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Two facts concerning the transformations which satisfy the weak Pinsker property

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Dedicated to the memory of Bill Parry

Abstract. We show that every ergodic, finite entropy transformation which satisfies the weak Pinsker property possesses a finite generator whose two-sided tail field is exactly the Pinsker algebra. This is proved by exhibiting a generator endowed with a block structure quite analogous to the one appearing in the construction of the Ornstein–Shields examples of non Bernoulli K-automorphisms. We also show that, given two transformations T_1 and T_2 in the previous class (i.e. satisfying the weak Pinsker property), and a Bernoulli shift B, if $T_1 \times B$ is isomorphic to $T_2 \times B$, then T_1 is isomorphic to T_2. That is, one can ‘factor out’ Bernoulli shifts.

1. *Introduction*

Let $(X, \mathcal{A}, m, T)$ be an ergodic, invertible, finite entropy measure-preserving transformation. It is well known that, given any finite partition P of X, the ‘remote past’ $\bigwedge_{N>0} \bigvee_{n\leq -N} T^n P$ is the zero entropy algebra of the factor generated by P (so that, if P generates, i.e. if $\bigvee_{n\in Z} T^n P = \mathcal{A}$ $(=(P)_T)$, then $\bigwedge_{N>0} \bigvee_{n\leq -N} T^n P$ is the Pinsker algebra of T, $\pi(T)$). The same holds true for the ‘distant future’ $\bigwedge_{N>0} \bigvee_{n\geq N} T^n P$. A striking result of Ornstein and Weiss [**2**] states that, for every transformation as above, there exists a finite partition P such that $\bigwedge_{N>0} \bigvee_{|n|\geq N} T^n P = \mathcal{A}$.

We consider the class WP of transformations which satisfy the weak Pinsker property, that is, those transformations $(X, \mathcal{A}, m, T)$ which are ergodic, invertible, with finite entropy and such that there exist two sequences of finite partitions of X, H_n, B_n, $n \geq 1$ verifying: (1) $(H_1)_T \vee (B_1)_T = \mathcal{A}$; (2) $(H_1)_T \perp (B_1)_T$; (3) for all $n > 0$ the $T^i B_n$, $i \in \mathbb{Z}$ are independent; (4) $(H_n)_T \vee (B_n)_T = (H_{n-1})_T$, $n \geq 2$; (5) $(H_n)_T \perp (B_n)_T$, $n \geq 1$; and (6) $E(H_n, T) \downarrow 0$, see [**6**]. We show that, for every transformation T in WP, there exists a finite partition P which generates and for which the ‘two-sided tail field’

$\bigwedge_{N>0} \bigvee_{|n|\geq N} T^n P$ is exactly the Pinsker algebra $\pi(T)$. This means that, opposite to what is taking place in the Ornstein and Weiss construction, the generator P behaves well under the $\vee$ operation of its more and more distant future with its more and more remote past. This also says that all the transformations in WP are 'hyperbolic' in Vershik's sense [**7**].

In §3, we come back to the still unanswered question A of [**4**]. If the answer to the question were positive, it would in particular imply the following. Consider two transformations T_1 and T_2 and take the direct product of both of them with the same Bernoulli transformation to obtain T_1' and T_2'. If T_1' and T_2' are isomorphic, then T_1 and T_2 are isomorphic. We show this last fact to be true if T_1 and T_2 are in WP.

2.

LEMMA 1. *We consider an ergodic, invertible, measure-preserving transformation* $(X, \mathcal{A}, m, T)$ *endowed with a finite generator* P, $P = (p_0, p_1, \ldots, p_k)$. *We assume that* $k \geq 4$. *Let* $\epsilon > 0$ *be given. Assume that there exist two finite partitions of* X, H_1 *and* B_1, $H_1 = (h_1, h_2)$, *such that* $(H_1)_T \perp (B_1)_T$, $(H_1 \vee B_1)_T = \mathcal{A}$, *the* $T^i B_1$, $i \in \mathbb{Z}$ *are independent, and* $E(H_1, T) < \epsilon/10$. *There then exists a partition* ${}^1P$ *of* X *for which the following is true.*

(1) $|{}^1P - P| < \epsilon$, *where* ${}^1P = ({}^1p_0, {}^1p_1, \ldots, {}^1p_k)$ *and* $|{}^1P - P| = \sum_i m({}^1p_i \triangle p_i)$.

