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New additive results for the *g*-Drazin inverse

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This paper studies additive properties of the generalized Drazin inverse (g-Drazin inverse) in a Banach algebra and finds an explicit expression for the g-Drazin inverse of the sum $a + b$ in terms of a and b and their g-Drazin inverses under fairly mild conditions on a and b.

1. Introduction

The ordinary inverse in rings or algebras does not behave well with respect to addition. If a, b are invertible elements of a unital ring, we cannot expect that the sum $a + b$ is invertible, and even less that $(a + b)^{-1} = a^{-1} + b^{-1}$ (under some restrictions on a, b). Thus it comes as a surprise that the Drazin inverse (or its generalized cousin) is quite amenable under addition. In his original paper [4], Drazin showed that $(a + b)^{\mathrm{D}} = a^{\mathrm{D}} + b^{\mathrm{D}}$ if the elements a, b of a ring are Drazin invertible and $ab = ba = 0$ (here, x^{D} is the Drazin inverse of x).

Admittedly, the above conditions on a, b are fairly restrictive, but Hartwig *et al*. [5] extended Drazin's result (this time for matrices) to the situation when only $ab = 0$, but with a more complicated formula for $(a + b)^{\mathrm{D}}$. Djordjević and Wei showed that this result is preserved when passing from matrices to bounded linear operators on Banach spaces. The present paper is motivated by results of Castro [1] for matrices in which much weaker conditions on a and b are used. We extend these results to the g-Drazin inverse for elements of a Banach algebra. Our main aim is to express $(a + b)^{\mathrm{D}}$ as a function of the elements a and b and their g-Drazin inverses. This investigation is facilitated by the use of matrices with elements in a Banach algebra developed in the next section.

Let $\mathcal{A}$ be a unital Banach algebra with unit 1. We use the following notation.

$\rho(a)$	the resolvent set of $a \in \mathcal{A}$
$\sigma(a)$	the spectrum of a
$R(\lambda; a)$	the resolvent of a: $R(\lambda; a) := (\lambda 1 - a)^{-1}$
a^{π}	the spectral idempotent of a at zero (definition 1.1)

a^{D}	the g-Drazin inverse of a (definition 1.1)
$\mathcal{A}^{-1}$	the group of all invertible elements of $\mathcal{A}$
$\mathcal{A}^{\mathrm{nil}}$	the set of all nilpotent elements of $\mathcal{A}$
$\mathcal{A}^{\mathrm{qnil}}$	the set of all quasi-nilpotent elements $a \in \mathcal{A}$; $\sigma(a) = \{0\}$
$\mathcal{A}^{\mathrm{D}}$	the set of all quasi-polar elements $a \in \mathcal{A}$; $\sigma(a) \setminus \{0\}$ is compact
S_n	$S_n(u, v) = \sum_{k=0}^{n} u^k v^{n-k}$, where $u, v \in \mathcal{A}$, $n \in \mathbb{Z}_+$

Definition 1.1. An element $p \in \mathcal{A}$ is a *spectral idempotent* of a if

$$p^2 = p, \qquad ap = pa \in \mathcal{A}^{\mathrm{qnil}}, \qquad a + p \in \mathcal{A}^{-1}. \tag{1.1}$$

Such an element is unique if it exists (see [6]), and will be denoted by $p = a^\pi$. If a^π exists, the *g-Drazin inverse* (or *generalized Drazin inverse*) of an element $a \in \mathcal{A}$ is defined by

$$a^{\mathrm{D}} = (a + a^\pi)^{-1}(1 - a^\pi) = (1 - a^\pi)(a + a^\pi)^{-1}. \tag{1.2}$$

From the definition, we can deduce that the g-Drazin inverse of $a \in \mathcal{A}$ is characterized as the (unique) element $c \in \mathcal{A}$ satisfying

$$ac = ca, \qquad ac^2 = c, \qquad a(1 - ac) \in \mathcal{A}^{\mathrm{qnil}}.$$

(Drazin's original definition [4] is a special case of the g-Drazin inverse for which $a(1 - ac) \in \mathcal{A}^{\mathrm{nil}}$. We refer to it as the *Drazin inverse*.) It is known [6, theorem 3.1] that $a \in \mathcal{A}$ possesses a spectral idempotent if and only if a is *quasi-polar* (this means that $\sigma(a) \setminus \{0\}$ is compact). We note that $a \in \mathcal{A}^{-1}$ if and only if a is quasi-polar and $a^\pi = 0$. The resolvent of an arbitrary quasi-polar element $a \in \mathcal{A}$ has a Laurent expansion,

$$R(\lambda; a) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n a^\pi - \sum_{n=0}^{\infty} \lambda^n (a^{\mathrm{D}})^{n+1}, \quad 0 < |\lambda| < r, \tag{1.3}$$

valid for a sufficiently small $r > 0$ [6, theorem 5.1]. We observe that a^{D} is the constant term of this expansion.

