n_2 B + n_3 \bar{C} = 0$ is a sufficient condition in order that system (1.2) has a centre at the origin and, proceeding as we did in Case 1, we can prove that $\mathrm{Re}(D) = \mathrm{Im}(BC) = \mathbb{L}_{A,C}^{K_2,N_2} = 0$ is also a sufficient condition in order that system (1.2) has a centre at the origin. Thus, the theorem is proved in this case.

Appendix A.

In this appendix we provide the results in [**10**, **11**, **23**].

Proposition A 1 (Llibre and Valls [11]). *For system (1.2) with $B = C = D = 0$ the following hold.*

(a) *If $j_1 \neq n_1 - 1$, the centre manifold variety of system (1.2) is the set $\{A \in \mathbb{C} \setminus \{0\}\}$.*

(b) *If $j_1 = n_1 - 1$, the centre manifold variety of system (1.2) is the set $\{A \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(A) = 0\}$.*

We set $M_{K_2,N_2} = (n-1)(K_2 + N_2)/2$ (which can be seen to be always an integer number) and $d = (1 - n_1 - j_1)(1 - n_2 - j_2)$.

Proposition A 2 (Llibre and Valls [11]). *For system (1.2) with $C = D = 0$ the following hold.*

(a) *If $j_1 = n_1 - 1$ and $j_2 = n_2 - 1$, the centre manifold variety of system (1.2) is the set $\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \mathrm{Re}(A) = \mathrm{Re}(B) = 0\}$.*

(b) *If $j_1 = n_1 - 1$ and $j_2 \neq n_2 - 1$, the centre manifold variety of system (1.2) is the set $\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \mathrm{Re}(A) = 0\}$.*

(c) *If $j_1 \neq n_1 - 1$ and $j_2 = n_2 - 1$, the centre manifold variety of system (1.2) is the set $\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \mathrm{Re}(B) = 0\}$.*

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(d) *If* $j_1 \neq n_1 - 1$ *and* $j_2 \neq n_2 - 1$, *we have that*

(i) $v_{2m+1} = 0$ *for* $m = 1, 2, \dots, M_{K_2,N_2} - 1$,

(ii) *there exists a constant* K_{n_1,j_1,n_2,j_2} *such that*

$$v_{2m+1} = \begin{cases} K_{n_1,j_1,n_2,j_2} \operatorname{Im}(\bar{A}^{K_2} B^{N_2}) & \text{if } d > 0 \text{ and } K_2 + N_2 \text{ even}, \\ K_{n_1,j_1,n_2,j_2} \operatorname{Re}(\bar{A}^{K_2} B^{N_2}) & \text{if } d > 0 \text{ and } K_2 + N_2 \text{ odd}, \\ K_{n_1,j_1,n_2,j_2} \operatorname{Im}(A^{K_2} B^{N_2}) & \text{if } d < 0 \text{ and } K_2 + N_2 \text{ even}, \\ K_{n_1,j_1,n_2,j_2} \operatorname{Re}(A^{K_2} B^{N_2}) & \text{if } d < 0 \text{ and } K_2 + N_2 \text{ odd}. \end{cases}$$

(iii) *If the constant* K_{n_1,j_1,n_2,j_2} *introduced in statement (ii) is non-zero, the centre manifold variety of system (1.2) is*

$$\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \operatorname{Im}(\bar{A}^{K_2} B^{N_2}) = 0\} \quad \text{if } d > 0 \text{ and } K_2 + N_2 \text{ even},$$
$$\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \operatorname{Re}(\bar{A}^{K_2} B^{N_2}) = 0\} \quad \text{if } d > 0 \text{ and } K_2 + N_2 \text{ odd},$$
$$\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \operatorname{Im}(A^{K_2} B^{N_2}) = 0\} \quad \text{if } d < 0 \text{ and } K_2 + N_2 \text{ even},$$
$$\{(A, B) \in (\mathbb{C} \setminus \{0\})^2 : \operatorname{Re}(A^{K_2} B^{N_2}) = 0\} \quad \text{if } d < 0 \text{ and } K_2 + N_2 \text{ odd}.$$