(2) ${}^1P$ *is a generator, i.e.* $({}^1P)_T = \mathcal{A}$.

(3) *There exist integers* n_0, n_1 *and* n *satisfying* $n_0 < n_1$ *and* $n_1 < \epsilon n$ *such that the set* $F = \{x \mid T^i x \in {}^1p_0, 0 \leq i \leq n_0 - 1\}$ *has the property that (a)* $F, TF, \ldots, T^{n-1}$ *are pairwise disjoint, (b)* $m(\bigcup_{i=0}^{i=n-1} T^i F) > 1 - (\epsilon/100)$ *and (c)* F *is* $(H_1)_T$ *measurable.*

(4) *The atoms* h_1 *and* h_2 *have on* $X - \bigcup_{i=0}^{i=n-1} T^i F$ *exactly the same trace as* ${}^1p_1$ *and* ${}^1p_2$, *respectively.*

(5) *The two partitions* $\bigvee_{i=0}^{i=-n+1} T^i(H_1)$ *and* $\bigvee_{i=n_0}^{i=n_1-1} T^{-i}({}^1P)$ *have the same trace on* F.

(6) *Let* ${}^1I = \bigcup_{i=n_1}^{i=n-1} T^i F$. *Let* ${}^1P_{{}^1I} = ({}^1p_0', {}^1p_1', \ldots, {}^1p_k', {}^1I)$ *be the partition which coincides with* ${}^1P$ *outside of* ${}^1I$, *so* ${}^1p_i' = p_i \cap {}^1I^c$. *(Note that (3), (4) and (5) imply that* $({}^1P_{{}^1I})_T = (H_1)_T$.*)*

We consider a point w *in* $({}^1P_{{}^1I})_T$ *given by its* ${}^1P_{{}^1I}$*-name (i.e.* $w = (w_i)$, $i \in \mathbb{Z}$, w_i *the element of the alphabet of* ${}^1P_{{}^1I}$ *in which* $T^i w$ *lies); for almost every* w, $\mathbb{Z}$ *contains a family of disjoint intervals* I_k *of length* $n - n_1 - 1$ *which have the property that, for all* k, *for all* $i \in I_k$, $w_i = {}^1I$. *We want the following to be satisfied by the partition* ${}^1P$*:*

(a) *for almost every* w, *the partitions* $\bigvee_{i \in I_k} T^i({}^1P)$, $k \in \mathbb{Z}$, *form an independent family for the conditional measure* m/w;

(b) *for almost every* w *as before, for every interval* $J \subset I_k$ *such that* $\natural J > n_0/2$, $m(\bigcap_{i \in J} T^i({}^1p_0)/w) = 0$.

Proof. (of the existence of ${}^1P$) (Here we have been trying to keep a proper balance between the length of the statement and the length of its proof.) Let $a = E(H_1, T)$, where $a < \epsilon/10$. Let $n_0 = \lfloor (1/10\epsilon) \rfloor$; let $n > \lfloor (1/\epsilon^3) \rfloor$ be such that $\bigvee_{i=0}^{i=n} T^{-i} H_1$ is made of atoms of size greater than 2^{-2na} on a set G of measure greater than $1 - \epsilon/200$. (That such an n exists is a consequence of the McMillan theorem and of the hypothesis made on the entropy of $E(H_1, T)$.) Let $n_1 = \epsilon n$. Because of the properties of H_1 and B_1, there exists an integer K such that $\bigvee_{-K}^{+K} T^i(H_1 \vee B_1) \supset^{\epsilon/4} P$. We call P_1 the $\bigvee_{-K}^{+K} T^i(H_1 \vee B_1)$ measurable partition which satisfies $|P_1 - P| < \epsilon/4$. We also assume that n is sufficiently large to ensure that $K/n < \epsilon^2$. We apply the strong Rokhlin lemma and obtain a set F_1 which is $(H_1)_T$ measurable, such that $F_1, TF_1, \ldots, T^{n-1}F_1$ are disjoint and have a union which fills $1 - \epsilon/200$ of the space X and such that $m(F_1 \cap G) > (1 - \epsilon/200)m(F_1)$. We call $F = F_1 \cap G$. We put in ${}^1p_0$ the set $\bigcup_{i=0}^{i=n_0-1} T^i F$. We now define ${}^1P$ on $\bigcup_{i=n_0}^{i=n_1-1} T^i F$. There are less than 2^{2na} sets in the trace of the partition $\bigvee_{i=0}^{i=n} T^{-i} H_1$ on F. Therefore:

