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COMPLEMENT OF THE ZERO DIVISOR GRAPH OF A LATTICE

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Abstract

In this paper, we determine when $(\Gamma_I(L))^c$, the complement of the zero divisor graph $\Gamma_I(L)$ with respect to a semiprime ideal I of a lattice L, is connected and also determine its diameter, radius, centre and girth. Further, a form of Beck's conjecture is proved for $\Gamma_I(L)$ when $\omega((\Gamma_I(L))^c) < \infty$.

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1. Introduction

The investigation of graphs associated with various algebraic structures is a very important topic, well known and established in modern algebra. The graphs have played crucial roles in the study of ring constructions and their applications in coding theory (see Alfaro and Kelarev [2, 3] and Bereg *et al.* [6]) and automata theory (see [16–18]), and in the study of commutative rings and semigroups (see [1, 4, 5, 19, 22, 23]).

The idea of associating a graph with the zero divisors of a commutative ring was introduced by Beck [5]. He was mainly interested in the colouring of such graphs. In recent years, zero divisor graphs have also been extensively studied by many mathematicians for ordered structures; see [8, 11–15, 20].

In this paper, we determine when $(\Gamma_I(L))^c$, the complement of the zero divisor graph $\Gamma_I(L)$ with respect to a semiprime ideal I of a lattice L, is connected and also determine its diameter, radius, centre and girth. Further, a form of Beck's conjecture is proved for $\Gamma_I(L)$ when $\omega((\Gamma_I(L))^c) < \infty$.

We begin with the necessary definitions.

A nonempty subset I of a lattice L is called a *semi-ideal* if for $a \in I$, $b \in L$, $b \leq a$ implies $b \in I$. A semi-ideal I of a lattice L is called an *ideal* of L if $a, b \in I$

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implies $a \vee b \in I$. Dually, we have the concepts of semi-filter and filter. An ideal I of L is said to be *principal ideal* generated by $a \in L$ if $I = (a] = \{x \mid x \leq a\}$. Dually, we have the concept of *principal filter* generated by a. A proper semi-ideal (ideal) I of a lattice L is called *prime* if $a, b \in L$ and $a \wedge b \in I$ imply $a \in I$ or $b \in I$. An ideal I of a lattice L is said to be *semiprime* if $a \wedge b \in I$ and $a \wedge c \in I$ together imply that $a \wedge (b \vee c) \in I$. Dually, we have the concept of semiprime filter and prime filter. A prime ideal P of a lattice L is said to be *minimal prime ideal belonging to an ideal* I if $I \subseteq P$ and there exists *no* prime ideal Q such that $I \subsetneq Q \subsetneq P$.

For an ideal I and a nonempty subset A of a lattice L, define a subset $I : A$ of L as follows: $I : A = \{z \in L \mid z \wedge a \in I, \forall\, a \in A\}$. If $A = \{x\}$ then we write $I : x$ instead of $I : \{x\}$. Note that, if $x \leq y$ for $x, y \in L$, then $I : y \subseteq I : x$. Observe that $I \subseteq I : A$ and $I : A = \bigcap_{x \in A} I : x$. Note that $I : A$ need not be an ideal but it is a semi-ideal. Further, if I is a semiprime ideal of L, then $I : A$ is an ideal of L. Moreover, if $I = (0]$ then $I : x$ is nothing but $\mathrm{Ann}(x) = \{y \mid y \wedge x = 0\}$.

Let L be a lattice and let I be an ideal in L. Denote by $Z_I(L) = \{x \in L \mid x \wedge y \in I$ for some $y \in L \setminus I\}$ and $Z_I(L)^* = \{x \in L \setminus I \mid x \wedge y \in I$ for some $y \in L \setminus I\}$. Clearly, $Z_I(L)^* \cup I = Z_I(L)$.

Heinzer and Ohm [10] introduced the concept of *N-prime* (Nagata prime) ideals in commutative rings. We extend this concept to lattices.

Let I be an ideal of a lattice L. A prime ideal P of a lattice L is said to be a *maximal N-prime* of I if P is maximal with respect to the property of being contained in $Z_I(L) = Z_I(L)^* \cup I$.

A prime ideal P of a lattice L is said to be a *B-prime* of I (Bourbaki prime) if $P = I : x$ for some $x \in L$. A B-prime ideal of I is also known as an associated prime ideal of I. More details on B-prime ideals in rings can be found in [7].

Rav's Theorem (Rav [21]). *Let L be a lattice, I be an ideal of L such that $I \cap F = \emptyset$ for a semiprime filter F in L. Then there exists a prime ideal $P \supseteq I$ and $P \cap F = \emptyset$.*

Lemma 1.1 (Rav [21]). *Every semiprime ideal is representable as an intersection of prime ideals.*

Remark 1.2. Let I be a semiprime ideal of a lattice L with 1. Let $S = L \setminus Z_I(L)$. Clearly, S is a nonempty subset of L. We prove that S is a semiprime filter of L. Let $a, b \in S$. Suppose, to the contrary, that $a \wedge b \notin S$, that is, $a \wedge b \in Z_I(L)$. Then there exists $c \notin I$ such that $a \wedge b \wedge c \in I$. Clearly, $b \wedge c \in I$, otherwise $a \in Z_I(L)$ which is not possible. Hence, $b \wedge c \in I$ together with $c \notin I$ gives $b \in Z_I(L)$, again a contradiction. Thus we have $a \wedge b \in S$. Now let $a \in S$ and $b \geq a$. Suppose $b \notin S$, that is, $b \in Z_I(L)$. Since $Z_I(L)$ is a semi-ideal, $a \in Z_I(L)$. This contradicts the fact that $a \in S$. This proves that S is a filter of L.

Now we prove that S is semiprime. Let $a \vee b, a \vee c \in S$. We claim that $a \vee (b \wedge c) \in S$. Suppose on the contrary that $a \vee (b \wedge c) \notin S$. Then $a \vee (b \wedge c) \in Z_I(L)$. Thus there exists $d \notin I$ such that $(a \vee (b \wedge c)) \wedge d \in I$. Hence $a \wedge d \in I$ and $b \wedge c \wedge d \in I$. If $b \wedge d \in I$, then using semiprimeness of I, we have $(a \vee b) \wedge d \in I$. Therefore, $a \vee b \in Z_I(L)$,

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a contradiction. Thus $b \wedge d \notin I$. Now again $a \wedge (b \wedge d) \in I$ and $c \wedge (b \wedge d) \in I$ with I a semiprime ideal of L, so we have $(a \vee c) \wedge (b \wedge d) \in I$. This gives $a \vee c \in Z_I(L)$, again a contradiction. Hence $a \vee (b \wedge c) \in S$. Thus S is a semiprime filter of L.

