# ANOTHER ENUMERATION OF TREES

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Given a set of vertices which have each been assigned one of the colours $C_1, C_2, \ldots, C_m$, with $n_j$ vertices $C_j$, a formula is derived for the number of oriented trees on these vertices, having a designated root, and subject to any number of restrictions of the form "no arc goes from a vertex of colour $C_i$ to a vertex of colour $C_j$". The formula is based on a combinatorial construction which defines a correspondence between such trees and certain sequences.

In 1889, A. Cayley (**2**) found that the number of oriented trees which can be constructed on $n$ vertices, having a specified root, is exactly $n^{n-2}$. Cayley's formula has been generalized in several interesting ways; see Raney (**8**), Riordan (**9**), Knuth (**5**), Good (**3**), Moon (**6**). In this paper we present a combinatorial construction which leads to another rather pleasant generalization of Cayley's formula.

The term "oriented tree" is used in this paper to distinguish the trees discussed here from "free trees" (which have no root and no orientation specified for the arcs) and from "ordered trees" (in which the relative order of the vertices pointing to a vertex is significant as well as the orientation of the arcs).

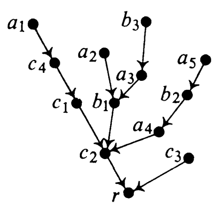


FIGURE 1. Oriented tree

Figure 1 shows an oriented tree on the vertices $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, c_1, c_2, c_3, c_4, r$, in which all arcs go from an "$a$" to a "$b$" or a "$c$", or from a "$b$" to an "$a$" or a "$c$", or from a "$c$" to a "$c$" or an "$r$". The admissible kinds of arcs just described are represented graphically in Figure 2. It is natural to ask: "How many ways are there to draw arcs on the specified vertices so that an oriented tree of this type is obtained?" In general, we will find that if there are $a$, $b$, and $c$ vertices of the corresponding types, then

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the total number of oriented trees subject to the restrictions of Figure 2 is exactly $(a + c)^{b-1}(b + c)^{a-1}(c + 1)^{c-1}(c^2 + ca + cb)$. The theorem below shows that similar formulas may be obtained when any diagram of "chromatic constraints" is considered in place of Figure 2.

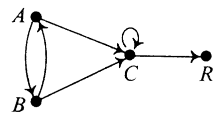


FIGURE 2. Chromatic constraints

In the following discussion, the notation $|X|$ stands for the number of elements in the (finite) set $X$. Furthermore, if $f$ is a function, we write $f^0(x) = x$ and $f^{r+1}(x) = f(f^r(x))$ when the latter is defined.

**1. The basic construction.** Let us say $(U, V, f)$ is a *T-graph* if $V$ is a finite set of vertices, $U \subseteq V$, and $f$ is a function from $U$ into $V$ such that there are no "cycles", i.e., no vertices $x$ with $f^m(x) = x$ for some $m > 0$. It follows that for all $x \in V$ there is a least integer $m \geqq 0$ such that $f^m(x) \notin U$, and in this case we write $f^\infty(x) = f^m(x)$. In terms of this notation, an *oriented tree* with root $r$ is a $T$-graph of the form $(U, U \cup \{r\}, f)$.

The enumeration formulas to be derived rest essentially on the following construction which generalizes a theorem due to Prüfer (**7**).

LEMMA. *Let $U$, $V$, and $W$ be sets of vertices, with $W$ disjoint from $U \cup V$. Let $f$ be a function from $V - U$ into $U$. The number of functions $h$ from $U$ into $V \cup W$, such that $(U \cup V, U \cup V \cup W, f \cup h)$ is a $T$-graph, is*

$$|V \cup W|^{|U|-1}|W|.$$

*Proof.* Let $n = |U|$. We will prove the more interesting result that there is a one-to-one correspondence between such functions $h$ and sequences of vertices $a_1, a_2, \ldots, a_n$ such that $a_k \in V \cup W$ for $1 \leqq k < n$ and $a_n \in W$. For this purpose, assume the set $U \cup V \cup W$ has been linearly ordered by some relation.

First, suppose such a function $h$ is given, and consider the directed graph $G$ with vertices $U \cup V \cup W$, with arcs from $v$ to $f(v)$ for $v \in V - U$, and with arcs from $u$ to $h(u)$ for $u \in U$. Let us say a vertex $u \in U$ is "free" with respect to $G$ if there is no oriented path from $u'$ to $u$ for any other $u' \in U$. Since there are no oriented cycles in $G$, there is at least one free vertex. Let $u_1$ be the lowest free vertex (in the assumed linear ordering). Once $u_1, \ldots, u_t$ have been defined, let $u_{t+1}$ be the lowest free vertex in the directed graph obtained from $G$ by removing $u_k$ and the arc from $u_k$ to $h(u_k)$ for $1 \leqq k \leqq t$. This rule defines a sequence $u_1, u_2, \ldots, u_n$ containing each of the $n$ vertices

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of $U$. Now let $a_k = h(u_k)$ for $1 \leqq k \leqq n$. Clearly, $a_k \in V \cup W$ for $1 \leqq k < n$ and $a_n \in W$.