(A) Because $a < \epsilon/10$, $3^{\epsilon n} > 2^{2na}$, and because of the choice of n_0, n_1 and n, we can assign in a one-to-one way to every atom h in $\bigvee_{i=0}^{i=n} T^{-i} H_1 \cap F$ a name h'_i, $0 \le i \le n_1 - n_0$, in the alphabet ${}^1p_1, {}^1p_2, {}^1p_3$. For every h as before, we put $T^{i+n_0}h$ in the element of the partition ${}^1P$ given by the symbol h'_i for all $0 \le i \le n_1 - n_0$. This and (4) make that ${}^1P$ is now defined on the complement of ${}^1I$ (given by (6)) and satisfies (5) and (6)(a). As a consequence ${}^1P_{1_I}$ is also defined.

It remains to construct ${}^1P$ on ${}^1I$. First, we associate to every name w of length n_1 in the alphabet of ${}^1P_{1_I}$, which starts with n_0 symbols ${}^1p_0$ (we call this family of words W), a distribution d_w on P^{n-n_1} (the family of words of length $n - n_1$ in the alphabet of P) for which we want the following:

(B) d_w restricted to $P_K^{n-n_1-K}$ (we only consider the preceding words between K and $n - n_1 - K$) is the conditional distribution of $\bigvee_{i=K}^{i=n-n_1-K} T^i P_1$ on the atom $h(w)$ of $\bigvee_{i=0}^{i=n} T^i H_1$ whose name corresponds to w by the assignment (A) modified in such a way that it gives zero measure to every word that contains a subword of length $n_0/2$ which is only made of the symbol p_0.

(C) $m(F)(\sum_{w\in W} E(d_w)m(h(w)) = E(P, T) - E(H_1, T)$.

In this part, it will frequently be the case that we shall consider sets and partitions in product spaces $(X \times Y, \mathcal{A} \times \mathcal{B}, m \times \mu)$ (for instance). By a slight abuse of notation, a set A which is measurable with respect to one algebra, say $X \times \mathcal{B}$, will still be called A and not $X \times A$. The same applies to partitions. There is never any ambiguity since the space to which the partition belongs is always recognized by the name of the transformation which acts upon it.

We call T_F the transformation which is the factor of T restricted to $({}^1P_{1_I})_T$ induced on F. Let $Z = (\prod_{w\in W}(P^{n-n_1}, d_w))^{\mathbb{Z}}$. That is, for every $w \in W$, we consider X_w the product $(P^{n-n_1}, d_w)^{\mathbb{Z}}$. The Bernoulli shift on this space is called S_w. For every $w_0 \in W$, let S'_{w_0} be the transformation on Z defined by $\prod_{w\in W; w\ne w_0} I_w \times S_{w_0}$ (I_w is the identity transformation on X_w). On $F \times Z$, we build the following skew product $\hat{T}_F$: $(f, z) \to (T_F f, S'_{w(f)}z)$. Here $w(f)$ is the element of W whose symbols between $n_0 + 1$ and n_1 are given by the $(n_0 + 1, n_1)$ name of f under the action of T for the partition ${}^1P_{1_I}$, that is, the sequence of elements of the partition ${}^1P_{1_I}$ in which the orbit of f falls between time n_0 and time n_1. Note that, since $f \in F$, the name of f between 0 and n_0 is always a sequence of symbols p_0. We use the same ceiling function above F as the one which allowed us to go from T_F to T, to go from $\hat{T}_F$ to $\hat{T}$. Let the partition $\hat{P}$ then be defined, given every h as in (A), on $\bigcup_{i=n_1}^{i=n} \hat{T}^i h$ by considering $h \times (P^{n-n_1}, d_w)$