Let $x \in Z_I(L)$. Then $(x] \cap S = \emptyset$. Therefore, by Rav's theorem, there exists a prime ideal P in L such that $(x] \subseteq P$ and $P \cap S = \emptyset$. Clearly, $P \subseteq Z_I(L)$. Using Zorn's lemma, we can obtain a maximal N-prime P_1 of I such that $x \in P_1$. Thus, for every $x \in Z_I(L)$ there exists a maximal N-prime P_i of I such that $x \in P_i$. Therefore, $Z_I(L) \subseteq \bigcup_{i\in\Lambda} P_i$. By the definition of maximal N-prime of I, each $P_i \subseteq Z_I(L)$ and hence $\bigcup_{i\in\Lambda} P_i \subseteq Z_I(L)$. Thus, if $\{P_i\}_{i\in\Lambda}$ denotes the set of all maximal N-primes of I in L, then $Z_I(L) = \bigcup_{i\in\Lambda} P_i$. Further, we observed that $I \subseteq \bigcap_{i\in\Lambda} P_i$. For this, if there is a maximal N-prime ideal P_1 of I such that $I \nsubseteq P_1$, then there exists $x \in I$ such that $x \notin P_1$. We prove that $P_1 \vee (x] \subseteq Z_I(L)$. Let t be any element in $P_1 \vee (x]$. Hence $t \leq p \vee x$ for some $p \in P_1$. As $P_1 \subseteq Z_I(L)$, we have $p \in Z_I(L)$. Therefore, there exists $q \notin I$ such that $p \wedge q \in I$. Since $x \in I$, $x \wedge q \in I$. By semiprimeness of I, we have $(p \vee x) \wedge q \in I$ and hence $t \wedge q \in I$. This together with $q \notin I$ gives $t \in Z_I(L)$. Thus $P_1 \vee (x] \subseteq Z_I(L)$. Now again $(P_1 \vee (x]) \cap S = \emptyset$, for a semiprime filter $S = L \setminus Z_I(L)$. By Rav's theorem, there exists a prime ideal Q in L such that $P_1 \vee (x] \subseteq Q$ and $Q \cap S = \emptyset$. Clearly, $Q \subseteq Z_I(L)$. Hence $P_1 \subsetneq P_1 \vee (x] \subseteq Q \subseteq Z_I(L)$, a contradiction to the fact that P_1 is maximal N-prime ideal of I.

2. The connectivity of $(\Gamma_I(L))^c$

Joshi [11] introduced the concept of the zero divisor graph of a poset P having the smallest element 0 with respect to an ideal I of P. We mention below this definition when the corresponding poset is a lattice.

Definition 2.1. Let I be an ideal of a lattice L with 0. We associate an undirected graph, called the *zero divisor graph of L with respect to the ideal I*, denoted by $\Gamma_I(L)$ in which the set of vertices is $V(\Gamma_I(L)) = Z_I(L)^* = \{x \in L \setminus I \mid x \wedge y \in I \text{ for some } y \in L \setminus I\}$ and two distinct vertices x, y are adjacent if and only if $x \wedge y \in I$.

It is clear from the definition that if an ideal I is prime then $Z_I(L)^* = \emptyset$. Throughout the paper, we assume that I is a semiprime ideal of a bounded lattice L and $Z_I(L)^* = V(\Gamma_I(L))$ is nonempty. Hence $|Z_I(L)^*| \geq 2$.

We recall the following concepts from graph theory.

Let $G = (V, E)$ be a simple graph. The complement of G, denoted by G^c, is defined by setting $V(G^c) = V(G) = V$ and two distinct vertices $u, v \in V$ are joined by an edge in G^c if and only if there exists no edge in G joining u and v. Let $x, y \in V$ be distinct vertices. We denote by $d(x, y)$ the length of a shortest path from x to y, if one exists, and put $d(x, y) = \infty$ if no such path exists. The *diameter* of G is zero if G is the graph on one vertex and is $\operatorname{diam}(G) = \sup\{d(x, y) \mid x, y \in V\}$ otherwise. For any $v \in V$, the *eccentricity* of v denoted by $e(v)$ is defined as $e(v) = \sup\{d(u, v) \mid u \in V\}$. The set of vertices of G with minimal eccentricity is called the *centre* of the graph and the minimum eccentric value is called the *radius* of G and is denoted by $r(G)$. The *girth*

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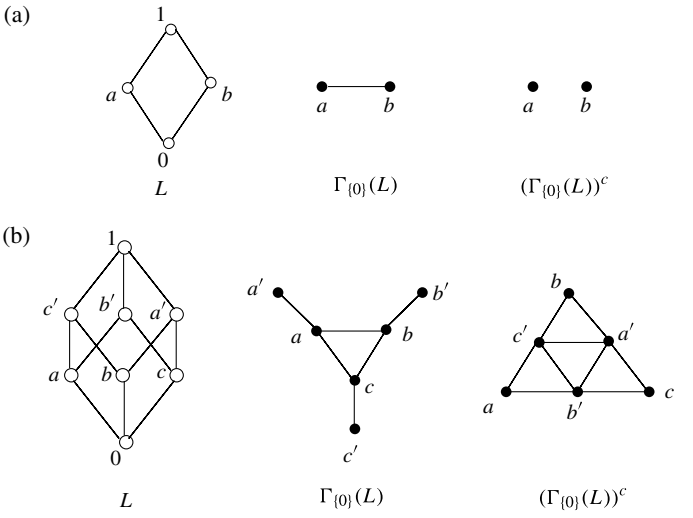

FIGURE 1. (a) Connected zero divisor graph whose complement is disconnected. (b) A zero divisor graph and its complement both connected.

of the graph G, denoted by $\mathrm{gr}(G)$, is defined as the length of a shortest cycle in G. If G does not contain any cycle, then we set $\mathrm{gr}(G) = \infty$. A graph G is called *hypertriangulated* (*triangulated*) if each edge (vertex) of G is an edge (vertex) of a triangle.

Undefined terms and notations are from Harary [9].

The following result is an easy consequence of Joshi [11, Theorem 2.4].

THEOREM 2.2 (Joshi [11]). *The zero divisor graph* $\Gamma_I(L)$ *is connected and* $\mathrm{diam}(\Gamma_I(L)) \leq 3$.

From Figure 1(a), it is clear that $\Gamma_{\{0\}}(L)$ is connected but $(\Gamma_{\{0\}}(L))^c$ is not, whereas in Figure 1(b), $\Gamma_{\{0\}}(L)$ and $(\Gamma_{\{0\}}(L))^c$ both are connected. Hence it is natural to ask the following question.

QUESTION. When is $(\Gamma_I(L))^c$ connected?

We answer this question in the following theorem.

MAIN THEOREM. *The graph* $(\Gamma_I(L))^c$ *is connected if and only if L does not have exactly two maximal N-primes* P_1, P_2 *of I such that* $P_1 \cap P_2 = I$.