2. Preliminary results

We say that $\mathcal{P} = (p_1, p_2, \dots, p_n)$ is a *total system of idempotents* in $\mathcal{A}$ if $p_i^2 = p_i$ for all $i \in \{1, \dots, n\}$, $p_i p_j = 0$ if $i \neq j$, and $\sum_{i=1}^{n} p_i = 1$. Given a total system $\mathcal{P}$ of idempotents in $\mathcal{A}$, we consider the set $\mathcal{M}_n(\mathcal{A}, \mathcal{P}) \subset \mathcal{M}_n(\mathcal{A})$ consisting of all matrices $A = [a_{ij}]$ with elements in $\mathcal{A}$ such that $a_{ij} \in p_i \mathcal{A} p_j$ for all $i, j \in \{1, \dots, n\}$. We observe that $\mathcal{M}_n(\mathcal{A}, \mathcal{P})$ is an algebra with unit $I(\mathcal{P}) = \mathrm{diag}(p_1, \dots, p_n)$ under the usual matrix operations; it becomes a unital Banach algebra with the norm

$$\|A\| = \Big\| \sum_{i,j=1}^{n} a_{ij} \Big\|, \qquad A = [a_{ij}] \in \mathcal{M}_n(\mathcal{A}, \mathcal{P}).$$

The subadditivity of $\|\cdot\|$ and $\|I(\mathcal{P})\| = 1$ are clear. To show that $\|\cdot\|$ is submultiplicative, consider matrices $A, B \in \mathcal{M}_n(\mathcal{A}, \mathcal{P})$. We have

$$\begin{aligned} \|AB\| &= \bigg\|\sum_{i,j=1}^{n}\bigg(\sum_{k=1}^{n} a_{ik}b_{kj}\bigg)\bigg\| \\ &= \bigg\|\bigg(\sum_{i,k=1}^{n} a_{ik}\bigg)\bigg(\sum_{m,j=1}^{n} b_{mj}\bigg)\bigg\| \\ &\leqslant \bigg\|\sum_{i,k=1}^{n} a_{ik}\bigg\|\bigg\|\sum_{m,j=1}^{n} b_{mj}\bigg\| \\ &= \|A\|\|B\| \end{aligned}$$

(as $a_{ik}b_{mj} = 0$ when $m \neq k$). The completeness of $\mathcal{M}_n(\mathcal{A}, \mathcal{P})$ will follow from lemma 2.1.

If $\mathcal{P}$ is a total system of idempotents, we define a mapping $\varphi : \mathcal{A} \to \mathcal{M}_n(\mathcal{A}, \mathcal{P})$ by

$$\varphi(x) = \begin{bmatrix} p_1xp_1 & p_1xp_2 & \cdots & p_1xp_n \\ p_2xp_1 & p_2xp_2 & \cdots & p_2xp_n \\ \cdots & \cdots & \cdots & \cdots \\ p_nxp_1 & p_nxp_2 & \cdots & p_nxp_n \end{bmatrix}. \tag{2.1}$$

Lemma 2.1. *The mapping φ is an isometric Banach algebra isomorphism from $\mathcal{A}$ onto $\mathcal{M}_n(\mathcal{A}, \mathcal{P})$.*

Proof. We observe that, for each $x \in \mathcal{A}$,

$$x = \sum_{i,j=1}^{n} p_ixp_j. \tag{2.2}$$

This proves that $\|x\| = \|\varphi(x)\|$. The linearity of φ is clear. Furthermore,

$$\varphi(1) = \operatorname{diag}(p_1, \ldots, p_n) = I(\mathcal{P}),$$

so that φ preserves unit. The property $\varphi(ab) = \varphi(a)\varphi(b)$ follows from

$$p_i(ab)p_j = p_ia\bigg(\sum_{k=1}^{n} p_k\bigg)bp_j = \sum_{k=1}^{n}(p_iap_k)(p_kbp_j). \tag{2.3}$$

The injectivity of φ follows from the norm preservation. Finally, to prove that φ is surjective, assume that $A = [a_{ij}] \in \mathcal{M}_n(\mathcal{A}, \mathcal{P})$ is given, and set $a = \sum_{i,j=1}^{n} a_{ij}$. Then $p_iap_j = a_{ij}$ for all i, j, and $\varphi(a) = A$. □

In view of this lemma, it is reasonable to identify $x \in \mathcal{A}$ with its image $\varphi(x)$, that is, to write

$$x = \begin{bmatrix} p_1xp_1 & p_1xp_2 & \cdots & p_1xp_n \\ p_2xp_1 & p_2xp_2 & \cdots & p_2xp_n \\ \cdots & \cdots & \cdots & \cdots \\ p_nxp_1 & p_nxp_2 & \cdots & p_nxp_n \end{bmatrix}; \tag{2.4}$$

this will afford us the convenience of working with matrix representations of elements of the Banach algebra for a suitable choice of a total system of idempotents. We have to remember that whenever an element $a_{ii} = p_i x p_i$ is considered in the context of the matrix (2.4), then a_{ii}^{-1} means the inverse in $\mathcal{A}_i = p_i \mathcal{A} p_i$, where $\mathcal{A}_i$ is an algebra with unit p_i; similarly, a_{ii}^{D} is the g-Drazin inverse of a_{ii} in $\mathcal{A}_i$.