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(in $F \times Z$) and putting, given an atom $x = x_{n_1}, \ldots, x_n$ of P^{n-n_1} (the sequences of length $n - n_1$ in the alphabet of P), $\hat{T}^i(x \cap h)$ in x_i, $n_1 \le i \le n$. It is a consequence of our construction that $\hat{P}$ coincides with ${}^1P$ on ${}^1I^c$ and that $(\hat{P}, \hat{T})$ satisfies (6) of the lemma (in particular, the $(\hat{P}_{1_I}, \hat{T})$ process is the same as the $({}^1P_{1_I}, T)$ process). This implies that $(\hat{P}, \hat{T})$ is $\hat{P}_{1_I}$ relatively very weak Bernoulli (for the definition, see [**5**, p. 223]). The same is true of (P, T) relative to ${}^1P_{1_I}$ (as a consequence of [**4**, Proposition 4] and [**3**]). It follows from (C) and from Abramov's formula that the relative entropies are also identical. An other important observation is that if one looks at the process (P_1, T) relative to ${}^1P_{1_I}$ at the times i such that $T^i w$ is in ${}^1I$, but not in the first or the last K elements of an ${}^1I$ block ($w \in ({}^1P_{1_I})_{\hat{T}}$), this process can be matched up to $\epsilon/4$ with $(\hat{P}, \hat{T})$. This follows from (B). It is therefore easy, using a suitable relatively independent joining (above (P_1, T)), to construct a relative joining of $(\hat{P}, \hat{T})$ with (P, T) which is diagonal when restricted to the product $({}^1P_{1_I})_{\hat{T}} \times (H_1)_T$ and such that $|P - \hat{P}| < \epsilon$. We can, using the relative version of the Ornstein copying lemma, construct in X a partition $\hat{P}'$ such that the process $(\hat{P}', T)$ is arbitrarily close in relative entropy and in distribution to the $(\hat{P}, \hat{T})$ process and satisfies $|\hat{P}' - P| < \epsilon$. The relative isomorphism theorem (cf. [**4**, Proposition 3]) now tells us that there exists ${}^1P$ such that ${}^1P$ is a generator for T, the $({}^1P, T)$ process is identical to the $(\hat{P}, \hat{T})$ process and $|{}^1P - P| < \epsilon$ (and ${}^1P_{1_I} = P_{1_I}$). All the conclusions of the lemma are therefore satisfied. □

Given a transformation $(X, \mathcal{A}, m, T)$, a factor algebra $\mathcal{A}_1$ and H an $\mathcal{A}_1$-measurable partition, we say that H splits off in $\mathcal{A}_1$ if there exists a partition B which is $\mathcal{A}_1$-measurable such that $(B)_T \perp (H)_T$, $(B)_T \vee (H)_T = \mathcal{A}_1$ and the $T^i B$, $i \in \mathbb{Z}$, are independent. In the case when $\mathcal{A}_1 = \mathcal{A}$, we just say that H splits off.

LEMMA 2. *We assume that we are given $(X, \mathcal{A}, m, T)$ as before, a finite generator P, $P = (p_0, p_1, \ldots, p_k)$, $k > 3$, and a sequence of l partitions $H_1, H_2, \ldots, H_l$ such that, for all $1 \le k \le l - 1$, $(H_k)_T \supset (H_{k+1})_T$, H_{k+1} splits off in $(H_k)_T$ and H_1 splits off. Let P satisfy the following: there exists a sequence of integers n_i^0, n_i^1, n_i, $1 \le i \le l$, such that:*

(1) *The sets $(x \in X \mid T^{-1}x \notin p_0, T^{n_i^0}x \notin p_0, T^j x \in p_0, 0 \le j \le n_i^0 - 1) = F_i$ are the bases of $(H_i)_T$ measurable Rokhlin towers T_i of height n_i, $1 \le i \le l$.*

(2) *If we call ${}^iI$ the sets which are the unions of the $n_i - n_i^1 + 1$ last levels of the towers T_i, $1 \le i \le l$, and ${}^iP$ the partitions whose elements are ${}^iI$ and the trace of P on ${}^iI^c$, $1 \le i \le l$, then, for all i, $(H_i)_T = ({}^iP)_T$.*

