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RANDOM SPARSE SAMPLING IN A GIBBS WEIGHTED TREE AND PHASE TRANSITIONS

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Abstract Let μ be the projection on $[0, 1]$ of a Gibbs measure on $\Sigma = \{0, 1\}^{\mathbb{N}}$ (or more generally a Gibbs capacity) associated with a Hölder potential. The thermodynamic and multifractal properties of μ are well known to be linked via the multifractal formalism. We study the impact of a random sampling procedure on this structure. More precisely, let $\{I_w\}_{w\in\Sigma^*}$ stand for the collection of dyadic subintervals of $[0, 1]$ naturally indexed by the finite dyadic words. Fix $\eta \in (0, 1)$, and a sequence $(p_w)_{w\in\Sigma^*}$ of independent Bernoulli variables of parameters $2^{-|w|(1-\eta)}$. We consider the (very sparse) remaining values $\widetilde{\mu} = \{\mu(I_w) : w \in \Sigma^*, p_w = 1\}$. We study the geometric and statistical information associated with $\widetilde{\mu}$, and the relation between $\widetilde{\mu}$ and μ. To do so, we construct a random capacity M_μ from $\widetilde{\mu}$. This new object fulfills the multifractal formalism, and its free energy is closely related to that of μ. Moreover, the free energy of M_μ generically exhibits one first order and one second order phase transition, while that of μ is analytic. The geometry of M_μ is deeply related to the combination of approximation by dyadic numbers with geometric properties of Gibbs measures. The possibility to reconstruct μ from $\widetilde{\mu}$ by using the almost multiplicativity of μ and concatenation of words is discussed as well.

Keywords: Hausdorff dimension; random sampling; thermodynamic formalism; phase transitions; metric number theory; ubiquity theory; multifractals; large deviations

2010 *Mathematics subject classification*: 28A78; 28A80; 37D35; 60F10; 60K35

1. Introduction

Statistical mechanics and multifractals are well known to be closely related. Typical situations are provided by the energy model associated with a Gibbs measure on the boundary Σ of the dyadic tree Σ^* in the context of the thermodynamic formalism [11, 29, 30], or the random energy model associated with a branching random walk on Σ^*, namely directed polymers on disordered trees [1, 2, 10, 13, 22, 27]. The purpose of this paper is to investigate the thermodynamic and geometric impact of a random sparse sampling on such structures.

Let us start by describing the interplay between thermodynamics and multifractals.

1.1. Free energy and singularity spectrum as a Legendre pair

For the sake of generality, we work on the d-dimensional dyadic tree and on $[0,1]^d$, $d \geqslant 1$. Let Σ_j be the set of words of length $j \geqslant 1$ over the alphabet $\{0,1\}^d$, i.e.,

$$\Sigma_j = \left\{(w_1 w_2 \cdots w_j) : \forall k \in \{1,\ldots,j\}, w_k = (w_k^{(1)}, w_k^{(2)}, \ldots, w_k^{(d)}) \in \{0,1\}^d\right\}.$$

The notation $|w| = j$ stands for the length (or the generation) of $w \in \Sigma_j$. Then, $\Sigma^* = \bigcup_{j\geqslant 1} \Sigma_j$ and $\Sigma = (\{0,1\}^d)^{\mathbb{N}_+}$ denote the set of finite words and infinite words over $\{0,1\}^d$, respectively. The set Σ is endowed with the standard ultra-metric distance, and $\Sigma^* \cup \Sigma$ is endowed with the shift operation denoted by σ.

If $w \in \Sigma^* \cup \Sigma$ and $1 \leqslant j \leqslant |w|$ is finite, $w_{|j}$ stands for the prefix of length j of w. If $W \in \Sigma^*$, $[W]$ is the *cylinder* of those words $w \in \Sigma$ such that $w_{||W|} = W$.

With each $w = w_1 \ldots w_j \in \Sigma_j$ is naturally associated the dyadic point

$$x_w = \left(\sum_{k=1}^{j} w_k^{(i)} 2^{-k}\right)_{1\leqslant i\leqslant d}, \tag{1}$$

of $[0,1]^d$, and the dyadic subcube $I_w = \prod_{i=1}^d \left[x_w^{(i)}, x_w^{(i)} + 2^{-j}\right]$ of $[0,1]^d$.

If $x = (x^{(1)}, x^{(2)}, , \ldots, x^{(d)}) \in [0,1]^d$ has no dyadic component, then x is encoded by a unique $w = w^{(1)} w^{(2)} \ldots w^{(d)} \in \Sigma$, and $I_j(x)$ stands for $I_{w_{|j}}$. When $x^{(i)}$ is dyadic, $w^{(i)}$ is chosen as the largest element of $\{0,1\}^{\mathbb{N}_+}$ in lexicographical order which encodes $x^{(i)}$. In both cases, $w_{|j}$ is also denoted by $x_{|j}$.

Definition 1. We call *capacity* a non-negative and non-decreasing function μ of the dyadic subcubes of $[0,1]^d$, i.e., for every $W, w \in \Sigma^*$ such that $I_w \subset I_W$, $0 \leqslant \mu(I_w) \leqslant \mu(I_W)$. The set of capacities is denoted by $\mathsf{Cap}([0,1]^d)$.

The support of $\mu \in \mathsf{Cap}([0,1]^d)$ is the set $\mathsf{supp}(\mu) = \displaystyle\bigcap_{j\geqslant 1} \bigcup_{w\in\Sigma_j:\mu(I_w)>0} I_w$.

We focus on two quantities especially relevant in the thermodynamic and geometric measure theoretic contexts.

- The *free energy* of a capacity $\mu \in \mathsf{Cap}([0,1]^d)$ with a non-empty support is defined as the thermodynamic (lower) limit given for $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j\to\infty} \tau_{\mu,j}(q), \quad \text{where } \tau_{\mu,j}(q) := \frac{-1}{j} \log_2 \sum_{w\in\Sigma_j:\mu(I_w)>0} \mu(I_w)^q, \tag{2}$$

and q is interpreted as the inverse of a temperature when it is positive (the precise connection with statistical mechanics terminology is that in finite volume j, $\tau_{\mu,j}(q)$ is the free energy associated with the potential $V(w) = -\log(\mu(I_w))$, $w \in \Sigma_j$).

When the free energy $\tau_\mu(q)$ is a limit (not only a liminf) and is differentiable with respect to q, it is possible to describe the asymptotical distribution properties of μ over

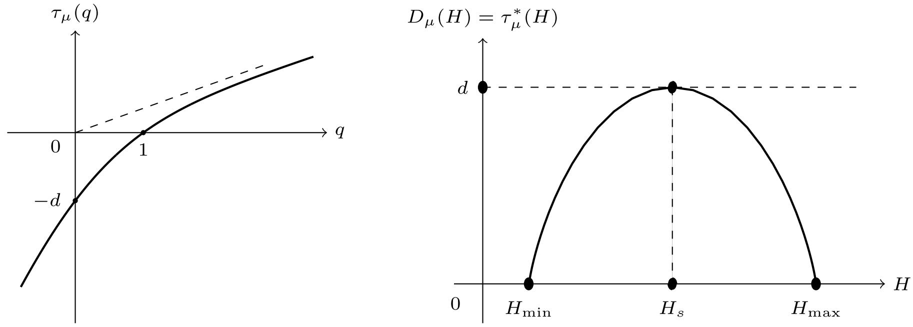

Figure 1. Left: Free energy function of a Gibbs measure μ on $[0,1]^d$. *Right:* The singularity spectrum of μ.

Σ_j thanks to large deviations theory, which roughly gives the approximation:

$$\forall H \in \mathbb{R}, \# \left\{ w \in \Sigma^* : |w| = j, \mu(I_w) \approx 2^{-jH} \right\} \approx 2^{j\tau_\mu^*(H)} \quad \text{as } j \to +\infty,$$

where τ_μ^* is the Legendre transform of τ_μ, i.e.,

$$\tau_\mu^*(H) := \inf_{q\in\mathbb{R}} \big(Hq - \tau_\mu(q)\big). \tag{3}$$

- The *singularity*, or *multifractal*, *spectrum* of μ is defined as

$$D_\mu : H \mapsto \dim \underline{E}_\mu(H), \quad H \in \mathbb{R},$$

where

$$\underline{E}_\mu(H) = \left\{ x \in \operatorname{supp}(\mu) : \liminf_{j\to\infty} \frac{\log_2 \big(\mu(I_{x|j})\big)}{-j} = H \right\}.$$

The Hausdorff dimension in $\mathbb{R}^d$ is denoted by dim, and by convention, $\dim \emptyset = -\infty$. The singularity spectrum provides a fine geometric description of the energy distribution at small scales by giving the Hausdorff dimension of the iso-Hölder sets $\underline{E}_\mu(H)$ of μ.

It turns out that when μ possesses nice scaling properties, one has

$$\forall H \in \mathbb{R}, \quad D_\mu(H) = \tau_\mu^*(H).$$

Definition 2. When the above formula is satisfied, τ_μ and D_μ are said to form a Legendre pair (see Figure 1). In this situation, one says that μ obeys the multifractal formalism at any $H \in \mathbb{R}$.

The validity of the multifractal formalism means that the geometric description of μ provided by its singularity spectrum D_μ matches with the asymptotic statistical description of the energy distribution μ provided by the free energy τ_μ and its Legendre transform.

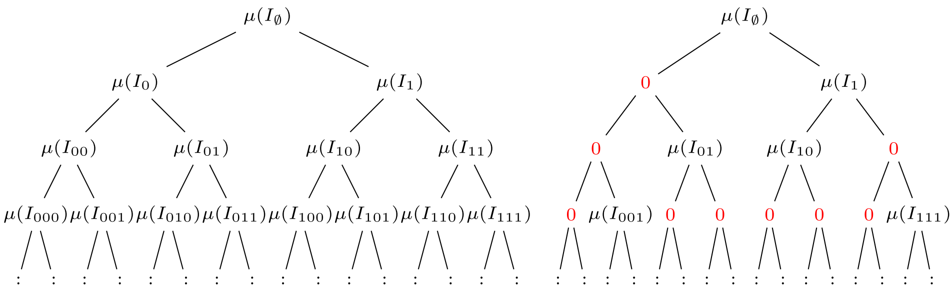

Figure 2. Left: Capacity μ on dyadic cubes. *Right:* Function $\widetilde{\mu}$ and surviving vertices after sampling.

1.2. Random sparse sampling operation on capacities

We perform on any capacity μ the natural random sampling process consisting in acting independently on the vertices of Σ^* by letting a vertex of generation j survive with probability $2^{-jd(1-\eta)}$, where $\eta \in (0, 1)$. It is a special case of decimation rule used in percolation theory on Σ^*. More formally:

Definition 3. Fix a real parameter $0 < \eta < 1$, called the *sampling index*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a sequence $(p_w)_{w\in\Sigma^*}$ a sequence of independent Bernoulli random variables so that $p_w \sim B(2^{-d(1-\eta)|w|})$, i.e.,

$$\mathbb{P}(p_w = 1) = 1 - \mathbb{P}(p_w = 0) = 2^{-d(1-\eta)|w|}. \tag{4}$$

When $p_w = 1$, w is said to be a surviving vertex (or a survivor).

For every $j \geqslant 1$, denote by $\mathcal{S}_j(\eta)$ the (random) set of surviving vertices in Σ_j:

$$\mathcal{S}_j(\eta) := \{w \in \Sigma_j : p_w = 1\}.$$

Let $\mu \in \mathsf{Cap}([0, 1]^d)$. We denote by $\widetilde{\mu} : \Sigma^* \to \mathbb{R}^+$ the function defined by

$$\forall\, w \in \Sigma^*, \quad \widetilde{\mu}(I_w) = \mu(I_w) \cdot p_w.$$

See Figure 2 for an illustration. The set of surviving vertices $\mathcal{S}_j(\eta)$ has a cardinality of expectation $2^{dj\eta}$ (which is exponentially less than 2^{dj}, the initial number of coefficients), and is very sparse. It is then natural to address the following questions about the remaining information after the sampling operation:

- What geometric and statistical information are associated with $\widetilde{\mu}$, and how are they related to the structure of μ?
- Can one recover the initial capacity (i.e., all the values $\mu(I_w)$, $w \in \Sigma^*$) from the sole knowledge of $\widetilde{\mu}$?

We bring answers when μ is the geometric realization on $[0, 1]^d$ of a Gibbs measure associated with a Hölder continuous potential on Σ, and more generally a non-trivial

Gibbs capacity. Specifically, the capacity μ satisfies that there exists a Gibbs measure ν on Σ, $K > 0$ and $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\mu(I_w) = K\nu([w])^{\alpha} 2^{-\beta|w|}, \quad \forall\, w \in \Sigma^*,$$

and μ is not constant, so $(\alpha, \beta) \neq (0, 0)$ (see § 2.1 for a precise definition of Gibbs measures and capacities).

We will mainly focus on the first question. Concerning the second question, the following alternative holds:

Theorem 1. *Suppose that μ is a Gibbs capacity associated with a Hölder potential on $[0, 1]^d$ not depending on only finitely many letters. With probability one, combining the almost multiplicativity property (8) of μ and concatenation of words, when $\eta \geqslant 1/2$ one can reconstruct all the values $\mu(I_w)$, $w \in \Sigma^*$, up to some multiplicative constant depending only on μ, while when $\eta < 1/2$ such a reconstruction is not possible.*

Remark 1. This exhibits a first phase transition phenomenon, at $\eta = 1/2$. See § 3, Theorem 4 for a more precise statement. In the case where μ is a Gibbs measure, one may wonder whether the ergodicity of μ could be exploited to get reconstruction through a more elaborate algorithm when $\eta < 1/2$. Though we do not have the answer, this seems unlikely in view of the fact that the remaining mass at small scales decreases almost surely exponentially. Indeed, $\mathbb{E}(\sum_{j \geqslant J} \sum_{|w|=j} \widetilde{\mu}(I_w)) = \sum_{j \geqslant J} 2^{-j(1-\eta)} \sim 2^{-J(1-\eta)}$.

Regarding the first question, we first reorganize the surviving information in a suitable and exploitable way, as follows. If $w, v \in \Sigma^*$, wv stands for the concatenation of w and v.

Definition 4. Let $\mu \in \mathsf{Cap}([0, 1]^d)$. Consider the random capacity $\mathsf{M}_\mu \in \mathsf{Cap}([0, 1]^d)$ associated with μ and the sequence $(p_w)_{w \in \Sigma^*}$ defined by

$$\mathsf{M}_\mu(I_w) = \max\left\{\mu(I_{wv}) : v \in \Sigma^* \text{ and } p_{wv} = 1\right\}. \tag{5}$$

See Figure 3 for the construction of M_μ. By construction, any capacity $\mu \in \mathsf{Cap}([0, 1]^d)$ satisfies $\mu(I_w) = \max\left\{\mu(I_{wv}) : v \in \Sigma^*\right\}$, hence (5) is the most natural formula to be used to build a capacity from $\widetilde{\mu}$.

It is not difficult to see that with probability 1, for every $w \in \Sigma^*$, the set $\{v \in \Sigma^* \text{ and } p_{wv} = 1\}$ is non-empty, so that M_μ is well defined. Observe that by our choice (4), most of the coefficients $\widetilde{\mu}(I_w)$ equal 0, hence typically one has $\mathsf{M}_\mu(I_w) \ll \mu(I_w)$ when μ has no atoms.

The definition of M_μ can be rephrased as

$$\mathsf{M}_\mu(I_w) = \max\{\mu(I_v) : v \text{ survives}, [v] \subset [w]\} = \max\left\{\widetilde{\mu}(I_{wv}) : v \in \Sigma^*\right\}.$$

Notice that M_μ and $\widetilde{\mu}$ are equivalent objects in the following sense. If μ is strictly positive, $\widetilde{\mu}$ can be recovered from M_μ since $\widetilde{\mu}(I_w) \neq 0$ if and only if $\mathsf{M}_\mu(I_w) > \mathsf{M}_\mu(I_{wv})$ for all $v \in \Sigma^*$ such that $|v| \geqslant 1$. From now on, we focus on the capacity M_μ, not on $\widetilde{\mu}$.

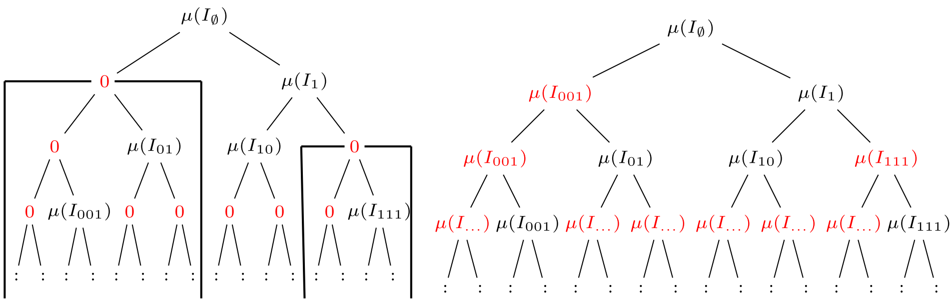

Figure 3. Left: surviving vertices after sampling, and the coefficients used to compute $\mathsf{M}_\mu(I_0)$ and $\mathsf{M}_\mu(I_{11})$. *Right:* The capacity M_μ.

Starting from a positive capacity μ whose free energy τ_μ and singularity spectrum D_μ form a Legendre pair, we consider the following questions in order to estimate the structural perturbations induced by the sampling process:

- Do the free energies in finite volume $\tau_{\mathsf{M}_\mu,j}$ (see equation (2)) converge to a thermodynamic limit τ_{M_μ} as $j \to +\infty$?
- Is it possible to conduct a fine analysis of the local behavior of M_μ so that D_{M_μ} is computable? If so, do τ_{M_μ} and D_{M_μ} form a Legendre pair?
- Are there explicit relations between the pair $(\tau_{\mathsf{M}_\mu}, D_{\mathsf{M}_\mu})$ and the original one (τ_μ, D_μ), so that one can recover all or any of the initial information (before sampling)?

When μ is a Gibbs capacity, these questions are answered positively. It is worth pointing now that the sampling deeply modifies and complexifies the initial structure, creating several phenomenological differences between μ and M_μ, both from thermodynamic (phase transitions appear) and geometric (metric approximation theory comes into play) viewpoints.

1.3. Statement of the results for the random capacity M_μ when μ is Gibbsian

Only capacities with full support, i.e., $\mu(I_w) > 0$ for all $w \in \Sigma^*$, are considered.

Definition 5. Let $\mu \in \mathcal{C}([0,1]^d)$ with full support. For $x \in [0,1]^d$, the lower and upper local dimensions of μ at x are respectively defined as

$$\underline{\dim}(\mu, x) = \liminf_{r \to 0^+} \frac{\log_2 \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{\dim}(\mu, x) = \limsup_{r \to 0^+} \frac{\log_2 \mu(B(x,r))}{\log r}.$$

When $\underline{\dim}(\mu, x) = \overline{\dim}(\mu, x)$, their common value is denoted by $\dim(\mu, x)$.

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For $H \in \mathbb{R}$, set

$$\underline{E}_\mu(H) = \left\{x \in [0,1]^d : \underline{\dim}(\mu, x) = H\right\},$$
$$\overline{E}_\mu(H) = \left\{x \in [0,1]^d : \overline{\dim}(\mu, x) = H\right\},$$
$$E_\mu(H) = \underline{E}_\mu(H) \cap \overline{E}_\mu(H).$$

Recall that the singularity spectrum of μ is the mapping

$$D_\mu : H \in \mathbb{R} \longmapsto \dim \underline{E}_\mu(H).$$

The lower local dimension is distinguished with respect to $\overline{\dim}(\mu, x)$ or $\dim(\mu, x)$, because it provides at any x the best local control of the capacity μ. Since μ is bounded, one has $\underline{\dim}(\mu, x) \geqslant 0$ at any x, hence $\underline{E}_\mu(H) = \emptyset = \overline{E}_\mu(H)$ for all $H < 0$.

The multifractal formalism states that for every $\mu \in \mathsf{Cap}([0,1]^d)$ with full support,

$$\dim \underline{E}_\mu(H) \leqslant \tau_\mu^*(H) := \inf_{q \in \mathbb{R}} \big(Hq - \tau_\mu(q)\big), \quad \forall\, H \in \mathbb{R}, \tag{6}$$

see for instance [7, 28], which deal with measures, but easily extend to capacities. Recall that the multifractal formalism holds for μ at $H \in \mathbb{R}$ when there is equality in (6).

Results are obtained for a non-homogeneous Gibbs capacity μ, i.e., associated with a Hölder potential non-cohomologous to a constant (see Definition 8 in § 2.1 for a precise description). For such an object, the following statement gathers information deduced from the study of Gibbs measures and almost-additive potentials [7, 11, 17, 18, 20, 29, 30]: Let

$$H_{\min} = \tau_\mu'(+\infty) \leqslant H_s = \tau_\mu'(0) \leqslant H_{\max} = \tau_\mu'(-\infty).$$

(1) The free energy function τ_μ is the limit of $(\tau_{\mu,j})_{j \geqslant 1}$ as $j \to +\infty$. The function τ_μ is analytic, increasing, and strictly concave on $\mathbb{R}$.

(2) The strictly concave function τ_μ^* is non-negative on its domain of definition, namely $[H_{\min}, H_{\max}] \subset \mathbb{R}_+^*$, and analytic on $(H_{\min}, H_{\max})$. It reaches its maximum at H_s, and $\tau_\mu^*(H_s) = d$.

(3) For all $H \geqslant 0$, we have $D_\mu(H) = \dim E_\mu(H) = \dim \overline{E}_\mu(H) = \tau_\mu^*(H)$. The multifractal formalism holds for μ, and (τ_μ, D_μ) forms a Legendre pair.

Let us describe our result on the random capacity M_μ obtained after the sampling of μ. For this, some additional notations are needed.

Definition 6. Let μ be a non-homogeneous Gibbs capacity. Given $\eta \in (0,1)$, one introduces the following parameters, which depend continuously on D_μ and η only:

- $H_\ell(\eta_\ell) = \min\{H \geqslant 0 : D_\mu(H) \geqslant d(1-\eta)\}$.
- $H_\ell(\widetilde{\eta})$ is the (unique) real number such that the tangent to the graph of D_μ at the point $(H_\ell(\widetilde{\eta}), D_\mu(H_\ell(\widetilde{\eta}))$ passes through $(0, d(1-\eta))$.
- $q_{\widetilde{\eta}} = D_\mu'(H_\ell(\widetilde{\eta}))$ and $q_{\eta_\ell} = D_\mu'(H_\ell(\eta_\ell))$.
- Finally, $\widetilde{H}_\ell(\widetilde{\eta}) = -\frac{\tau_\mu(q_{\widetilde{\eta}})}{q_{\widetilde{\eta}}}$.

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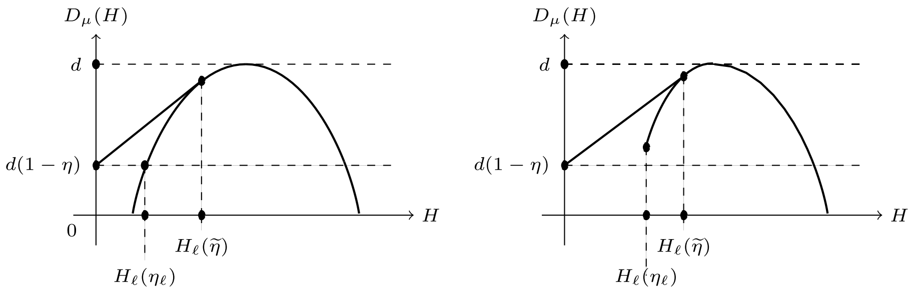

Figure 4. Values of $H_\ell(\eta_\ell)$ and $H_\ell(\widetilde{\eta})$ depending on D_μ and η: *Left:* when $D_\mu(H_{\min}) \leqslant d(1-\eta)$. *Right:* when $D_\mu(H_{\min}) > d(1-\eta)$.

See Figure 4 for an illustration. The origin and roles of these parameters, as well as the notations themselves, will be explained in §§ 4 and next.

Theorem 2. *Let μ be a non-homogeneous Gibbs capacity on $[0,1]^d$. Let $0 < \eta < 1$ be a sampling parameter. With probability 1:*

(1) *The singularity spectrum of M_μ reads:*

$$D_{\mathsf{M}_\mu}(H) = \begin{cases} D_\mu(H) - d(1-\eta) & \text{if } H_\ell(\eta_\ell) \leqslant H \leqslant H_\ell(\widetilde{\eta}), \\ q_{\widetilde{\eta}} \cdot H & \text{if } H_\ell(\widetilde{\eta}) \leqslant H \leqslant H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), \\ D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big) & \text{if } H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) \leqslant H \leqslant H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}), \\ -\infty & \text{otherwise.} \end{cases}$$

(2) *The free energy function of M_μ is the limit of $(\tau_{M_\mu,j})_{j\geqslant 1}$ as $j \to \infty$, and $(D_{\mathsf{M}_\mu}, \tau_{\mathsf{M}_\mu})$ forms a Legendre pair. One has*

$$\tau_{\mathsf{M}_\mu}(q) = \begin{cases} \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta}) \cdot q & \text{if } q \leqslant q_{\widetilde{\eta}}, \\ \tau_\mu(q) + d(1-\eta) & \text{if } q_{\widetilde{\eta}} < q < q_{\eta_\ell}, \\ H_\ell(\eta_\ell) \cdot q & \text{if } q_{\eta_\ell} < +\infty \text{ and } q \geqslant q_{\eta_\ell}. \end{cases}$$

Let us explain in a word the formula for D_{M_μ}, which is quite easy to graphically interpret: to build D_{M_μ}, separate in three parts the multifractal spectrum D_μ of μ (see the left part of Figure 5). The left part $H < H_\ell(\eta_\ell)$ does not appear in D_{M_μ}. Then the second part $H_\ell(\eta_\ell) \leqslant H \leqslant H_\ell(\widetilde{\eta})$ (drawn in black) is translated along the vertical direction so that its left endpoint touches the horizontal axis. The intermediary part (drawn in blue) is a segment. The third part $H \geqslant H_\ell(\widetilde{\eta})$ is translated along the horizontal direction to the right, so that the final graph (see the right part of Figure 5) is differentiable.