Furthermore, if $(\Gamma_I(L))^c$ *is connected then the following statements hold.*

(1) $\mathrm{diam}((\Gamma_I(L))^c) = 2$.
(2) $e(x) = 2$ *for all* $x \in V((\Gamma_I(L))^c)$. *Hence* $r((\Gamma_I(L))^c) = 2$ *and the centre of* $(\Gamma_I(L))^c$ *is* $V((\Gamma_I(L))^c)$.
(3) $\mathrm{gr}(\Gamma_I(L))^c) = 3$. *In fact,* $(\Gamma_I(L))^c)$ *is hypertriangulated.*

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We present the proof of the Main Theorem in three subsections, as follows:

(1) L has exactly one maximal N-prime of I;
(2) L has exactly two maximal N-primes of I;
(3) L has at least three maximal N-primes of I.

We first prove the following two lemmas which will be used frequently in subsequent sections.

LEMMA 2.3. *If* $(\Gamma_I(L))^c$ *is connected, then* $\operatorname{diam}((\Gamma_I(L))^c) \geq 2$.

PROOF. Let $a, b \in Z_I(L)^*$, $a \neq b$. By Theorem 2.2, $\Gamma_I(L)$ is connected; hence there exists $c \in Z_I(L)^*$ with $c \neq a$ such that $c \wedge a \in I$. Hence, if $(\Gamma_I(L))^c$ is connected, then $d(a, c) \geq 2$ in $(\Gamma_I(L))^c$ and so $\operatorname{diam}((\Gamma_I(L))^c) \geq 2$. □

LEMMA 2.4. *Let* Q *be a maximal* N*-prime of* I *in* L. *Let* $x, y \in L \setminus I$ *be any two elements such that* $Q \subseteq I : x \cup I : y$. *Then either* $Q = I : x$ *or* $Q = I : y$.

PROOF. Let Q be a maximal N-prime of I in L. Let $x, y \in L \setminus I$ be any two elements such that $Q \subseteq I : x \cup I : y$. Note that $I : x \cup I : y \subseteq Z_I(L)$. Let $S = L \setminus Z_I(L)$. Clearly, $I : x \cap S = \emptyset$ and $I : y \cap S = \emptyset$. As $I : x$, $I : y$ are ideals in the lattice L and S is the semiprime filter, by Rav's theorem, there exist prime ideals Q_1 and Q_2 such that $I : x \subseteq Q_1$, $I : y \subseteq Q_2$ and $Q_1 \cap S = \emptyset = Q_2 \cap S$. Since $Q \subseteq I : x \cup I : y$, then either $Q \subseteq I : x \subseteq Q_1$ or $Q \subseteq I : y \subseteq Q_2$. Further, Q is a maximal N-prime of I in L, which implies that either $Q = Q_1 = I : x$ or $Q = Q_2 = I : y$. □

When L has exactly one maximal *N*-prime of *I*. Our aim in this subsection is to prove the following theorem.

THEOREM 2.5. *If* P *is the only maximal* N*-prime of* I *in* L, *then* $(\Gamma_I(L))^c$ *is connected and* $\operatorname{diam}(\Gamma_I(L))^c = 2$.

PROOF. First, we claim that P is not a B-prime of I. On the contrary, suppose that $P = I : a$ for some $a \in L$. If $a \in I$, then $L = I : a = P$, a contradiction. Hence $a \in L \setminus I$. By the hypothesis, $Z_I(L) = P$ gives $P \setminus I = Z_I(L)^*$ and $|Z_I(L)^*| \geq 2$, and there exists an element $(a \neq)b \in Z_I(L)^* = P \setminus I$. Then $a \in Z_I(L)^* \subseteq P = I : a$, a contradiction. Hence $P \neq I : x$ for any $x \in L \setminus I$, that is, P is not a B-prime of I.

Let $a, b \in V((\Gamma_I(L))^c)$ be any two nonadjacent vertices in $(\Gamma_I(L))^c$. Clearly, $P \neq I : a$ and $P \neq I : b$. By Lemma 2.4, $P \nsubseteq I : a \cup I : b$. Thus there exists $w \in P$ such that $w \notin I : a \cup I : b$. Since $P = Z_I(L)$, we have $w \in Z_I(L)^* = V((\Gamma_I(L))^c)$ such that $w \wedge a \notin I$ and $w \wedge b \notin I$. This gives that $a - w - b$ is a path in $(\Gamma_I(L))^c$. Hence $(\Gamma_I(L))^c$ is connected and this together with Lemma 2.3 gives $\operatorname{diam}(\Gamma_I(L))^c = 2$. □

EXAMPLE 2.6. We now provide an example of a distributive lattice with exactly one maximal N-prime ideal of I. Let $\mathbb{N}$ be the set of natural numbers. Consider $L = F \cup \{\mathbb{N}\}$ where $F = \{X \subseteq \mathbb{N} \mid |X| < \infty\}$. Consider the semiprime ideal $(0] = \{\emptyset\}$ in L. Then $F = Z_{(0]}(L)$ and F is the only maximal N-prime ideal of L which is not a B-prime of $(0]$.

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When L has exactly two maximal N-primes of I

LEMMA 2.7. *Let L have exactly two maximal N-primes of I, say P_1 and P_2. If $P_1 \cap P_2 \neq I$ then there exist $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$ such that $a \wedge b \notin I$.*

PROOF. Since $P_1 \neq P_2$, there exist $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. Clearly, $a \vee b \notin P_1 \cup P_2 = Z_I(L)$. Hence $I : a \vee b = I$. Suppose $a \wedge b \in I$. Clearly, $I \subseteq P_1 \cap P_2$ follows from Remark 1.2. Let $x \in P_1 \cap P_2$ with $x \notin I$. Since $I : a \vee b = I$, either $a \wedge x \notin I$ or $b \wedge x \notin I$. If $a \wedge x \notin I$, then the elements $a \in P_1 \setminus P_2$ and $b \vee x \in P_2 \setminus P_1$ satisfy our claim. Otherwise, the elements $a \vee x \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$ will satisfy our claim. □

LEMMA 2.8. *Let P, Q be distinct B-prime ideals of I in L with $P = I : x$ and $Q = I : y$ for some $x, y \in L \setminus I$. Then $x \wedge y \in I$.*

PROOF. Suppose to the contrary that $x \wedge y \notin I$. Then $x \notin I : y$ and $y \notin I : x$. By the primeness of $I : x$ and $I : y$, we have $I : x \subseteq I : y$ and $I : y \subseteq I : x$, a contradiction. □

The following theorem is essentially due to Joshi [11, Theorem 2.14].