(3) *For almost every element $w = (w_s)$, $s \in \mathbb{Z}$ of $({}^iP)_T$ (where w_s is the element of the partition ${}^iP$ in which $T^s w$ lies), if we consider the family $A(w)$ of intervals of $\mathbb{Z}$ of length $n_i - n_i^1 + 1$ corresponding to the values of s such that $T^s w \in {}^iI$, then, for the conditional measure m/w, the family of partitions $\bigvee_{s \in J} T^s({}^{i-1}P)$, $J \in A(w)$, is independent. Let $\delta > 0$ be given. Then there exists $\mu > 0$ such that if H_{l+1} is an $(H_l)_T$ measurable partition which splits off in $(H_l)_T$ such that $E(H_{l+1}, T) < \mu$, one can construct a partition P' which will still be a generator for T such that $|P - P'| < \delta$, (1) and (2) are valid for P' with l changed to $l + 1$, the sets ${}^iI$ being turned into ${}^iI'$ and $m({}^iI \Delta {}^iI') < \delta$, $1 \le i \le l$.*

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Proof. For the proof we are going to proceed in the same way as in the proof of Lemma 1. We work first in $(H_l)_T$ and consider ${}^lP$ as a generator for the action of T restricted to this factor. We pick δ_1 which will be made precise later and we apply Lemma 1 (using δ_1 instead of ϵ) to get n^0_{l+1}, n^1_{l+1}, n_{l+1}, F_{l+1}, H_{l+1}, ${}^l_1P'$ and ${}^{l+1}I$ satisfying the conclusions (1), (2), (3), (4), (5) and (6) of Lemma 1. Given δ_2, if δ_1 was chosen sufficiently small, we can modify ${}^l_1P'$ into ${}^lP'$ on $\bigcup_{i=n^1_{l+1}}^{i=n_{l+1}} T^i F_{l+1}$ in such a way that:

(1) The traces of $\bigvee_{k=0}^{k=n_{l+1}} T^{-k}({}^l_1P')$ and of $\bigvee_{k=0}^{k=n_{l+1}} T^{-k}({}^lP')$ on F_{l+1} are identical.

(2) Given any atom h in the trace of $\bigvee_{k=0}^{k=n_{l+1}} T^{-k}({}^lP')$ on F_{l+1}, the ${}^lP'$-name of its points between time 0 and time n_{l+1} has the property that the p_0 symbols, as soon as they appear in blocks of size $>n^0_{l-1}$, appear in fact in blocks of length n^0_l (we call these blocks (B)) followed by a block of symbols of length $n^1_l - n^0_l - 1$ (we call (C) the family of the names which appear in these blocks) followed by a block of length $n_l - n^1_l - 1$ of symbols ${}^lI'$. For a proportion δ_2 of these atoms h, for more than a proportion $1 - \delta_2$ of the blocks in (C), the ${}^l_1P'$-name and the ${}^lP'$-name differ in a proportion less than δ_2 of the time.

(3) Because of the way ${}^lI'$ has just been built, the set $F'_l = (x \in X \mid T^{-1}x \notin p_0$, $T^jx \in p_0, 0 \le j \le n^0_l - 1)$ is the basis of a Rokhlin tower of height n_l. We then have $|F_l - F'_l| < \delta_2$ (partitioning F'_l according to the ${}^lP'$ name of its points between $n^0_l + 1$ and n^1_l, the labels of the elements of the partition are exactly described by (C)).

(4) $|{}^lP' - {}^lP| < \delta_2$ (as a consequence of (2)) and $({}^lP')_T = ({}^lP)_T$ (as a consequence of (1)). We can associate to every w in (C) a distribution d'_w on the words of length n_{l-1} in the alphabet of ${}^{l-1}P$ in such a way that for these w whose (C) names in ${}^{l-1}P$ differ from their ${}^{l-1}P'$ name a proportion less than δ_2 of the time (which, by (2) occupy more than $1 - \delta_2$ of the set $\bigcup_{i=n^1_l}^{i=n_l} T^i F'_l$), $d(d'_w, d_w) < \delta_2$ (the conditional distributions of P_{l-1} names and P'_{l-1} names of length n_{l-1} are close) and also that $m(F_l)(\sum_{w\in(C)} E(d_w)m(w)) = E(H_{l-1}, T) - E(H_l, T)$.