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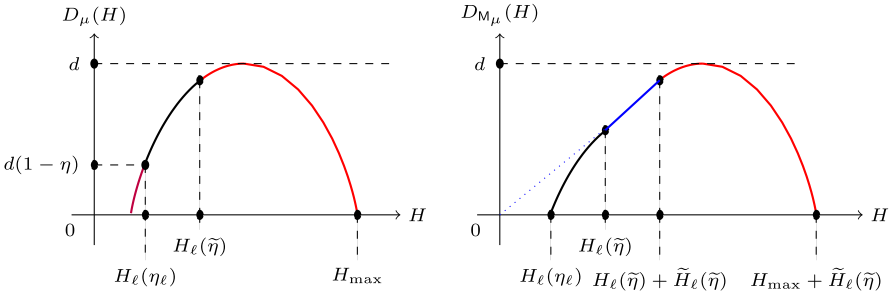

Figure 5. Case $D_\mu(H_{\min}) \leqslant d(1-\eta)$: *Left:* singularity spectrum of μ. *Right:* Almost sure singularity spectrum of M_μ. The parts drawn with same color are translated copies of each other. The left part $H \leqslant H_\ell(\widetilde{\eta})$ of the spectrum of μ (drawn in purple) does not appear in the singularity spectrum of M_μ, and a linear part appears in D_{M_μ} which was not present in D_μ. Observe that the slope of D_{M_μ} at $H_\ell(\eta_\ell)$ is finite.

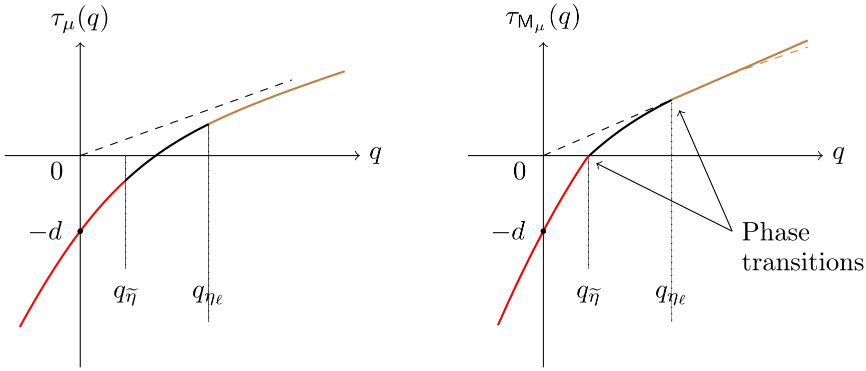

Figure 6. Case $D_\mu(H_{\min}) \leqslant d(1-\eta)$: *Left:* free energy τ_μ of μ; *Right:* free energy τ_{M_μ} of M_μ.

1.4. Comments

• It is quite easy to see that the lower local dimension of M_μ at any x must be greater than $H_\ell(\eta_\ell)$ (see Lemma 6). It is much more involved to define and to understand the role of the other parameters.

• From τ_{M_μ}, one recovers the initial free energy τ_μ, except for $q \geqslant q_{\eta_\ell}$ whenever $q_{\eta_\ell} < \infty$ (i.e., $H_{\min} < H_\ell(\eta_\ell)$); see Figure 6. This situation is generic, since for a generic set of Hölder potentials one has $\tau_\mu^*(H_{\min}) = 0$ [32]. Similarly, one recovers D_μ from D_{M_μ} for $H \geqslant H_\ell(\eta_\ell)$. In this sense, the sampling procedure implies a loss of information on the local dimensions if $H_{\min} < H_\ell(\eta_\ell)$, since the values $D_\mu(H)$ are 'lost' when $H < H_\ell(\eta_\ell)$. Interestingly, this does not contradict the fact that after Theorem 1, when $\eta \geqslant 1/2$, one can reconstruct μ up to a multiplicative constant. Also, when $\eta < 1/2$, the reconstruction by concatenation is not possible, nevertheless one fully recovers (τ_μ, D_μ) from $(\tau_{M_\mu}, D_{M_\mu})$. Figure 7 describes the situation when $D_\mu(H_{\min}) > d(1-\eta)$.

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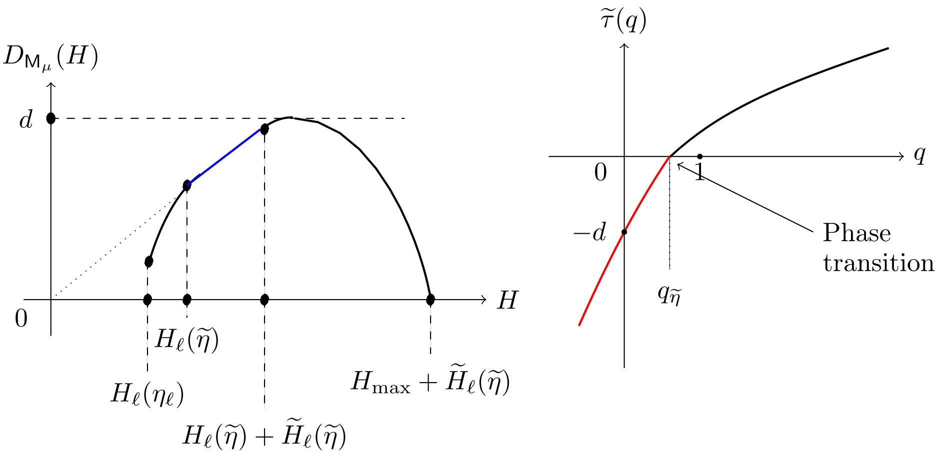

Figure 7. Case $D_\mu(H_{\min}) > d(1-\eta)$: Observe that D_μ and D_{M_μ} have an infinite slope at $H_\ell(\eta_\ell) = H_{\min}$.

• The free energy τ_{M_μ} is not differentiable at $q_{\widetilde{\eta}}$, and it is differentiable but not twice differentiable at q_{η_ℓ} when $d(1-\eta) > D_\mu(H_{\min})$. Moreover, τ_{M_μ} is analytic outside these singularities. In the thermodynamics language, τ_{M_μ} presents a first order phase transition at the inverse temperature $q_{\widetilde{\eta}}$, and a second order phase transition at the inverse temperature q_{η_ℓ} whenever $d(1-\eta) > D_\mu(H_{\min})$ (this is the generic situation).

Let us mention that in thermodynamic formalism theory, the study of phase transitions associated with continuous potentials has a long history and is still an active domain of research, in which nice classes of examples of potentials providing phase transitions are listed (see e.g., [8, 9, 16, 19, 21, 23, 30, 31]).

• In most of the usual situations, upper bounds for dimensions of 'fractal sets' are easily deduced from covering arguments, and lower bounds are more difficult to derive. The structure of M_μ, combining random and dynamical phenomena, makes both the derivation of the sharp upper bound *and* lower bound for D_{M_μ} delicate. It is too soon in the paper to give an intuition of the proofs. Let us only say that they follow from a careful analysis of the distribution and the scaling behavior (with respect to μ) of the surviving vertices. Also, results on large deviations for Gibbs measures, heterogeneous mass transference principles (which combines ergodic and approximation theories) and percolation theory, are involved.

• One may also want to describe the asymptotical statistical distribution of M_μ through the notion of large deviations, as is often the case in statistical physics.

Definition 7. Let $\mu \in \mathcal{C}([0,1]^d)$ with full support. For every set $I \subset \mathbb{R}^+$, and every integer $j \geqslant 1$, set

$$\mathcal{E}_\mu(j, I) = \left\{ w \in \Sigma_j : \frac{\log_2 \mu(I_w)}{-j} \in I \right\}.$$

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If $H \geqslant 0$ and $\varepsilon > 0$, we introduce the notation

$$\mathcal{E}_\mu(j, H \pm \varepsilon) = \left\{ w \in \Sigma_j : \frac{\log_2 \mu(I_w)}{-j} \in [H - \varepsilon, H + \varepsilon] \right\}.$$

Then, the lower and upper large deviations spectra of μ are respectively

$$\begin{aligned} \underline{f}_\mu(H) &= \lim_{\varepsilon \to 0} \liminf_{j \to +\infty} \frac{\log_2 \# \mathcal{E}_\mu(j, H \pm \varepsilon)}{j} \\ \text{and} \quad \overline{f}_\mu(H) &= \lim_{\varepsilon \to 0} \limsup_{j \to +\infty} \frac{\log_2 \# \mathcal{E}_\mu(j, H \pm \varepsilon)}{j}. \end{aligned}$$

Heuristically, one should have in mind that the number of words of length j satisfying $\mu(I_w) \sim 2^{-jH}$ is between $2^{j \underline{f}_\mu(H)}$ and $2^{j \overline{f}_\mu(H)}$. Next theorem states that M_μ behaves nicely with respect to the large deviations theory, as the Gibbs capacity μ does.

Theorem 3. *Under the same assumptions as in Theorem 2, with probability 1,*

$$\text{for all } H \geqslant 0, \quad \underline{f}_{\mathsf{M}_\mu}(H) = \overline{f}_{\mathsf{M}_\mu}(H) = D_{\mathsf{M}_\mu}(H).$$

1.5. Conclusion and further perspectives

The hierarchical structure of the initial capacity μ is so robust that, although it is highly sampled, the remaining coefficients still possess a rich structure, especially in terms of scaling properties and multifractal formalism. For instance, one consequence of Theorem 2 is that no matter how close to 0 η is (i.e., even if only a very small logarithmic proportion of vertices is kept), it is always possible to reconstruct, from the knowledge of τ_{M_μ}, all the dimensions of the set of points with local dimension greater than H_s. This phenomenon is remarkable, since at the same time, a lot of information on the dimensions of the set of points with local dimension smaller than H_s is lost. This asymmetry was, at least from our point of view, unexpected. Let us finish with some perspectives:

- Theorem 2 can certainly be extended to capacities obtained after sampling of branching random walks.
- Other sampling procedures shall be investigated: one may allow correlations between the p_w, or make η depend on the vertex w. One may also multiply $\mu(I_w)$ by some positive r.v. when $p_w = 1$. Other phase transitions would certainly occur.
- An interesting question is whether the capacity M_μ is equivalent, after a natural renormalization procedure, to a measure, as it is the case for Gibbs capacities.

The paper is organized as follows. Section 2 contains details on Gibbs measures and capacities, and gathers some information about large deviations and multifractal analysis. Section 3 focuses on the reconstruction of the original capacity μ from its sample $\widetilde{\mu}$. The rest of the paper is devoted to the investigation of the structure of M_μ. Some definitions are introduced to explain the origin of the parameters used in Theorem 2 (§ 4). There, we first explain that we will work with a slight, and natural, modification of M_μ possessing the same statistical and geometric properties as M_μ, but necessary to get an application of

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our result to wavelet series. In § 5, the scaling and distribution properties of the surviving vertices are investigated. A key decomposition of the value of $\mu(I_w)$ when w survives, is proved (see Proposition 4). Section 6 is dedicated to the computation of the free energy function and the lower and upper large deviations spectra of M_μ, from which follows the sharp upper bound for the singularity spectrum D_{M_μ}, while in § 7 we establish the sharp lower bound. The case of homogeneous Gibbs capacities (i.e., when the associated Gibbs measure is the Lebesgue measure) is dealt with in § 8.

Notational conventions:

- *Capital letters* ($E_{\mathsf{M}_\mu}(H)$, $F_\mu, \ldots$) are sets of points $x \in [0, 1]^d$ having some properties.
- *Curved letters* are used for sets of finite words having specific properties (e.g., $\mathcal{S}_j(\eta, W)$ for some surviving coefficients, $\mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon)$ or $\mathcal{T}_\mu(j, \eta', \varepsilon)$ for words with specific properties, see next Definition 17).
- *Calligraphic letters* ($\mathscr{A}$, $\mathscr{B}, \ldots$) stand for probabilistic events.
- For every $W \in \Sigma_J$, $\mathcal{N}(W)$ is the set of $3^d - 1$ words corresponding to the neighboring cubes of generation J of I_W. Sometimes we write $\mathcal{N}_J(W)$ when the length J of W is specified.

2. Complements on Gibbs measures and capacities

2.1. Formal definition of Gibbs measures and capacities

Let $\psi : \Sigma \to \mathbb{R}$ be a Hölder continuous mapping. Then, the function Ψ defined as

$$\Psi([w]) = \sup_{t \in [w]} \sum_{i=0}^{|w|-1} \psi(\sigma^i t), \quad \forall\, w \in \Sigma^*$$

is almost additive: there exists $C_1 > 0$ such that for all $u, v \in \Sigma^*$,

$$|\Psi([u]) + \Psi([v]) - \Psi([uv])| \leqslant C_1;$$

see [30]. This almost additivity property implies that the topological pressure

$$P(\sigma, \phi) = \lim_{j \to \infty} \frac{1}{j} \log \sum_{w \in \Sigma_j} \exp(\Psi([w]))$$

exists in $\mathbb{R}$, and there exists a fully supported Gibbs measure ν on Σ such that for another constant $C_2 \geqslant 1$ one has

$$C_2^{-1} \exp(\Psi([w]) - nP(\sigma, \psi)) \leqslant \nu([w]) \leqslant C_2 \exp(\Psi([w]) - nP(\sigma, \psi)), \quad \forall\, w \in \Sigma^*.$$

Also, there is a unique choice of such a ν so that ν is ergodic. Moreover, the mapping $q \in \mathbb{R} \mapsto P(\sigma, q\psi)$ is convex, analytic, and it is strictly convex if and only if ψ is not cohomologous to a constant, i.e., there is no continuous function φ on Σ and constant $c \in \mathbb{R}$ such that $\psi = c + \varphi - \varphi \circ \sigma$. These are important facts from thermodynamic formalism (see e.g., [30]).

Definition 8. A capacity $\mu \in \mathrm{Cap}([0, 1]^d)$ is called a Gibbs capacity if

$$\mu(I_w) = K\nu([w])^\alpha e^{-|w|\beta}, \quad \forall\, w \in \Sigma^*, \tag{7}$$

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where $K > 0$, $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(0, 0)\}$, and ν is a Gibbs measure associated with a Hölder continuous potential ψ has above.

Equivalently, one says that μ is associated with the Hölder potential

$$\phi = \alpha\psi - \alpha P(\sigma, \psi) - \beta.$$

The capacity μ is said to be *homogeneous* when ψ is cohomologous to a constant or $\alpha = 0$, i.e., when ϕ is cohomologous to a constant, and *non-homogeneous* otherwise.

Observe that if $(\alpha, \beta) = (1, 0)$, μ reduces to the Gibbs measure associated with ψ, and that

$$\tau_\mu(q) = \frac{1}{\log(2)}\Big((\beta + \alpha P(\sigma, \psi))q - P(\sigma, \alpha q\psi)\Big), \quad \forall q \in \mathbb{R}.$$

The following fact is key: the capacity μ possesses self-similarity properties expressed through the following almost multiplicative property (easy to check): there exists a constant $C \geqslant 1$ such that

$$\text{for all words } v \text{ and } w, \quad C^{-1}\mu(I_w)\mu(I_v) \leqslant \mu(I_{wv}) \leqslant C\mu(I_w)\mu(I_v). \tag{8}$$

2.2. Large deviations and multifractal properties

Let $\mu \in \mathcal{C}([0, 1]^d)$ with non-empty support. The concave function τ_μ^* is called Legendre spectrum of μ (recall that the Legendre transform τ_μ^* is given by (3)). For a non-homogeneous Gibbs capacity μ, one always has:

- τ_μ is strictly concave and analytic, and D_μ is strictly concave, and real analytic over $(H_{\min}, H_{\max})$. Also, $D_\mu = \tau_\mu^*$ and $(D_\mu^*)^* = D_\mu$.
- If $H = \tau'_\mu(q)$, then $\tau_\mu(q) = D_\mu^*(q) = qH - D_\mu(H) = q\tau'_\mu(q) - D_\mu(\tau'_\mu(q))$.
- If $q = D'_\mu(H)$, then $D_\mu(H) = \tau_\mu^*(H) = Hq - \tau_\mu(q) = HD'_\mu(H) - \tau_\mu(D'_\mu(H))$.

These relationships will be used repeatedly in the following.

Definition 9. For any fully supported capacity $\mu \in \mathcal{C}([0, 1]^d)$, define the level sets

$$E_\mu^{\leqslant}(H) = \{x \in [0, 1]^d : \dim(\mu, x) \leqslant H\}$$
$$\text{and } E_\mu^{\geqslant}(H) = \{x \in [0, 1]^d : \dim(\mu, x) \geqslant H\}.$$

The sets $\underline{E}_\mu^{\leqslant}(H)$, $\underline{E}_\mu^{\geqslant}(H)$, $\overline{E}_\mu^{\leqslant}(H)$, $\overline{E}_\mu^{\geqslant}(H)$ are defined similarly using the lower and upper local dimensions, respectively.

If $j \geqslant 1$ and $w \in \Sigma_j$, denote by $\mathcal{N}_j(w)$ the set of at most 3^d elements $v \in \Sigma_{|w|}$ such that I_v is a neighbor of I_w in $\mathbb{R}^d$. Also, for $x \in [0, 1]^d$ and $j \geqslant 1$, set $\mathcal{N}_j(x) = \mathcal{N}(x_{|j})$. One defines the set

$$\widetilde{E}_\mu(H) = \left\{x \in [0, 1]^d : \lim_{j\to+\infty} \frac{\log_2 \max_{w\in\mathcal{N}_j(x)} \mu(I_w)}{j} = \lim_{j\to+\infty} \frac{\log_2 \min_{w\in\mathcal{N}_j(x)} \mu(I_w)}{j} = H\right\}.$$

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Obviously $\widetilde{E}_\mu(H) \subset E_\mu(H)$. This refinement of $E_\mu(H)$ is needed when looking for the lower bound of the Hausdorff dimensions of some sets in § 7.

A direct consequence of large deviations theory (see e.g., [7, 28]) is the next property, true for all capacities; see [3, 7, 26, 28].

Proposition 1. *Let $\mu \in \mathcal{C}([0,1]^d)$ with full support. For all $H \leqslant \tau_\mu'(0^+)$, one has*

$$\limsup_{j\to\infty} \frac{1}{j} \log_2 \#\mathcal{E}_\mu(j,[0,H]) \leqslant \tau_\mu^*(H).$$

Next proposition describes additional multifractal and large deviation features of a Gibbs measure μ.

Proposition 2. *Let μ be a non-homogeneous Gibbs capacity. Recall that $H_{\min} = \tau_\mu'(+\infty) < H_s = \tau_\mu'(0) < H_{\max} = \tau_\mu'(-\infty)$.*

(1) *For every $H \geqslant 0$ and $F \in \{E, \underline{E}, \overline{E}\}$, one has*
$$\dim F_\mu(H) = \underline{D}_\mu(H) = \overline{D}_\mu(H) = \tau_\mu^*(H) = D_\mu(H),$$
with $F_\mu(H) = \emptyset$ if and only if $D_\mu(H) = -\infty$.

(2) *For every $H \in [H_{\min}, H_s]$ (i.e., in the increasing part of the singularity spectrum D_μ), one has*
$$\dim E_\mu^{\leqslant}(H) = \dim \underline{E}_\mu^{\leqslant}(H) = \dim \overline{E}_\mu^{\leqslant}(H) = D_\mu(H).$$

(3) *For every $H \in [H_s, H_{\max}]$ (i.e., in the decreasing part of D_μ), one has*
$$\dim E_\mu^{\geqslant}(H) = \dim \underline{E}_\mu^{\geqslant}(H) = \dim \overline{E}_\mu^{\geqslant}(H) = D_\mu(H).$$

(4) *For every possible local dimension $H \in (H_{\min}, H_{\max})$, there exists a unique $q \in \mathbb{R}$ such that $H = \tau_\mu'(q)$. The Gibbs measure μ_H associated with the potential $q\phi$ is exact dimensional with dimension $D_\mu(H)$, and $\mu_H\big(E_\mu(H)\big) = \mu\big(\widetilde{E}_\mu(H)\big) = 1$.*

(5) *For every $\varepsilon > 0$ and every interval $I \subset \mathbb{R}_+$, there exists an integer J_I such that for every $j \geqslant J_I$,*
$$\left| \frac{\log_2 \mathcal{E}_\mu(j,I)}{j} - \sup_{h\in I} D_\mu(h) \right| \leqslant \varepsilon.$$

(6) *There exists $K > 0$ such that for every finite word $w \in \Sigma^*$, $\left| \dfrac{\log_2 \mu(I_w)}{-|w|} \right| \leqslant K$.*

Items (1) and (3) of the last proposition state in particular that the Hausdorff dimension of the sets of points at which $\dim(\mu,x) = H$ is the same as the Hausdorff dimension of the set of points at which $\underline{\dim}(\mu,x) = H$. This will be of particular importance.

We often use item (5) under the following form. Recall the formula for $\mathcal{E}_\mu(j, H\pm\varepsilon)$ in Definition 7: heuristically, $\mathcal{E}_\mu(j,H\pm\varepsilon)$ contains those words of length j such that $\mu(I_w) \sim 2^{-j(H\pm\varepsilon)}$. For every $H_{\min} \leqslant H \leqslant H_{\max}$ and $\varepsilon, \widetilde{\varepsilon} > 0$, there exists a generation J such that $j \geqslant J$ implies

$$\left| \frac{\log_2 \#\mathcal{E}_\mu(j,H\pm\varepsilon)}{j} - \sup_{h\in[H-\varepsilon,H+\varepsilon]} D_\mu(h) \right| \leqslant \widetilde{\varepsilon}. \tag{9}$$

One needs to keep in mind that $\#\mathcal{E}_\mu(j, H \pm \varepsilon) \approx 2^{jD_\mu(H)}$.

3. Reconstruction of the initial capacity μ

Fix a Gibbs capacity μ. We investigate the possibility to reconstitute the whole Gibbs tree $(\mu(I_w))_{w\in\Sigma^*}$ from the sole knowledge of $\tilde{\mu}$ (or equivalently, from M_μ).

Assume first that the capacity μ is associated with a Bernoulli measure, i.e., there exists $q_0, q_1 > 0$ such that for any word $w \in \Sigma_*$ one has $\mu(I_{w1}) = q_1\mu(I_w)$ and $\mu(I_{w0}) = q_0\mu(I_w)$. Hence, in order to reconstitute μ, it is enough to find q_0 and q_1. Assume that there exist two surviving vertices w and w' having different proportions of zeros and ones in their dyadic decomposition. It is easy to check that this event has probability one. Then the knowledge of $\mu(I_w)$ and $\mu(I_{w'})$ leads to two linearly independent equations with unknowns q_0 and q_1, hence to their values.

This idea generalizes directly to the case where μ is constructed from a Markov measure, i.e., there exist an integer $k \geqslant 0$ and $((q_{v0}, q_{v_1}))_{v\in\Sigma_k} \in (0, \infty)^{2^{k+1}}$ such that for all $w \in \Sigma_*$ and $v \in \Sigma_k$ one has $\mu(I_{wv0}) = q_{v0}\mu(I_{wv})$ and $\mu(I_{wv1}) = q_{v1}\mu(I_{wv})$.

When μ is associated with a general Gibbs measure, the situation is not that simple. The answer we propose uses the basic tools we have at our disposal, namely concatenation of words and quasi-Bernoulli property (8); it depends on the value of η, and there is a phase transition at $\eta = 1/2$.

Definition 10. Let $k \in \mathbb{N}^*$. A word $u \in \Sigma^*$ is *k-reconstructible* when there is a finite sequence of words $(w_1, u_1, w_2, u_2, \dots w_k, u_k)$ in Σ^* such that

- for every $i \in \{1, \dots, k\}$, $p_{w_i} = p_{w_iu_i} = 1$,
- $u = u_1u_2 \cdots u_k$.

One says that $S \subset \Sigma^*$ is k-reconstructible when every word $u \in S$ is k-reconstructible.

This definition follows from the idea that when u is k-reconstructible, after sampling of the initial tree one has access to the value of the weights $\mu(I_{w_i})$ and $\mu(I_{w_iu_i})$ for every i. Hence, by the quasi-Bernoulli property (8), one estimates, up to the constant $C > 1$, the value of $\mu(I_{u_i})$, and by concatenation of the words $u_1, \dots, u_k$ and (8) again, one reconstructs the value of $\mu(I_{u_i})$ up to the constant C^{k+1}. Next theorem completes Theorem 1 in the introduction.

Theorem 4. *When $\eta \geqslant 1/2$, Σ^* is 1-reconstructible, while when $\eta < 1/2$, Σ^* is not k-reconstructible, for any integer $k \geqslant 1$.*

Proof. Fix a generation $\ell \geqslant 1$, and a word $u \in \Sigma_\ell$. By construction, for any word $w \in \Sigma_j$,

$$\mathbb{P}(p_w p_{wu} = 1) = 2^{-j(1-\eta)}2^{-(j+\ell)(1-\eta)} = 2^{-\ell(1-\eta)}2^{-j2(1-\eta)}. \tag{10}$$

- Assume first that $\eta \geqslant 1/2$. Consider the random variable $Z_j = \#\{w \in \Sigma_j : p_w p_{wu} = 1\}$ and the event $\mathcal{Z}_j = \{Z_j = 0\}$. By independence, one has $\mathbb{P}(\mathcal{Z}_j) = \big(1 - 2^{-\ell(1-\eta)}2^{-j2(1-\eta)}\big)^{2^j}$, which tends superexponentially fast to zero when $\eta > 1/2$. When

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$\eta = 1/2$, $\mathbb{P}(\mathcal{Z}_j) \leqslant c < 1$ for some constant c, uniformly in $j \geqslant 1$. Hence, in all cases,

$$\sum_{n=1}^{+\infty} \mathbb{P}(\mathcal{Z}^c_{n(\ell+1)}) = +\infty.$$

Since the events $(\mathcal{Z}^c_{n(\ell+1)})_{n\geqslant 1}$ are mutually independent, the Borel–Cantelli Lemma gives the almost sure existence of infinitely many words $w \in \Sigma^*$ such that $p_w p_{wu} = 1$. Consequently, u is 1-reconstructible.