Conversely, assume such a sequence $a_1, a_2, \ldots, a_n$ is given. Let us now say a vertex $u \in U$ is "free" with respect to the sequence if there is no $j$ for which $a_j = u$ or for which $a_j \in V - U$ and $f(a_j) = u$. Since $a_n \in W$, there must be at least one free vertex. Let $u_1$ be the lowest free vertex (in the assumed linear ordering). Once $u_1, \ldots, u_t$ have been defined, let $u_{t+1}$ be the lowest free vertex with respect to $a_{t+1}, \ldots, a_n$ which is different from $u_1, \ldots, u_t$. This rule defines a sequence $u_1, u_2, \ldots, u_n$ containing each of the $n$ vertices of $U$. Now let $h(u_k) = a_k$ for $1 \leqq k \leqq n$. Then $(U \cup V, U \cup V \cup W, f \cup h)$ is a $T$-graph, since $(f \cup h)^m(u_k) = u_r$ for $m > 0$ implies $r > k$.

The two constructions just given are obviously inverse to each other, so the stated one-to-one correspondence has been established. (Prüfer essentially published the special case in which $U = V$ and $|W| = 1$.) We may note also from the construction that if $u$ is the highest vertex of $U$, in the assumed linear ordering, then $(f \cup h)^\infty(u) = a_n$; for in the rule for determining the sequence $u_1, u_2, \ldots, u_n$, the vertex $u$ becomes free only when there is an oriented path from $u$ to all remaining vertices.

**2. The main construction.** Let $\mathscr{C}$ be a family of non-empty, disjoint sets, and let $V = \bigcup \mathscr{C}$, i.e., $V = \bigcup_{C \in \mathscr{C}} C$. We will assume $V$ is a finite set of vertices, partitioned into the classes represented by $\mathscr{C}$. Let $\mathscr{G}$ be a directed graph on the elements of $\mathscr{C}$ (cf. Figure 2); and for $C \in \mathscr{C}$, let $\mathscr{G}(C)$ be the set of all $C' \in \mathscr{C}$ such that there is an arc from $C$ to $C'$ in $\mathscr{G}$. The family $\mathscr{C}$ and the directed graph $\mathscr{G}$ will be fixed throughout this section.

If $\mathscr{K}$ is a subset of $\mathscr{C}$, we say a $\mathscr{K}$-*structure* is a $T$-graph $(U, V, f)$ for which $U = \bigcup \mathscr{K}$, $V = \bigcup \mathscr{C}$, and if $u \in C \in \mathscr{K}$, then $f(u) \in \bigcup \mathscr{G}(C)$. In other words, we are considering the set of $T$-graphs on $V$ satisfying the "chromatic constraints" of $\mathscr{G}$, where we think of $\mathscr{C}$ as a set of colours. Our goal is to enumerate the number of possible $\mathscr{K}$-structures, i.e., the number of functions $f$ satisfying the restrictions just mentioned.

THEOREM. *The number of possible* $\mathscr{K}$*-structures is equal to*

$$\sum_g \left( \prod_{C \in \mathscr{K}} |\bigcup \mathscr{G}(C)|^{|C|-1} |g(C)| \right),$$

*where the sum is over all functions* $g$ *such that* $(\mathscr{K}, \mathscr{C}, g)$ *is a* $T$*-graph and* $g(C) \in \mathscr{G}(C)$ *for all* $C \in \mathscr{K}$.

(Note: Using a theorem of Tutte (**11**), this formula can also be written as

$$\det A \cdot \prod_{C \in \mathscr{K}} |\bigcup \mathscr{G}(C)|^{|C|-1},$$

where $A$ is a matrix whose rows and columns are indexed by the elements of $\mathscr{K}$; $A_{CC'} = -|C' \cap \bigcup \mathscr{G}(C)|$ when $C \neq C'$, and $A_{CC} = |\bigcup \mathscr{G}(C) - C|$.)