THEOREM 2.9. *The graph $\Gamma_I(L)$ is a complete bipartite graph if and only if there exist prime ideals P_1 and P_2 such that $I = P_1 \cap P_2$.*

THEOREM 2.10. *Let L have exactly two maximal N-primes of I, say P_1 and P_2. Then the following statements hold:*

(1) *$(\Gamma_I(L))^c$ is connected if and only if $P_1 \cap P_2 \neq I$;*
(2) *if $(\Gamma_I(L))^c$ is connected then* $\operatorname{diam}((\Gamma_I(L))^c) = 2$.

PROOF. (1) Let $(\Gamma_I(L))^c$ be connected. Suppose $P_1 \cap P_2 = I$. By Theorem 2.9, $\Gamma_I(L)$ is a complete bipartite graph and hence $(\Gamma_I(L))^c$ is disconnected, a contradiction.

Conversely, assume that $P_1 \cap P_2 \neq I$. There are only two possibilities: either both P_1 and P_2 are B-primes of I or at least one of them is not a B-prime of I.

Case (1). Both P_1 and P_2 are B-primes of I. Hence $P_1 = I : x$ and $P_2 = I : y$ for some $x, y \in L \setminus I$. By Lemma 2.8, $x \wedge y \in I$ and hence $x, y \in Z_I(L)^* = P_1 \cup P_2 \setminus I$. Since $P_1 \cap P_2 \neq I$, there exists $t \in P_1 \cap P_2 \setminus I$. Thus $t \wedge x, t \wedge y \in I$ and by the semiprimeness of I, we have $t \wedge (x \vee y) \in I$. This gives $x \vee y \in Z_I(L) = P_1 \cup P_2$. Hence $x \vee y \in P_1$ or $x \vee y \in P_2$, a contradiction. Thus this case is not possible.

Case (2). At least one of P_1, P_2 is not a B-prime of I. Without loss of generality, we may assume that P_1 is not a B-prime of I.

We claim that there exist $a, b \in V((\Gamma_I(L))^c)$ such that $a \wedge b \in I$. If $a \wedge b \notin I$ for any $a, b \in V((\Gamma_I(L))^c)$, then $(\Gamma_I(L))^c$ is a complete graph, a contradiction to Lemma 2.3. Hence assume that $a \wedge b \in I$ for some $a, b \in V((\Gamma_I(L))^c)$. Clearly, $P_1 \neq I : a$ and $P_1 \neq I : b$. By Lemma 2.4, $P_1 \nsubseteq I : a \cup I : b$. Thus there exists $t \in P_1$ such that $t \notin I : a \cup I : b$. Clearly, $t \in V((\Gamma_I(L))^c)$ such that $a - t - b$ is a path in $(\Gamma_I(L))^c$. This gives that $d(a, b) = 2$, for any two nonadjacent vertices $a, b \in V((\Gamma_I(L))^c)$ and hence $(\Gamma_I(L))^c$ is connected.

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(2) Let $(\Gamma_I(L))^c$ be connected. By Lemma 2.3 and by the first part of this theorem, it follows that $\operatorname{diam}((\Gamma_I(L))^c) = 2$. □

THEOREM 2.11. *Suppose L has exactly two maximal N-primes of I, say P_1 and P_2. Then the following statements are equivalent:*

(1) $(\Gamma_I(L))^c$ *is connected;*
(2) $P_1 \cap P_2 \neq I$;
(3) $\operatorname{diam}(\Gamma_I(L)) = 3$.

PROOF. (1) ⇔ (2) follows from Theorem 2.10.

(2) ⇒ (3): Let $P_1 \cap P_2 \neq I$. By Lemma 2.7, there exist $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$ such that $a \wedge b \notin I$. It is easy to see that $a, b \in Z_I(L)^*$. Further, $I : a \vee b = I$. Clearly, $a - b$ is not an edge in $\Gamma_I(L)$, that is, $d(a, b) \neq 1$ in $\Gamma_I(L)$. If there exists $c \in Z_I(L)^*$ such that $a - c - b$ is a path in $\Gamma_I(L)$, then by the semiprimeness of I, we have $c \in I : a \vee b = I$, a contradiction. Thus $d(a, b) \neq 2$ in $\Gamma_I(L)$. By Theorem 2.2, $\Gamma_I(L)$ is connected and $\operatorname{diam}(\Gamma_I(L)) \leq 3$, so we have $\operatorname{diam}(\Gamma_I(L)) = 3$.

(3) ⇒ (2): Assume that $\operatorname{diam}(\Gamma_I(L)) = 3$. Therefore, there exists a path $a - b - c - d$ of length three in $\Gamma_I(L)$. Hence $a \wedge b, b \wedge c, c \wedge d \in I$ in L. Since $a \in Z_I(L) = P_1 \cup P_2$, $a \in P_1$ or $a \in P_2$. If $a \in P_1 \cap P_2$, then we are done. Suppose $a \notin P_1$ and since $a \wedge b \in I \subseteq P_1 \cap P_2$, $b \in P_1$. If $b \in P_2$, then again we are done. Let $b \notin P_2$. Since $b \wedge c \in I \subseteq P_1 \cap P_2$, $c \in P_2$. If $c \in P_1$, then again $P_1 \cap P_2 \neq I$. Let $c \notin P_1$. As $c \wedge d \in I \subseteq P_1 \cap P_2$, this gives $d \in P_1$. Thus $a \in P_2$ and $d \in P_1$ implies that $a \wedge d \in P_1 \cap P_2$. Since $a \wedge d \notin I$, $P_1 \cap P_2 \neq I$. □

We now provide examples of lattices having exactly two maximal N-primes of I.

EXAMPLE 2.12. Let $L = \{0, 1, a, b\}$ with $a \wedge b = 0$ and $a \vee b = 1$. Consider a semiprime ideal $(0]$ in L. Then $(a] = P_1$, $(b] = P_2$ are the only maximal N-primes of $(0]$ and which are B-prime of $(0]$ also and $P_1 \cap P_2 = (0]$. Note that in this case, $(\Gamma_{(0]}(L))^c$ is not connected.

EXAMPLE 2.13. Let $\mathbb{N}$ be the set of natural numbers. Let $L = \{X \subseteq \mathbb{N} \mid |X| < \infty\} \cup \{\mathbb{N} - \{1\}\} \cup \{\mathbb{N}\}$. Consider a semiprime ideal $(0] = \{\emptyset\}$ in L. Further $P_1 = (\mathbb{N} - \{1\}]$ and $P_2 = \{X \subseteq \mathbb{N} \mid |X| < \infty\}$ are the only maximal N-prime ideals of $(0]$ such that $P_1 \cap P_2 \neq (0]$. Note that $P_1 = (0] : \{1\}$ is a B-prime of $(0]$, whereas P_2 is not a B-prime of $(0]$. From Theorem 2.10, it is clear that $(\Gamma_{(0]}(L))^c$ is connected.