• Assume now that $\eta < 1/2$. For every $u \in \Sigma^*$, denote by r_u the random variable equal to 1 if u is 1-reconstructible, and 0 otherwise. Hence, r_u is a Bernoulli variable, with parameter $\tilde{p}_{|u|}$, the probability that there exists $w \in \Sigma^*$ such that $p_w p_{wu} = 1$ (which depends only on $|u|$). By (10),

$$\forall \ell \geqslant 1,\ \tilde{p}_\ell \leqslant \sum_{w\in\Sigma^*} \mathbb{P}(p_w p_{wu} = 1) = \sum_{j\geqslant 1} 2^j 2^{-2j(1-\eta)} 2^{-\ell(1-\eta)} = C_1 2^{-\ell(1-\eta)}.$$

Fix $\varepsilon > 0$ so small that $\eta + \varepsilon < 1$, and $(\varepsilon_j)_{j\geqslant 1}$ a positive sequence converging to zero, such that $0 < \varepsilon_j \leqslant \varepsilon$ and $\sum_{j\geqslant 1} 2^{-j\varepsilon_j} < +\infty$.

Let us introduce $\widetilde{Z}^1_j = \sum_{u\in\Sigma_j} r_u$, the number of 1-reconstructible words at generation j. The random variable $\widetilde{Z}^1_j$ is a sum of non-independent random variables with common law the Bernoulli law with parameter $\tilde{p}_j$. Markov's inequality yields $\mathbb{P}(\widetilde{Z}^1_j \geqslant 2^{j\varepsilon_j} 2^j \tilde{p}_j) \leqslant 2^{-j\varepsilon_j}$, and Borel–Cantelli's lemma implies that almost surely, for j large enough, we have

$$\widetilde{Z}^1_j \leqslant 2^{j\varepsilon_j} 2^j \tilde{p}_j \leqslant C_1 2^{j(\eta+\varepsilon_j)}. \tag{11}$$

This implies that Σ^* is not 1-reconstructible, since at most $C_1 2^{j(\eta+\varepsilon_j)} \ll 2^j$ words can be reconstructed.

Assume that for $k \geqslant 2$, the number $\widetilde{Z}^k_j$ of k-reconstructible words at any generation j is bounded from above by $C_k j^k 2^{j(\eta+\varepsilon)}$ for some constant C_k. Let $J \geqslant k+1$. Any $(k+1)$-reconstructible word u in Σ_J is the concatenation of a k-reconstructible word and a 1-reconstructible word. Hence, by (11), for the constant $C_{k+1} = C_1 C_k$, one has

$$\widetilde{Z}^{k+1}_J \leqslant \sum_{i=1}^{J-k} \widetilde{Z}^1_i \widetilde{Z}^k_{J-i} \leqslant C_1 C_k \sum_{i=1}^{J-k} 2^{i(\eta+\varepsilon_i)} (J-i)^k 2^{(J-i)(\eta+\varepsilon)} \leqslant C_{k+1} J^{k+1} 2^{J(\eta+\varepsilon)}.$$

One concludes that Σ^* is not k-reconstructible, for any k, since $\widetilde{Z}^k_J \ll 2^J$. □

4. Modified version of M_μ. New parameters, and alternative definitions for the parameters $H_\ell(\eta_\ell)$, $H_\ell(\widetilde{\eta})$ and $\widetilde{H}(\widetilde{\eta})$

From now on, we consider a non-homogeneous Gibbs capacity μ. The homogeneous case will be dealt with at the all end of the paper (§ 8).

We work with the $\| \ \|_\infty$ over $\mathbb{R}^d$, so that balls are Euclidean cubes.

4.1. Modified version of the capacity M_μ

We study a slight modification of M_μ.

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Definition 11. Let $\mu \in \mathrm{Cap}([0,1]^d)$. We set

$$\widetilde{\mathsf{M}}_\mu(I_w) = \max_{u \in \mathcal{N}_j(w)} \mathsf{M}_\mu(I_u) = \max\left\{\mu(I_{uv}) : u \in \mathcal{N}_j(w), v \in \Sigma^*, p_{uv} = 1\right\}.$$

Thus, the difference between the capacities M_μ and $\widetilde{\mathsf{M}}_\mu$ is that $\widetilde{\mathsf{M}}_\mu(I_j(x))$ carries information about the behavior of μ in the neighborhood of x, not only in the dyadic cube $I_j(x)$ of generation j containing x.

We consider $\widetilde{\mathsf{M}}_\mu$ for two reasons. First it is natural to extend M_μ to balls: for $x \in [0,1]^d$ and $r > 0$ one denotes $B(x,r)$ the closed ball of radius r centered at x, and defines $\mathsf{M}_\mu(B(x,r)) = \max\{\mathsf{M}_\mu(I_w) : I_w \subset B(x,r)\}$. Then the multifractal analysis of M_μ using the more intrinsic logarithmic density $\frac{\log(\mu(B(x,r))}{\log(r)}$ to define the local dimensions of M_μ is given by the multifractal analysis of $\widetilde{\mathsf{M}}_\mu$. Second, knowing the multifractal nature of $\widetilde{\mathsf{M}}_\mu$ allows one to find the multifractal properties of the sparse wavelets series weighted by using the random sample $\widetilde{\mu}$ of μ (see [25] for an account of multifractal analysis of functions).

From now on, only $\widetilde{\mathsf{M}}_\mu$ *is considered. For simplicity, we merge the notations* $\widetilde{\mathsf{M}}_\mu$ *and* M_μ*, so that*

$$\mathsf{M}_\mu(I_w) = \max\left\{\mu(I_{uv}) : u \in \mathcal{N}_j(w), v \in \Sigma^*, p_{uv} = 1\right\}. \tag{12}$$

The reader will check that our proofs to study the capacity defined by (12) are easily adapted to the case where the capacity is defined by (5). In fact, the case we study is a little bit more complicated, since it involves a control of all the immediate neighbors.

4.2. New parameters

Definition 12. The real number $\eta_\ell \in [0,\eta]$ is defined as

$$\eta_\ell = \begin{cases} 0 & \text{if } 0 \leqslant D_\mu(H_{\min}) \leqslant d(1-\eta) \\ 1 - \dfrac{d(1-\eta)}{D_\mu(H_{\min})} & \text{otherwise.} \end{cases}$$

For all $\eta' \in [\eta_\ell, \eta]$, there exists a unique $H_\ell(\eta') \in [H_{\min}, H_s]$ such that

$$D_\mu\big(H_\ell(\eta')\big) = \frac{d(1-\eta)}{1-\eta'}. \tag{13}$$

See Figures 8 and 9 for a geometrical interpretation of $H_\ell(\eta')$, which makes it easier to understand. By construction one has:

- $H_\ell(\eta) = H_s$;
- if $D_\mu(H_{\min}) \leqslant d(1-\eta)$, $\eta_\ell = 0$ and $H_\ell(\eta_\ell)$ is the unique solution of $D_\mu(H) = d(1-\eta)$ in $[H_{\min}, H_s]$;
- if $D_\mu(H_{\min}) > d(1-\eta)$, $\eta_\ell > 0$ and $H_\ell(\eta_\ell) = H_{\min}$.

Definition 13. For $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$, let

$$\widetilde{H}_\ell(\eta') = \left(\frac{1}{\eta'} - 1\right) H_\ell(\eta')$$

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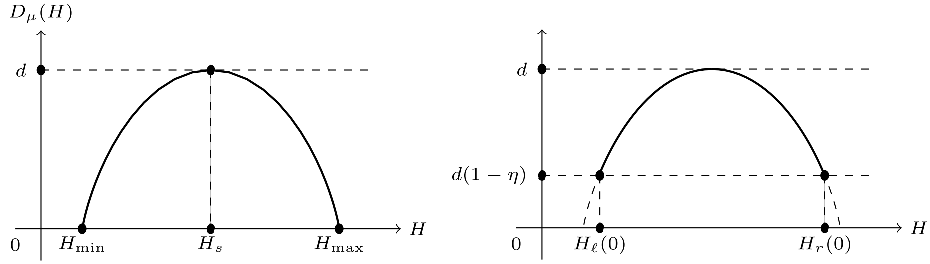

Figure 8. Left: Typical singularity spectrum of a Gibbs measure. *Right:* Parameters $H_\ell(0)$ and $H_r(0)$.

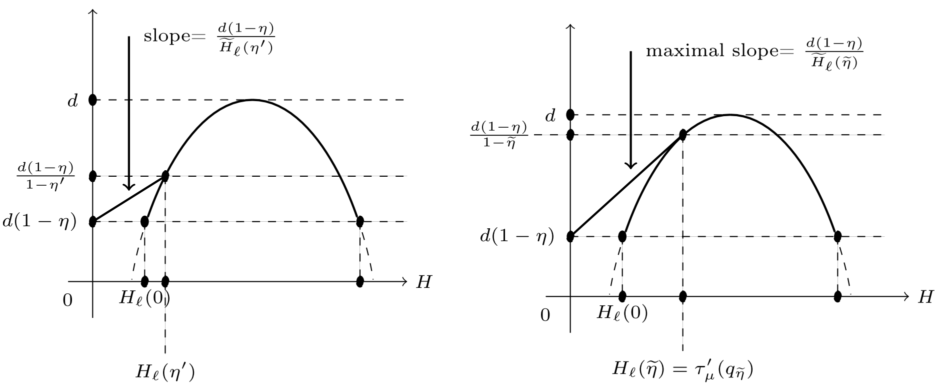

Figure 9. Left: Parameters $H_\ell(\eta')$ and $\widetilde{H}_\ell(\eta')$. *Right:* Optimal parameter $\widetilde{\eta}$.

$$\text{and} \quad \widetilde{\eta} = \operatorname{argmin}_{\eta' \in [\eta_\ell, \eta] \setminus \{0\}} \widetilde{H}_\ell(\eta'). \tag{14}$$

Again, see Figure 9 for a geometrical interpretation of these parameters (in the case we discard at the moment, i.e., when μ is homogeneous, the function τ_μ is linear so that $H_{\min} = H_s = H_{\max}$ and $\widetilde{\eta} = \eta_\ell = \eta$). It is easily seen that by definition the value $\widetilde{\eta}$ is so that the straight line passing through the points $(0, d(1-\eta))$ and $\left(H_\ell(\widetilde{\eta}), \frac{d(1-\eta)}{1-\widetilde{\eta}}\right)$ is tangent to the singularity spectrum of μ. This value always exists and is unique. Since D_μ is strictly concave, $D'_\mu(H_{\min}+) = \infty$ and $D'_\mu(H_s) = 0$, one has $\widetilde{\eta} \in (\eta_\ell, \eta)$.

Definition 14. Let $q_{\widetilde{\eta}}$ be the unique solution to the equation

$$H_\ell(\widetilde{\eta}) = \tau'_\mu(q_{\widetilde{\eta}}), \tag{15}$$

$$\text{and let} \quad q_{\eta_\ell} = \sup\left\{q \geqslant 0 : \tau^*_\mu(\tau'_\mu(q)) \geqslant d(1-\eta)\right\}. \tag{16}$$

Let $\widetilde{\tau}:\mathbb{R}\to\mathbb{R}$ be the mapping defined as

$$\widetilde{\tau}(q)=\begin{cases}\tau_\mu(q)+\widetilde{H}_\ell(\widetilde{\eta})q & \text{if } q\leqslant q_{\widetilde{\eta}},\\ \tau_\mu(q)+d(1-\eta) & \text{if } q_{\widetilde{\eta}}<q<q_{\eta_\ell},\\ H_\ell(0)q & \text{if } q_{\eta_\ell}<\infty \text{ and } q\geqslant q_{\eta_\ell}.\end{cases} \tag{17}$$

See Figure 6 for a representation of $\widetilde{\tau}$.

Observe that $q_{\eta_\ell}<+\infty$ if and only if $\eta_\ell=0$, and in this case $\tau'_\mu(q_{\eta_\ell})=H_\ell(0)$.

Definition 15. The real number $\eta_r\in[0,\eta]$ is defined as

$$\eta_r=\begin{cases}0 & \text{if } 0\leqslant D_\mu(H_{\max})\leqslant d(1-\eta)\\ 1-\dfrac{d(1-\eta)}{D_\mu(H_{\max})} & \text{otherwise.}\end{cases}$$

For all $\eta'\in[\eta_r,\eta]$, there exists a unique $H_r(\eta')\in[H_s,H_{\max}]$ such that

$$D_\mu\big(H_r(\eta')\big)=\frac{d(1-\eta)}{1-\eta'}.$$

By construction one has:

- $H_r(\eta)=H_s$;
- if $D_\mu(H_{\max})\leqslant d(1-\eta)$, $\eta_r=0$ and $H_r(\eta_r)$ is the unique solution of $D_\mu(H)=d(1-\eta)$ in $[H_s,H_{\max}]$;
- if $D_\mu(H_{\max})>d(1-\eta)$, $\eta_r>0$ and $H_r(\eta_r)=H_{\max}$.

The existence of H_ℓ and H_r is ensured by the continuity of the Legendre spectrum D_μ on its support.

As $\widetilde{H}_\ell(\eta')$ was defined in Definition 13, we can also define a parameter $\widetilde{H}_r(\eta')$ as follows: for every $\eta'\in[\eta_r,\eta]\setminus\{0\}$, let

$$\widetilde{H}_r(\eta')=\left(\frac{1}{\eta'}-1\right)H_r(\eta').$$

The geometrical interpretation is the same as the one for $\widetilde{H}_\ell(\eta')$ (see Figure 9), except now that everything is done on the decreasing part of the spectrum.

The following lemma provides us with another interpretation of the exponent $H_\ell(\widetilde{\eta})$ (see (14)), which is useful to simplify some formulas and to understand its role.

Lemma 1. *One has*

$$H_\ell(\widetilde{\eta})=\operatorname{argmax}_H\left(\frac{D_\mu(H)}{H+\widetilde{H}_\ell(\widetilde{\eta})}\right). \tag{18}$$

Proof. Due to the unimodal character of D_μ, the maximum we seek for is reached at $H\in[H_{\min},H_s]$. Also, a rapid calculation shows that since D_μ is strictly concave and differentiable over $(H_{\min},H_s]$ with $D'_\mu(H_{\min}+)=+\infty$ and $D'_\mu(H_s)=0$, then for any

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$\gamma > 0$, $H \mapsto \frac{D_\mu(H)}{H+\gamma}$ reaches its maximum at a unique point of $(H_{\min}, H_s)$. Notice that from its definition the function $\eta' \mapsto H_\ell(\eta')$ is differentiable.

Let us introduce the function $\varphi(\eta') = \eta' \widetilde{H}_\ell(\eta') = (1-\eta')H_\ell(\eta')$. Recall that by (13), one has $D_\mu\big(H_\ell(\eta')\big) = \dfrac{d(1-\eta)}{1-\eta'}$. So $D'_\mu\big(H_\ell(\eta')\big)H_\ell{}'(\eta') = \dfrac{d(1-\eta)}{(1-\eta')^2} = \dfrac{D_\mu\big(H_\ell(\eta')\big)}{1-\eta'}$. One deduces that

$$\varphi'(\eta') = -H_\ell(\eta') + (1-\eta')H_\ell{}'(\eta') = -H_\ell(\eta') + \frac{D_\mu\big(H_\ell(\eta')\big)}{D'_\mu\big(H_\ell(\eta')\big)} = -\frac{D^*_\mu\big(D'_\mu\big(H_\ell(\eta')\big)\big)}{D'_\mu\big(H_\ell(\eta')\big)},$$

since $D^*_\mu(H) = HD'_\mu(H) - D_\mu(H)$.

On the other hand, the derivative of $H \mapsto \dfrac{D_\mu(H)}{H + \widetilde{H}_\ell(\widetilde{\eta})}$ vanishes at $\alpha = \operatorname{argmax}_H \left(\frac{D_\mu(H)}{H+\widetilde{H}_\ell(\widetilde{\eta})}\right)$. This yields

$$D'_\mu(\alpha)(\alpha + \widetilde{H}_\ell(\widetilde{\eta})) - D_\mu(\alpha) = 0,$$

i.e.,

$$\widetilde{H}_\ell(\widetilde{\eta}) = -\frac{D^*_\mu(D'_\mu(\alpha))}{D'_\mu(\alpha)}.$$

Since $\widetilde{\eta}$ is chosen so that $\widetilde{H}_\ell(\eta')$ is minimal at $\widetilde{\eta}$, we have $\widetilde{H}'_\ell(\widetilde{\eta}) = 0$. This implies that $\varphi'(\widetilde{\eta}) = \widetilde{H}_\ell(\widetilde{\eta})$, so finally

$$-\frac{D^*_\mu\big(D'_\mu(\alpha)\big)}{D'_\mu(\alpha)} = -\frac{D^*_\mu\big(D'_\mu\big(H_\ell(\widetilde{\eta})\big)\big)}{D'_\mu\big(H_\ell(\widetilde{\eta})\big)}. \tag{19}$$

Recalling that D_μ is the Legendre transform of τ_μ, we know that $q \in \mathbb{R}^*_+ \mapsto \tau'_\mu(q)$ is a bijection onto $(H_{\min}, H_s)$. Hence, since the mapping $q > 0 \mapsto -\dfrac{\tau_\mu(q)}{q}$ is injective (τ_μ strictly concave), the identification $\Big(H, q, D^*_\mu\big(D'_\mu(H)\big)\Big) = \Big(\tau'_\mu(q), D'_\mu(H), \tau_\mu(q)\Big)$ implies that $H \in (H_{\min}, H_s) \mapsto -\dfrac{D^*_\mu\big(D'_\mu(H)\big)}{D'_\mu(H)}$ is injective as well. Equation (19) yields finally $\alpha = H_\ell(\widetilde{\eta})$. □

A last property of the previous parameters is also needed.

Lemma 2. *One has $q_{\widetilde{\eta}} = \widetilde{\eta}\tau^*_\mu(H_\ell(\widetilde{\eta}))/H_\ell(\widetilde{\eta})$. Also, $\widetilde{\tau}(q_{\widetilde{\eta}}) = 0$ and $\widetilde{\tau}$ is continuous at $q_{\widetilde{\eta}}$.*

Proof. By definition of $H_\ell(\widetilde{\eta})$ (see Figure 9), $q_{\widetilde{\eta}}$ is the slope of the tangent line to the graph of τ^*_μ at $H_\ell(\widetilde{\eta})$, and this tangent line passes through the point $(0, d(1-\eta))$. Hence $\tau^*_\mu(H_\ell(\widetilde{\eta})) - d(1-\eta) = q_{\widetilde{\eta}}H_\ell(\widetilde{\eta})$. Recalling that $\tau^*_\mu(H_\ell(\widetilde{\eta})) = d\frac{1-\eta}{1-\widetilde{\eta}}$, one deduces that

$$\widetilde{\eta}\tau^*_\mu(H_\ell(\widetilde{\eta})) = q_{\widetilde{\eta}}H_\ell(\widetilde{\eta}) = q_{\widetilde{\eta}}\tau'_\mu(q_{\widetilde{\eta}}),$$

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from which $\widetilde{\tau}(q_{\widetilde{\eta}}) = 0$ follows. But by definition of the Legendre transform, one has $\tau_\mu^*(H_\ell(\widetilde{\eta})) = \tau_\mu^*(\tau_\mu'(q_{\widetilde{\eta}})) = q_{\widetilde{\eta}}\tau_\mu'(q_{\widetilde{\eta}}) - \tau_\mu(q_{\widetilde{\eta}})$. We deduce that

$$\tau_\mu(q_{\widetilde{\eta}}) = \tau_\mu^*(H_\ell(\widetilde{\eta}))(\widetilde{\eta} - 1) = -d(1-\eta),$$

hence $\widetilde{\tau}(q_{\widetilde{\eta}}{}^+) = 0$. □

The previous definitions and discussion clarify the origin of the parameters introduced to state Theorem 2. The rest of the paper is devoted to the proof of the multifractal properties of M_μ defined by (11).

5. Analysis of the surviving vertices

5.1. Basic properties of the distribution of the surviving vertices

Recall Definition 3 in which $\mathcal{S}_j(\eta)$ is defined, and also that x_w, defined by (1), is the dyadic point corresponding to the projection of the finite word $w \in \Sigma_j$ to $[0,1]^d$. The first question concerns the distribution of the points x_w, for $w \in \mathcal{S}_j(\eta)$.

Definition 16. For every $j \geqslant 1$, and every finite word $W \in \Sigma^*$, one sets

$$\mathcal{S}_j(\eta, W) = \{w \in \mathcal{S}_j(\eta) : I_w \subset I_W\}.$$

The set $\mathcal{S}_j(\eta, W)$ describes the *surviving* coefficients at generation j included in I_W. Obviously, for every $J \leqslant j$,

$$\mathcal{S}_j(\eta) = \bigcup_{W \in \Sigma_J} \mathcal{S}_j(\eta, W).$$

Lemma 3. *There exists a positive sequence $(\varepsilon_j)_{j\geqslant 1}$ converging to 0 such that, with probability 1, for every j large enough, for every $W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}$, one has $\mathcal{S}_j(\eta, W) \neq \emptyset$.*

In other words, every cylinder of generation $\lfloor j(\eta - \varepsilon_j)\rfloor$ contains a surviving vertex w of generation j.

Proof. Fix a positive sequence $(\varepsilon_j)_{j\geqslant 1}$ converging to 0. For each $j \geqslant 1$ and $W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}$, the cylinder $[W]$ contains exactly $2^{j-\lfloor j(\eta-\varepsilon_j)\rfloor}$ distinct cylinders $[w]$, with $w \in \Sigma_j$. Denote this set by $S(W)$. The probability of the event $\mathscr{E}(W) = \{\forall\, w \in S(W), p_w = 0\}$ is given by $(1-2^{-j(1-\eta)})^{2^{j-\lfloor j(\eta-\varepsilon_j)\rfloor}}$. Thus,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{W\in\Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}} \mathscr{E}(W)\right) &\leqslant 2^{\lfloor j(\eta-\varepsilon_j)\rfloor}(1-2^{-j(1-\eta)})^{2^{j-\lfloor j(\eta-\varepsilon_j)\rfloor}}\\ &\leqslant 2^{\lfloor j(\eta-\varepsilon_j)\rfloor}\exp(-2^{j\varepsilon_j}).\end{aligned}$$

If we choose $\varepsilon_j = (\log^2(j))/j$, we get $\sum_{j\geqslant 1}\mathbb{P}\Big(\bigcup_{W\in\Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}}\mathscr{E}(W)\Big) < \infty$. So the Borel–Cantelli lemma yields that, with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor j(\eta-\varepsilon_j)\rfloor}$, there exists $w \in \Sigma_j$ such that $I_w \subset I_W$ and $p_w = 1$, i.e., $w \in \mathcal{S}_j(\eta, W)$. □

The sequence $(\varepsilon_j)_{j\geqslant 1}$ is now fixed.

Lemma 3 has the following consequence: Almost surely, the set of points belonging to an infinite number of balls of the form $B(x_w, 2^{-\lfloor |w|(\eta-\varepsilon_{|w|})\rfloor})$ with $p_w = 1$ is exactly the whole cube $[0, 1]^d$, i.e.,

$$[0,1]^d = \limsup_{j\to+\infty} \bigcup_{w\in\mathcal{S}_j(\eta)} B(x_w, 2^{-\lfloor |w|(\eta-\varepsilon_{|w|})\rfloor}). \tag{20}$$

Next we obtain an upper bound for the cardinality of $\mathcal{S}_j(\eta, W)$ when $W \in \Sigma_{\lfloor \eta j\rfloor}$.

Lemma 4. *With probability one, for every large j and every $W \in \Sigma_{\lfloor \eta j\rfloor}$, $\#\mathcal{S}_j(\eta, W) \leqslant j$.*

Proof. This is standard computations. Denote for every $j \geqslant 1$ and every word $W \in \Sigma_{\lfloor \eta j\rfloor}$, the random variable

$$B_W = \sum_{w\in\Sigma_j : I_w\subset I_W} p_w$$

is equal to the cardinality of $\mathcal{S}_j(\eta, W)$.

With this formulation, the $(B_W)_{W\in\Sigma_{\lfloor \eta j\rfloor}}$ are i.i.d. random variables with common law the binomial law $B(n_j, \rho_j)$ of parameters $n_j = 2^{d(j-\lfloor \eta j\rfloor)}$ and $\rho_j = 2^{-dj(1-\eta)}$. We have

$$\begin{aligned}\mathbb{P}\big(B(n_j,\rho_j) \geqslant j\big) &= \sum_{l=j}^{n_j} \binom{n_j}{l} (\rho_j)^l (1-\rho_j)^{n_j-l}\\ &= \sum_{l=j}^{n_j} \binom{n_j}{l} 2^{-djl(1-\eta)} (1-2^{-dj(1-\eta)})^{2^{d(j-\lfloor \eta j\rfloor)}-l}.\end{aligned}$$

Observe that

$$\binom{n_j}{l} 2^{-djl(1-\eta)} = (l!)^{-1}(2^{d(\eta j-[\eta j])})\cdots(2^{d(\eta j-[\eta j])}-(l-1)2^{-dj(1-\eta)}) \leqslant \frac{2^{dl}}{l!}.$$

Finally,

$$\mathbb{P}\big(B(n_j,\rho_j) \geqslant j\big) \leqslant \sum_{l=j}^{n_j} \frac{2^{dl}}{l!}(1-2^{-dj(1-\eta)})^{2^{d(j-\lfloor \eta j\rfloor)}-l} \leqslant \sum_{l=j}^{+\infty} \frac{2^{dl}}{l!} \leqslant 2^{-dj}$$

for j large enough. We deduce that $\sum_{j\geqslant 1} 2^{d\lfloor \eta j\rfloor}\mathbb{P}\big(B(n_j,\rho_j) \geqslant j\big) < +\infty$. Then the Borel–Cantelli lemma yields that almost surely there exists $J \geqslant 1$ such that for all $j \geqslant J$, for all $W \in \Sigma_{\lfloor \eta j\rfloor}$, one has $B_W < j$. □

As a conclusion, one keeps in mind the intuition that every cylinder $W \in \Sigma_{\lfloor \eta j\rfloor}$ contains at least one, but not much more than one surviving vertex $w \in \mathcal{S}_j(\eta)$.