Now, given δ_3, if δ_2 was chosen sufficiently small, because of (4) we can construct as in Lemma 1 a partition ${}^{l-1}P'$ which behaves in $(H_{l-1})_T$ with respect to ${}^{l-1}P$ exactly in the same way as ${}^lP'$ was behaving with respect to ${}^lP$ (with δ_2 changed into δ_3). Iterating the previous process l times, the last condition being δ, we work all the way backward and take for the announced μ the corresponding value of δ_1. □

We can now prove the proposition which provides the existence of generators with a nice block structure (as in the examples in [**1**]) for all the transformations in WP.

PROPOSITION 1. *Let $(X\mathcal{A}, m, T)$ be an ergodic invertible measure-preserving transformation with finite entropy which satisfies the weak Pinsker property. Then call $(H_j)_T$, $j \in N$, a decreasing sequence of factor algebras which split off such that $E(H_j, T) \downarrow 0$. There exists a finite generator P, $P = (p_0, p_1, \ldots, p_k)$, $k \ge 4$, an increasing subsequence k_j of integers, and sequences of integers n^0_j, n^1_j, n_j, $j \in N$, $n^0_j < n^1_j < n_j$, such that the following are satisfied:*

(1) *for all j, $(x \in X \mid T^{-1}x \notin p_0, T^ix \in p_0, i \in [0, n^0_j - 1], T^{n^0_j} \notin p_0)$ is the basis F_j of an $(H_{k_j})_T$ measurable Rokhlin tower of height n_j;*

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(2) *if we call* I^j *the* $(H_{k_j})_T$*-measurable set which is* $\bigcup_{l=n_j^1}^{l=n_j} T^l F_j$ *and* P^j *the partition whose elements are* I^j *and the trace of* P *on* $(I^j)^c$*, then* $(P^j)_T = (H_{k_j})_T$*;*

(3) *for almost every* $w \in (P^j)_T$ *(for all* $j \in N$*) if* $I_k(w)$ *is the family of intervals of length* $n_j - n_j^1 + 1$ *in the name of* w *which are* n_j^1 *away to the right of a sequence of exactly* n_j^0 *symbols* p_0*, then for the conditional measure* m/w *the sequence of partitions* $\bigvee_{s\in I_k} T^s P^{j-1}$*,* $k \in \mathbb{Z}$*, is independent.*

Proof. We fix a finite generator H_0 in X. Assume that we have a partition P_j such that the conditions of Lemma 2 are satisfied for H_i, $1 \le i \le j$, and furthermore such that $\bigvee_{-K^i_{j,s}}^{K^i_{j,s}} T^l P_j \supset^{\epsilon^i_{j,s}} H_{k_i}$, $0 \le i \le j$, $1 \le s \le j$ $(*)$. We apply the lemma with adequate δ to produce k_{j+1} (which controls $E(H_{k_{j+1}}, T)$) and P_{j+1} for which the conclusions of the lemma will be satisfied with j changed into $j+1$, and the conditions $(*)$ being kept with $K_{j+1,s} = K^i_{j,s}$, $\epsilon^i_{j+1,s} = \epsilon^i_{j,s} \times (1 + 1/2^{j+1})$, $1 \le s \le j$ and $K^i_{j+1,j+1}$ such that

$$\bigvee_{-K^i_{j+1,j+1}}^{K^i_{j+1,j+1}} T^l P_{j+1} \supset^{1/2^{j+1}} H_{k_i}, \quad 0 \le i \le j+1.$$

This gives the induction procedure which ensures that P_j converges towards P which satisfies the conclusions of the proposition. □

As a corollary we find the announced property for the two-sided tail field.

COROLLARY 1. *Let* $(X\mathcal{A}, m, T)$ *be ergodic invertible with finite entropy satisfying the weak Pinsker property. Then there exists a generator* P *whose two-sided tail field* $\bigwedge_{N>0} \bigvee_{|n|\ge N} T^n P$ *is exactly* $\pi(T)$*, the Pinsker algebra of* T*.*

Proof. We consider the generator P constructed in the proposition. We have the property that, for all j, $\bigwedge_{N>0} (P^j)_T \bigvee_{|n|\ge N} T^n P = (P^j)_T$ as a consequence of the property (3) of this generator which says exactly that for almost every $w \in (P^j)_T$, for the measure m/w, the algebra $\bigwedge_{N>0} \bigvee_{|n|\ge N} T^n P$ is trivial. Therefore $\bigwedge_{N>0} \bigvee_{|n|\ge N} T^n P \subset (P^j)_T$ for all j and this implies the conclusion. □

3.