Remark 2.2. In the case of $\mathcal{A} = L(X)$, the Banach algebra of all bounded linear operators on a Banach space X, we compare operator matrices relative to a topological direct sum $X = X_1 \oplus \cdots \oplus X_n$ of (closed) subspaces with the matrix representation φ. Every $A \in L(X)$ has an operator matrix representation

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix},$$

with each $A_{ij} : X_j \to X_i$ a bounded linear operator. Let $\mathcal{P} = (P_1, \dots, P_n)$ be the total system of idempotent operators in $L(X)$ associated with the direct sum $X = X_1 \oplus \cdots \oplus X_n$. Then $P_i A P_j$ is represented by an operator matrix with A_{ij} in the position (i, j) and with all other entries zero. For instance, for $n = 3$, instead of the operator A_{23} customary in operator matrices, we will work with the operator

$$P_2 A P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

which acts on X rather than on subspaces. For the diagonal operators, we have the equality of the spectra $\sigma_i(P_i A P_i) = \sigma(A_{ii})$, $i = 1, \dots, n$, where σ_i is taken relative to the algebra $P_i L(X) P_i$.

From the definition, it follows that $a \in \mathcal{A}$ is g-Drazin invertible if and only if there exists a total system of idempotents (p_1, p_2) for $\mathcal{A}$ such that

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},$$

where $a_1 \in \mathcal{A}_1^{-1}$ and $a_2 \in \mathcal{A}_2^{\mathrm{qnil}}$. Indeed, this total system is $(p_1, p_2) = (1 - a^{\pi}, a^{\pi})$, and the g-Drazin inverse of a is then given by

$$a^{\mathrm{D}} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the inverse a_1^{-1} is taken in $\mathcal{A}_1$, we may have $a_1^{-1} = 0$ if $p_1 = 0$. If $p_1 = 1$, $a \in \mathcal{A}^{-1}$.

Part (i) of the following result is well known for matrices [7]; it was generalized to bounded linear operators in [2, 3]. We refer to our convention of writing $\mathcal{A}_i$ for the algebra $p_i \mathcal{A} p_i$.

Theorem 2.3. *Let $x, y \in \mathcal{A}$, and let*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix} \tag{2.5}$$

relative to a total system of idempotents (p_1, p_2) for x, and (p_2, p_1) for y.

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(i) *If* $a \in \mathcal{A}_1^{\mathrm{D}}$ *and* $b \in \mathcal{A}_2^{\mathrm{D}}$*, then* x *and* y *are g-Drazin invertible, and*

$$x^{\mathrm{D}} = \begin{bmatrix} a^{\mathrm{D}} & 0 \\ u & b^{\mathrm{D}} \end{bmatrix}, \qquad y^{\mathrm{D}} = \begin{bmatrix} b^{\mathrm{D}} & u \\ 0 & a^{\mathrm{D}} \end{bmatrix}, \tag{2.6}$$

where

$$u = \sum_{n=0}^{\infty} (b^{\mathrm{D}})^{n+2} c a^n a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} b^n c (a^{\mathrm{D}})^{n+2} - b^{\mathrm{D}} c a^{\mathrm{D}}. \tag{2.7}$$

(ii) *If* $x \in \mathcal{A}^{\mathrm{D}}$ *and* $a \in \mathcal{A}_1^{\mathrm{D}}$*, then* $b \in \mathcal{A}_2^{\mathrm{D}}$*, and* x^{D} *and* y^{D} *are given by (2.6) and (2.7).*

Proof. We prove the result for x. We write $\sigma_i(s)$ and $\rho_i(s)$, respectively, for the spectrum and resolvent set of $s \in \mathcal{A}_i$ relative to the algebra $\mathcal{A}_i$, $i = 1, 2$. From

$$\lambda 1 - x = \begin{bmatrix} \lambda p_1 - a & 0 \\ -c & \lambda p_2 - b \end{bmatrix},$$

we deduce that

$$\lambda \in \rho_1(a) \cap \rho_2(b) \implies \lambda \in \rho(x) \quad \text{and} \quad \lambda \in \rho(x) \cap \rho_1(a) \implies \lambda \in \rho_2(b),$$

that is,

$$\sigma(x) \subset \sigma_1(a) \cup \sigma_2(b) \quad \text{and} \quad \sigma_2(b) \subset \sigma(x) \cup \sigma_1(a).$$

Recall that a, b, x are Drazin invertible in the corresponding algebras if and only if the non-zero spectra $\sigma_1(a) \setminus \{0\}$, $\sigma_2(b) \setminus \{0\}$, $\sigma(x)$ are compact. The preceding inclusions then imply that

$$a \in \mathcal{A}_1^{\mathrm{D}} \text{ and } b \in \mathcal{A}_2^{\mathrm{D}} \implies x \in \mathcal{A}^{\mathrm{D}}, \qquad x \in \mathcal{A}^{\mathrm{D}} \text{ and } a \in \mathcal{A}_1^{\mathrm{D}} \implies b \in \mathcal{A}_2^{\mathrm{D}}.$$