5.2. Definition of sets of words with specific scaling properties

The above lemmas give some hints about the possible values for $\mu(I_w)$ for $w \in \mathcal{S}_j(\eta)$. Indeed, observe that any word w can be written as the concatenation $w = w_{|\lfloor \eta j\rfloor} \cdot \sigma^{\lfloor \eta j\rfloor} w$ (σ is the shift operation on Σ). Further, by the almost multiplicativity property (8) of μ, one has

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta j\rfloor}})\mu(I_{\sigma^{\lfloor \eta j\rfloor} w}).$$

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Lemmas 3 and 4 assert that all the possible values for $\mu(I_{w_{|\lfloor \eta j\rfloor}})$ are reached, since (heuristically) every cylinder of generation ηj contains a survivor at generation j. Hence, in order to describe the values of $\mu(I_w)$, it is necessary to investigate the possible values for $\mu(I_{\sigma^{\lfloor \eta j\rfloor} w})$ when $w \in \mathcal{S}_j(\eta)$. A quick analysis could lead to the intuition that since most of the coefficients are put to zero, only the most frequent local dimension H_s survive, i.e., $\mu(I_{\sigma^{\lfloor \eta j\rfloor} w}) \approx 2^{-\lfloor j\eta\rfloor H_s}$.

The goal of this section is to prove that this intuition is neither true, nor absolutely false. In fact, we explain that in order to investigate the values of $\mu(I_w)$ for $w \in \mathcal{S}_j(\eta)$, one needs to look at all the decompositions

$$w = w_{|\lfloor \eta' j\rfloor} \cdot \sigma^{\lfloor \eta' j\rfloor} w, \tag{21}$$

and to use that

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j\rfloor}})\mu(I_{\sigma^{\lfloor \eta' j\rfloor} w}),$$

for all possible values of $\eta' \in [\eta_\ell, \eta] \cup [\eta_r, \eta]$, and that the most frequent behaviors for $\mu(I_{\sigma^{\lfloor \eta' j\rfloor} w})$ are related to the local dimensions $H_\ell(\eta')$ and $H_r(\eta')$, H_s corresponding to $H_\ell(\eta) = H_r(\eta)$.

These considerations lead to the following definition.

Definition 17. Let $\alpha, \varepsilon \geqslant 0$ be two real numbers, and let $\eta' \in [0, \eta]$.

When $w \in \Sigma_j$, the prefix $w_{|\lfloor \eta' j\rfloor}$ is referred to as the η'-*root* of w, and the suffix $\sigma^{\lfloor \eta' j\rfloor} w$ is the η'-*tail* of w.

We introduce the following sets:

- $\mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon)$ is the set of those finite words $w \in \Sigma_j$ whose η'-root $w_{|\lfloor \eta' j\rfloor}$ belongs to $\mathcal{E}_\mu(\lfloor \eta' j\rfloor, \alpha \pm \varepsilon)$, i.e.,
$$\frac{\log_2 \mu(I_{w_{|\lfloor \eta' j\rfloor}})}{-\lfloor \eta' j\rfloor} \in [\alpha - \varepsilon, \alpha + \varepsilon].$$
- When $W \in \Sigma^*$, $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon, W)$ is the set of those finite words $w \in \Sigma_j$ satisfying $I_w \subset I_W$ and whose η'-tail $\sigma^{\lfloor \eta' j\rfloor} w$ belongs to $\mathcal{E}_\mu(j - \lfloor \eta' j\rfloor, H_\ell(\eta') \pm \varepsilon)$, i.e.,
$$\frac{\log_2 \mu(I_{\sigma^{\lfloor \eta' j\rfloor} w})}{j - \lfloor \eta' j\rfloor} \in [H_\ell(\eta') - \varepsilon, H_\ell(\eta') + \varepsilon]. \tag{22}$$
- the set $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon)$ is the set of all finite words $w \in \Sigma_j$ satisfying (22), so for every $J \leqslant j$,
$$\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon) = \bigcup_{W \in \Sigma_J} \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon, W).$$
- the sets $\mathcal{T}_{\mu,r}(j, \eta', \varepsilon, W)$ and $\mathcal{T}_{\mu,r}(j, \eta', \varepsilon)$ are defined as $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon, W)$ and $\mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon)$ by replacing $H_\ell(\eta')$ by $H_r(\eta')$.
- $\mathcal{T}_\mu(j, \eta', \varepsilon) = \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon) \cup \mathcal{T}_{\mu,r}(j, \eta', \varepsilon)$.

Recall the decomposition (21) of any word w. The idea, illustrated by Figure 10, is that the sets $\mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon)$ describe the scaling behavior of the η'-root $w_{|\lfloor \eta' j\rfloor}$ of the word $w \in \Sigma_j$, while $\mathcal{T}_\mu(j, \eta', \varepsilon)$ describe the scaling behavior of the η'-tail $\sigma^{\lfloor \eta' j\rfloor} w$ of w.

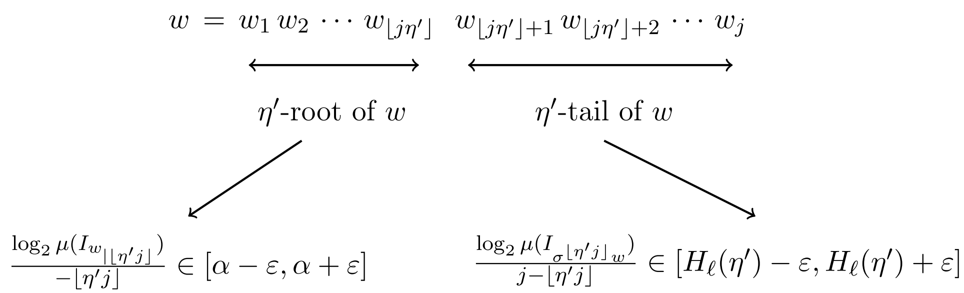

Figure 10. Decomposition of a word $w \in \mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon)$ into its η'-tail and its η'-root.

Observe that we focus on the cases where the η'-tail of w behaves with a local dimension close to some $H_\ell(\eta')$ or $H_r(\eta')$. Indeed, these specific behaviors of the η'-tail will turn out to be key to explain the structure of the local dimensions of M_μ (see Propositions 3, and 4 to 6).

Observe that the knowledge of which sets $\mathcal{R}_\mu(j, \eta', \alpha \pm \varepsilon)$ and $\mathcal{T}_\mu(j, \eta', \varepsilon)$ a given word w belongs to, yields $\mu(I_w)$ up to a multiplicative factor of order $2^{\pm \varepsilon j}$.

5.3. Analysis of the values of μ at the surviving vertices

The first proposition gives upper and lower bounds for the possible values of $\mu(I_w)$, when w survives after sampling.

Proposition 3. *Almost surely, there exists a positive sequence $(\varepsilon_j^1)_{j\geqslant 1}$ converging to 0 such that for j large enough, for all $w \in \mathcal{S}_j(\eta)$, one has*

$$j(H_\ell(\eta_\ell) - \varepsilon_j^1) \leqslant -\log_2 \mu(I_w) \leqslant j(H_r(\eta_r) + \varepsilon_j^1).$$

Proof. This is a consequence of the large deviations properties of Gibbs capacities. Fix an integer $p \geqslant 1$. Consider the interval $I_p = [0, H_\ell(\eta_\ell) - 2^{-p}] \cup [H_r(\eta_r) + 2^{-p}, +\infty)$. By definition of H_ℓ and H_r, one has $\sup\{D_\mu(h) : h \in I_p\} < d(1-\eta)$. Let us call $\xi_p = d(1-\eta) - \sup\{D_\mu(h) : h \in I_p\}$.

By item (5) of Proposition 2, there exists a generation J_p such that $j \geqslant J_p$ implies

$$\left| \frac{\log \#\mathcal{E}_\mu(j, I_p)}{-j} \log 2^j - \sup_{h \in I_p} D_\mu(h) \right| \leqslant \xi_p/2.$$

Using the definition of ξ_p, this rephrases as

$$\#\mathcal{E}_\mu(j, I_p) \leqslant 2^{j(\sup_{h\in I_p} D_\mu(h) + \xi_p/2)} \leqslant 2^{j(d(1-\eta) - \xi_p/2)}.$$

Let us compute the probability of the event $\mathscr{A}_j^p = \left\{\mathcal{S}_j(\eta) \cap \mathcal{E}_\mu(j, I_p) \neq \emptyset\right\}$. One has

$$\begin{aligned} \forall j \geqslant J_p, \mathbb{P}(\mathscr{A}_j^p) &\leqslant 1 - (1 - 2^{-dj(1-\eta)})^{\#\mathcal{E}_\mu(j, I_p)} \\ &\leqslant 1 - (1 - 2^{-dj(1-\eta)})^{2^{j(d(1-\eta) - \xi_p/2)}} \\ &\leqslant 2^{-j\xi_p/4}. \end{aligned}$$

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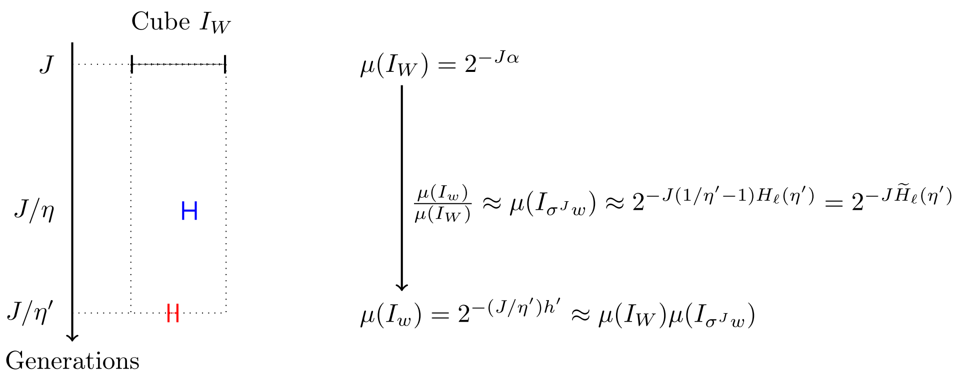

Figure 11. Behavior of the surviving vertices inside a cube I_W.

The Borel–Cantelli lemma implies that, almost surely, $\mathscr{A}_j^p$ is not realized when j becomes greater than some integer $J'_p \geqslant J_p$. In the construction, one can ensure that J_{p+1} is always strictly greater than J_p, for all integers p.

Choosing now $\varepsilon_j^1 = 2^{-p}$ for $j \in [J_p, J_{p+1})$ yields the result. □

Next proposition is crucial, emphasizing the role of the parameters η', $H_\ell(\eta')$ and $H_r(\eta')$ in our problem. The underlying idea is the following: The almost multiplicativity property implies that for every word $w \in \Sigma$, for every $\eta' \in [\eta_\ell, \eta] \cup [\eta_r, \eta]$, one has

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j\rfloor}})\mu(I_{w_{\sigma^{\lfloor \eta' j\rfloor} w}}).$$

But if a vertex w survives after sampling, i.e., if $w \in \mathcal{S}_j(\eta)$, then we are going to prove that $\mu(I_w)$ can be decomposed as

$$\mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j\rfloor}})2^{-(j-\lfloor \eta' j\rfloor)H_\ell(\eta')} \quad \text{or} \quad \mu(I_w) \approx \mu(I_{w_{|\lfloor \eta' j\rfloor}})2^{-(j-\lfloor \eta' j\rfloor)H_r(\eta')},$$

for some suitable choice of η' (depending on w). So there is a quite explicit formula for its η'-tail. We will then establish a complementary information (Proposition 5): η' being fixed, with probability one for j large enough, for $W \in \Sigma_{\lfloor \eta' j\rfloor}$, there is necessarily at least one word $w \in \mathcal{S}_j(\eta, W)$ such that the above decomposition holds.

Proposition 4. *With probability one, there exists a positive sequence $(\varepsilon_j^2)_{j\geqslant 1}$ converging to 0 such that for all $w \in \mathcal{S}_j(\eta)$, there exists $\eta' \in [\eta_\ell, \eta] \cup [\eta_r, \eta]$ such that $w \in \mathcal{T}_\mu(j, \eta', \varepsilon_j^2)$.*

Proof. We fix $w \in \mathcal{S}_j(\eta)$, and we look for a suitable η'. See Figure 11.

Let us denote, for all $j \geqslant 1$, $\alpha_j := -\dfrac{\log_2 \mu(I_w)}{j}$, and for all $\eta' \in [0, \eta]$, $\alpha_j(\eta') = -\dfrac{\log_2 \mu(I_{w_{|\lfloor \eta' j\rfloor}})}{\lfloor \eta' j\rfloor}$ and $H_j(\eta') = \dfrac{-\log_2 \mu(I_{\sigma^{\lfloor \eta' j\rfloor} w})}{j - \lfloor \eta' j\rfloor}$. By the almost multiplicativity property of μ, we have

$$\alpha_j j = \alpha_j(\eta')\lfloor \eta' j\rfloor + H_j(\eta')(j - \lfloor \eta' j\rfloor) + O(1), \tag{23}$$

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where $O(1)$ is bounded independently on w, j and η' (it depends only on the constant C involved in (8)).

On the other hand, for $\eta', \eta'' \in [0, \eta]$ we have

$$H_j(\eta'')(j - \lfloor \eta'' j \rfloor) - H_j(\eta')(j - \lfloor \eta' j \rfloor) = -\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w}) + \log_2 \mu(I_{\sigma^{\lfloor \eta'' j \rfloor} w}),$$

which is bounded above by $c|\lfloor \eta' j \rfloor - \lfloor \eta'' j \rfloor|$ for some constant $c > 0$ by (8). Also, by item (6) of Proposition 2, $H_j(\eta')$ and $H_j(\eta'')$ are bounded by a constant $K > 0$ independently of j, w and η'. Subsequently,

$$\begin{aligned} |H_j(\eta'') - H_j(\eta')| &\leqslant \left| H_j(\eta'') - H_j(\eta') \frac{j - \lfloor \eta' j \rfloor}{j - \lfloor \eta'' j \rfloor} \right| + H_j(\eta') \left| 1 - \frac{j - \lfloor \eta' j \rfloor}{j - \lfloor \eta'' j \rfloor} \right| \\ &\leqslant (c + K) \frac{|\lfloor \eta' j \rfloor - \lfloor \eta'' j \rfloor|}{j - \lfloor \eta'' j \rfloor} \leqslant (c + K) \frac{|\eta'' - \eta'| + 1/j}{1 - \eta}. \end{aligned}$$

From this inequality, one deduces that there exists a continuous function $\widetilde{H}_j : [0, \eta] \to \mathbb{R}^+$ such that

$$s_j = \sup\{|H_j(\eta') - \widetilde{H}_j(\eta')| : \eta' \in [0, \eta]\} = O(1/j)$$

independently of w as $j \to \infty$, and (23) holds with $\widetilde{H}_j$ instead of H_j.

• Suppose that $\widetilde{H}_j(\eta) = H_s$. Since $H_\ell(\eta) = H_s$, Proposition 4 is proved with $\eta' = \eta$.

• Suppose now that $\widetilde{H}_j(\eta) < H_s = H_\ell(\eta)$.

- Suppose first that $\eta_\ell = 0$. Recall that $H_\ell(0) = H_{\min}$.

If $\widetilde{H}_j(0) \leqslant H_\ell(0)$, then we see that $j\alpha_j = j\widetilde{H}_j(0) + O(1) \leqslant j(H_\ell(0) + O(1/j))$, which due to Proposition 3 implies that $H_\ell(0) - \varepsilon_j^1 \leqslant \widetilde{H}_j(0) + O(1/j) \leqslant H_\ell(0) + O(1/j)$. Hence (24) below is satisfied with $\eta' = 0$.

If $\widetilde{H}_j(0) > H_\ell(0)$, observe that the mapping $\eta' \mapsto (\widetilde{H}_j - H_\ell)(\eta')$ is continuous, positive at $\eta' = 0$, negative at $\eta' = \eta$. The continuity ensures the existence of $\eta' \in (0, \eta)$ such that $\widetilde{H}_j(\eta') = H_\ell(\eta')$, and (24) is satisfied with this η'.

- Suppose now that $\eta_\ell > 0$ and recall that $H_\ell(\eta')$ ranges in $[H_\ell(\eta_\ell), H_s]$. Notice that for any η', $j - \lfloor \eta' j \rfloor \geqslant j - \lfloor \eta j \rfloor$ which tends to $+\infty$ when $j \to +\infty$. Hence, by Proposition 3, for j large enough we have $H_j(\eta') \geqslant H_\ell(\eta_\ell) - \varepsilon^1_{j - \lfloor \eta' j \rfloor}$, so that for all $\eta' \in [\eta_\ell, \eta]$,

$$\widetilde{H}_j(\eta') \geqslant H_\ell(\eta_\ell) - \varepsilon^1_{j - \lfloor \eta' j \rfloor} - s_j.$$

By continuity of $\eta' \mapsto (\widetilde{H}_j - H_\ell)(\eta')$, there exists $\eta' \in [\eta_\ell, \eta]$ such that $|\widetilde{H}_j(\eta') - H_\ell(\eta')| \leqslant \varepsilon^1_{j - \lfloor \eta' j \rfloor} + s_j$. In all cases, we found $\eta' \in [\eta_\ell, \eta]$ such that $|\widetilde{H}_j(\eta') - H_\ell(\eta')| \leqslant \varepsilon^1_{j - \lfloor \eta' j \rfloor} + 2s_j + O(1/j)$. Since H_j and $\widetilde{H}_j$ differ by $o(1)$, the result follows.

• Finally suppose that $\widetilde{H}_j(\eta) > H_s$. Similar arguments as above yield $\eta' \in [\eta_r, \eta]$ such that $|H_j(\eta') - H_r(\eta')| \leqslant \varepsilon^1_{j - \lfloor \eta' j \rfloor} + 2s_j + O(1/j)$. We let the reader check the details.

Since the bound $\varepsilon^1_{j - \lfloor \eta' j \rfloor} + 2s_j + O(1/j)$ tends to 0 uniformly in $\eta' \in [0, \eta]$ as $j \to +\infty$, the sequence $(\varepsilon_j^2 := \varepsilon^1_{j - \eta j} + 2s_j + O(1/j))_{j \geqslant 1}$ fulfills the conditions of Proposition 4. □

The previous proposition tells us that every surviving vertex $w \in \mathcal{S}_j(\eta)$ is such that, either for some $\eta' \in [\eta_\ell, \eta]$ (depending on w), its η'-tail satisfies

$$(j - \lfloor \eta' j \rfloor)(H_\ell(\eta') - \varepsilon_j^2) \leqslant -\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w}) \leqslant (j - \lfloor \eta' j \rfloor)(H_\ell(\eta') + \varepsilon_j^2), \tag{24}$$

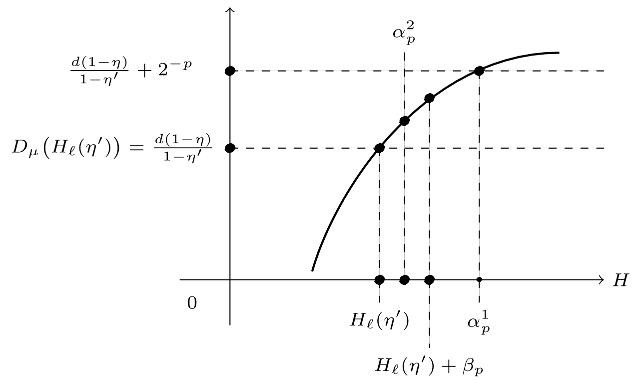

Figure 12. Relative positions of $H_\ell(\eta')$, $H_\ell(\eta')+\beta_p$, α_p^1, α_p^2.

or for some $\eta' \in [\eta_r, \eta]$ (also depending on w), its η'-tail satisfies

$$(j - \lfloor \eta' j \rfloor)(H_r(\eta') - \varepsilon_j^2) \leqslant -\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w}) \leqslant (j - \lfloor \eta' j \rfloor)(H_r(\eta') + \varepsilon_j^2).$$

Next proposition shall be understood as a renewal property for the local dimensions $H_\ell(\eta')$. It claims that η' being fixed in $[\eta_\ell, \eta]$, almost surely, for j large enough, for all $W \in \Sigma_{\lfloor \eta' j \rfloor}$, there is a surviving vertex $w \in \mathcal{S}_j(\eta, W)$ such that $\dfrac{-\log_2 \mu(I_{\sigma^{\lfloor \eta' j \rfloor} w})}{j - \lfloor \eta' j \rfloor} \approx H_\ell(\eta')$. See Figure 10 for an illustration of this decomposition.

Proposition 5. *Given $\eta' \in [\eta_\ell, \eta)$, there exists a positive sequence $(\varepsilon_j^3)_{j \geqslant 1}$ converging to 0 such that, with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor \eta' j \rfloor}$, $\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon_j^3) \neq \emptyset$.*

Of course, the same holds true for $\mathcal{T}_{\mu,r}(j, \eta', \varepsilon_j^3)$, but this second property is not needed.

Proof. Fix $\eta' \in [\eta_\ell, \eta)$, which implies that $D_\mu(H_\ell(\eta')) < d = \|D_\mu\|_\infty$. For every integer $p \geqslant 1$ so large that $D_\mu\big(H_\ell(\eta')\big) + 2^{-p} < d$, let $\alpha_p^1, \alpha_p^2, \beta_p$ be such that

- α_p^1 is the unique real number in $[H_\ell(\eta_\ell), H_s]$ such that $D_\mu(\alpha_p^1) = D_\mu(H_\ell(\eta')) + 2^{-p}$;
- $\beta_p = (\alpha_p^1 - H_\ell(\eta'))/2$;
- α_p^2 is such that $H_\ell(\eta') < \alpha_p^2 < H_\ell(\eta') + \beta_p = \alpha_p^1 - \beta_p$.

Observe that $D_\mu(H_\ell(\eta')) < D_\mu(\alpha_p^2) < D_\mu(\alpha_p^1 - \beta_p) < D_\mu(\alpha_p^1)$ (see Figure 12).

For every integer $p \geqslant 1$, due to the large deviations properties of μ (part (5) of Proposition 2 and equation (9)), we can fix an integer j_p such that for all $j \geqslant j_p$,

$$\#\mathcal{E}_\mu(j, \alpha_p^1 \pm \beta_p) \geqslant 2^{j D_\mu(\alpha_p^2)}.$$

Using the definition of our parameters, this implies that

$$\#\mathcal{E}_\mu(j, H_\ell(\eta') \pm \widetilde{\varepsilon}_p) \geqslant 2^{j\left(D_\mu(H_\ell(\eta')) + \widehat{\varepsilon}_p\right)},$$

where $\widetilde{\varepsilon}_p = 3\beta_p$ and $\widehat{\varepsilon}_p = D_\mu(\alpha_p^2) - D_\mu(H_\ell(\eta')) > 0$.

It is clear from the continuity and the monotonicity of the mapping D_μ that $(\widetilde{\varepsilon}_p)_{p\geqslant 1}$ and $(\widehat{\varepsilon}_p)_{p\geqslant 1}$ are two positive decreasing sequences, and that $\lim_{p\to+\infty}\widetilde{\varepsilon}_p = \lim_{p\to+\infty}\widehat{\varepsilon}_p = 0$.

For $j \geqslant j_p/(1-\eta')$ (hence so that $j - \lfloor \eta' j\rfloor \geqslant j_p$) and $W \in \Sigma_{\lfloor \eta' j\rfloor}$, consider the event

$$\mathscr{A}(\eta', \widetilde{\varepsilon}_p, W) = \Big\{\forall\, w' \in \mathcal{E}_\mu\big(j - \lfloor \eta' j\rfloor, H_\ell(\eta') \pm \widetilde{\varepsilon}_p\big), p_{Ww'} = 0\Big\}. \tag{25}$$

One has

$$\begin{aligned}\mathbb{P}\Big(\mathscr{A}(\eta', \widetilde{\varepsilon}_p, W)\Big) &= (1-2^{-d(1-\eta)j})^{\#\mathcal{E}_\mu\left(j-\lfloor \eta' j\rfloor, H_\ell(\eta')\pm\widetilde{\varepsilon}_p\right)}\\ &\leqslant \exp\left(-2^{-d(1-\eta)j}\#\mathcal{E}_\mu(j-\lfloor \eta' j\rfloor, H_\ell(\eta')\pm\widetilde{\varepsilon}_p)\right)\\ &\leqslant \exp\left(-2^{-d(1-\eta)j+(j-\lfloor \eta' j\rfloor)\left(D_\mu(H_\ell(\eta'))+\widehat{\varepsilon}_p\right)}\right).\end{aligned}$$

Recalling that $D_\mu\big(H_\ell(\eta')\big) = \frac{d(1-\eta)}{1-\eta'}$, we get

$$\mathbb{P}\Big(\mathscr{A}(\eta', \widetilde{\varepsilon}_p, W)\Big) \leqslant \exp(-2^{(j-\lfloor \eta' j\rfloor)\widehat{\varepsilon}_p} + O(1)) \leqslant C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_p}).$$

We choose the sequence $(\varepsilon_j^3)_{j\geqslant 1}$ as follows: we first build some sequences of integers by induction. Pick an integer p_0 so large that the previous inequality holds true for $j \geqslant j_{p_0}/(1-\eta')$. Also, choose $\widetilde{j}_{p_0} > j_{p_0}$ so large that for all $j \geqslant \widetilde{j}_{p_0}/(1-\eta')$, one has $C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0}}) \leqslant 2^{-dj}$. Set $\widetilde{j}_{p_0-1} = 0$.