(iv) *For* $K_2 + N_2 = 2$, $K_{n_1,j_1,n_2,j_2} = -2\pi$ *and for* $K_2 + N_2 = 3$,

$$K_{n_1,j_1,n_2,j_2} = \begin{cases} 3\pi(n_1 - 1)/(1 + j_1 - n_1) & \text{if } d < 0 \text{ and } K_2 = 1, \\ 3\pi(n_1 - 1 - 3j_1)/(4(1 + j_1 - n_1)) & \text{if } d < 0 \text{ and } K_2 = 2, \\ j\pi/(1 + j_1 - n_1) & \text{if } d > 0 \text{ and } K_2 = 1, \\ (3j_1 - n_1 + 1)\pi/(4(1 + j_1 - n_1)) & \text{if } d > 0 \text{ and } K_2 = 2. \end{cases}$$

Proposition A 3 (Llibre and Valls [10]). *For system (1.2) with* $D = 0$, $A, B, C \in \mathbb{C} \setminus \{0\}$ *and* $j_3 = n_3 - 1$ *(note that then* n *must be odd), the following hold.*

(a) $\operatorname{Re}(C) = 0$ *is a necessary condition to have a centre at the origin and* $v_{2k+1} = 0$ *for* $k = 1, \dots, M_{K_2,N_2}$.

(b) *There exists a constant* K_{n_1,j_1,n_2,j_2} *such that*

$$v_{2m+1} = \begin{cases} K_{n_1,j_1,n_2,j_2} \operatorname{Im}(\bar{A}^{K_2} B^{N_2}) & \text{if } d > 0 \text{ and } K_2 + N_2 \text{ even}, \\ K_{n_1,j_1,n_2,j_2} \operatorname{Re}(\bar{A}^{K_2} B^{N_2}) & \text{if } d > 0 \text{ and } K_2 + N_2 \text{ odd}, \\ K_{n_1,j_1,n_2,j_2} \operatorname{Im}(A^{K_2} B^{N_2}) & \text{if } d < 0 \text{ and } K_2 + N_2 \text{ even}, \\ K_{n_1,j_1,n_2,j_2} \operatorname{Re}(A^{K_2} B^{N_2}) & \text{if } d < 0 \text{ and } K_2 + N_2 \text{ odd}. \end{cases}$$

(c) *If the constants* K_{n_1,j_1,n_2,j_2} *introduced in statement (b) are non-zero, then the centre manifold variety of system (1.2) is the subset of* $\{(A, B, C) \in (\mathbb{C} \setminus \{0\})^3\}$ *such that*

$$\operatorname{Re}(C) = \begin{cases} \operatorname{Im}(\bar{A}^{K_2} B^{N_2}) = 0 & \text{if } d > 0 \text{ and } K_2 + N_2 \text{ even}, \\ \operatorname{Re}(\bar{A}^{K_2} B^{N_2}) = 0 & \text{if } d > 0 \text{ and } K_2 + N_2 \text{ odd}, \\ \operatorname{Im}(A^{K_2} B^{N_2}) = 0 & \text{if } d < 0 \text{ and } K_2 + N_2 \text{ even}, \\ \operatorname{Re}(A^{K_2} B^{N_2}) = 0 & \text{if } d < 0 \text{ and } K_2 + N_2 \text{ odd}. \end{cases}$$

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(d) *If* $\mathrm{Re}(C)=0$, *then for* $K_2+N_2=2$ *we have* $K_{n_1,j_1,n_2,j_2}=-2\pi$ *and for* $K_2+N_2=3$, *we have that*

$$K_{n_1,j_1,n_2,j_2}=\begin{cases}3\pi(n_1-1)/(1+j_1-n_1) & \text{if } d<0 \text{ and } K_2=1,\\ 3\pi(n_1-1-3j_1)/(4(1+j_1-n_1)) & \text{if } d<0 \text{ and } K_2=2,\\ j\pi/(1+j_1-n_1) & \text{if } d>0 \text{ and } K_2=1,\\ (3j_1-n_1+1)\pi/(4(1+j_1-n_1)) & \text{if } d>0 \text{ and } K_2=2.\end{cases}$$

Proposition A 4 (Llibre and Valls [23]). *For system (1.2) with* $D=0$, $A,B,C\in\mathbb{C}\setminus\{0\}$, *with* $j_3\neq n_3-1$, *the following hold.*

(a) $v_k=0$ *for any* k *such that* $2k/(n_1+j_1-1)\notin\mathbb{N}$.