When L has at least three maximal *N*-primes of *I*.

THEOREM 2.14. *Let L have at least three maximal N-primes of I. Then* $(\Gamma_I(L))^c$ *is connected and* $\operatorname{diam}((\Gamma_I(L))^c) = 2$.

PROOF. We claim that there exist $x, y \in V((\Gamma_I(L))^c)$ such that $x \wedge y \in I$. If $x \wedge y \notin I$ for any $x, y \in V((\Gamma_I(L))^c)$, then $\operatorname{diam}((\Gamma_I(L))^c) = 1$, a contradiction to $\operatorname{diam}((\Gamma_I(L))^c) \geq 2$, by Lemma 2.3. Thus, we have $x \wedge y \in I$ for some $x, y \in V((\Gamma_I(L))^c)$. Then x and y are not adjacent in $(\Gamma_I(L))^c$. Let P_1 be a maximal N-prime of I. If $P_1 \nsubseteq I : x \cup I : y$, then there exists $w \in P_1$ such that $w \notin I : x \cup I : y$. It is easy to see that $w \in Z_I(L)^*$

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such that $x - w - y$ is a path of length two in $(\Gamma_I(L))^c$ between x and y. In such a case, we are done. Hence, assume that $P_1 \subseteq I : x \cup I : y$. By Lemma 2.4, either $P_1 = I : x$ or $P_1 = I : y$. Without loss of generality, we may assume that $P_1 = I : x$. By the hypothesis, L has at least three maximal N-primes of I. Let P_2 be a maximal N-prime of I such that $P_2 \neq P_1$. Again, if $P_2 \nsubseteq I : x \cup I : y$, then we are done. So we assume that $P_2 \subseteq I : x \cup I : y$, then again by Lemma 2.4, we obtain that $P_2 = I : y$. Let P_3 be a maximal N-prime of I such that $P_3 \notin \{P_1, P_2\}$. Note that P_3 is not a subset of $P_1 \cup P_2 = I : x \cup I : y$. Hence, there exists $t \in P_3$ such that $t \wedge x \notin I$ and $t \wedge y \notin I$, that is, there exists a path $x - t - y$ of length two in $(\Gamma_I(L))^c$ between x and y. This proves that $(\Gamma_I(L))^c$ is connected and $\operatorname{diam}((\Gamma_I(L))^c) \leq 2$. From Lemma 2.3, we have $\operatorname{diam}((\Gamma_I(L))^c) = 2$. □

Example 2.15. Let X be a set such that $|X| = n$, $n \geq 3$. Consider $L = P(X)$, the power set of X and the semiprime ideal $(0] = \{\emptyset\}$ in L. Then the lattice L has at least three maximal N-primes of $(0]$.

Lemma 2.16. *Let $G = (V, E)$ be a simple connected graph such that G^c is connected. If x is any element of V with $|V| \geq 2$, then $e(x) \geq 2$ in G^c.*

Proof. Let $x \in V$. Since $|V| \geq 2$ and G is connected, there exists $y \in V$ such that $x - y$ is an edge in G. Hence $x - y$ is not an edge in G^c. Thus $d(x, y) \geq 2$ in G^c. Therefore, $e(x) \geq 2$ in G^c. □

Proof of the Main Theorem. From Theorem 2.5, Theorem 2.10 and Theorem 2.14, it follows that $(\Gamma_I(L))^c$ is connected and $\operatorname{diam}((\Gamma_I(L))^c) = 2$ if and only if L does not have exactly two maximal N-primes P_1, P_2 of I such that $P_1 \cap P_2 = I$.

Now we claim that $r((\Gamma_I(L))^c) = 2$. Since $(\Gamma_I(L))^c$ is connected, it follows from Lemma 2.16 and $\operatorname{diam}(\Gamma_I(L))^c = 2$ that for any $x \in Z_I(L)^*$, $e(x) = 2$ in $(\Gamma_I(L))^c$, proving $r((\Gamma_I(L))^c) = 2$. Hence, the centre of $(\Gamma_I(L))^c$ is the set of all vertices of $(\Gamma_I(L))^c$.

Now we prove that $\operatorname{gr}((\Gamma_I(L))^c) = 3$. For this, let $x \in Z_I(L)^*$, as $Z_I(L)^* \neq \emptyset$. Hence, there exists $y \notin I$ such that $x \wedge y \in I$. Clearly, $x, y \in V((\Gamma_I(L))^c)$ such that x is not adjacent to y in $(\Gamma_I(L))^c$. Since $(\Gamma_I(L))^c$ is connected and $\operatorname{diam}((\Gamma_I(L))^c) = 2$, we have $d(x, y) = 2$ in $(\Gamma_I(L))^c$, that is, there exists $z \in V((\Gamma_I(L))^c)$ such that $x - z - y$ is a path of length two in $(\Gamma_I(L))^c$. As $y \notin I$ and $x \wedge z \wedge y \in I$, this implies that $x \wedge z \in Z_I(L)^*$. If $x \wedge z \notin \{x, z\}$, then $x - x \wedge z - z - x$ is a cycle of length three in $(\Gamma_I(L))^c$ and hence we are done. So we may assume that $x \wedge z \in \{x, z\}$. Without loss of generality, we assume that $x \wedge z = x$, that is, $x < z$. Since $z \in Z_I(L)^*$, there exists $t \notin I$ such that $t \wedge z \in I$. Clearly, $t \in V((\Gamma_I(L))^c) \setminus \{x, y, z\}$ such that $t \wedge x \leq t \wedge z \in I$. Hence $d(x, t) \neq 1$ in $(\Gamma_I(L))^c$. Since $(\Gamma_I(L))^c$ is connected, therefore $d(x, t) = 2$ in $(\Gamma_I(L))^c$. Then there exists $w \in V((\Gamma_I(L))^c)$ such that $x - w - t$ is a path of length two in $(\Gamma_I(L))^c$. Hence $x \wedge w \notin I$ which yields $z \wedge w \notin I$. Thus $x - w - z - x$ is a cycle of length three in $(\Gamma_I(L))^c$. Hence $\operatorname{gr}((\Gamma_I(L))^c) = 3$.