Then, assume that for $m \geqslant 0$, integers and $\widetilde{j}_{p_0}, \ldots, \widetilde{j}_{p_0+m}$ are found such that for $n = 0, \ldots, m$;

- $\widetilde{j}_{p_0+n} > \max(j_{p_0+n}, \widetilde{j}_{p_0+n-1})$;
- for $j(1-\eta') \geqslant \widetilde{j}_{p_0+n}$ one has $C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0+n}}) \leqslant 2^{-dj}$.

Then we choose $\widetilde{j}_{p_0+m+1} > \max(j_{p_0+m+1}, \widetilde{j}_{p_0+m})$ so large that for all $j \geqslant \widetilde{j}_{p_0+m+1}/(1-\eta')$, one has $C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0+m+1}}) \leqslant 2^{-dj}$.

Finally, for every $j \geqslant \widetilde{j}_{p_0}/(1-\eta')$, there is a unique integer m_j such that

$$\widetilde{j}_{p_0+m_j}/(1-\eta') \leqslant j < \widetilde{j}_{p_0+m_j+1}/(1-\eta'), \tag{26}$$

and we set $\varepsilon_j^3 = \widetilde{\varepsilon}_{p_0+m_j}$. By construction we obtain

$$\mathbb{P}\Big(\mathscr{A}(\eta', \varepsilon_j^3, W)\Big) \leqslant C\exp(-2^{(1-\eta')j\widehat{\varepsilon}_{p_0+m_j}}) \leqslant 2^{-dj}. \tag{27}$$

Subsequently,

$$\begin{aligned}&\mathbb{P}\Big(\Big\{\exists\, j \geqslant \widetilde{j}_{p_0}/(1-\eta') \text{ and } \exists\, W \in \Sigma_{\lfloor \eta' j\rfloor} : \mathscr{A}(\eta', \varepsilon_j^3, W) \text{ holds}\Big\}\Big)\\ &\leqslant \sum_{j\geqslant \widetilde{j}_{p_0}/(1-\eta')}\ \sum_{W\in\Sigma_{\lfloor \eta' j\rfloor}} \mathbb{P}\Big(\mathscr{A}(\eta', \varepsilon_j^3, W)\Big)\\ &\leqslant \sum_{j\geqslant \widetilde{j}_{p_0}/(1-\eta')} 2^{d\lfloor \eta' j\rfloor}2^{-dj} < +\infty.\end{aligned}$$

We conclude thanks to the Borel–Cantelli lemma. □

Last proposition can be realized simultaneously on several $\eta' \in [\eta_\ell, \eta]$.

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Corollary 1. *For all integers $N \geqslant 1$ and $0 \leqslant k \leqslant N-1$, let $\eta_{N,k} = \eta_\ell + \frac{k}{N}(\eta - \eta_\ell)$. There exists a positive sequence $(\varepsilon_j^{4,N})_{j\geqslant 1}$ converging to 0 when j tends to infinity, such that with probability 1, for $N \geqslant 2$ and j large enough, for all $0 \leqslant k \leqslant N-1$ and all $W \in \Sigma_{\lfloor \eta_{N,k} j \rfloor}$, $\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta_{N,k}, \varepsilon_j^{4,N}) \neq \emptyset$.*

Proof. Fix $N \geqslant 2$. For each $k \in \{0, \ldots, N-1\}$, we apply Proposition 4, so that we get a sequence $(\varepsilon_j^3(k))_{j\geqslant 1}$ and a sequence $(\widetilde{j}_p(k))_{p \geqslant p_0(k)}$, such that (27) holds true, i.e., for every $j \geqslant \widetilde{j}_{p_0(k)}/(1-\eta')$ and $W \in \Sigma_{\lfloor \eta_{N,k} j \rfloor}$,

$$\mathbb{P}\Big(\mathscr{A}(\eta_{N,k}, \varepsilon_j^3(k), W)\Big) \leqslant 2^{-dj}, \tag{28}$$

where the event $\mathscr{A}$ is defined in (25).

One has $\mathscr{A}(\eta', \varepsilon', W) \subset \mathscr{A}(\eta', \varepsilon, W)$ whenever $0 < \varepsilon < \varepsilon'$. Hence, one can choose the integer $p = \max(p_0(0), \ldots, p_0(N-1))$, the sequences $\varepsilon_j^{4,N} := \max(\varepsilon_j^3(0), \ldots, \varepsilon_j^3(N-1))$ and $\widetilde{j}_p := \max(\widetilde{j}_p(0), \ldots, \widetilde{j}_p(N-1))$, so that the following property holds: for all $0 \leqslant k \leqslant N-1$, for all $j \geqslant \widetilde{j}_{p_0}/(1-\eta)$, for all $W \in \Sigma_{\lfloor \eta_{N,k} j \rfloor}$, (28) holds true with $\varepsilon_j^{4,N}$ instead of $\varepsilon_j^3(k)$.

Thus,

$$\begin{aligned}
&\mathbb{P}\left(\left\{\exists \left\{\begin{array}{l} j \geqslant \widetilde{j}_{p_0}/(1-\eta) \\ k \in \{0, \ldots, N-1\}, \quad \mathscr{A}(\eta_{N,k}, \varepsilon_j^{4,N}, W) \text{ holds} \\ W \in \Sigma_{\lfloor \eta_{N,k} j \rfloor} \end{array}\right\}\right\}\right) \\
&\quad \leqslant \sum_{k=0}^{N-1} \sum_{j \geqslant \widetilde{j}_{p_0}/(1-\eta)} \sum_{W \in \Sigma_{\lfloor \eta_{N,k} j \rfloor}} \mathbb{P}\Big(\mathscr{A}(\eta_{N,k}, \varepsilon_j^3(k), W)\Big) \\
&\quad \leqslant \frac{1}{1 - 2^{-d(1-\eta)}} \sum_{k=0}^{N-1} 2^{-d(1-\eta_{N,k})\widetilde{j}_{p_0}/(1-\eta)} < +\infty.
\end{aligned}$$

The result follows again by the Borel–Cantelli lemma. □

Next proposition completes the previous corollary by showing (roughly speaking), that for a fixed $W \in \Sigma_J$ with J large enough, for η' in some interval $[\eta_0, \eta]$ fixed in advance, the probability to find $w \in \bigcup_{J/\eta \leqslant j \leqslant J/\eta_0} \mathcal{S}_j(\eta, W)$ with a η'-tail having a local dimension smaller than $H_\ell(\eta')$ decreases exponentially with J.

Proposition 6. *Let* $\eta_0 = \begin{cases} \frac{H_{\min}}{H_{\min} + \widetilde{H}_\ell(\widetilde{\eta})} & \text{if } \eta_\ell = 0 \\ \eta_\ell & \text{if } \eta_\ell > 0. \end{cases}$

For all integers $N \geqslant 1$ and $k \in \{-1, 0 \ldots, N-1\}$, set $\widetilde{\eta}_{N,k} = \eta - (\eta - \eta_0)\frac{k}{N}$.

For $J \geqslant 1$ and $W \in \Sigma_J$, consider the event $\mathscr{C}(N, J, W)$ defined as

$$\mathscr{C}(N, J, W) = \left\{\begin{array}{l} \exists\, k \in \{-1, 0, \ldots, N-1\}, \exists\, j \in [J/\widetilde{\eta}_{N,k}, J/\widetilde{\eta}_{N,k+1}], \\ \exists\, w \in \mathcal{S}_j(\eta, W) \text{ such that } \mu(I_{\sigma^J w}) > 2^{-J(\widetilde{H}_\ell(\widetilde{\eta}_{N,k}) - \varepsilon_N)} \end{array}\right\},$$

with the convention that $\widetilde{H}_\ell(\widetilde{\eta}_{N,-1}) = \widetilde{H}_\ell(\widetilde{\eta}_{N,0}) = \widetilde{H}_\ell(\eta)$.

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With probability one, there exists a positive sequence $(\varepsilon_N)_{N\geqslant 1}$ *converging to 0 such that for all* $N \geqslant 1$*, for* J *large enough and* $W \in \Sigma_J$*, one has* $\mathbb{P}\big(\mathscr{C}(N, J, W)\big) \leqslant 2^{-J\varepsilon_N}$*.*

The proof uses arguments similar to those developed earlier, and is left to the reader.

Proposition 5 asserts that for all $W \in \Sigma_{\lfloor \eta' j\rfloor}$, $\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon_j^3)$ is not empty when j becomes large. The last proposition of this section shows that its cardinality cannot be very large. This fact will be interpreted geometrically as a *weak redundancy* property from the viewpoint of ubiquity theory [5, 6] and has nice geometric consequences for our study.

Proposition 7. (1) *For all* $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$*, for all* $\varepsilon \in (0, 1)$*, there exists* $\beta > 0$ *such that with probability 1, for every* j *large enough and all* $W \in \Sigma_{\lfloor \eta' j\rfloor}$*,*

$$1 \leqslant \#\big(\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \beta)\big) \leqslant 2^{\eta' j \varepsilon}. \tag{29}$$

(2) *The same holds true for* $\eta' \in [\eta_r, \eta] \setminus \{0\}$ *and the sets* $\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,r}(j, \eta', \beta)$*.*

Proof. (1) Obviously, it is enough to get the conclusion for ε small. Fix $\varepsilon \in (0, 1)$ and $\eta' \in [\eta_\ell, \eta] \setminus \{0\}$. Due to the almost multiplicativity property of μ, and equation (9), there exist $\beta > 0$ and J_0 such that for $j \geqslant J_0$, for each $W \in \Sigma_{\lfloor \eta' j\rfloor}$,

$$\#\mathcal{T}_{\mu,\ell}(j, \eta', \beta, W) \leqslant 2^{\left(D_\mu(H_\ell(\eta')) + d\varepsilon^2\right)(j - \lfloor \eta' j\rfloor)}. \tag{30}$$

Notice that the cardinality $n_j = \#\mathcal{T}_{\mu,\ell}(j, \eta', \beta, W)$ is independent of W. Since $D_\mu(H_\ell(\eta')) = d(1-\eta)/(1-\eta') \leqslant d$, $\varepsilon < 1$ and $\eta' \leqslant \eta < 1$, for $j \geqslant J_0$, one has

$$n_j \leqslant 2^{\left(D_\mu(H_\ell(\eta')) + d\varepsilon^2\right)(j - \lfloor \eta' j\rfloor)} \leqslant 2^{d(1-\eta)j} 2^{d\varepsilon^2 j + d}.$$

By definition,

$$\#\big(\mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon_j^3)\big) = \sum_{w \in \mathcal{T}_{\mu,\ell}(j, \eta', \beta, W)} p_w.$$

Denote this random variable by $B(j, \eta', \beta, W)$, whose law is binomial with parameters $(n_j, 2^{-d(1-\eta)j})$. Thus

$$\begin{aligned}
\mathbb{P}\Big(B(j, \eta', \beta, W) \geqslant 2^{\varepsilon \eta' j}\Big) &\leqslant \sum_{2^{\varepsilon\eta' j} \leqslant l \leqslant n_j} \binom{n_j}{l} (2^{-d(1-\eta)j})^l \\
&\leqslant \sum_{2^{\varepsilon\eta' j} \leqslant l \leqslant n_j} \frac{(n_j 2^{-d(1-\eta)j})^l}{l!} \\
&\leqslant \sum_{2^{\varepsilon\eta' j} \leqslant l \leqslant n_j} \frac{2^{dj\varepsilon^2 l + dl}}{l!} \\
&\leqslant \sum_{l \geqslant 2^{\varepsilon\eta' j}} \left(\frac{e 2^{dj\varepsilon^2 + d}}{l}\right)^l
\end{aligned}$$

for j large enough by Stirling's formula. Then, if $\varepsilon \leqslant \eta'/(4d)$, there is another integer J_0' such that for $j \geqslant J_0'$, for all $l \geqslant 2^{\varepsilon\eta' j}$, we have $\dfrac{e2^{dj\varepsilon^2+d}}{l} \leqslant 2^{-\varepsilon\eta' j/2} \leqslant 1/2$, hence

$$\mathbb{P}\Big(B(j,\eta',\beta,W) \geqslant 2^{\varepsilon\eta' j}\Big) \leqslant 2\cdot 2^{-\lfloor 2^{\varepsilon\eta' j}\rfloor \varepsilon\eta' j/2},$$

and

$$\sum_{j\geqslant J_0'} \sum_{W\in\Sigma_{\lfloor \eta' j\rfloor}} \mathbb{P}\Big(B(j,\eta',\beta,W) \geqslant 2^{\varepsilon\eta' j}\Big) \leqslant \sum_{j\geqslant J_0'} 2^{d\lfloor \eta' j\rfloor} 2\cdot 2^{-\lfloor 2^{\varepsilon\eta' j}\rfloor \varepsilon\eta' j/2} < \infty.$$

The desired conclusion follows from the Borel–Cantelli lemma.
(2) The computations are identical for $\eta' \in [\eta_r,\eta]\setminus\{0\}$ and $\#\big(\mathcal{S}_j(\eta,W)\cap\mathcal{T}_{\mu,\ell}(j,\eta',\beta)\big)$. □

6. Free energy and large deviations for M_μ; upper bound for D_{M_μ}

Recall the definitions (15) and (16) for $q_{\widetilde{\eta}}$ and q_{η_ℓ}, and also formula (17) for $\widetilde{\tau}(q)$ that we reproduce for convenience:

$$\widetilde{\tau}(q) = \begin{cases} \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta})q & \text{if } q \leqslant q_{\widetilde{\eta}}, \\ \tau_\mu(q) + d(1-\eta) & \text{if } q_{\widetilde{\eta}} < q < q_{\eta_\ell}, \\ H_\ell(0)q & \text{if } q_{\eta_\ell} < \infty \text{ and } q \geqslant q_{\eta_\ell}. \end{cases}$$

In this section, we first compute $\widetilde{\tau}^*$ (§ 6.1). Then we show that with probability 1:

$$\tau_{\mathsf{M}_\mu} \geqslant \widetilde{\tau} \qquad (\S\,6.2), \tag{31}$$

$$\underline{f}_{\mathsf{M}_\mu} \geqslant \widetilde{\tau}^* \qquad (\S\,6.3). \tag{32}$$

Using that $D_{\mathsf{M}_\mu} \leqslant \tau^*_{\mathsf{M}_\mu}$ holds true, (31) and Lemma 5 below yield the desired upper bound $D_{\mathsf{M}_\mu} \leqslant \widetilde{\tau}^*$ for the multifractal spectrum of M_μ.

Also, since it is always true that $\underline{f}_{\mathsf{M}_\mu} \leqslant \overline{f}_{\mathsf{M}_\mu} \leqslant \tau_{\mathsf{M}_\mu}$, (31) and (32) yield $\widetilde{\tau}^* = \tau^*_{\mathsf{M}_\mu} = \underline{f}_{\mathsf{M}_\mu} = \overline{f}_{\mathsf{M}_\mu}$. Then, Varadhan's integral lemma (see [12, Theorem 4.3.1]) (or in our situation very simple estimates) imply that the free energy $\tau_{\mathsf{M}_\mu}(q)$ exists as a limit (not only as a liminf) for all $q \in \mathbb{R}$, and that it equals $\widetilde{\tau}$.

Finally, in § 6.4 we provide alternative direct arguments showing that the range of $\underline{\dim}(\mathsf{M}_\mu,\cdot)$ is contained in $[\widetilde{\tau}'(+\infty),\widetilde{\tau}'(-\infty)]$.

6.1. The Legendre transform of $\widetilde{\tau}^*$

The first lemma is computational.

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Lemma 5. *One has*

$$\widetilde{\tau}^*(H) = \begin{cases} \tau_\mu^*(H) - d(1-\eta) & \text{if } H_\ell(\eta_\ell) \leqslant H \leqslant H_\ell(\widetilde{\eta}), \\ q_{\widetilde{\eta}} H & \text{if } H_\ell(\widetilde{\eta}) \leqslant H \leqslant H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), \\ \tau_\mu^*\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big) & \text{if } H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) \leqslant H \leqslant H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}), \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. At first we notice that there is a first order phase transition at $q_{\widetilde{\eta}}$, since $H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}) = \widetilde{\tau}'(q_{\widetilde{\eta}}{}^-) > \widetilde{\tau}'(q_{\widetilde{\eta}}{}^+) = H_\ell(\widetilde{\eta})$.

• When $H \geqslant H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})$: Since $\widetilde{\tau}$ and τ_μ differ by a linear term of slope $\widetilde{H}_\ell(\widetilde{\eta})$ over $(-\infty, q_{\widetilde{\eta}}]$, their Legendre transform are translated versions of each other by $\widetilde{H}_\ell(\widetilde{\eta})$ over the interval $[\widetilde{\tau}'(q_{\widetilde{\eta}}), +\infty) = [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), +\infty)$. Hence, for $H \geqslant H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})$, one has $\widetilde{\tau}^*(H) = \tau_\mu^*(H - \widetilde{H}_\ell(\widetilde{\eta}))$.

• When $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$: The discontinuity of $(\widetilde{\tau})'$ at $q_{\widetilde{\eta}}$ implies that for H in the interval $[\widetilde{\tau}'(q_{\widetilde{\eta}}{}^+), \widetilde{\tau}'(q_{\widetilde{\eta}}{}^-)] = [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$, one has

$$\widetilde{\tau}^*(H) = \inf_{q\in\mathbb{R}} (qH - \widetilde{\tau}(q)) = q_{\widetilde{\eta}} H - \widetilde{\tau}(q_{\widetilde{\eta}}) = q_{\widetilde{\eta}} H.$$

• When $\eta_\ell = 0$ and $H \leqslant H_\ell(\widetilde{\eta})$: In this case we have $q_{\eta_\ell} < +\infty$. Since $\widetilde{\tau}$ and τ_μ differ by the constant $d(1-\eta)$ over $[q_{\widetilde{\eta}}, q_{\eta_\ell}]$, for $H \in \big[\widetilde{\tau}'(q_{\eta_\ell}), \widetilde{\tau}'(q_{\widetilde{\eta}})\big] = \big[H_\ell(0), H_\ell(\widetilde{\eta})\big]$ one has $\widetilde{\tau}^*(H) = \tau_\mu^*(H) - d(1-\eta)$. Then, when $q \geqslant q_{\eta_\ell}$, $\widetilde{\tau}$ is linear with slope $\widetilde{\tau}'(q_{\eta_\ell}^-)$, so $\widetilde{\tau}^*(H) = -\infty$ for all $H < H_\ell(0)$.

• When $\eta_\ell > 0$ and $H \leqslant H_\ell(\widetilde{\eta})$: Here $q_{\eta_\ell} = +\infty$ and $H_\ell(\eta_\ell) = H_{\min}$. The same argument as above yields $\widetilde{\tau}^*(H) = \tau_\mu^*(H) - d(1-\eta)$ for all $H \leqslant H_\ell(\widetilde{\eta})$. □

6.2. Lower bound for τ_{M_μ}: for every q, $\tau_{\mathsf{M}_\mu}(q) \geqslant \tilde{\tau}(q)$

6.2.1. When $q_{\widetilde{\eta}} < q < q_{\eta_\ell}$. The submultiplicativity property (8) of μ gives for $J \geqslant 1$

$$\begin{aligned} \sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q &= \sum_{W\in\Sigma_J} \left(\max_{W'\in\mathcal{N}_J(W)} \max_{w\in\mathcal{S}(\eta,W')} \mu(I_w) \right)^q \\ &\leqslant 3^d \sum_{W\in\Sigma_J} \max_{w\in\mathcal{S}(\eta,W)} \mu(I_w)^q \\ &\leqslant 3^d C^q \sum_{W\in\Sigma_J} \mu(I_W)^q \sum_{w\in\Sigma^*, p_{Ww}=1} \mu(I_w)^q \\ &= 3^d C^q \sum_{W\in\Sigma_J} \mu(I_W)^q \sum_{k\geqslant 0} \sum_{w\in\Sigma_k} \mu(I_w)^q p_{Ww}. \end{aligned}$$

The random variables p_{Ww} being independent, with law $B(2^{-d(J+k)(1-\eta)})$, this yields

$$\mathbb{E}\left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \right) \leqslant C_q \left(\sum_{W\in\Sigma_J} \mu(I_W)^q \right) \sum_{k\geqslant 0} 2^{-(J+k)d(1-\eta)} \sum_{w\in\Sigma_k} \mu(I_w)^q,$$

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with $C_q = 3^d C^q$. Observe that a direct consequence of (8) is that for some $C'_q > 0$,

$$\sup_{k\geqslant 1} 2^{k\tau_\mu(q)} \sum_{w\in\Sigma_k} \mu(I_w)^q \leqslant C'_q. \tag{33}$$

Consequently, setting $\widetilde{C}_q = C_q C'_q$, we get

$$\mathbb{E}\left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \leqslant \widetilde{C}_q \left(2^{-Jd(1-\eta)} \sum_{W\in\Sigma_J} \mu(I_W)^q\right) \sum_{k\geqslant 0} 2^{-k(\tau_\mu(q)+d(1-\eta))}.$$

Since $q > q_{\widetilde{\eta}}$, we have $\tau_\mu(q)+d(1-\eta) > 0$. Hence for some constant $\widehat{C}_q$, we have

$$\mathbb{E}\left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \leqslant \widehat{C}_q \left(2^{-Jd(1-\eta)} \sum_{W\in\Sigma_J} \mu(I_W)^q\right).$$

Finally, for every $\varepsilon > 0$, applying (33) again, we get

$$\mathbb{E}\left(\sum_{J\geqslant 1} 2^{J(\tau_\mu(q)+d(1-\eta)-\varepsilon)} \sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q\right) \leqslant 3^d \widehat{C}_q C'_q \sum_{J\geqslant 1} 2^{-J\varepsilon},$$

which is finite. We conclude that with probability 1,

$$\limsup_{J\to+\infty} \frac{1}{J} \log_2 \sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \leqslant -\tau_\mu(q) - d(1-\eta),$$

i.e., $\tau_{\mathsf{M}_\mu}(q) \geqslant \tau_\mu(q)+d(1-\eta) = \tilde{\tau}(q)$. This holds for each $q_{\widetilde{\eta}} < q < q_{\eta_\ell}$ almost surely, and by concavity (hence continuity) of τ_{M_μ} and $\widetilde{\tau}$, almost surely for all $q_{\widetilde{\eta}} \leqslant q \leqslant q_{\eta_\ell}$.

6.2.2. When $q \in (0, q_{\widetilde{\eta}})$. • Suppose for a while that both η_ℓ and η_r are positive.

Fix $0 < \varepsilon < H_\ell(\eta_\ell)$, such that, due to Proposition 4 and the continuity of the mappings H_ℓ and H_r, there exist $j_0 \geqslant 1$, and two sets of parameters $\eta_\ell = \eta_{\ell,1} < \cdots < \eta_{\ell,N_\ell} = \eta$ and $\eta_r = \eta_{r,1} < \cdots < \eta_{r,N_r}$ such that for $j \geqslant j_0$, if $w \in \mathcal{S}_j(\eta)$, then $w \in \mathcal{T}_\mu(j, \eta_{i,k}, \varepsilon)$ for some $i \in \{\ell, r\}$ and $1 \leqslant k \leqslant N_i$, i.e.,

$$(j - \lfloor \eta_{i,k} j\rfloor)(H_i(\eta_{i,k}) - \varepsilon) \leqslant -\log_2 \mu(I_{\sigma^{\lfloor \eta_{i,k} j\rfloor} w}) \leqslant (j - \lfloor \eta_{i,k} j\rfloor)(H_i(\eta_{i,k}) + \varepsilon). \tag{34}$$

For J large enough, given $W \in \Sigma_J$, assume that $\mathsf{M}_\mu(I_W)$ is realized at w, i.e., $\mathsf{M}_\mu(I_W) = \mu(I_w)$, for some word of length $|w| = j \geqslant J$, $I_w \subset I_{W'}$ and $W' \in \mathcal{N}_J(W)$. In this case, there exists $\eta_{i,k}$ such that (34) holds. One distinguishes two possibilities linked to the parameters $\eta_{i,k}$:

- if $\lfloor \eta_{i,k} j\rfloor \leqslant J$, then $I_W \subset \bigcup_{u\in\mathcal{N}_{\lfloor \eta_{i,k} j\rfloor}(w_{|\lfloor \eta_{i,k} j\rfloor})} I_u$;
- if $\lfloor \eta_{i,k} j\rfloor > J$, then

$$\mathsf{M}_\mu(I_W) \leqslant C\mu(I_w)\mu(I_{\sigma^{j-\lfloor \eta_{i,k} j\rfloor} w}) \leqslant C\mu(I_W) 2^{(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)},$$

where (34) has been used.