Then

$$R(\lambda; x) = \begin{bmatrix} \lambda p_1 - a & 0 \\ -c & \lambda p_2 - b \end{bmatrix}^{-1} = \begin{bmatrix} R(\lambda; a) & 0 \\ R(\lambda; b) c R(\lambda; a) & R(\lambda; b) \end{bmatrix},$$

with the resolvents in appropriate algebras. Comparing the constant terms of the Laurent expansions (1.3) on both sides, we get

$$x^{\mathrm{D}} = \begin{bmatrix} a^{\mathrm{D}} & 0 \\ u & b^{\mathrm{D}} \end{bmatrix}, \tag{2.8}$$

with

$$u = \sum_{n=0}^{\infty} (b^{\mathrm{D}})^{n+2} c a^n a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} b^n c (a^{\mathrm{D}})^{n+2} - b^{\mathrm{D}} c a^{\mathrm{D}}.$$

The proof for y is analogous. □

LEMMA 2.4. *Let* $b, q \in \mathcal{A}^{\mathrm{qnil}}$ *and let* $qb = 0$*. Then* $q + b \in \mathcal{A}^{\mathrm{qnil}}$*.*

Proof. This follows from the equation $(\lambda 1 - q)(\lambda 1 - b) = \lambda(\lambda 1 - (q + b))$. □

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3. Main results

As mentioned in the introduction, Hartwig *et al*. in [5] for matrices and Djordjević and Wei in [3] for operators used the condition $ab = 0$ to derive a formula for $(a+b)^{\mathrm{D}}$. Following [1], we substantially relax this hypothesis by assuming three conditions symmetric in $a, b \in \mathcal{A}^{\mathrm{D}}$,

$$a^\pi b = b, \qquad ab^\pi = a, \qquad b^\pi a b a^\pi = 0. \tag{3.1}$$

First we show that $ab = 0$ implies conditions (3.1). From $ab = 0$, we get $a^{\mathrm{D}}ab = 0$ and $(1-a^\pi)b = 0$, which yields $a^\pi b = b$. Symmetrically, $ab^\pi = a$; the third condition is clear. To prove that our conditions are strictly weaker than $ab = 0$, we construct matrices a, b satisfying (3.1), but not $ab = 0$ (or $ba = 0$).

Example 3.1. We take $\mathcal{A}$ to be the algebra of all complex 3×3 matrices, and set

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then

$$a^{\mathrm{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad a^\pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $b^2 = 0$, we have $b^{\mathrm{D}} = 0$ and $b^\pi = 1$. It is now easy to see that the matrices a, b satisfy conditions (3.1), while

$$ab = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0, \qquad ba = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \neq 0.$$

The following equivalences shed light on conditions (3.1),

$$a^\pi b = b \iff a^{\mathrm{D}}b = 0 \iff b\mathcal{A} \subset a^\pi \mathcal{A}, \tag{3.2}$$

$$ab^\pi = a \iff ab^{\mathrm{D}} = 0 \iff \mathcal{A}a \subset \mathcal{A}b^\pi, \tag{3.3}$$

$$b^\pi a b a^\pi = 0 \iff b a^\pi \mathcal{A} \subset (b^\pi a)^\circ, \tag{3.4}$$

where $u^\circ := \{x \in \mathcal{A} : ux = 0\}$, $u \in \mathcal{A}$. Indeed, if $a^\pi b = b$, then $a^{\mathrm{D}}b = a^{\mathrm{D}}a^\pi b = 0$; if $a^{\mathrm{D}}b = 0$, then $a^\pi b = (1 - aa^{\mathrm{D}})b = b$. That $a^\pi b = b$ implies $b\mathcal{A} \subset a^\pi\mathcal{A}$ is clear. For the reverse implication, we note that $b\mathcal{A} \subset a^\pi\mathcal{A}$ implies $b = a^\pi u$ for some $u \in \mathcal{A}$, and $a^\pi b = a^\pi u = b$. This proves (3.2). Equivalences (3.3) are proved similarly; (3.4) is clear.

It is not difficult to show that, for matrices and bounded linear operators on a Banach space, equivalences (3.2)–(3.4) are equivalent to

$$\mathcal{R}(b) \subset \mathcal{R}(a^\pi), \qquad \mathcal{N}(b^\pi) \subset \mathcal{N}(a), \qquad \mathcal{R}(ba^\pi) \subset \mathcal{N}(b^\pi a),$$

respectively, where $\mathcal{R}$ denotes the range and $\mathcal{N}$ the nullspace of an operator.

Let us also observe that if a, b satisfy (3.1) and one of the elements is invertible, then the other is zero. As expected, our theorems give no information about the ordinary inverse.

We start with an important special case of our main theorem. First we note the following result involving $S_n(u,v)=\sum_{k=0}^n u^k v^{n-k}$, where $u,v\in\mathcal{A}$.