Now, we show that $(\Gamma_I(L))^c$ is hypertriangulated. Let $x - z$ be any edge in $(\Gamma_I(L))^c$. Proceeding as above, it can be shown that there exists $w \in V((\Gamma_I(L))^c) \setminus \{x, z\}$ such

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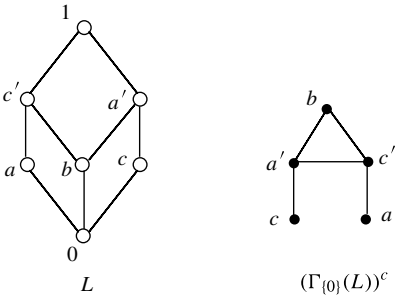

FIGURE 2. A graph $(\Gamma_{\{0\}}(L))^c$ with $\operatorname{diam}((\Gamma_{\{0\}}(L))^c) = 3$.

that $w \wedge x, w \wedge z \notin I$. This shows that any edge of $(\Gamma_I(L))^c$ is an edge of a triangle in $(\Gamma_I(L))^c$. Thus $(\Gamma_I(L))^c$ is hypertriangulated. □

REMARK 2.17. From the Main Theorem, it is clear that if I is a semiprime ideal in L, then $\operatorname{diam}((\Gamma_I(L))^c) = 2$. Note that if we drop the condition that I is a semiprime ideal in L, then the assertion of the Main Theorem need not be true. Consider the lattice depicted in Figure 2. Then it is easy to see that $\operatorname{diam}((\Gamma_{\{0\}}(L))^c) = 3$ for a nonsemiprime ideal (0].

Hence we raise the following question.

QUESTION. Find the class $\mathcal{I}$ of ideals of a lattice L such that for $I \in \mathcal{I}$, $\operatorname{diam}((\Gamma_I(L))^c) = 3$.

From the Main Theorem, it is clear that $(\Gamma_I(L))^c$ is connected and $\operatorname{gr}((\Gamma_I(L))^c) = 3$ whenever there exist exactly two maximal N-prime ideals P_1, P_2 such that $P_1 \cap P_2 \neq I$. Hence it is natural ask the following question.

QUESTION. Is $\operatorname{gr}((\Gamma_I(L))^c) = 3$ when $P_1 \cap P_2 = I$?

We answer this question in the following result.

LEMMA 2.18. *Suppose that L has exactly two maximal N-primes of I, say P_1 and P_2. If $P_1 \cap P_2 = I$, then $(\Gamma_I(L))^c$ contains a cycle if and only if either $|P_1 \setminus P_2| \geq 3$ or $|P_2 \setminus P_1| \geq 3$ if and only if $\operatorname{gr}((\Gamma_I(L))^c) = 3$.*

PROOF. By the hypothesis, P_1 and P_2 are the only two maximal N-primes of I in L. By Remark 1.2, it follows that $Z_I(L) = P_1 \cup P_2$. By Theorem 2.9, it is clear that $\Gamma_I(L)$ is a complete bipartite graph with vertex set $V_1 = P_1 \setminus P_2$ and $V_2 = P_2 \setminus P_1$. Then it is clear that $\operatorname{gr}((\Gamma_I(L))^c) = 3$ if and only if $|P_1 \setminus P_2| \geq 3$ or $|P_2 \setminus P_1| \geq 3$. □

We close this section by proving a sufficient condition for $\operatorname{gr}((\Gamma_I(L))^c) = 3$.

LEMMA 2.19. *If there exist distinct elements $a, b, c \in Z_I(L)^* \setminus P$ for some prime ideal P belonging to I, then $\operatorname{gr}((\Gamma_I(L))^c) = 3$.*

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PROOF. As P is a prime ideal of L belonging to I, that is, $I \subseteq P$ and $a, b, c \in Z_I(L)^* \setminus P$, we obtain that $a \wedge b, b \wedge c, c \wedge a \notin P$ and so $a \wedge b, b \wedge c, c \wedge a \in L \setminus I$. Hence, it follows that $a - b - c - a$ is a cycle of length three in $(\Gamma_I(L))^c$. Thus $\mathrm{gr}((\Gamma_I(L))^c) = 3$. □

3. Beck's conjecture for $(\Gamma_I(L))^c$

Beck [5] introduced the notion of colouring in a commutative ring R as follows. Let G be a simple graph whose vertices are the elements of R and two vertices x and y are adjacent if $xy = 0$. Then the graph G is known as the *zero divisor graph* of R. The *chromatic number* of G is denoted by $\chi(G)$. Thus, $\chi(G)$ is the minimum number of colours which can be assigned to the elements of R such that adjacent elements receive different colours. A *clique* of a graph G is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $\omega(G)$, is called the *clique number* of G. We always have $\chi(G) \geq \omega(G)$. Beck [5] conjectured that $\chi(G) = \omega(G)$ but Anderson and Naseer [4] gave an example of a commutative local ring R with 32 elements for which $\chi(G) > \omega(G)$.

A form of Beck's conjecture is proved for the zero divisor graph of a poset with 0 by Halaš and Jukl [8] and for the zero divisor graph of a poset having the smallest element 0 with respect to an ideal by Joshi [11] under the assumption that the corresponding zero divisor graph does not contain an infinite clique. Therefore, we raise the following question.

QUESTION. Is Beck's conjecture true for the complement of the zero divisor graph of a poset P?

We answer this question partially in the following theorem.

THEOREM 3.1. *Let L be a finite Boolean lattice such that $|L| = 2^n$, $n \in \mathbb{N}$. Then $\omega((\Gamma_{\{0\}}(L))^c) = \chi((\Gamma_{\{0\}}(L))^c) = 2^{n-1} - 1$.*

PROOF. Consider a Boolean lattice $L = \mathbf{2}^n$ for some $n \in \mathbb{N}$. Clearly, $V((\Gamma_{\{0\}}(L))^c) = L \setminus \{0, 1\}$. Hence $|V((\Gamma_{\{0\}}(L))^c)| = 2(2^{n-1} - 1)$. If p is an atom in L, then we observe that $C_p = \{x \neq 1 \mid x \geq p\} = [p) \setminus \{1\}$ forms a clique in $(\Gamma_{\{0\}}(L))^c$. Further, if $y \in V((\Gamma_{\{0\}}(L))^c) \setminus C_p$, then $y \wedge p = 0$. Hence C_p is a maximal clique in $(\Gamma_{\{0\}}(L))^c$. Thus $\omega((\Gamma_{\{0\}}(L))^c) = |C_p| = 2^{n-1} - 1$. Thus $V((\Gamma_{\{0\}}(L))^c)$ contains a complete subgraph $K_{2^{n-1}-1}$.

Now we claim that either x or x' (the complement of x) is a member of C_p.

If $x, x' \in V((\Gamma_{\{0\}}(L))^c) \setminus C_p$, then $p \wedge x = 0 = p \wedge x'$. By the distributivity of L, $p \wedge (x \vee x') = 0$. This gives $p = 0$, a contradiction. Hence either x or x' belong to C_p.

Hence, for colouring $(\Gamma_{\{0\}}(L))^c$, we first assign the colours to the elements of C_p. Therefore, we need $2^{n-1} - 1$ colours. Now for any element $y \notin C_p$, we have $y' \in C_p$ and we assign the colour of y' to y. Thus $\omega((\Gamma_{\{0\}}(L))^c) = \chi((\Gamma_{\{0\}}(L))^c) = |C_p| = 2^{n-1} - 1$. □

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The following theorem is essentially due to Joshi [11].