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In the second case, some information is lost between the generations J and $\lfloor \eta' j \rfloor$. We deduce from these observations and the quasi-Bernoulli property of μ that

$$\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q$$

$$\leqslant \sum_{i\in\{\ell,r\}} \sum_{\substack{k=1\\ \lfloor \eta_{i,k} j\rfloor \leqslant J}}^{N_i} \sum_{w\in\mathcal{T}_\mu(j,\eta_{i,k},\varepsilon)} \mu(I_w)^q + \sum_{i\in\{\ell,r\}} \sum_{\substack{k=1\\ \lfloor \eta_{i,k} j\rfloor > J}}^{N_i} \sum_{w\in\mathcal{T}_\mu(j,\eta_{i,k},\varepsilon)} \mu(I_w)^q$$

$$\leqslant \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 3^d \sum_{J\leqslant j\leqslant J/\eta_{i,k}} \sum_{u\in\Sigma_{\lfloor j\eta_{i,k}\rfloor}} C^q \mu(I_u)^q 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}$$

$$+ \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 3^d \sum_{W\in\Sigma_J} \sum_{j>J/\eta_{i,k}} C^q \mu(I_W)^q 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}.$$

Recalling (33) and the fact that $\widetilde{H}_i(\eta') = H_i(\eta')(\eta'^{-1}-1)$ for every η', there exists a positive constant $\widetilde{C}'_q$ such that the first term in the last sum is bounded from above by

$$\widetilde{C}_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} \sum_{J\leqslant j\leqslant J/\eta_{i,k}} 2^{-\lfloor j\eta_{i,k}\rfloor \tau_\mu(q)} 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}$$

$$\leqslant \widetilde{C}'_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 2^{qJ\varepsilon/\eta_{i,k}} \sum_{J\leqslant j\leqslant J/\eta'_{i,k}} 2^{-j\eta_{i,k}(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))}$$

and the second one by

$$\widetilde{C}_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 2^{-J\tau_\mu(q)} \sum_{j>J/\eta_{i,k}} 2^{-q(j-\lfloor j\eta_{i,k}\rfloor)(H_i(\eta_{i,k})-\varepsilon)}$$

$$\leqslant \widetilde{C}'_q \sum_{i\in\{\ell,r\}} \sum_{k=1}^{N_i} 2^{-J(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))} 2^{qJ({\eta_{i,k}}^{-1}-1)\varepsilon}.$$

Since $q \geqslant 0$, $\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k})$ is bounded from below by $\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta})$, which is negative. Consequently,

$$\sum_{J\leqslant j\leqslant (J+1)/\eta_{i,k}} 2^{-j\eta_{i,k}(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))} = O(2^{-J(\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta}))}).$$

In addition, one always has $\eta_{i,k} \geqslant \eta_i$, hence

$$2^{-J(\tau_\mu(q)+q\widetilde{H}_i(\eta_{i,k}))} 2^{qJ({\eta_{i,k}}^{-1}-1)\varepsilon} \leqslant 2^{-J(\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta}))} 2^{qJ({\eta_i}^{-1}-1)\varepsilon}.$$

Putting everything together we get for some $C''_q > 0$

$$\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \leqslant C''_q \left(\sum_{i\in\{\ell,r\}} N_i\big(2^{qJ\varepsilon/\eta_i} + 2^{qJ({\eta_i}^{-1}-1)\varepsilon}\big)\right) 2^{-J(\tau_\mu(q)+q\widetilde{H}_\ell(\widetilde{\eta}))}.$$

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This yields $\tau_{\mathsf{M}_\mu}(q) \geqslant \tau_\mu(q) + q\widetilde{H}_\ell(\widetilde{\eta}) + O(\varepsilon)$, and letting ε tend to 0 gives the desired conclusion.

• Now we deal with the case where at least one of parameters η_ℓ and η_r equals zero.

According to the value of η_i, we construct a subset $\Sigma_J^{(i)}$ of words of length J having specific properties:

First case: $\eta_i > 0$*:* set $\Sigma_J^{(i)} = \emptyset$.

Second case: $\eta_i = 0$*:* in this case, $D_\mu(H_i(\eta_i)) = d(1-\eta)$. Heuristically, $\Sigma_J^{(i)}$ contains those words W such that $\mathsf{M}_\mu(I_W) = \mu(I_w)$ for some surviving vertex $w \in \mathcal{S}_j(\eta)$ having an 'extreme' behavior, i.e., $\mu(I_w) \sim 2^{-jH_i(\eta_i)}$. We proceed as follows:

At first, let $K \geqslant H_{\max}$ be as in Proposition 2(6). For $\eta_i' \in [\eta_i, \eta]$ close to η_i, we denote by $\widehat{\eta_i}$ the unique real number in $[\eta_i, \eta]$ such that $H_i(\widehat{\eta_i}) = H_i(\eta_i') + K\eta_i'$ if $i = \ell$ and $H_i(\widehat{\eta_i}) = H_i(\eta_i') - K\eta_i'$ if $i = r$. Notice that $\widehat{\eta_i} > \eta_i'$.

Now, fix $\varepsilon = qH_i(\eta_i)/4$, and choose η_i' small enough so that $(1-\eta_i')(H_i(\widehat{\eta_i}) - K\widehat{\eta_i}) > H_i(\eta_i)/2$ if $i = \ell$ and $H_i(\eta_i') - 2K\widehat{\eta_i} > H_i(\eta_i)/2$ if $i = r$, and

$$\begin{cases} H_i(\eta_i') + 2K\widehat{\eta_i} < H_s \text{ and } D_\mu(H_i(\eta_i') + 2K\widehat{\eta_i}) \leqslant D_\mu(H_i(\eta_i) + \varepsilon/2 & \text{if } i = \ell, \\ H_i(\eta_i') - 2K\widehat{\eta_i} > H_s \text{ and } D_\mu(H_i(\eta_i') - 2K\widehat{\eta_i}) \leqslant D_\mu(H_i(\eta_i)) + \varepsilon/2 & \text{if } i = r. \end{cases}$$

By item (5) of Proposition 2, there exists an integer J_i such that for $j \geqslant J_i$, if $i = \ell$ we have

$$\#\mathcal{E}_\mu(j, [0, H_\ell(\eta_i') + 2K\widehat{\eta_i}]) \leqslant 2^{j(D_\mu(H_i(\eta_i') + 2K\eta_i') + \varepsilon/2)} \leqslant 2^{j(D_\mu(H_i(\eta_i)) + \varepsilon)}, \tag{35}$$

and if $i = r$ we have

$$\#\mathcal{E}_\mu(j, [H_i(\eta_i') - 2K\widehat{\eta_i}, +\infty)) \leqslant 2^{j(D_\mu(H_i(\eta_i') - 2K\eta_i') + \varepsilon/2)} \leqslant 2^{j(D_\mu(H_i(\eta_i)) + \varepsilon)}.$$

It is also possible to choose J_i such that $\varepsilon_j^2 \leqslant K\eta_i'/2 \leqslant K\widehat{\eta_i}/2$ for $j \geqslant J_i$, where $(\varepsilon_j^2)_{j\geqslant 1}$ is the sequence introduced in Proposition 4.

For $J \geqslant J_i$, take $\Sigma_J^{(i)}$ as the set of those words $W \in \Sigma_J$ such that $\mathsf{M}_\mu(I_W) = \mu(I_w)$, where $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,i}(j, \eta', \varepsilon_j^2, W)$ for some η' satisfying

$$\begin{cases} H_\ell(\eta') + \varepsilon_j^2 \leqslant H_\ell(\eta_\ell') + K\eta_\ell' & \text{if } i = \ell, \\ H_r(\eta') - \varepsilon_j^2 \geqslant H_r(\eta_r') - K\eta_r' & \text{if } i = r. \end{cases}$$

In particular, $\eta' \leqslant \widehat{\eta_i}$. The words $W \in \Sigma_J^{(i)}$ are the ones that may cause problems when compared to the case where $\eta_\ell, \eta_r > 0$. The other words W are such that $\mathsf{M}_\mu(I_W)$ is reached at some w associated with η' satisfying $H_\ell(\eta') \in [H_\ell(\eta_\ell') + K\eta_\ell'/2, H_r(\eta_r') - K\eta_r'/2]$, i.e., η' stays bounded away from 0.

When $J \geqslant J_i$ and $W \in \Sigma_J^{(i)}$, for the associated word $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,i}(j, \eta', \varepsilon_j^2, W)$ (according to the previous notations), using (8) and the definition of K, one has:

$$C^{-1}2^{-\lfloor j\eta' \rfloor K} \mu(I_{\sigma^{\lfloor \eta' j \rfloor}w}) \leqslant \mathsf{M}_\mu(I_W) = \mu(I_w) \leqslant C\mu(I_{\sigma^{\lfloor \eta' j \rfloor}w}),$$

which yields, due to the property of (W, w) and the fact that $\eta' \leqslant \widehat{\eta_i}$:

$$C^{-1}2^{-j\widehat{\eta_i}K}2^{-j(H_i(\eta') + \varepsilon_j^2)} \leqslant \mathsf{M}_\mu(I_W) = \mu(I_w) \leqslant C2^{-(j - \lfloor \widehat{\eta_i} j \rfloor)(H_i(\eta') - \varepsilon_j^2)}.$$

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This yields, for J large enough,

$$\begin{cases} 2^{-j(H_i(\eta_i')+2K\widehat{\eta}_i)} \leqslant \mathsf{M}_\mu(I_W) = \mu(I_w) \leqslant 2^{-j(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i)} & \text{if } i = \ell, \\ \mathsf{M}_\mu(I_W) = \mu(I_w) \leqslant 2^{-j(H_i(\eta_i')-2K\widehat{\eta}_i)} & \text{if } i = r. \end{cases}$$

Hence, each such word W is associated with one surviving word $w \in \mathcal{E}_\mu(j, [0, H_i(\eta_i') + 2K\widehat{\eta}_i])$, for some $j \geqslant J$ if $i = \ell$, and one surviving word $w \in \mathcal{E}_\mu(j, [H_i(\eta_i') - 2K\widehat{\eta}_i, +\infty))$, for some $j \geqslant J$ if $i = r$.

Then, if $i = \ell$, writing $j = J + k$ one gets:

$$\sum_{W \in \Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q \leqslant \sum_{k=0}^{+\infty} \sum_{w \in \mathcal{E}_\mu(J+k,[0,H_i(\eta_i')+2K\widehat{\eta}_i])} p_w C^q 2^{-(J+k)q(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i)}.$$

Taking expectations and recalling (35), one gets

$$\begin{aligned} &\mathbb{E}\left(\sum_{W \in \Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q\right) \\ &\leqslant 3^d C^q \sum_{k \geqslant 0} 2^{-(J+k)d(1-\eta)} 2^{(J+k)(d(1-\eta)+\varepsilon)} 2^{-q(J+k)(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i)} \\ &= 3^d C^q \sum_{k \geqslant 0} 2^{(J+k)(\varepsilon - q(1-\eta_i')(H_i(\widehat{\eta}_i)-K\widehat{\eta}_i))}. \end{aligned}$$

The choice for ε and η_i' implies $\varepsilon - q(1-\eta_i')(H_i(\widehat{\eta}_i) - K\widehat{\eta}_i) \leqslant -\varepsilon$. One deduces that

$$\mathbb{E}\left(\sum_{W \in \Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q\right) \leqslant C_{q,\varepsilon} 2^{-J\varepsilon}$$

for some constant $C_{q,\varepsilon} > 0$. Finally, applying the Borel–Cantelli lemma, we deduce that with probability 1, for J large enough we have

$$\sum_{W \in \Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q \leqslant 1. \tag{36}$$

Observe that (36) holds true even if $\eta_i > 0$ (in which case $\Sigma_J^{(i)}$ is empty). If $i = r$, similar computations yield

$$\mathbb{E}\left(\sum_{W \in \Sigma_J^{(i)}} \mathsf{M}_\mu(I_W)^q\right) \leqslant 3^d C^q \sum_{k \geqslant 0} 2^{(J+k)(\varepsilon - q(H_i(\eta_i')-2K\widehat{\eta}_i))},$$

with a similar conclusion.

Finally, the same estimates as when both η_ℓ and η_r are strictly positive yield

$$\liminf_{J \to +\infty} \frac{-1}{J} \log_2 \sum_{W \in \Sigma_J \setminus (\Sigma_J^{(\ell)} \cup \Sigma_J^{(r)})} \mathsf{M}_\mu(I_W)^q \geqslant \tau_\mu(q) + q\widetilde{H}_\ell(\widetilde{\eta}) = \tilde{\tau}(q).$$

Since $\tilde{\tau}(q) < 0$ and (36) holds for J large enough, one concludes that $\tau_{\mathsf{M}_\mu}(q) \geqslant \tilde{\tau}(q)$.

6.2.3. When $q < 0$. Applying Proposition 5 with $\eta' = \widetilde{\eta}$, there exists a positive sequence $(\varepsilon_j^3)_{j\geqslant 1}$ converging to 0 such that with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor \widetilde{\eta} j\rfloor}$, there exists $w \in \mathcal{S}_j(\eta, W)$ such that the $\widetilde{\eta}$-tail of w satisfies

$$2^{-(j-\lfloor \widetilde{\eta} j\rfloor)(H_\ell(\widetilde{\eta})+\varepsilon_j^3)} \leqslant \mu(I_{\sigma^{\lfloor \widetilde{\eta} j\rfloor} w}).$$

The quasi-Bernoulli property (8) implies

$$\mathsf{M}_\mu(I_{Ww}) \geqslant C^{-1}\mu(I_W) 2^{-j(1-\widetilde{\eta})(H_\ell(\widetilde{\eta})+\varepsilon_j)},$$

which for $q < 0$ yields

$$\begin{aligned}\sum_{W\in\Sigma_{\lfloor \widetilde{\eta} j\rfloor}} \mathsf{M}_\mu(I_W)^q &\leqslant C^q 2^{-jq(1-\widetilde{\eta})(H_\ell(\widetilde{\eta})+\varepsilon_j)} \sum_{W\in\Sigma_{\lfloor \widetilde{\eta} j\rfloor}} \mu(I_W)^q \\ &\leqslant C^{q+1} 2^{-\lfloor \widetilde{\eta} j\rfloor q(\widetilde{H}_\ell(\widetilde{\eta})+\varepsilon_j/\widetilde{\eta})} \sum_{W\in\Sigma_{\lfloor \widetilde{\eta} j\rfloor}} \mu(I_W)^q.\end{aligned}$$

One concludes that $\tau_{\mathsf{M}_\mu}(q) \geqslant \tau_\mu(q) + \widetilde{H}_\ell(\widetilde{\eta})q = \tilde{\tau}(q)$.

6.2.4. When $q_{\eta_\ell} < +\infty$ and $q > q_{\eta_\ell}$. Recall that this implies $\eta_\ell = 0$. We have already shown that $\tau_{\mathsf{M}_\mu}(q) \geqslant \tau_\mu(q) + d(1-\eta)$ when $q \in [q_{\widetilde{\eta}}, q_{\eta_\ell}]$.

The tangent to the graph of $q \mapsto \tau_{\mathsf{M}_\mu}(q)$ at $(q_{\eta_\ell}, \tau_{\mathsf{M}_\mu}(q_{\eta_\ell}))$ is the affine line passing through $(0, 0)$, whose slope is $\tau'_{\mathsf{M}_\mu}(q_{\eta_\ell}) = H_\ell(0)$. Consequently, the concavity of τ_{M_μ} implies that $\tau_{\mathsf{M}_\mu}(q) \leqslant q H_\ell(0)$ for all $q \geqslant q_{\eta_\ell}$. On the other hand, if $q \geqslant q_{\eta_\ell}$, for all $J \geqslant 1$,

$$\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^q \leqslant \left(\sum_{W\in\Sigma_J} \mathsf{M}_\mu(I_W)^{q_{\eta_\ell}}\right)^{q/q_{\eta_\ell}},$$

from which it follows that $\tau_{\mathsf{M}_\mu}(q) \geqslant \frac{q}{q_{\eta_\ell}} \tau_{\mathsf{M}_\mu}(q_{\eta_\ell}) = q H_\ell(0)$.

6.3. Lower bound for the lower large deviations spectrum $\underline{f}_{\mathsf{M}_\mu}(H)$: for every $H \geqslant 0$, $\underline{f}_{\mathsf{M}_\mu}(H) \geqslant \tilde{\tau}^*(H)$

Let us check that (32) holds. It is enough to deal with a dense countable subset of the support $[H_\ell(\eta_\ell), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$ of $\tilde{\tau}^*$, since $\underline{f}_{\mathsf{M}_\mu}$ is lower semi-continuous and $\tilde{\tau}^*$ is continuous (see the graph of $\tilde{\tau}^*$ on Figure 13).

- Suppose first that $H \in [H_\ell(\eta_\ell), H_\ell(\widetilde{\eta})]$.

Let $\eta_H \in [\eta_\ell, \widetilde{\eta})$ be the unique real number such that

$$H = H_\ell(\eta_H). \tag{37}$$

By item (4) of Proposition 2, for every $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that when j becomes large,

$$\#\mathcal{E}_\mu(\lfloor \eta_H j\rfloor, [0, H_\ell(\eta_H) + \varepsilon]) \geqslant 2^{\lfloor \eta_H j\rfloor(D_\mu(H_\ell(\eta_H))-\beta(\varepsilon))}.$$

One also knows that $\beta(\varepsilon)$ can be taken so that $\beta(\varepsilon) \to 0$ when $\varepsilon \to 0$.

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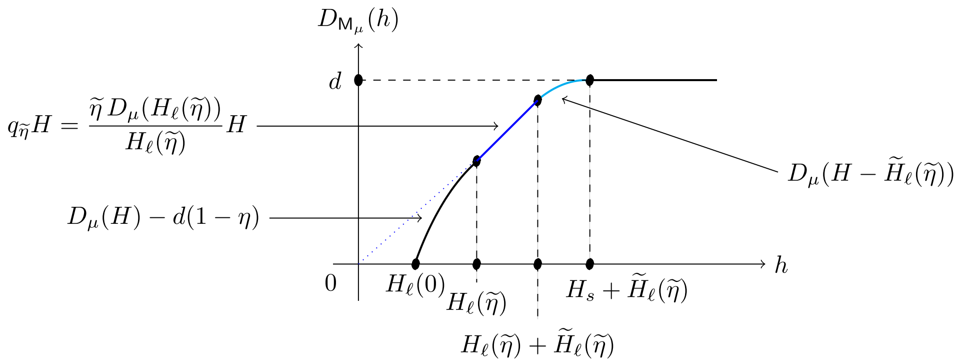

Figure 13. The mapping $H \mapsto \widetilde{\tau}^*(H)$.

In addition, by Proposition 5, there is a positive sequence $(\varepsilon_j^3)_{j\geqslant 1}$ converging to 0 such that, with probability 1, for j large enough, each cube I_W (with $W \in \mathcal{E}_\mu(\lfloor \eta_H j \rfloor, [0, H_\ell(\eta_H)+\varepsilon]))$ contains a smaller cube I_w, with $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta_H, \varepsilon_j^3)$.

By the quasi-Bernoulli property (8) of μ,

$$\begin{aligned}\mathsf{M}_\mu(I_w) &\geqslant \mu(I_w) \geqslant C^{-1} 2^{-\lfloor \eta_H j\rfloor (H_\ell(\eta_H)+\varepsilon)} 2^{-(j-\lfloor \eta_H j \rfloor)(H_\ell(\eta_H)+\varepsilon_j^3)} \\ &\geqslant 2^{-j(H_\ell(\eta_H)+2\varepsilon)}\end{aligned}$$

when j becomes large. Thus,

$$\liminf_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H_\ell(\eta_H)+2\varepsilon]) \geqslant \eta_H \Big(D_\mu(H_\ell(\eta_H)) - \beta(\varepsilon)\Big).$$

Since by construction $H = H_\ell(\eta_H)$ and $\eta_H D_\mu(H_\ell(\eta_H)) = D_\mu(H) - d(1-\eta)$, letting ε go to zero gives

$$\lim_{\varepsilon\to 0^+} \liminf_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H+2\varepsilon]) \geqslant D_\mu(H) - d(1-\eta) = \widetilde{\tau}^*(H).$$

One concludes that $\underline{f}_{\mathsf{M}_\mu}(H) \geqslant \widetilde{\tau}^*(H)$, for otherwise there would exist $H' < H$ such that

$$\limsup_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H']) \geqslant \widetilde{\tau}^*(H) > \widetilde{\tau}^*(H').$$

This contradicts the fact that by Proposition 1, for all $H' \leqslant H_s + \widetilde{H}_\ell(\widetilde{\eta}) = \tau'_{\mathsf{M}_\mu}(0)$,

$$\limsup_{j\to+\infty} \frac{1}{j} \log_2 \#\mathcal{E}_{\mathsf{M}_\mu}(j, [0, H']) \leqslant \tau^*_{\mathsf{M}_\mu}(H') \leqslant \widetilde{\tau}^*(H').$$

- For $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\eta_\ell)]$, the same idea is used: There exists a positive sequence $(\beta_j)_{j\geqslant 1}$ converging to 0 such that, for j large enough, at generation $\lfloor j\widetilde{\eta} \rfloor$ there are at least $2^{\lfloor j\widetilde{\eta}\rfloor (D_\mu(H_\ell(\widetilde{\eta})) - \varepsilon_j))}$ words W in $\mathcal{E}_\mu(\lfloor j\widetilde{\eta} \rfloor, [0, H_\ell(\widetilde{\eta}) + \beta_j])$.

In addition, by Proposition 5, with probability 1, for j large enough, each of these I_W contains a smaller cube I_w, with $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \widetilde{\eta}, \varepsilon_j^3)$.

Then, let w' be the word of generation $j' = \lfloor j H_\ell(\widetilde{\eta})/H \rfloor$ such that $I_w \subset I_{w'} \subset I_W$. One has $-\log_2 \mathsf{M}_\mu(I_{w'}) \leqslant -\log_2 \mathsf{M}_\mu(I_w) \sim j H_\ell(\eta') \sim j' H$. It follows that for any $\varepsilon > 0$,

$$\liminf_{j' \to +\infty} \frac{1}{j'} \log_2 \# \mathcal{E}_{\mathsf{M}_\mu}(j', [0, H + \varepsilon]) \geqslant \frac{\widetilde{\eta} D_\mu \left(H_\ell(\widetilde{\eta})\right)}{H_\ell(\widetilde{\eta})} H = \widetilde{\tau}^*(H).$$

The conclusion is the same as in the previous case.

- For $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, Remark 2 in § 7 will give the result.

6.4. Alternative approach to the bounds of $\underline{\dim}(\mathsf{M}_\mu, \cdot)$

We give another proof of the fact that $\underline{\dim}(\mathsf{M}_\mu, \cdot)$ is bounded by $\widetilde{\tau}'(+\infty)$ and $\widetilde{\tau}'(-\infty)$. Next proposition, which provides the upper bound, will be used in § 7.

Proposition 8. *Almost surely, for every $x \in [0, 1]^d$,*

$$\underline{\dim}(\mathsf{M}_\mu, x) \leqslant \overline{\dim}(\mathsf{M}_\mu, x) \leqslant \overline{\dim}(\mu, x) + \widetilde{H}_\ell(\widetilde{\eta}).$$

As a consequence, for every $x \in [0, 1]^d$, $\underline{\dim}(\mathsf{M}_\mu, x) \leqslant H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}) = \widetilde{\tau}'(-\infty)$.

Proof. Let $x \in [0, 1]^d$. Proposition 5 applied with $\eta' = \widetilde{\eta}$, for each j large enough gives

$$\mathsf{M}_\mu(I_{\lfloor j\widetilde{\eta} \rfloor}(x)) \geqslant C^{-1} \mu(I_{\lfloor j\widetilde{\eta} \rfloor}(x)) 2^{-(j - \lfloor j\widetilde{\eta} \rfloor)(H_\ell(\widetilde{\eta}) + \varepsilon_j^3)}.$$

Taking logarithm on both sides, dividing by $-\lfloor j\widetilde{\eta} \rfloor \log(2)$, and taking the $\liminf$ as $j \to \infty$ yields the desired conclusion. Since for every x, $\overline{\dim}(\mu, x) \leqslant H_{\max}$, the result follows. □

Notice that this property immediately yields $D_{\mathsf{M}_\mu}(H) \leqslant D_\mu\big(H - \widetilde{H}_\ell(\widetilde{\eta})\big)$ for all $H \in [H_s + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, and $\dim \underline{E}^{\geqslant}_{\mathsf{M}_\mu}(H) = -\infty$ for all $H > H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})$, a fact which already followed from the inequality $\tau_{\mathsf{M}_\mu} \geqslant \widetilde{\tau}$.

Now we deal with the lower bound $\underline{\dim}(\mathsf{M}_\mu, x) \geqslant H_\ell(\eta_\ell) = \widetilde{\tau}'(+\infty)$.

Lemma 6. *With probability 1, for every $x \in [0, 1]^d$, one has $\underline{\dim}(\mathsf{M}_\mu, x) \geqslant H_\ell(\eta_\ell)$.*

Proof. By Proposition 3, with probability 1, for j large enough, the surviving vertices $w \in \mathcal{S}_j(\eta)$ all satisfy $\mu(I_w) \leqslant 2^{-j(H_\ell(\eta_\ell) - \varepsilon_j^1)}$. Hence, for every large integer J and every word $W \in \Sigma_J$, $\mathsf{M}_\mu(I_W) \leqslant 2^{-J(H_\ell(\eta_\ell) - \varepsilon_J^1)}$, since $\mathsf{M}_\mu(I_W)$ is the maximum of $\mu(I_w)$ over all surviving words w such that $I_w \subset I_W$. Subsequently, for every x, $\underline{\dim}(\mathsf{M}_\mu, x) \geqslant H_\ell(\eta_\ell)$. □

7. Lower bound for the singularity spectrum and validity of the multifractal formalism for M_μ

For each admissible local dimension H, we are going to exhibit an auxiliary probability measure ν (which depends on H) such that $\nu\big(\underline{E}_{\mathsf{M}_\mu}(H)\big) = 1$, and such that the dimension of ν equals the announced value for $D_{\mathsf{M}_\mu}(H)$, i.e., $\widetilde{\tau}^*(H)$. Due to the results obtained in § 6, this implies the validity of the multifractal formalism for M_μ.