(b) $\mathrm{Im}(BC)=0$ *is a necessary condition for system (1.2) to have a centre at the origin.*

(c) *If* $\mathrm{Im}(BC)=0$, *then* $v_{2k+1}=0$ *for* $k=2,3,\dots,m$ *with* $2m<(n_1+j_1-1)(K_2+N_2)$.

(d) *If* $\mathrm{Im}(BC)=0$, *there exist real constants* L_k, $k=0,\dots,N_2$, *such that, for* $2m=(n_1+j_1-1)(K_2+N_2)$,

$$v_{2m+1}=\begin{cases}\mathbb{L}_{A,C}^{K_2,N_2}\displaystyle\sum_{k=0}^{N_2}L_k\left(\frac{B}{\bar{C}}\right)^k, & n_1\neq 0,\\ \dfrac{1}{n_2\bar{C}}\mathbb{L}_{A,C}^{K_2,N_2}(n_2B+n_3\bar{C})\displaystyle\sum_{k=1}^{N_2}L_k\sum_{j=0}^{k-1}\left(\frac{B}{\bar{C}}\right)^{k-1-j}\left(\frac{-n_3}{n_2}\right)^j, & n_1=0.\end{cases}$$

(e) *If* $\mathrm{Im}(BC)=0$ *and for some* $k=0,\dots,N_2$, *the constants* L_k *in statement (d) are non-zero, and* $\mathbb{L}_{A,C}^{K_2,N_2}=0$, *the origin of (1.2) is a centre.*

With the constants L_k introduced in Proposition A 4, we introduce the set

$$S_1=\left\{C,B\in\mathbb{C}\setminus\{0\}:\sum_{k=0}^{N_2}L_k\left(\frac{B}{\bar{C}}\right)^k=0\right\}\qquad \text{if } n_1\neq 0,$$

and

$$S_1=\left\{C,B\in\mathbb{C}\setminus\{0\}:\sum_{k=1}^{N_2}L_k\sum_{j=0}^{k-1}\left(\frac{B}{\bar{C}}\right)^{k-1-j}\left(\frac{-n_3}{n_2}\right)^j=0\right\}\quad \text{if } n_1=0.$$

Proposition A 5 (Valls [23]). *For system (1.2) with* $D=0$, $A,B,C\in\mathbb{C}\setminus\{0\}$, $j_3\neq n_3-1$ *and* $\mathrm{Im}(BC)=0$, *the following hold.*

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(a) *If there exists* $k_1 \in \{0, \dots, N_2\}$ *such that the constants* L_k *in Proposition A 4 satisfy that* $L_{k_1} \neq 0$ *and* $B, C \in S_1$*, then there exist constants* $D_1, D_2 \in \mathbb{C}$ *such that for* $2m = (n_1 + j_1 - 1)(K_2 + N_2 + 2)$,

$$v_{2m+1} = (D_1 A\bar{A} + D_2 C\bar{C})\mathbb{L}_{A,C}^{K_2,N_2}.$$

(b) *Under the same hypothesis as statement* (*a*)*, if* $N_2 + K_2 \geqslant 5$*,* $D_1, D_2 \neq 0$ *and* C *satisfies* $D_2 C\bar{C} = -D_1 A\bar{A}$*, there exists a constant* L *such that for* $2m = (n_1 + j_1 - 1)(N_2 + K_2 + 4)$ *we have*