LEMMA 3.2. *Let $a\in\mathcal{A}$ and $b\in\mathcal{A}^{\mathrm{D}}$ satisfy conditions (3.1). Then, for any non-negative integer n,*

$$b^\pi(a+b)^n a^\pi = b^\pi S_n(b,a)a^\pi, \qquad a(a+b)^n b = aS_n(b,a)b. \tag{3.5}$$

Proof. A proof by induction is based on the observation that $b^\pi a^k b a^\pi = 0$ for all $k>1$. □

THEOREM 3.3. *Let $b\in\mathcal{A}^{\mathrm{D}}$, $a\in\mathcal{A}^{\mathrm{qnil}}$, and let $ab^\pi=a$ and $b^\pi ab=0$. Then a, b satisfy conditions (3.1), $a+b\in\mathcal{A}^{\mathrm{D}}$ and*

$$(a+b)^{\mathrm{D}} = b^{\mathrm{D}} + \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a(a+b)^n = b^{\mathrm{D}} + \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}aS_n(b,a). \tag{3.6}$$

Proof. Assume first that $b\in\mathcal{A}^{\mathrm{qnil}}$. Then $b^\pi=1$, and the second condition gives $ab=0$. By lemma 2.4, $a+b\in\mathcal{A}^{\mathrm{qnil}}$. Then (3.6) holds as $(a+b)^{\mathrm{D}}=b^{\mathrm{D}}=0$.

If b is not quasi-nilpotent, we use a matrix representation relative to the total system $(p_1,p_2)=(1-b^\pi,b^\pi)$ of idempotents, where $p_1\neq 0$. We have

$$b=\begin{bmatrix} b_1 & 0\\ 0 & b_2\end{bmatrix}, \qquad a=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},$$

where $b_1\in\mathcal{A}_1^{-1}$ and $b_2\in\mathcal{A}_2^{\mathrm{qnil}}$. Expressing the condition $ab^\pi=a$ in matrix form,

$$a(1-b^\pi)=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix} a_{11} & 0\\ a_{21} & 0\end{bmatrix}=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix},$$

we obtain $a_{11}=a_{21}=0$ (since $p_1\neq 0$). For the sake of brevity, we write $a_1:=a_{12}$, $a_2:=a_{22}$. (Here and elsewhere in the paper, we write $\left[\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right]$ instead of $\left[\begin{smallmatrix} p_1 & 0\\ 0 & 0\end{smallmatrix}\right]$ in equations involving multiplication of $\mathcal{M}_2(\mathcal{A},\{p_1,p_2\})$ matrices, provided $p_1\neq 0$, as it gives the same product. A similar convention is used for p_2.)

We now express the condition $b^\pi ab=0$ in matrix form,

$$b^\pi ab=\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} 0 & a_1\\ 0 & a_2\end{bmatrix}\begin{bmatrix} b_1 & 0\\ 0 & b_2\end{bmatrix}=\begin{bmatrix} 0 & 0\\ 0 & a_2b_2\end{bmatrix}=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}.$$

This yields the condition $a_2b_2=0$. We have

$$a+b=\begin{bmatrix} 0 & a_1\\ 0 & a_2\end{bmatrix}+\begin{bmatrix} b_1 & 0\\ 0 & b_2\end{bmatrix}=\begin{bmatrix} b_1 & a_1\\ 0 & a_2+b_2\end{bmatrix},$$

where $a_2+b_2\in\mathcal{A}_2^{\mathrm{qnil}}$ in view of lemma 2.4. (Note that $a_2=p_2ap_2\in\mathcal{A}_2^{\mathrm{qnil}}$, since $a\in\mathcal{A}^{\mathrm{qnil}}$.) We apply theorem 2.3 (i) to obtain

$$(a+b)^{\mathrm{D}}=\begin{bmatrix} b_1 & a_1\\ 0 & a_2+b_2\end{bmatrix}^{\mathrm{D}}=\begin{bmatrix} b_1^{-1} & u\\ 0 & 0\end{bmatrix},$$

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where $u = \sum_{n=0}^{\infty}(b_1^{-1})^{n+2}a_1(a_2+b_2)^n$. Since

$$(b^{\mathrm{D}})^{n+2}[a(a+b)^n] = \begin{bmatrix}(b_1^{-1})^{n+2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & a_1(a_2+b_2)^n\\ 0 & a_2^{n+1}\end{bmatrix} = \begin{bmatrix}0 & (b_1^{-1})^{n+2}a_1(a_2+b_2)^n\\ 0 & 0\end{bmatrix},$$

we have

$$(a+b)^{\mathrm{D}} = \begin{bmatrix}b_1^{-1} & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & u\\ 0 & 0\end{bmatrix} = b^{\mathrm{D}} + \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a(a+b)^n.$$

The second part of equation (3.6) follows from lemma 3.2, since $a^\pi = 1$. □

Corollary 3.4. *Let $b \in \mathcal{A}^{\mathrm{D}}$, $a \in \mathcal{A}^{\mathrm{qnil}}$ and let $ab = 0$. Then $a+b \in \mathcal{A}^{\mathrm{D}}$ and*

$$(a+b)^{\mathrm{D}} = \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+1}a^n. \tag{3.7}$$

Proof. At the beginning of the section we showed that $ab = 0$ implies (3.1). Therefore, the preceding theorem applies to give $a+b \in \mathcal{A}^{\mathrm{D}}$, and to furnish equation (3.6) for $(a+b)^{\mathrm{D}}$. By induction, $a(a+b)^n = a^{n+1}$, and (3.7) follows. □

Specializing the preceding corollary (with multiplication reversed) to bounded linear operators, we recover [3, theorem 2.2].