PROPOSITION 2. *Let* $(X_1, \mathcal{A}_1, m_1, T_1)$ *and* $(X_2, \mathcal{A}_2, m_2, T_2)$ *be two ergodic, invertible finite entropy measure-preserving transformations which satisfy the weak Pinsker property and let* $(Y, \mathcal{B}, \mu, S)$ *be a Bernoulli shift for which the following is true: the two product transformations* $(X_i \times Y, \mathcal{A}_i \times \mathcal{B}, m_i \times \mu, T_i \times S)$*,* $i = 1, 2$*, are isomorphic. Then* $(X_1, \mathcal{A}_1, m_1, T_1)$ *is isomorphic to* $(X_2, \mathcal{A}_2, m_2, T_2)$*.*

Proof. Let P_1 and P_2 be finite generators for T_1 and T_2, respectively. The hypothesis says that there exists a transformation $(X, \mathcal{A}, m, T)$ and four partitions of X, P_1', P_2', B_1 and B_2, such that the following hold.

(1) The process (P_1', T) is a copy of the process (P_1, T_1).

(2) The process (P_2', T) is a copy of (P_2, T_2).

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(3) T restricted to (B_i, T), $i = 1, 2$, is a copy of $(Y, \mathcal{B}, \mu, S)$.
(4) $(P_1')_T$ and $(B_1)_T$ are independent as are $(P_2')_T$ and $(B_2)_T$.
(5) $(P_1')_T \vee (B_1)_T = \mathcal{A} = (P_2')_T \vee (B_2)_T$.

Because of (1) and the hypothesis, there exist two partitions of X, H_1 and B_1', such that $(H_1)_T$ is independent of $(B_1')_T$, $(H_1)_T \vee (B_1')_T = (P_1')_T$, and, in the same way, H_2 and B_2' such that $(H_2)_T$ is independent of $(B_2')_T$, $(H_2)_T \vee (B_2)_T = (P_2')_T$. The hypothesis implies that T_1 and T_2 have the same entropy a. We can assume that $E(H_i, T) \leq a/3$, $i = 1, 2$. It is obvious that H_1 and H_2 split off. Now [**4**, Proposition 4] implies that H_1 is H_2 conditionally finitely determined, and also that H_2 is H_1 conditionally finitely determined. This, as a consequence of [**4**, Proposition 3], implies that both H_1 and H_2 split off in $(H_1 \vee H_2)_T$. If we make the direct product of T restricted to $(H_1 \vee H_2)_T$ with a Bernoulli shift of entropy $a - E(H_1 \vee H_2, T)$, we find two generators of this transformation, one generating a copy of the (P_1, T_1) process, the other generating a copy of the (P_2, T_2) process, yielding the announced isomorphism. Note that this proof covers the case where a or any of the quantities $E(H_1, T)$, $E(H_2, T)$, $E(H_1, T|(H_2)_T)$, and $E(H_2, T|(H_1)_T)$ are equal to zero (see [**4**, Corollary 5.1]). □

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REFERENCES

[1] D. S. Ornstein and P. C. Shields. An uncountable family of K-automorphisms. *Adv. Math.* **10** (1973), 103–120.
[2] D. S. Ornstein and B. Weiss. Every transformation is bilaterally deterministic. *Israel J. Math.* **21**(2–3) (1975), 154–158.
[3] M. Rahe. Relatively finitely determined implies relatively very weak Bernoulli. *Canad. J. Math.* **30**(3) (1978), 531–548.
[4] J.-P. Thouvenot. Quelque propriétés des systèmes dynamiques qui se décomposent en un produit de deux systèmes dont l'un est un schema de Bernoulli. *Israel J. Math.* **21**(2–3) (1975), 177–203.
[5] J.-P. Thouvenot. Remarques sur les systèmes dynamiques donnés avec plusieurs facteurs. *Israel J. Math.* **21**(2–3) (1975), 215–232.
[6] J.-P. Thouvenot. On the stability of the weak Pinsker property. *Israel J. Math.* **27**(2) (1977), 150–162.
[7] A. Vershik. Towards the definition of metric hyperbolicity. *Moscow Math. J.* **5**(3) (2005), 721–737.