$$v_{2m+1} = L(A\bar{A})^2\mathbb{L}_{A,C}^{K_2,N_2}.$$

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References

1. A. Andronov, E. Leontovich, I. Gordon and A. Maier, *Theory of bifurcations of dynamic systems on a plane* (Halsted Press, Wiley, New York, 1973).
2. T. Blows and N. Lloyd, The number of small-amplitude limit cycles of Liénard equations, *Math. Proc. Camb. Phil. Soc.* **95** (1984), 751–758.
3. J. Chavarriga, Integrable systems in the plane with a centre linear part, *Applic. Math.* **22** (1994), 285–309.
4. A. Cima, A. Gasull, V. Mañosa and F. Mañosas, Algebraic properties of the Liapunov and period constants, *Rocky Mt. J. Math.* **27** (1997), 471–501.
5. A. Cima, J. Llibre and J. Medrado, New family of centers for polynomial vector fields of arbitrary degree, *Commun. Appl. Nonlin. Analysis*, in press.
6. J. Françoise and R. Pons, Une approche algorithmique du problème du centre pour des perturbations homogènes, *Bull. Sci. Math.* **120** (1996), 1–17.
7. V. Kertész and R. Kooij, Degenerate Hopf bifurcation in two dimensions, *Nonlin. Analysis* **17** (1991), 267–83.
8. Y.-R. Liu and J.-B. Li, Theory of values of singular point in complex autonomous differential systems, *Sci. China* A **33** (1990), 10–23.
9. J. Llibre and C. Valls, Classification of the centers, their cyclicity and isochronicity for a class of polynomial differential systems generalizing the linear systems with cubic homogeneous nonlinearities, *J. Diff. Eqns* **246** (2009), 2192–2204.
10. J. Llibre and C. Valls, Centers for a 6-parameter family of polynomial vector fields of arbitrary degree, *Bull. Sci. Math.* **132** (2008), 40–53.
11. J. Llibre and C. Valls, Centers for polynomial vector fields of arbitrary degree, *Commun. Pure Appl. Analysis* **8** (2009), 725–742.
12. N. Lloyd, C. Christopher, J. Devlin, J. Pearson and N. Yasmin, Quadratic-like cubic systems, *Diff. Eqns Dynam. Syst.* **5** (1997), 329–345.
13. N. Lloyd and J. Pearson, Computing centre conditions for certain cubic systems, *J. Computat. Appl. Math.* **40** (1992), 323–336.
14. N. Lloyd and J. Pearson, Five limit cycles for a simple cubic system, *Publ. Mat.* **41** (1997), 199–208.
15. N. Lloyd and J. Pearson, Bifurcation of limit cycles and integrability of planar dynamical systems in complex form, *J. Phys.* A **32** (1999), 1973–1984.
16. N. Lloyd, J. Pearson and V. Romanovsky, Computing integrability conditions for a cubic differential system, *Comput. Math. Appl.* **32** (1996), 99–107.

17. H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, *Oeuvres de Henri Poincaré*, Volume I, pp. 95–114 (Gauthier-Villars, Paris, 1951).
18. M. Popa and K. Sibirskii, Focal cyclicity of critical points of a differential system, *J. Diff. Eqns* **22** (1986), 1539–1545.
19. D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, *Trans. Am. Math. Soc.* **338** (1993), 799–841.
20. V. Sheplev, N. Bryxina and Slin'ko, The algorithm of calculation of Liapunov's coefficients by analysis of chemical self-sustained oscillations, *Dokl. Akad. Nauk SSSR* **359** (1998), 789–792.
21. K. Sibirskii, On the number of limit cycles arising from a singular point of focus or center type, *Dokl. Akad. Nauk SSSR* **161** (1965), 304–307 (in Russian; English transl.: *Sov. Math. Dokl.* **6** (1965), 428–431).
22. K. Sibirskii, *Algebraic invariants of differential equations and matrices* (Stiintsa, Kishinev, 1976; in Russian).
23. C. Valls, Bifurcation of limit cycles and integrability conditions for 6-parameter families of polynomial vector fields of arbitrary degree, *Nonlin. Analysis TMA* **69** (2008), 256–275.
24. C. Valls, New arbitrary-parameter families of centres for polynomial vector fields of arbitrary degree, *Dynam. Sys. Int. J.* **23**(2) (2008), 163–166.
25. Y. Wang and J. Merino, Self-organisational origin of agates: banding, fibre twisting, composition and dynamic crystallization model, *Geochim. Cosmochim. Acta* **54** (1990), 1627–1638.
26. H. Żołądek, Quadratic systems with center and their perturbations, *J. Diff. Eqns* **109** (1994), 223–273.
27. C. Zuppa, Order of cyclicity of the singular point of Liénard's polynomial vector field, *Bol. Soc. Bras. Mat.* **12** (1981), 105–111.