After all the preparation, our main theorem follows.

Theorem 3.5. *Let $a, b \in \mathcal{A}^{\mathrm{D}}$ satisfy conditions (3.1). Then $a+b \in \mathcal{A}^{\mathrm{D}}$ and*

$$\begin{aligned}(a+b)^{\mathrm{D}} = {}& b^{\mathrm{D}}a^\pi + b^\pi a^{\mathrm{D}} + \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a(a+b)^n a^\pi + \sum_{n=0}^{\infty}b^\pi(a+b)^n b(a^{\mathrm{D}})^{n+2}\\ &- \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a(a+b)^n ba^{\mathrm{D}} - \sum_{n=0}^{\infty}b^{\mathrm{D}}a(a+b)^n b(a^{\mathrm{D}})^{n+2}\\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathrm{D}})^{k+2}a(a+b)^{n+k+1}b(a^{\mathrm{D}})^{n+2}.\end{aligned} \tag{3.8}$$

The terms of the form $(a+b)^m$ in this formula can be replaced by $S_m(b,a)$.

Proof. The case of $a \in \mathcal{A}^{\mathrm{qnil}}$ is covered by theorem 3.3. If $a \in \mathcal{A}^{-1}$, then $b = a^\pi b = 0 = b^{\mathrm{D}}$, $b^\pi = 1$ and equation (3.8) holds. Thus we assume that a is neither quasinilpotent nor invertible, and use matrix representation of elements relative to the total system $(p_1, p_2) = (1-a^\pi, a^\pi)$ of idempotents, where $p_1 \neq 0$ and $p_2 \neq 0$.

We have

$$a = \begin{bmatrix}a_1 & 0\\ 0 & a_2\end{bmatrix}, \qquad b = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix},$$

where $a_1 \in \mathcal{A}_1^{-1}$ and $a_2 \in \mathcal{A}_2^{\mathrm{qnil}}$. Condition $a^\pi b = b$ expressed in matrix form yields

$$(1-a^\pi)b = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} = \begin{bmatrix}b_{11} & b_{12}\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

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This gives $b_{11} = b_{12} = 0$. For brevity, we write $b_1 := b_{21}$ and $b_2 := b_{22}$. Applying theorem 2.3 (ii) to the matrix

$$b = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix},$$

we deduce that $b_2 \in \mathcal{A}_2^{\mathrm{D}}$ and that

$$b^{\mathrm{D}} = \begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}, \qquad b^{\pi} = 1 - bb^{\mathrm{D}} = \begin{bmatrix} 1 & 0 \\ -b_2^{\mathrm{D}} b_1 & b_2^{\pi} \end{bmatrix}. \tag{3.9}$$

Expressing the condition $ab^{\pi} = a$ in matrix form, we get

$$\begin{aligned} a(1 - b^{\pi}) &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_2^{\mathrm{D}} b_1 & 1 - b_2^{\pi} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ a_2 b_2^{\mathrm{D}} b_1 & a_2(1 - b_2^{\pi}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which gives $a_2 b_2^{\pi} = a_2$ (and $a_2 b_2^{\mathrm{D}} b_1 = 0$). When we express $b^{\pi} a b a^{\pi} = 0$ in matrix form, we obtain

$$\begin{aligned} b^{\pi} a b a^{\pi} &= \begin{bmatrix} 1 & 0 \\ -b_2^{\mathrm{D}} b_1 & b_2^{\pi} \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & b_2^{\pi} a_2 b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

From this, we get $b_2^{\pi} a_2 b_2 = 0$.

Applying theorem 3.3 to the elements a_2, b_2, we conclude that $a_2 + b_2$ is g-Drazin invertible in $\mathcal{A}_2$ with

$$(a_2 + b_2)^{\mathrm{D}} = b_2^{\mathrm{D}} + \sum_{n=0}^{\infty} (b_2^{\mathrm{D}})^{n+2} a_2 (a_2 + b_2)^n.$$

By theorem 2.3 (i), $a + b \in \mathcal{A}^{\mathrm{D}}$ and

$$(a + b)^{\mathrm{D}} = \begin{bmatrix} a_1 & 0 \\ b_1 & a_2 + b_2 \end{bmatrix}^{\mathrm{D}} = \begin{bmatrix} a_1^{-1} & 0 \\ z & (a_2 + b_2)^{\mathrm{D}} \end{bmatrix},$$

where

$$\begin{aligned} z = \sum_{n=0}^{\infty} b_2^{\pi} (a_2 + b_2)^n b_1 a_1^{-n-2} &- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b_2^{\mathrm{D}})^{k+1} a_2 (a_2 + b_2)^{n+k} b_1 a_1^{-n-2} \\ &- \biggl(b_2^{\mathrm{D}} + \sum_{n=0}^{\infty} (b_2^{\mathrm{D}})^{n+2} a_2 (a_2 + b_2)^n \biggr) b_1 a_1^{-1}, \end{aligned} \tag{3.10}$$

noting that $a_1^{\pi} = 0$.