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These auxiliary measures do not always have the same nature, depending on H. They can be taken as a Gibbs measure when $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta}))$, but not for the other values of H.

We introduce two families of measures in § 7.1, whose properties are established in § 7.5. Then we obtain the sharp lower bound for D_{M_μ} in §§ 7.2 to 7.4.

7.1. Two families of measures

The first family is used to obtain a sharp lower bound for $D_{\mathsf{M}_\mu}(H)$ when $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$. It is based on the following result.

Recall Proposition 6 in which the event $\mathscr{C}(N, J, W)$ is defined.

Theorem 5. *With probability 1, for all $\alpha \in [H_{\min}, H_{\max}]$, there exists an exact dimensional Borel probability measure ν_α of Hausdorff dimension $D_\mu(\alpha)$ supported on $\widetilde{E}_\mu(\alpha)$ (i.e., $\nu_\alpha(\widetilde{E}_\mu(\alpha)) = 1$), such that:*

(1) *for all $\delta > 1$;*

$$\nu_\alpha\left(\bigcap_{J\geqslant 1}\bigcup_{j\geqslant J}\bigcup_{w\in\mathcal{S}_j(\eta)} B\big(x_w, (2\cdot 2^{-\lfloor \eta j\rfloor})^\delta\big)\right) = 0.$$

(2) *for all $N > 1/\eta$, for ν_α-almost every x, there exists an integer $J_{N,\alpha,x} \geqslant 1$ such that for all $J \geqslant J_{N,\alpha,x}$, the event $\mathscr{C}(N, J, x_{|J})$ is not realized.*

Theorem 5 is proved at the end of this Section (§ 7.5). Observe that the result holds simultaneously for all $\alpha \in [H_{\min}, H_{\max}]$.

In the first item, the limsup set contains those points $x \in [0, 1]^d$ that are very close to the surviving coefficients, i.e., those x satisfying for some $\delta > 1$

$$|x - x_w| < 2\cdot 2^{-\lfloor |w|\eta\rfloor \delta}$$

for infinitely many surviving words w. By the covering Lemma 3, when $\delta < 1$, every $x \in [0, 1]^d$ satisfies the last inequality infinitely many times. Part (1) of Theorem 5 states that this is no longer true when $\delta > 1$, in the sense that the ν_α-measure of these sets of points is always 0.

The second part of the theorem is technical, and used in the proofs below.

The second family of measures allows us to compute the value of $D_{\mathsf{M}_\mu}(H)$ when $H \in [H_\ell(\eta_\ell), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$. These measures are built thanks to the theory of heterogeneous ubiquity theory, developed in [4–6, 15], whose main results can be resumed as follows.

Theorem 6. *Let $\mathcal{F} = ((x_n, r_n))_{n\geqslant 1}$ be a sequence of couples such that $(x_n)_{n\geqslant 1}$ is a sequence of points in $[0, 1]^d$, and $(r_n)_{n\geqslant 1}$ is a positive sequence converging to zero. Assume that*

$$(0, 1)^d \subset \limsup_{n\to+\infty} B(x_n, r_n). \tag{38}$$

Let $\alpha \in (H_{\min}, H_{\max})$. Recall that the Gibbs measure μ_α was defined in Proposition 2(4).

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For every $\delta \geqslant 1$ *and any positive sequence* $\widetilde{\beta} := (\widetilde{\beta}_n)_{n\geqslant 1}$ *converging to zero, define*

$$U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big) := \bigcap_{N\geqslant 1} \bigcup_{\substack{n\geqslant N:\\ (r_n)^{\alpha+\widetilde{\beta}_n} \leqslant \mu_\alpha(B(x_n,r_n)) \leqslant (r_n)^{\alpha-\widetilde{\beta}_n}}} B\Big(x_n, (r_n)^{\delta}\Big). \tag{39}$$

For every $\delta \geqslant 1$*, there exists a Borel probability measure* $\nu_{\alpha,\delta}$ *and a positive sequence* $\widetilde{\beta} := (\widetilde{\beta}_n)_{n\geqslant 1}$ *converging to zero such that*

$$\nu_{\alpha,\delta}\left(U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big)\right) = 1,$$

and $\nu_{\alpha,\delta}(E) = 0$ *for every set* E *such that* $\dim E < D_\mu(\alpha)/\delta$*.*

In particular, one has

$$\dim U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big) \geqslant \dim \nu_{\alpha,\delta} \geqslant \frac{D_\mu(\alpha)}{\delta}.$$

Moreover, if $(\alpha^{(p)}, \mathcal{F}^{(p)}, \delta^{(p)})_{p\geqslant 1}$ *stands for a sequence of parameters satisfying the above conditions, there exists a measure* $\widetilde{\nu}$ *and sequences* $\widetilde{\beta}^{(p)} := (\widetilde{\beta}_n^{(p)})_{p\geqslant 1, n\geqslant 1}$ *converging to zero satisfying*

$$\widetilde{\nu}\left(\bigcap_{p\geqslant 1} U_\mu\big(\alpha^{(p)}, \delta^{(p)}, \mathcal{F}^{(p)}, \widetilde{\beta}^{(p)}\big)\right) = 1,$$

and $\widetilde{\nu}(E) = 0$ *for every set* E *such that* $\dim E < \inf_{p\geqslant 1} \frac{D_\mu(\alpha^{(p)})}{\delta^{(p)}}$*.*

In particular,

$$\dim \bigcap_{p\geqslant 1} U_\mu\big(\alpha^{(p)}, \delta^{(p)}, \mathcal{F}^{(p)}, \widetilde{\beta}^{(p)}\big) \geqslant \inf_{p\geqslant 1} \frac{D_\mu(\alpha^{(p)})}{\delta^{(p)}}.$$

The last property is due to the fact that the sets $U_\mu\big(\alpha, \delta, \mathcal{F}, \widetilde{\beta}\big)$ enjoy the large intersection property, i.e., when intersecting a countable number of them, the Hausdorff dimension of the resulting set is at least the infimum of all the dimensions; see [6, 15].

We are going to apply Theorem 6 with well-chosen families $(x_n, r_n)_{n\geqslant 1}$:

Let $\mathcal{D}_\ell$ be a dense countable subset of $[\eta_\ell, \eta] \setminus \{0\}$, such that $\widetilde{\eta} \in \mathcal{D}_\ell$. With probability 1, for all $\eta' \in \mathcal{D}_\ell$, Proposition 5 proves the existence of words $w \in \mathcal{S}_j(\eta, W) \cap \mathcal{T}_{\mu,\ell}(j, \eta', \varepsilon_j^3)$, for j large enough, for all $W \in \Sigma_{\lfloor \eta' j \rfloor}$. For such a surviving word w, we set $r_w = 2 \cdot 2^{-\lfloor \eta' j \rfloor}$. The sequence of couples (x_w, r_w) obtained in this way is denoted

$$\mathcal{F}_{\eta'} := \big(x_n(\eta'), r_n(\eta')\big)_{n\geqslant 1}$$

after being reordered so that the sequence of radii $(r_n(\eta'))_{n\geqslant 1}$ is non-increasing. By construction, the covering property (38) is satisfied for the family $\mathcal{F}_{\eta'}$, so that the second part of Theorem 6 can be applied with the countable number of families $\big(\mathcal{F}_{\eta'}\big)_{\eta' \in \mathcal{D}_\ell}$.

7.2. The right part of the spectrum D_{M_μ}

For $H \in \big[H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})\big]$, set $\alpha_H = H - \widetilde{H}_\ell(\widetilde{\eta})$. We are going to use the measure ν_{α_H} built in Theorem 5.

Lemma 7. *With probability 1, for all* $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$*, there exists a set* $G_H \subset \widetilde{E}_\mu(\alpha_H)$ *such that:*

- $\nu_{\alpha_H}(G_H) = 1$;
- *for all* $x \in G_H$*, for all integers* $N > 1/\eta$*, there exists* $J_N(x) \geqslant 1$ *such that for all* $J \geqslant J_N(x)$*, for all* $J \leqslant j < J/(\widetilde{\eta}_{N,-1})$*, one has*

$$\bigcup_{W \in \mathcal{N}_J(x)} \mathcal{S}_j(\eta, W) = \emptyset.$$

Proof. Notice that for all $N \geqslant 1$ we have $1/\widetilde{\eta}_{N,-1} < 1/\eta$.

Assume toward contradiction that with positive probability, there exists $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, a set F_H of positive ν_{α_H}-measure, and $N \geqslant 1$ such that

$$F_H \subset \bigcap_{J \geqslant 1} \bigcup_{j \geqslant J} \bigcup_{w \in \mathcal{S}_j(\eta)} B\big(x_w, (2 \cdot 2^{-\lfloor \eta j \rfloor})^\delta\big)$$

for all $\delta \in (1, \widetilde{\eta}^{-1}\widetilde{\eta}_{N,-1})$. This contradicts Theorem 5.

Consequently, with probability 1, for all $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$ there exists a set G_H such that both items of the statement hold. Moreover, G_H can be taken a subset of $\widetilde{E}_\mu(\alpha_H)$ since $\nu_{\alpha_H}\big(\widetilde{E}_\mu(\alpha_H)\big) = 1$. □

Now, we prove the inequality $D_{\mathsf{M}_\mu}(H) \geqslant D_\mu\big(H - \widetilde{H}(\widetilde{\eta})\big)$. Consider a set Ω' of probability 1 over which the conclusions of Theorem 5 and Lemma 7 hold true.

Lemma 8. *For all* $\omega \in \Omega'$ *and* $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$*, one has* $G_H \subset \underline{E}_{\mathsf{M}_\mu}(H)$.

Proof. Take $\omega \in \Omega'$, and fix an integer $N > 1/\eta$. Fix $x \in G_H$. We focus on the values of $M_\mu(I_J(x))$, by analyzing the values of $\mu(I_w)$ when $w \in \mathcal{S}_j(\eta)$ is a surviving vertex such that I_w is included in the neighborhood $\mathcal{N}_J(x)$ of x.

First, combining part (2) of Theorem 5 and Lemma 7, for all J large enough, for all $W \in \mathcal{N}_J(x)$, one has:

- for all $J \leqslant j \leqslant J/\widetilde{\eta}_{N,-1}$, $\bigcup_{W \in \mathcal{N}_J(x)} \mathcal{S}_j(\eta, W) = \emptyset$;
- for all $-1 \leqslant k \leqslant N-1$, for all $J/\widetilde{\eta}_{N,k} \leqslant j \leqslant J/\widetilde{\eta}_{N,k+1}$, for all $w \in \mathcal{S}_j(\eta, W)$,

$$\mu(I_{\sigma^J w}) \leqslant 2^{-J(\widetilde{H}_\ell(\eta_{N,k}) - \varepsilon_N)} \leqslant 2^{J\varepsilon_N} 2^{-J\widetilde{H}_\ell(\widetilde{\eta})};$$

- if $j > J/\eta_0$, for all $w \in \Sigma_j$ such that $I_w \subset I_W$,

$$\begin{aligned}\mu(I_{\sigma^J w}) &\leqslant 2^{-J(\eta_0^{-1}-1)(H_{\min}-\varepsilon_N)} \\ &\leqslant \begin{cases} 2^{-J\left(\widetilde{H}_\ell(\widetilde{\eta}) - (\eta_0^{-1}-1)\varepsilon_N\right)} & \text{if } \eta_\ell = 0, \\ 2^{-j(\eta_\ell^{-1}-1)(H_\ell(\eta_\ell)-\varepsilon_N)} \leqslant 2^{-J\left(\widetilde{H}_\ell(\widetilde{\eta}) - (\eta_\ell^{-1}-1)\varepsilon_N\right)} & \text{if } \eta_\ell > 0. \end{cases}\end{aligned}$$

One used the fact that $\mu(L) \leqslant |L|^{H_{\min} - \epsilon_N}$ for any small enough interval, as well as the definition of η_0 in Proposition 6.

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Second, since $x \in G_H \subset \widetilde{E}_\mu(\alpha_H)$, there exists a sequence $(\widetilde{\varepsilon}_J)_{J\geqslant 1}$ (depending on x) tending to 0 as $J \to +\infty$ such that $x_{|J} \in \mathcal{E}_\mu(J, \alpha_H \pm \widetilde{\varepsilon}_J)$. In particular, one has for $I_W \in \mathcal{N}_J(x)$,

$$\mu(I_W) \leqslant 2^{-J(\alpha_H - \widetilde{\varepsilon}_J)}.$$

When $\eta_\ell = 0$, combining the previous inequalities and (8), one gets

$$\begin{aligned} \mathsf{M}_\mu(I_J(x)) &= \max\{\mu(I_w) : w \in \mathcal{S}_j(\eta, W), W \in \mathcal{N}_J(x)\} \\ &\leqslant C \cdot \max\{\mu(I_W) : I_W \in \mathcal{N}_J(x)\} \\ &\qquad \cdot \max\{\mu(I_{\sigma^J w}) : w \in \mathcal{S}_j(\eta, W), W \in \mathcal{N}_J(x)\} \\ &\leqslant C 2^{-J\left(\alpha_H - \widetilde{\varepsilon}_J + \widetilde{H}_\ell(\widetilde{\eta}) - \eta_0^{-1}\varepsilon_N\right)}. \end{aligned}$$

Consequently,

$$\underline{\dim}(\mathsf{M}_\mu, x) \geqslant \alpha_H + \widetilde{H}_\ell(\widetilde{\eta}) - \eta_0^{-1}\varepsilon_N = H - \eta_0^{-1}\varepsilon_N.$$

This holds for all $N > 1/\eta$ hence $\underline{\dim}(\mathsf{M}_\mu, x) \geqslant H$, for every $x \in G_H$. The same estimate is true when $\eta_\ell > 0$ by replacing η_0 by η_ℓ.

On the other hand, by Proposition 8 we know that $\overline{\dim}(\mathsf{M}_\mu, x) \leqslant \overline{\dim}(\mu, x) + \widetilde{H}_\ell(\widetilde{\eta}) = \alpha_H + \widetilde{H}_\ell(\widetilde{\eta}) = H$, hence $\underline{\dim}(\mathsf{M}_\mu, x) = H$ (in fact we obtained that $G_H \subset E_{\mathsf{M}_\mu}(H)$). □

One now concludes. Recall that with probability 1, simultaneously for all $H \in [H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, one has $\nu_{\alpha_H}(G_H) = 1$, so one has $\dim G_H \geqslant D_\mu(\alpha_H) = D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$. Finally, since $G_H \subset \underline{E}_{\mathsf{M}_\mu}(H)$,

$$\dim \underline{E}_{\mathsf{M}_\mu}(H) \geqslant \dim G_H \geqslant D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta})).$$

Remark 2. Observe that the previous arguments give the lower bound for the Hausdorff dimension of the level sets of the limit local dimension: for any $H \in [H_{\min} + \widetilde{H}_\ell(\widetilde{\eta}), H_{\max} + \widetilde{H}_\ell(\widetilde{\eta})]$, $\dim E_{\mathsf{M}_\mu}(H) \geqslant D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta}))$. Also, notice that this implies $\underline{f}_{\mathsf{M}_\mu}(H) \geqslant D_\mu(H - \widetilde{H}_\ell(\widetilde{\eta})) = \widetilde{\tau}^*(H)$, hence the lower bound for $\underline{f}_{\mathsf{M}_\mu}(H)$ we claimed in § 6.3.

7.3. The middle part of the spectrum D_{M_μ}

Let $H \in [H_\ell(\widetilde{\eta}), H_\ell(\widetilde{\eta}) + \widetilde{H}_\ell(\widetilde{\eta})]$. We apply Theorem 6 with the parameters:

- $\eta' = \widetilde{\eta}$;
- the family $\mathcal{F}_{\widetilde{\eta}} = (x_n(\widetilde{\eta}), r_n(\widetilde{\eta}))_{n\geqslant 1}$;
- $\alpha = H_\ell(\widetilde{\eta})$;
- $\delta = H_\ell(\widetilde{\eta})/(\widetilde{\eta} H)$ (which does belong to $[1, 1/\widetilde{\eta}]$).

There exists a sequence $\widetilde{\beta} := (\widetilde{\beta}_n)_{n\geqslant 1}$ and a Borel probability measure $\nu_{\alpha,\delta}$ supported on the set $U_\mu\big(H_\ell(\widetilde{\eta}), \delta, \mathcal{F}_{\widetilde{\eta}}, \widetilde{\beta}\big)$ and such that

$$\dim \nu_{\alpha,\delta} \geqslant \frac{\dim \mu_{H_\ell(\widetilde{\eta})}}{\delta} = \widetilde{\eta}\frac{D_\mu(H_\ell(\widetilde{\eta}))}{H_\ell(\widetilde{\eta})} H = q_{\widetilde{\eta}} H = \widetilde{\tau}^*(H),$$

where Lemmas 2 and 5 have been used.

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Lemma 9. *One has* $U_\mu\big(H_\ell(\widetilde{\eta}),\delta,\mathcal{F}_{\widetilde{\eta}},\widetilde{\beta}\big)\subset \underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H)$.

Proof. Let $x\in U_\mu\big(H_\ell(\widetilde{\eta}),\delta,\mathcal{F}_{\widetilde{\eta}},\widetilde{\beta}\big)$. By definition of this limsup set, there is an increasing sequence of integers $(j_k)_{k\geqslant 1}$ and words $w_k\in\mathcal{S}_{j_k}(\eta)\cap\mathcal{R}_\mu(j_k,\widetilde{\eta},H_\ell(\widetilde{\eta})\pm\widetilde{\beta}_{j_k})\cap\mathcal{T}_{\mu,\ell}(j_k,\widetilde{\eta},\varepsilon^3_{j_k})$ such that for each $k\geqslant 1$, $x\in B\big(x_{w_k},(2\cdot 2^{-\lfloor\widetilde{\eta}j_k\rfloor})^\delta\big)$. In other words, w_k satisfies

$$\begin{cases} 2^{-\lfloor\widetilde{\eta}j_k\rfloor(H_\ell(\widetilde{\eta})+\widetilde{\beta}_{\lfloor\widetilde{\eta}j_k\rfloor})}\leqslant\mu(I_{w_{k|\lfloor\widetilde{\eta}j_k\rfloor}})\leqslant 2^{-\lfloor\widetilde{\eta}j_k\rfloor(H_\ell(\widetilde{\eta})-\widetilde{\beta}_{\lfloor\widetilde{\eta}j_k\rfloor})}\\ 2^{-(j_k-\lfloor\widetilde{\eta}j_k\rfloor)(H_\ell(\widetilde{\eta})+\varepsilon^3_{j_k})}\leqslant\mu(I_{\sigma^{\lfloor\widetilde{\eta}j_k\rfloor}w_k})\leqslant 2^{-(j_k-\lfloor\widetilde{\eta}j_k\rfloor)(H_\ell(\widetilde{\eta})-\varepsilon^3_{j_k})}.\end{cases}$$

Consider for each $k\geqslant 1$ the largest integer J_k such that $2^{-J_k}\geqslant(2\cdot 2^{-\lfloor\widetilde{\eta}j_k\rfloor})^\delta$. With such a choice, one has $I_{w_k}\subset\mathcal{N}_{J_k}(x)$, so that $\mathsf{M}_{J_k}(x)\geqslant\mu(I_{w_k})$. Since $J_k=\delta\widetilde{\eta}j_k+o(1/k)$, one concludes that

$$\begin{aligned}\mathsf{M}_{J_k}(x)\geqslant\mu(I_{w_k})&\geqslant C^{-1}2^{-\lfloor\widetilde{\eta}j_k\rfloor(H_\ell(\widetilde{\eta})+\widetilde{\beta}_{\lfloor\widetilde{\eta}j_k\rfloor})}2^{-(j_k-\lfloor\widetilde{\eta}j_k\rfloor)(H_\ell(\widetilde{\eta})+\varepsilon^3_{j_k})}\\ &\geqslant 2^{-\frac{J_k}{\delta}(H_\ell(\widetilde{\eta})+\widehat{\beta}_k)},\end{aligned}$$

for some sequence $\widehat{\beta}_k$ converging to 0 as $k\to+\infty$. Taking the liminf as $k\to+\infty$ on both sides yields $\underline{\dim}(\mathsf{M}_\mu,x)\leqslant H$. □

From the previous lemma, we deduce that

$$\nu_{\alpha,\delta}\Big(E^{\leqslant}_{\mathsf{M}_\mu}(H)\Big)\geqslant\nu_{\alpha,\delta}\,(U_\mu\big(H_\ell(\widetilde{\eta}),\delta,\mathcal{F}_{\widetilde{\eta}},\widetilde{\beta}\big)>0.$$

Moreover, $\dim E^{\leqslant}_{\mathsf{M}_\mu}(H')\leqslant\tau^*_{\mathsf{M}_\mu}(H')\leqslant\widetilde{\tau}^*_\mu(H')<\widetilde{\tau}^*_\mu(H)$, for any $H'<H$. Consequently, Theorem 6 implies that $\nu_{\alpha,\delta}\big(E^{\leqslant}_{\mathsf{M}_\mu}(H')\big)=0$. We deduce that

$$\nu_{\alpha,\delta}\left(E^{\leqslant}_{\mathsf{M}_\mu}(H)\Big\backslash\bigcup_{n\geqslant 1}\underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H-1/n)\right)=1.$$

Since $\underline{E}_{\mathsf{M}_\mu}(H)=E^{\leqslant}_{\mathsf{M}_\mu}(H)\setminus\bigcup_{n\geqslant 1}\underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H-1/n)$, we get $\nu_{\alpha,\delta}\big(\underline{E}_{\mathsf{M}_\mu}(H)\big)=1$, i.e., $D_{\mathsf{M}_\mu}(H)\geqslant\widetilde{\tau}^*(H)$. Since we already proved the converse inequality, the equality holds.

7.4. The left part of the spectrum D_{M_μ}

Let $H\in\big[H_\ell(\eta_\ell),H_\ell(\widetilde{\eta})\big)$. Recall that $\eta_H\in[\eta_\ell,\widetilde{\eta})$ defined in (37) satisfies $H=H_\ell(\eta_H)$.

Let $(H^{(p)})_{p\geqslant 1}$ be a decreasing sequence of real numbers in the interval $\big(H_\ell(\eta_\ell),H_\ell(\widetilde{\eta})\big)$ converging to H, with the constraint that $\eta_{H^{(p)}}\in\mathcal{D}_\ell$. For each $p\geqslant 1$, consider any sequence $(\delta^{(p)})_{p\geqslant 1}$ converging to $1/\eta_H$ as $p\to+\infty$, and such that the sequence of real numbers $\big(\frac{D_\mu(H^{(p)})}{\delta^{(p)}}\big)_{p\geqslant 1}$ is non-increasing.

We apply the second part of Theorem 6: there exists a collection of positive sequences $\big(\widetilde{\beta}^{(p)}:=(\widetilde{\beta}^{(p)}_n)_{n\geqslant 1}\big)_{p\geqslant 1}$ converging to 0, such that the set $\bigcap_{p\geqslant 1}U_\mu\big(H^{(p)},\delta^{(p)},\mathcal{F}_{\eta_{H^{(p)}}},\widetilde{\beta}^{(p)}\big)$ supports a measure $\widetilde{\nu}_H$, whose dimension is greater than or equal to

$$\inf_{p\geqslant 1}\frac{D_\mu(H^{(p)})}{\delta^{(p)}}=\eta_H D_\mu(H)=\widetilde{\tau}^*(H),$$

where Lemma 5 and the definition of η_H are used to get the last equality.

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Also, similarly to what was done in § 7.3, one gets

$$\bigcap_{p\geqslant 1} U_\mu\big(H^{(p)}, \delta^{(p)}, \mathcal{F}_{\eta_{H^{(p)}}}, \widetilde{\beta}^{(p)}\big) \subset \underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H)$$

and $\widetilde{\nu}_H\left(\underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H-1/n)\right) = 0$ for all $n \geqslant 1$ if $\widetilde{\tau}^*(H) > 0$. This yields

$$\widetilde{\nu}_H\left(E^{\leqslant}_{\mathsf{M}_\mu}(H) \setminus \bigcup_{n\geqslant 1} \underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H-1/n)\right) = 1,$$

and finally $D_{\mathsf{M}_\mu}(H) = \widetilde{\tau}^*(H)$, when $\widetilde{\tau}^*(H) > 0$. If $\widetilde{\tau}^*(H) = 0$, one has $H = H_\ell(\eta_\ell)$ and $\eta_\ell = 0$, and Lemma 6 directly yields $\bigcup_{n\geqslant 1} \underline{E}^{\leqslant}_{\mathsf{M}_\mu}(H-1/n) = \emptyset$, so the desired conclusion holds as well.

7.5. Proof of Part (1) of Theorem 5

If $\alpha \in (H_{\min}, H_{\max})$, taking ν_α as the Gibbs measure μ_α of item (4) of Proposition 2, it is not too difficult to prove the desired property by using natural coverings because the Hausdorff dimension of μ_α is positive.

We give a construction of a measure ν_α that works for $\alpha \in \{H_{\min}, H_{\max}\}$, based on a concatenation method. It is also possible to adapt this method to get another choice for ν_α when $\alpha \in (H_{\min}, H_{\max})$ (as explained at the end of the proof).

Due to Lemma 4, one can fix a positive sequence $(\varepsilon_j)_{j\geqslant 1}$ converging to 0, such that, with probability 1, for j large enough, for all $W \in \Sigma_{\lfloor \eta j \rfloor}$,

$$\#\mathcal{S}_j(\eta, W) \leqslant 2^{\eta j \varepsilon_j}. \tag{40}$$

Without loss of generality, one assumes that $(\varepsilon_j)_{j\geqslant 1}$ is non-increasing, $1/j \leqslant \varepsilon_j \leqslant d$ for all $j \geqslant 1$, and $\varepsilon_{j+1}/\varepsilon_j$ converges to 1 as $j \to +\infty$.