THEOREM 3.2 (Joshi [11]). *If $\omega(\Gamma_I(L))$ is finite, then L has only a finite number of minimal prime semi-ideals belonging to I and if n is this number, then $\chi(\Gamma_I(L)) = \omega(\Gamma_I(L)) = n$.*

In the following, we prove a form of Beck's conjecture (Theorem 3.3) for zero divisor graphs with respect to an ideal I under the assumption that the complement of the corresponding zero divisor graph with respect to an ideal I does not contain an infinite clique. Explicitly, we prove the following result.

THEOREM 3.3. *If $\omega((\Gamma_I(L))^c)$ is finite, then L has only a finite number of minimal prime ideals belonging to I and if n is this number, then $\chi(\Gamma_I(L)) = \omega(\Gamma_I(L)) = n$.*

First, we give bounds for the clique number of $(\Gamma_I(L))^c$.

LEMMA 3.4. *Let P be any prime ideal of L belonging to I (that is, $I \subseteq P$). Then the following statements hold.*

(1) *If $(\Gamma_I(L))^c$ does not contain any infinite clique, then $Z_I(L) \setminus P$ is finite.*
(2) *If $\omega((\Gamma_I(L))^c)$ is finite, then $Z_I(L) \setminus P$ is finite and indeed $|Z_I(L) \setminus P| \leq \omega((\Gamma_I(L))^c)$.*

PROOF. (1) Let $Z_I(L) \setminus P$ be an infinite set. We choose an infinite sequence of distinct elements $x_i \in Z_I(L) \setminus P$. Since P is a prime ideal of L belonging to I and as $x_i \notin P$ for $i = 1, 2, 3, \ldots$, it follows that $x_i \wedge x_j \notin I$ for all $i, j \in \{1, 2, 3, \ldots\}$. Observe that the subgraph of $(\Gamma_I(L))^c$ induced on $\{x_i \mid i = 1, 2, 3, \ldots\}$ is an infinite clique. This contradicts the assumption that $(\Gamma_I(L)^c)$ does not contain any infinite clique. Hence, $Z_I(L) \setminus P$ is finite.
(2) Let $\omega((\Gamma_I(L))^c) = n$. We assert that $|Z_I(L) \setminus P| \leq n$. Suppose to the contrary that $|Z_I(L) \setminus P| \geq n + 1$. Let $\{x_1, x_2, \ldots, x_{n+1}\} \subseteq Z_I(L) \setminus P$. Then it is clear that the subgraph of $(\Gamma_I(L))^c$ induced on $\{x_1, x_2, \ldots, x_{n+1}\}$ is a clique, a contradiction to $\omega((\Gamma_I(L))^c) = n$. Thus $|Z_I(L) \setminus P| \leq n = \omega((\Gamma_I(L))^c)$. □

LEMMA 3.5. *Let P be any prime ideal of L belonging to I. Let $A = \{Q \mid Q$ is a prime ideal of L belonging to I such that $Q \subseteq Z_I(L)$ but $Q \nsubseteq P\}$. Then the following statements hold.*

(1) *If $(\Gamma_I(L))^c$ does not contain any infinite clique, then A can admit only a finite number of elements which are pairwise incomparable under inclusion.*
(2) *If $\omega((\Gamma_I(L))^c)$ is finite, then A can admit at most $\omega((\Gamma_I(L))^c)$ elements which are pairwise incomparable under inclusion.*

PROOF. (1) Suppose that $(\Gamma_I(L))^c$ does not contain any infinite clique. Suppose on the contrary that there exist infinitely many elements in A which are pairwise incomparable under inclusion. Hence there exist $Q_i \in A$ for $i = 1, 2, 3, \ldots$ with Q_i and Q_j such that $Q_i \nsubseteq Q_j$ and $Q_j \nsubseteq Q_i$ for all $i, j \in \{1, 2, 3, \ldots\}$ with $i \neq j$. Now $Q_i \nsubseteq P$ for $i = 1, 2, 3, \ldots$. Hence there exists $x_1 \in Q_1 \setminus P$. Further, $Q_2 \nsubseteq P \cup Q_1$, so there exist $x_2 \in Q_2 \setminus (P \cup Q_1)$. Continuing in this way we get $x_i \in Q_i \setminus (P \cup Q_1 \cup Q_2 \cup \cdots \cup Q_{i-1})$

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for all $i = 1, 2, 3, \ldots$. Observe that $\{x_i \mid i = 1, 2, 3, \ldots\} \subseteq Z_I(L) \setminus P$. Hence $Z_I(L) \setminus P$ is infinite, a contradiction to Lemma 3.4(1).

(2) Let $\omega((\Gamma_I(L))^c) = n$. Suppose that A admits more than n elements which are pairwise incomparable under inclusion. Let $\{Q_1, \ldots, Q_{n+1}\} \subseteq A$ be such that Q_i and Q_j are not comparable for all $i, j \in \{1, 2, \ldots, n+1\}$ with $i \neq j$. Let $x_1 \in Q_1 \setminus P$. As in (1), we can choose $x_i \in Q_i \setminus (P \cup Q_1 \cup \cdots \cup Q_{i-1})$ for $i = 2, \ldots, n+1$. Observe that $\{x_i \mid i = 1, 2, \ldots, n+1\} \subseteq Z_I(L) \setminus P$. This implies that $|Z_I(L) \setminus P| \geq n + 1 > \omega((\Gamma_I(L))^c)$, a contradiction to Lemma 3.4(2). □

Lemma 3.6. *The following statements hold.*

(1) *If $(\Gamma_I(L))^c$ does not contain any infinite clique, then (i) the set of maximal N-primes of I in L is finite, (ii) the set of minimal prime ideals of L belonging to I is finite.*

(2) *If $\omega((\Gamma_I(L))^c)$ is finite, then (i) L can admit at most $\omega((\Gamma_I(L))^c) + 1$ maximal N-primes of I and if L admits exactly k maximal N-primes of I with $k \geq 3$, then $k \leq \omega((\Gamma_I(L))^c)$, (ii) L can admit at most $\omega((\Gamma_I(L))^c) + 1$ minimal prime ideals belonging to I, and if k is the number of minimal prime ideals of L belonging to I with $k \geq 3$, then $k \leq \omega((\Gamma_I(L))^c)$.*

Proof. (1) Let P be a maximal N-prime of I in L. Let $A = \{Q \mid Q$ is a maximal N-prime of I in L and $Q \neq P\}$. Since any maximal N-prime of I in L is a subset of $Z_I(L)$ and as distinct maximal N-primes of I in L are not comparable under inclusion, it follows from Lemma 3.5(1) that A is finite. It is clear that the set of all maximal N-primes of I in L is $A \cup \{P\}$. Thus L can admit only a finite number of maximal N-primes of I.