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To verify (3.8), we divide the right-hand side of (3.8) into four summands and calculate each separately in matrix form,

$$\begin{aligned}
A_1 &= \left(b^{\mathrm{D}} + \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a(a+b)^n\right)a^{\pi} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & b_2^{\mathrm{D}} \end{bmatrix} + \sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{n+2}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ b_1 & a_2+b_2 \end{bmatrix}^n\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & b_2^{\mathrm{D}} \end{bmatrix} + \sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{n+2}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_2)^n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & b_2^{\mathrm{D}} \end{bmatrix} + \sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{n+2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & a_2(a_2+b_2)^n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & b_2^{\mathrm{D}} \end{bmatrix} + \sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ 0 & (b_2^{\mathrm{D}})^{n+2} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & a_2(a_2+b_2)^n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & b_2^{\mathrm{D}} + \sum_{n=0}^{\infty}(b_2^{\mathrm{D}})^{n+2}a_2(a_2+b_2)^n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_2)^{\mathrm{D}} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
A_2 &= b^{\pi}\left(a^{\mathrm{D}} + \sum_{n=0}^{\infty}(a+b)^n b(a^{\mathrm{D}})^{n+2}\right) \\
&= \begin{bmatrix} a_1^{-1} & 0 \\ -b_2^{\mathrm{D}}b_1a_1^{-1} & 0 \end{bmatrix} \\
&\qquad + \sum_{n=0}^{\infty}\begin{bmatrix} 1 & 0 \\ -b_2^{\mathrm{D}}b_1 & b_2^{\pi} \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ b_1 & a_2+b_2 \end{bmatrix}^n\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} a_1^{-n-2} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_1^{-1} & 0 \\ -b_2^{\mathrm{D}}b_1a_1^{-1} & 0 \end{bmatrix} + \sum_{n=0}^{\infty}\begin{bmatrix} 1 & 0 \\ -b_2^{\mathrm{D}}b_1 & b_2^{\pi} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_2)^n \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b_1a_1^{-n-2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_1^{-1} & 0 \\ -b_2^{\mathrm{D}}b_1a_1^{-1} - \sum_{n=0}^{\infty}b_2^{\pi}(a_2+b_2)^n b_1 a_1^{-n-2} & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
A_3 &= -\sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a(a+b)^n ba^{\mathrm{D}} \\
&= -\sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{n+2}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ b_1 & a_2+b_2 \end{bmatrix}^n\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
&= -\sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{n+2}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_2)^n \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b_1a_1^{-1} & 0 \end{bmatrix} \\
&= -\sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{n+2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ a_2(a_2+b_2)^n b_1 a_1^{-1} & 0 \end{bmatrix}
\end{aligned}$$

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$$= -\sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & (b_2^{\mathrm{D}})^{n+2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_2(a_2+b_2)^n b_1 a_1^{-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -\sum_{n=0}^{\infty}(b_2^{\mathrm{D}})^{n+2} a_2(a_2+b_2)^n b_1 a_1^{-1} & 0 \end{bmatrix},$$

$$A_4 = -\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(b^{\mathrm{D}})^{k+1} a(a+b)^{n+k} b(a^{\mathrm{D}})^{n+2}$$

$$= -\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{k+1} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \times \begin{bmatrix} a_1 & 0 \\ b_1 & a_2+b_2 \end{bmatrix}^{n+k} \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} a_1^{-n-2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= -\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{k+1} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \times \begin{bmatrix} a_1 & 0 \\ b_1 & a_2+b_2 \end{bmatrix}^{n+k} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_1 a_1^{-n-2} & 0 \end{bmatrix}$$

$$= -\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ (b_2^{\mathrm{D}})^2 b_1 & b_2^{\mathrm{D}} \end{bmatrix}^{k+1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_2(a_2+b_2)^{n+k} b_1 a_1^{-n-2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(b_2^{\mathrm{D}})^{k+1} a_2(a_2+b_2)^{n+k} b_1 a_1^{-n-2} & 0 \end{bmatrix}.$$

It then follows that

$$A_1 + A_2 + A_3 + A_4 = \begin{bmatrix} a_1^{-1} & 0 \\ z & (a_2+b_2)^{\mathrm{D}} \end{bmatrix} = (a+b)^{\mathrm{D}}.$$

The last statement of the theorem follows from lemma 3.2. □

Specializing the preceding theorem to matrices, we recover [1, theorem 2.4].

4. Special cases

In this section we look for hypotheses stronger than (3.1), which will provide a simplification of equation (3.8) for $(a+b)^{\mathrm{D}}$. The results of the preceding section, in particular the matrix representations, suggest that we should retain the condition $a^{\pi}b = b$, while replacing $ab^{\pi} = a$ and $b^{\pi}aba^{\pi} = 0$ by a stronger hypothesis.

We have shown that the condition $ab = 0$ employed in [3] and [5] is stronger than conditions (3.1) of our main theorem. In our first result of this section we assume conditions that are weaker than $ab = 0$ but stronger than $ab^2 = 0$.