We treat the case $H_{\min}$, the case $H_{\max}$ is identical.

7.5.1. Construction of the measure $\nu_{H_{\min}}$ and an associated Cantor set $\mathcal{C}_{H_{\min}}$.

Let $(q_k)_{k\geqslant 1}$ be an increasing sequence of real numbers, and let $\alpha_k := \tau'_\mu(q_k)$.

- If $D_\mu(H_{\min}) = 0$, we choose q_k such that $D_\mu(\alpha_k) = \sqrt{\varepsilon_k}$, for every $k \geqslant 1$. Hence $(q_k)_{k\geqslant 1}$ is such that $\lim_{k\to+\infty} \alpha_k = H_{\min}$ and $\lim_{k\to\infty} D_\mu(\alpha_k) = 0$.
- If $D_\mu(H_{\min}) > 0$, we choose q_k such that $\lim_{k\to+\infty} \alpha_k = H_{\min}$. Also, $\lim_{k\to+\infty} D_\mu(\alpha_k) = D_\mu(H_{\min})$.

In all cases, by construction one has

$$|D_\mu(\alpha_{k+1}) - D_\mu(\alpha_k)| = \theta_k \sqrt{\varepsilon_k} \leqslant \theta_k D_\mu(\alpha_k), \tag{41}$$

with $\lim_{k\to+\infty} \theta_k = 0$.

Start by selecting as follows some intervals at which μ and μ_{α_k} have the desired scaling properties. Recall that by item (4) of Proposition 2, the measure Gibbs μ_{α_k} satisfies

$$\mu_{\alpha_k}\big(\widetilde{E}_\mu(\alpha_k)\big) = \mu_{\alpha_k}\big(\widetilde{E}\mu_{\alpha_k}\big(D_\mu(\alpha_k)\big)\big) = 1.$$

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Hence, for all $k \geqslant 1$, the sets

$$\mathcal{A}_J^k = \left\{W \in \Sigma_J : \forall\, W' \in \mathcal{N}(W),\, W' \in \mathcal{E}_\mu\big(J, \alpha_k \pm \varepsilon_k\big)\right\}$$
$$\text{and} \quad \mathcal{B}_J^k = \left\{W \in \Sigma_J : \forall\, W' \in \mathcal{N}(W),\, W' \in \mathcal{E}_{\mu_{\alpha_k}}\big(J, D_\mu(\alpha_k) \pm \varepsilon_k\big)\right\}$$

satisfy $\lim_{J\to+\infty} \mu_{\alpha_k}(\mathcal{A}_J^k) = \lim_{J\to+\infty} \mu_{\alpha_k}(\mathcal{B}_J^k) = 1$. Up to extraction of a subsequence, one deduces that there exists an integer $J_k \in \mathbb{N}_+$ and a collection $\mathcal{W}_k$ of words of generation J_k such that the cubes I_W, $W \in \mathcal{W}_k$, are pairwise disjoint, $\sum_{W\in\mathcal{W}_k} \mu_{\alpha_k}(I_W) \geqslant e^{-\varepsilon_k}$, and

$$\forall\, W \in \mathcal{W}_k, \forall\, W' \in \mathcal{N}(W),\, W' \in \mathcal{E}_\mu(J, \alpha_k \pm \varepsilon_k) \cap \mathcal{E}_{\mu_{\alpha_k}}(J, D_\mu(\alpha_k) \pm \varepsilon_k). \tag{42}$$

Now, let $(N_k)_{k\geqslant1}$ be an increasing sequence of integers such that for all $k \geqslant 1$,

$$\sum_{p=1}^{k-1} N_p J_p \max\big(1, \alpha_p + 2\varepsilon_p, D_\mu(\alpha_p) + 2\varepsilon_p\big) \leqslant \varepsilon_k N_k J_k, \tag{43}$$
$$\frac{J_{k+1}}{N_k J_k} \max\big(1, \alpha_{k+1} + 2\varepsilon_{k+1}, D_\mu(\alpha_{k+1}) + 2\varepsilon_{k+1}\big) \leqslant \varepsilon_{k+1}\alpha_k.$$

Let us also introduce the integer $\widetilde{J}_k = \sum_{p=1}^k N_p J_p$, which satisfies

$$N_k J_k \leqslant \widetilde{J}_k \leqslant N_k J_k (1+\varepsilon_k).$$

Then we define recursively a Cantor-like set $\mathcal{C}_{H_{\min}}$ and simultaneously a Borel probability measure $\nu_{H_{\min}}$ on $[0,1]^d$ supported on $\mathcal{C}_{H_{\min}}$. To do so, we use a construction by concatenation: the measure $\nu_{H_{\min}}$ behaves like μ_{α_k} between the generations $\widetilde{J}_{k-1}+1$ and $\widetilde{J}_k$. More precisely:

- Set $I_\emptyset = [0,1]^d$ and $\nu_{H_{\min}}([0,1]^d) = 1$.
- For every $k \geqslant 1$, write $\widetilde{W}_k \in \mathcal{W}_k^{N_k}$ as $\widetilde{W}_k = W_{k,1} \cdots W_{k,N_k}$ where $W_{k,i} \in \mathcal{W}_k \subset \Sigma_{J_k}$.
- The Cantor set is

$$\mathcal{C}_{H_{\min}} = \bigcap_{k\geqslant1} \bigcup_{(\widetilde{W}_1,\dots,\widetilde{W}_k)\in\mathcal{W}_1^{N_1}\times\cdots\times\mathcal{W}_k^{N_k}} I_{\widetilde{W}_1\cdots\widetilde{W}_k}.$$

- The measure $\nu_{H_{\min}}$ is defined recursively as follows: for every $k \geqslant 1$, for every $(\widetilde{W}_1,\dots,\widetilde{W}_k) \in \mathcal{W}_1^{N_1}\times\cdots\times\mathcal{W}_k^{N_k}$, set for every $i \in \{1,\dots,N_k\}$

$$\nu_{H_{\min}}\Big(I_{\widetilde{W}_1\cdots\widetilde{W}_{k-1}W_{k,1}\cdots W_{k,i-1}W_{k,i}}\Big) = \nu_{H_{\min}}\Big(I_{\widetilde{W}_1\cdots\widetilde{W}_{k-1}W_{k,1}\cdots W_{k,i-1}}\Big)\cdot\frac{\mu_{\alpha_k}(I_{W_{k,i}})}{\sum_{W'_k\in\mathcal{W}_k}\mu_{\alpha_k}(I_{W'_k})}.$$

It is clear that this measure $\nu_{H_{\min}}$, defined only on the cubes appearing in Cantor's construction, uniquely extends to a Borel probability measure on the cube $[0,1]^d$.

7.5.2. Properties of the measure $\nu_{H_{\min}}$. We first prove that the Cantor set contains only $x \in [0,1]^d$ satisfying simultaneously $\underline{\dim}(\mu, x) = H_{\min}$ and $\underline{\dim}(\nu_{H_{\min}}, x) = D_\mu(H_{\min})$.

Lemma 10. *One has* $\mathcal{C}_{H_{\min}} \subset \widetilde{E}_\mu(H_{\min}) \cap \widetilde{E}_{\nu_{H_{\min}}}(D_\mu(H_{\min}))$.

Proof. If $k \geqslant 2$ and $\widetilde{J}_k < J \leqslant \widetilde{J}_{k+1}$, set $k_J = k$.

Fix $x \in \mathcal{C}_{H_{\min}}$. Using the quasi-Bernoulli property (8) of μ and the inequalities (42), one gets that for every $k \geqslant 2$ and $\widetilde{J}_k < J \leqslant \widetilde{J}_{k+1}$, and for every cube $W \in \mathcal{N}_J(x)$,

$$2^{-\left(N_1 J_1(\alpha_1+\varepsilon_1)+\cdots+N_k J_k(\alpha_k+\varepsilon_k)+(J-\widetilde{J}_k)(\alpha_{k+1}+\varepsilon_{k+1})\right)} \\ \leqslant \mu(I_W) \leqslant 2^{-\left(N_1 J_1(\alpha_1-\varepsilon_1)+\cdots+N_k J_k(\alpha_k-\varepsilon_k)+(J-\widetilde{J}_k)(\alpha_{k+1}-\varepsilon_{k+1})\right)}.$$

The equations (43) give

$$2^{-\left((\alpha_k+2\varepsilon_k)N_k J_k+(J-\widetilde{J}_k)(\alpha_{k+1}+\varepsilon_{k+1})\right)} \leqslant \mu(I_W) \leqslant 2^{-\left((\alpha_k-2\varepsilon_k)N_k J_k+(J-\widetilde{J}_k)(\alpha_{k+1}-\varepsilon_{k+1})\right)},$$

which yields

$$2^{-\left((\alpha_k+2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(\alpha_{k+1}+\varepsilon_{k+1})\right)} \leqslant \mu(I_W) \leqslant 2^{-\left((\alpha_k-2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(\alpha_{k+1}-\varepsilon_{k+1})\right)}.$$

Since $\alpha_k \to H_{\min}$ when $k \to +\infty$, one deduces that $\lim_{J\to+\infty} \frac{\log_2 \mu(I_W)}{-J} = H_{\min}$, where $W \in \mathcal{N}_J(x)$. This proves that $x \in \widetilde{E}_\mu(H_{\min})$.

Similarly, the same arguments show that for every $k \geqslant 2$ and $\widetilde{J}_k < J \leqslant \widetilde{J}_{k+1}$, and for every cube $W \in \mathcal{N}_J(x)$,

$$2^{-\left((D_\mu(\alpha_k)+2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(D_\mu(\alpha_{k+1})+\varepsilon_{k+1})\right)} \\ \leqslant \nu_{H_{\min}}(I_W) \leqslant 2^{-\left((D_\mu(\alpha_k)-2\varepsilon_k)(1+\varepsilon_k)\widetilde{J}_k+(J-\widetilde{J}_k)(D_\mu(\alpha_{k+1})-\varepsilon_{k+1})\right)}.$$

The equation (41) then gives

$$2^{-JD_\mu(\alpha_k)(1+\widetilde{\theta}_k)} \leqslant \nu_{H_{\min}}(I_W) \leqslant 2^{-JD_\mu(\alpha_k)(1-\widetilde{\theta}_k)}, \tag{44}$$

for some decreasing sequence $\widetilde{\theta}_k$, tending to 0 when $k \to +\infty$.

This yields that $x \in \widetilde{E}_{\nu_{H_{\min}}}(D_\mu(H_{\min}))$, since $D_\mu(\alpha_k) \to D_\mu(H_{\min})$ when $k \to +\infty$. □

Observe also that (44) implies that for each j large enough,

$$\#\{W \in \Sigma_j : I_W \cap \mathcal{C}_{H_{\min}} \neq \emptyset\} \leqslant 2^{jD_\mu(\alpha_{k_j})(1+\widetilde{\theta}_{k_j})}. \tag{45}$$

7.5.3. Proof that well-approximated points have $\nu_{H_{\min}}$-measure 0. Fix an approximation rate $\delta > 1$. To get the result, one focuses first on the value of

$$\nu_{H_{\min}}\left(\bigcup_{w\in\mathcal{S}_j(\eta)} B(x_w, (2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\right) = \bigcup_{W\in\Sigma_{\lfloor \eta j\rfloor}} \bigcup_{w\in\mathcal{S}_j(\eta,W)} B(x_w, (2\cdot 2^{-\lfloor \eta j\rfloor})^\delta).$$

For each j large enough, consider $W \in \Sigma_{\lfloor j\eta\rfloor}$ such that $I_W \cap \mathcal{C}_{H_{\min}} \neq \emptyset$. One looks for points $x \in I_W \cap \mathcal{C}_{H_{\min}}$ such that $x \in B(x_w, (2\cdot 2^{-\lfloor \eta' j\rfloor})^\delta)$ for some surviving word $w \in \mathcal{S}_j(\eta, W)$. Hence, one sees that

$$\nu_{H_{\min}}\left(\bigcup_{W\in\Sigma_{\lfloor \eta j\rfloor}} \bigcup_{w\in\mathcal{S}_j(\eta,W)} B(x_w, (2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\right) \\ \leqslant \sum_{W\in\Sigma_{\lfloor \eta j\rfloor}: I_W\cap\mathcal{C}_{H_{\min}}\neq\emptyset} \ \sum_{\substack{w\in\mathcal{S}_j(\eta,W),\\ I_w\cap\mathcal{C}_{H_{\min}}\neq\emptyset}} \nu_{H_{\min}}\left(B(x_w, (2\cdot 2^{-\lfloor \eta j\rfloor})^\delta)\right).$$

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Recall that by (40), the number of such possible surviving vertices w (in the second sum above) is bounded from above by $2^{\eta j \varepsilon_j}$. Applying (45) to $B(x_w, (2 \cdot 2^{-\lfloor \eta j \rfloor})^\delta)$ for the generation $J = \lfloor \eta j \delta \rfloor$, we get

$$\begin{aligned}
&\nu_{H_{\min}}\left(\bigcup_{W \in \Sigma_{\lfloor \eta j \rfloor}} \bigcup_{w \in \mathcal{S}_j(\eta, W)} B(x_w, (2 \cdot 2^{-\lfloor \eta j \rfloor})^\delta)\right) \\
&\leqslant \left(\#\{W \in \Sigma_{\lfloor \eta j \rfloor} : I_W \cap \mathcal{C}_{H_{\min}} \neq \emptyset\}\right) \cdot 2^{\eta j \varepsilon_j} \cdot 2^{-\lfloor \eta j \delta \rfloor D_\mu(\alpha_{k_{\lfloor \eta j \delta \rfloor}})(1-\widetilde{\theta}_{k_{\lfloor \eta j \delta \rfloor}})} \\
&\leqslant 2^{\eta j \varepsilon_j + \lfloor \eta j \rfloor D_\mu(\alpha_{k_{\lfloor \eta j \rfloor}})(1+\widetilde{\theta}_{k_{\lfloor \eta j \rfloor}}) - \lfloor \eta j \delta \rfloor (D_\mu(\alpha_{k_{\lfloor \eta j \delta \rfloor}})(1-\widetilde{\theta}_{k_{\lfloor \eta j \delta \rfloor}})}.
\end{aligned}$$

It follows now from the properties imposed to the sequences $(\varepsilon_k)_{k \geqslant 1}$ and $(\alpha_k)_{k \geqslant 1}$ that

$$\xi_j := \nu_{H_{\min}}\left(\bigcup_{w \in \mathcal{S}_j(\eta)} B(x_w, (2 \cdot 2^{-\lfloor \eta j \rfloor})^\delta)\right) \leqslant C' 2^{\eta j (1-\delta) D_\mu(\alpha_{k_{\lfloor \eta j \rfloor}})(1+o(1))},$$

where C is another constant coming from the fact that we dropped some integer parts.

- When $D_\mu(H_{\min}) > 0$, it is direct that the series $\sum_j \xi_j$ converges.
- When $D_\mu(H_{\min}) = 0$, for large values of j one has by construction $j > k_{\lfloor \eta j \rfloor}$, so $j^{-1} \leqslant \varepsilon_j \leqslant \varepsilon_{k_{\lfloor \eta j \rfloor}}$, and $D_\mu(\alpha_{k_{\lfloor \eta j \rfloor}}) = \sqrt{\varepsilon_{k_{\lfloor \eta j \rfloor}}}$. Thus, for j large enough we get
$$2^{\eta j (1-\delta) D_\mu(\alpha_{k_{\lfloor \eta j \rfloor}})(1+o(1))} \leqslant 2^{-\sqrt{j}\eta(1-\delta)(1+o(1))},$$
hence the series $\sum_j \xi_j$ still converges.

Finally, the Borel–Cantelli lemma proves Part (1) of Theorem 5.

Observe that everything works similarly if we replace $H_{\min}$ with $H_{\max}$ and change the sequence α_k accordingly. When $\alpha \in (H_{\min}, H_{\max})$, we can even take the sequence $(\alpha_k)_{k \geqslant 1}$ to be constant (if not, this process gives other measures sitting on $\widetilde{E}_\mu(\alpha)$).

7.6. Proof of Part (2) of Theorem 5

Recall Proposition 6. Applying the Borel–Cantelli lemma, it is enough to prove that for all integers $N \geqslant 1$ and $p > 2(H_{\max} - H_{\min})^{-1}$,

$$\mathbb{E}\left(\sup_{\alpha \in \{H_{\min}, H_{\max}\} \cup I_p} \sum_{J \geqslant 1} \sum_{W \in \Sigma_J} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)}\right) < +\infty. \tag{46}$$

where $I_p = [H_{\min} + 1/p, H_{\max} - 1/p]$.

At first, notice that for any Borel probability measure ν on $[0, 1]$, one has

$$\begin{aligned}
\mathbb{E}\left(\sum_{J \geqslant 1} \sum_{W \in \Sigma_J} \nu(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)}\right) &= \sum_{J \geqslant 1} \sum_{W \in \Sigma_J} \nu(I_W) \mathbb{P}(\mathscr{C}(N, J, W)) \\
&\leqslant \sum_{J \geqslant 1} 2^{-J\varepsilon_N} \sum_{W \in \Sigma_J} \nu(I_W) < \infty.
\end{aligned}$$

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Applying this to $\nu_{H_{\min}}$ and $\nu_{H_{\max}}$ constructed above, it remains us to prove (46) only with the interval I_p.

Recall that when $\alpha \in I_p \subset (H_{\min}, H_{\max})$, one can take $\nu_\alpha = \mu_\alpha$ (where μ_α is the Gibbs measure of Proposition 2).

Let us write the interval I_p as $I_p = \tau'_\mu([q'_p, q_p])$, for some real numbers $q_p > q'_p$. Recall that the Gibbs capacity μ is associated with a Hölder potential ϕ which belongs to the C^β Hölder class, for some $\beta > 0$. Standard arguments based on the bounded distorsion property give that for $\kappa = 2\|\phi\|_\infty/\log(2)$ and $C_{q,q'} = e^{\frac{(|q|+|q'|)C}{(1-2^{-\beta})}}$, for all $q, q' \in [q'_p, q_p]$ and $W \in \Sigma^*$, setting $\alpha_q = \tau'_\mu(q)$, we have

$$\mu_{\alpha_{q'}}(I_W) \leqslant C_{q,q'} 2^{\kappa|q-q'|\cdot|W|} \mu_{\alpha_q}(I_W).$$

The interval I_p being compact, one can extract a finite collection of intervals $[\alpha_{\widetilde{q}_{k-1}}, \alpha_{\widetilde{q}_k}]$, $1 \leqslant k \leqslant K$, such that $q_p = \widetilde{q}_0 > \cdots > \widetilde{q}_K = q'_p$ and $|\widetilde{q}_k - \widetilde{q}_{k-1}| \leqslant \epsilon_N/(2\kappa)$.

Setting $C_k = \sup_{q' \in [\widetilde{q}_k, \widetilde{q}_{k-1}]} C_{q', \widetilde{q}_k}$, one rewrites the above properties as follows: for all $W \in \Sigma^*$, for all $1 \leqslant k \leqslant K$,

$$\sup_{q' \in [\widetilde{q}_k, \widetilde{q}_{k-1}]} \mu_{\alpha_{q'}}(I_W) \leqslant C_k 2^{|W|\varepsilon_N/2} \mu_{\alpha_{q_k}}(I_W).$$

From these considerations, we get for $1 \leqslant k \leqslant K$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{\alpha \in [\alpha_{\widetilde{q}_{k-1}}, \alpha_{\widetilde{q}_k}]} \sum_{J \geqslant 1} \sum_{W \in \Sigma_J} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)}\right) \\
&\quad \leqslant \mathbb{E}\left(\sum_{J \geqslant 1} \sum_{W \in \Sigma_J} \sup_{\alpha \in [\alpha_{\widetilde{q}_{k-1}}, \alpha_{\widetilde{q}_k}]} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)}\right) \\
&\quad \leqslant C_k \sum_{J \geqslant 1} \sum_{W \in \Sigma_J} 2^{J\varepsilon_N/2} \nu_{\alpha_{q_k}}(I_W) \mathbb{P}(\mathscr{C}(N, J, W)) \\
&\quad \leqslant C_k \sum_{J \geqslant 1} 2^{-J\varepsilon_N/2} < +\infty.
\end{aligned}$$

It follows that

$$\mathbb{E}\left(\sup_{\alpha \in I_p}\left(\sum_{J \geqslant 1} \sum_{W \in \Sigma_J} \nu_\alpha(I_W) \mathbf{1}_{\mathscr{C}(N,J,W)}\right)\right) \leqslant \sum_{k=1}^{K} C_k \sum_{J \geqslant 1} 2^{-J\varepsilon_N/2} < +\infty,$$

i.e., (46) holds.

8. Case of a homogeneous Gibbs measure

Let us rapidly deal with the case of a homogeneous capacity, denoted by λ (see Figure 14 for the graphs of τ_λ, D_λ, $\tau_{\mathsf{M}_\lambda}$ and D_{M_λ}). We assume without loss of generality that for some $\beta > 0$, for every finite word $w \in \Sigma^*$, $\lambda(I_w) \sim 2^{-\beta|w|}$, in the sense that the ratio of the two quantities is lower and upper bounded by fixed positive constants.

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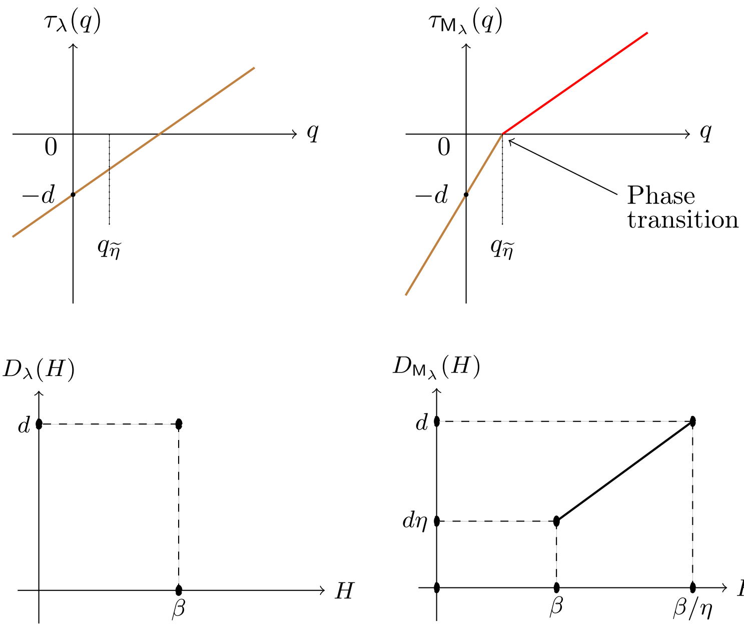

Figure 14. Left: Free energy function (top) and associated multifractal spectrum (bottom) of a homogeneous capacity λ. *Right:* The almost sure free energy (top) and the multifractal spectrum (bottom) of M_λ.

In this situation, $H_{\min} = H_s = H_{\max} = \beta$, so $\eta_\ell = \eta_r = \widetilde{\eta} = \eta$, and $H_\ell(\eta_\ell) = H_r(\eta_r) = \beta$. One also has $\widetilde{H}_\ell(\widetilde{\eta}) = \beta(1/\eta - 1)$. Moreover, $q_{\eta_\ell} = +\infty$.

The free energy function $\tau_\lambda(q) = \beta q - d$ is linear, and $q_{\widetilde{\eta}}$ is the solution to $\tau_\lambda(q) = -d(1-\eta)$, i.e., $q_{\widetilde{\eta}} = d\eta/\beta$.

The proof follows exactly the same lines as in the previous sections, except that most of the arguments are trivial. Indeed, all the survivors at a given generation j satisfy $\lambda(I_w) \sim 2^{-j\beta}$ (there is no dependence of the value $\lambda(I_w)$ on the location of w). The sets $\mathcal{R}_\lambda$, $\mathcal{T}_\lambda$ are similarly defined, but are also trivial.

The obtained energy function is

$$\tau_{\mathsf{M}_\lambda}(q) = \begin{cases} \tau_\lambda(q) + \beta(1/\eta - 1)q = q\beta/\eta - d & \text{if } q \leqslant d\eta/\beta, \\ \tau_\lambda(q) + d(1-\eta) = q\beta - d(1-\eta) & \text{if } q > d\eta/\beta, \end{cases}$$

and the associated multifractal spectrum is

$$D_{\mathsf{M}_\lambda}(H) = \begin{cases} \dfrac{d\eta}{\beta} H & \text{if } H \in [\beta, \beta/\eta], \\ -\infty & \text{otherwise.} \end{cases}$$

Actually, this question has already been studied by Jaffard in the context of 'lacunary wavelet series' and multifractal analysis of functions [24]. More precisely, Jaffard computes

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the singularity spectrum of wavelet series whose wavelet coefficients are defined as follows: Fix $(\psi_{j,k})_{j,k\in\mathbb{Z}}$, a wavelet basis of $L^2(\mathbb{R})$ associated with a smooth mother wavelet and normalized so that all its elements have the same L^∞ norm. Fix $\beta > 0$, and for each $j \geqslant 1$ select uniformly and independently $2^{\lfloor \eta j \rfloor}$ intervals among the 2^j dyadic subintervals of $[0, 1]$ of generation j. Then assign the coefficient $2^{-\beta j}$ to $\psi_{j,k}$ if $[k2^{-j}, (k+1)2^{-j}]$ has been selected; otherwise assign the coefficient 0. Though different, this sparse collection of coefficients is close to that obtained by sampling the homogeneous capacity λ as above in the special situation where $\lambda(I_w) = 2^{-\beta|w|}$ for all $w \in \Sigma^*$. It turns out that the multifractal analysis of the resulting sparse wavelet series is essentially reducible to that of M_λ, which in this case follows from quite a direct application of homogeneous ubiquity theory [14, 24]. Of course, Jaffard obtained the same multifractal spectrum, although he did not compute the free energy function.

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