If P is any minimal prime ideal of L belonging to I, then $P \subseteq Z_I(L)$. Using similar arguments as above, it is easy to see that L can admit only a finite number of minimal prime ideals of L belonging to I.

(2) Let P be any maximal N-prime of I in L and let $A = \{Q \mid Q$ is a maximal N-prime of I in L and $Q \neq P\}$. Further, assume that $\omega((\Gamma_I(L))^c) = n$. Then by Lemma 3.5(1), $|A| \leq n$ and, using the same arguments as in the proof of (1), the set of all maximal N-primes of I in L is $A \cup \{P\}$. Thus L can admit at most $n + 1$ maximal N-primes of I.

Suppose that L admits exactly k maximal N-primes of I with $k \geq 3$. Let $\{P_1, P_2, \ldots, P_k\}$ be the set of all maximal N-primes of I. Then $Z_I(L) = \bigcup_{i=1}^{k} P_i$. It is easy to see that there exists $x_i \in P_i \setminus \bigcup_{j \in \{1,2,\ldots,k\} \setminus \{i\}} P_j$. Clearly, $x_i \neq x_j$, for all $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. Since $k \geq 3$, there exists at least one $t \in \{1, 2, \ldots, k\}$ such that both x_i and x_j are not in P_t and hence $x_i \wedge x_j \notin I$. Thus the subgraph of $(\Gamma_I(L))^c$ induced on $\{x_i \mid i = 1, 2, \ldots, k\}$ is a clique and so $k \leq \omega((\Gamma_I(L))^c)$.

Using similar arguments as above and using Lemma 3.5(2), the statement about minimal prime ideals can be proved. □

Proof of Theorem 3.3. It follows from Lemma 3.6 that L has only a finite number of minimal prime ideals of L belonging to I, say P_i, $1 \leq i \leq n$. Since I is a semiprime ideal in L, by Lemma 1.1, we have $I = \bigcap_{i=1}^{n} P_i$. Now we define a colouring f on vertices of $\Gamma_I(L)$ as $f(x) = \min\{i \mid x \notin P_i\}$. If $x, y \in V(\Gamma_I(L))$ such that x and y are adjacent in $\Gamma_I(L)$, then $x \wedge y \in I = \bigcap_{i=1}^{n} P_i$. Clearly, $x \notin P_i$ and $y \notin P_j$ for some minimal

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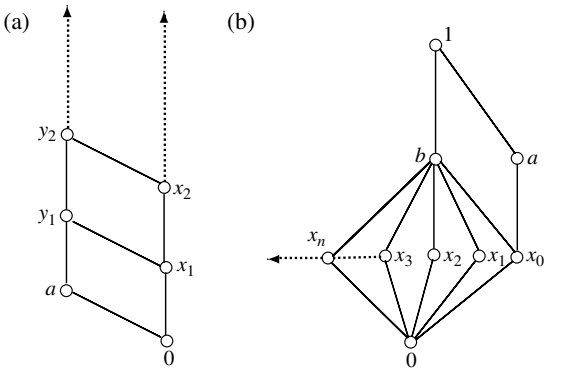

FIGURE 3. (a): $\omega((\Gamma_I(L))) < \infty \nRightarrow \omega((\Gamma_I(L))^c) < \infty$ for semiprime ideal I. (b) $\omega((\Gamma_I(L))^c) < \infty \nRightarrow \omega(\Gamma_I(L)) < \infty$ for non-semiprime ideal I.

prime ideals P_i and P_j belonging to I. Since $x \wedge y \in I = \bigcap_{i=1}^{n} P_i$, we conclude that $y \in P_i$ and $x \in P_j$. Thus $f(x) \neq f(y)$ and so f is a colouring on $\Gamma_I(L)$. This implies $\chi(\Gamma_I(L)) \leq n$. Thus $\omega(\Gamma_I(L)) \leq \chi(\Gamma_I(L)) \leq n$. Hence $\omega(\Gamma_I(L))$ is finite. Now we claim that, for a semiprime ideal I of L every minimal prime semi-ideal belonging to I is a minimal prime ideal belonging to I. For this let P be any minimal prime semi-ideal belonging to I. Let $x, y \in P$ be such that $x \vee y \notin P$ (such elements exist as P is a semi-ideal). Clearly, $F = L \setminus P$ is a maximal filter with respect to the property that $I \cap F = \emptyset$. Thus $I \cap (F \vee [x)) \neq \emptyset$ and $I \cap (F \vee [y)) \neq \emptyset$. Let $u \in I \cap (F \vee [x))$ and $v \in I \cap (F \vee [y))$. Then $u \geq f_1 \wedge x$ and $v \geq f_2 \wedge y$ for some $f_1, f_2 \in F$. Also, $u, v \in I$ give $(f_1 \wedge f_2) \wedge x, (f_1 \wedge f_2) \wedge y \in I$. By semiprimeness of I, we get $(f_1 \wedge f_2) \wedge (x \vee y) \in I$. Using this fact and $(f_1 \wedge f_2) \wedge (x \vee y) \in F$, we have $I \cap F \neq \emptyset$, a contradiction. Hence P is a prime ideal belonging to I. Clearly, P is a minimal prime ideal belonging to I. Thus, if I is a semiprime ideal of L, then every minimal prime semi-ideal belonging to I is a minimal prime ideal belonging to I. Hence, L has only a finite number of minimal prime semi-ideals belonging to I and if n is this number, then this together with $\omega(\Gamma_I(L)) < \infty$ and Theorem 3.2 yields that $\chi(\Gamma_I(L)) = \omega(\Gamma_I(L)) = n$. □

REMARK 3.7. From Theorem 3.3, it is clear that for a semiprime ideal I of a lattice L with 0, if $\omega((\Gamma_I(L))^c) < \infty$ then $\omega(\Gamma_I(L)) < \infty$. But the converse need not be true. Consider the lattice L as shown in Figure 3(a). Then we observe that $\Gamma_{(0]}(L)$ has a finite clique while $(\Gamma_{(0]}(L))^c$ does not. Also note that if we drop the condition that I is a semiprime ideal in L then the assertion that $\omega((\Gamma_I(L))^c) < \infty$ implies $\omega(\Gamma_I(L)) < \infty$ need not be true. Consider the lattice L depicted in Figure 3(b). Clearly, (0] is not a semiprime ideal. Further it is easy to observe that $\{x_0, x_1, x_2, \ldots\}$ forms an infinite clique in $\Gamma_{(0]}(L)$ while $\{x_0, a\}$ is the only finite clique in $(\Gamma_{(0]}(L))^c$.

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