Theorem 4.1. *Let $a, b \in \mathcal{A}^{\mathrm{D}}$ satisfy conditions*

$$a^{\pi}b = b, \qquad aba^{\pi} = 0. \tag{4.1}$$

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Then conditions (3.1) are satisfied, $ab^2 = 0$, $a + b \in \mathcal{A}^{\mathrm{D}}$ and

$$(a+b)^{\mathrm{D}} = b^{\mathrm{D}}a^{\pi} + b^{\pi}a^{\mathrm{D}} + \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a^{n+1}a^{\pi} + \sum_{n=0}^{\infty} b^{\pi}S_n(b,a)b(a^{\mathrm{D}})^{n+2}$$
$$- \sum_{n=0}^{\infty}(b^{\mathrm{D}})^{n+2}a^{n+1}ba^{\mathrm{D}} - \sum_{n=0}^{\infty} b^{\mathrm{D}}a^{n+1}b(a^{\mathrm{D}})^{n+2}$$
$$- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathrm{D}})^{k+2}a^{n+k+2}b(a^{\mathrm{D}})^{n+2}. \tag{4.2}$$

Proof. From $a^{\pi}b = b$, it follows that $a^{\pi}b^{\mathrm{D}} = b^{\mathrm{D}}$ $(a^{\pi}b^{\mathrm{D}} = a^{\pi}b(b^{\mathrm{D}})^2 = b(b^{\mathrm{D}})^2 = b^{\mathrm{D}})$ and

$$ab^{\pi} = a(1 - bb^{\mathrm{D}}) = a(1 - ba^{\pi}b^{\mathrm{D}}) = a.$$

Hence theorem 3.5 is applicable. Furthermore, $ab^2 = abb = [aba^{\pi}]b = 0$. From $ab^2 = 0$, we deduce that

$$aS_n(b,a)a^{\pi} = a^{n+1}a^{\pi}, \quad aS_n(b,a)b = a^{n+1}b, \quad n = 0, 1, \dots.$$

The result then follows by substituting in theorem 3.5. □

We present some special cases of the preceding theorem. In particular, in the following examples, we assume (4.1) while specializing the elements $a, b \in \mathcal{A}^{\mathrm{D}}$.

Example 4.2. Let $a^{\pi}b = b$, $aba^{\pi} = 0$ and $b^2 = b$. Then $b^{\mathrm{D}} = b$ and $b^{\pi} = 1 - b$, while $ab = ab^2 = 0$ by theorem 4.1, and $a^{\mathrm{D}}b = 0$. Hence (4.2) specializes to

$$(a+b)^{\mathrm{D}} = (1-b)a^{\mathrm{D}} + \sum_{n=0}^{\infty} ba^n a^{\pi}.$$

For this case, we can estimate the perturbation error. Since $aa^{\pi} \in \mathcal{A}^{\mathrm{qnil}}$, we have

$$\sum_{n=0}^{\infty} ba^n a^{\pi} = b\sum_{n=0}^{\infty}(aa^{\pi})^n a^{\pi} = b(1 - aa^{\pi})^{-1}a^{\pi}.$$

We then obtain

$$\frac{\|(a+b)^{\mathrm{D}} - a^{\mathrm{D}}\|}{\|a^{\mathrm{D}}\|} \leqslant \|b\|\frac{\|-a^{\mathrm{D}} + (1-aa^{\pi})^{-1}a^{\pi}\|}{\|a^{\mathrm{D}}\|} \leqslant \left(1 + \frac{\|(1-aa^{\pi})^{-1}a^{\pi}\|}{\|a^{\mathrm{D}}\|}\right)\|b\|.$$

Example 4.3. If $a^{\pi}b = b$, $aba^{\pi} = 0$ and $b \in \mathcal{A}^{\mathrm{qnil}}$, we get from (4.2) a result symmetrical to theorem 3.3,

$$(a+b)^{\mathrm{D}} = a^{\mathrm{D}} + \sum_{n=0}^{\infty} S_n(b,a)b(a^{\mathrm{D}})^{n+2}. \tag{4.3}$$

Example 4.4. Let $a^{\pi}b = b$, $aba^{\pi} = 0$ and $b^2 = 0$. Then (4.3) becomes

$$(a+b)^{\mathrm{D}} = a^{\mathrm{D}} + \sum_{n=0}^{\infty} a^n b(a^{\mathrm{D}})^{n+2} + \sum_{n=1}^{\infty} ba^n b(a^{\mathrm{D}})^{n+3}.$$

EXAMPLE 4.5. Let $a, b \in \mathcal{A}^{\mathrm{D}}$ and let $ab = 0$. As proved earlier, conditions (4.1) are satisfied. We then recover [5, theorem 2.1] for matrices and [3, theorem 2.3] for bounded linear operators,

$$(a+b)^{\mathrm{D}} = \sum_{n=0}^{\infty} (b^{\mathrm{D}})^{n+1} a^n a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} b^n (a^{\mathrm{D}})^{n+1}. \tag{4.4}$$

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