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*Proof*. Let $n = |\mathscr{K}|$. We will prove in fact that there is a one-to-one correspondence between $\mathscr{K}$-structures and sets of $n$ sequences of the form

$$(*) \qquad a_{C1}, a_{C2}, \ldots, a_{Cr}, \qquad r = |C|,$$

where $a_{Ck} \in \bigcup \mathscr{G}(C)$, $1 \leqq k < r$, and $a_{Cr} \in g(C)$ for all $C \in \mathscr{K}$, where $(\mathscr{K}, \mathscr{C}, g)$ is a $T$-graph contained in $\mathscr{G}$. Assume that the vertices $V$ are linearly ordered, and so are the "colours" $\mathscr{C}$.

First suppose that a $\mathscr{K}$-structure $(U, V, f)$ is given. We will define a function $g$ as required, and a set of sequences (*), and a sequence $C_1, C_2, \ldots, C_n$ representing the colours of $\mathscr{K}$, and also a sequence $u_1, u_2, \ldots, u_n$ with $u_k \in C_k$. Start with $C_1$, the lowest colour in the assumed linear order, and $u_1$ the highest vertex of $C_1$. If $t < n$ and if $C_t$ and $u_t$ have been chosen, we define $C_{t+1}$ and $u_{t+1}$ as follows: Let $m$ be maximal such that $f^m(u_t) \in C_t$, and $f^k(u_t) \notin \{C_1, \ldots, C_{t-1}\}$ for $0 \leqq k \leqq m$. Let $v_t = f^{m+1}(u_t)$ and let $g(C_t)$ be the class in $\mathscr{G}(C_t)$ such that $v_t \in g(C_t)$. Now if $g(C_t) = C_k$ for some $k$, $1 \leqq k < t$, or if $g(C_t) \notin \mathscr{K}$, choose $C_{t+1}$ to be the lowest colour of $\mathscr{K} - \{C_1, \ldots, C_t\}$ and let $u_{t+1}$ be the highest vertex of that colour. Otherwise, let $C_{t+1} = g(C_t)$, $u_{t+1} = v_t$.

We wish to prove that $(\mathscr{K}, \mathscr{C}, g)$ is a $T$-graph. Note that if $g(C_t) = C_k$ for $t < k \leqq n$, then $k = t + 1$. If $(\mathscr{K}, \mathscr{C}, g)$ is not a $T$-graph, there is some $t$ such that $g^r(C_t) = C_t$ and $g(C_t) = C_k$ for some $r > 0$ and $k < t$. We can find $s \leqq t$ such that $g(C_k) = C_{k+1}$ for $s \leqq k < t$ but $C_s \neq g(C_k)$ for $k < s$; it follows that $g(C_t) = C_k$ for some $k$ such that $s \leqq k < t$. Consider the values

$$u_s, f(u_s), f^2(u_s), \ldots, v_s = u_{s+1}, f(u_{s+1}), \ldots, u_t, f(u_t), \ldots, v_t, \ldots, u,$$

where $u$ is the first element encountered that is in

$$\bigcup(\mathscr{C} - \mathscr{K}) \cup C_1 \cup \ldots \cup C_{s-1}.$$

By construction, none of the elements of this sequence after $u_{s+1}$ are in $C_s$; none of the elements after $u_{s+2}$ are in $C_{s+1}$; and so on. It is therefore impossible for $v_t$ to be an element of $C_k$ for $s \leqq k < t$. This contradiction proves $(\mathscr{K}, \mathscr{C}, g)$ is a $T$-graph.

Finally, for $t = n, n-1, \ldots, 1$, we successively construct the sequence (*) for $C = C_t$. Consider the $T$-graph $(U_t, V, f_t)$, where $U_t = C_{t+1} \cup \ldots \cup C_n$ and $f_t$ is $f$ restricted to $U_t$. Reorder the elements of $C_t$, if necessary, so that $u_t$ is the highest element, and apply the construction of the lemma with $U$, $V$, $W$, and $f$ replaced, respectively, by $C_t$, $V_t$, $\bigcup \mathscr{G}(C_t) - V_t$, and $\phi_t$, where $V_t = \{v \in \bigcup \mathscr{G}(C_t) \mid f_t^\infty(v) \in C_t\}$, and $\phi_t = f_t^\infty$ restricted to $V_t$. The values of $f$ restricted to $C_t$ now correspond to a function $h$ as stated in the lemma, so we obtain a sequence (*) in which the last element is $v_t$.

Conversely, let us suppose we are given a set of sequences (*) for each $C \in \mathscr{K}$, defining a function $g$ of the required type. We will define a function $f$ such that $(U, V, f)$ is a $T$-graph of the required type, and we will also define a sequence $C_1, C_2, \ldots, C_n$ representing the colours of $\mathscr{K}$, and a sequence

$u_1, u_2, \ldots, u_n$ with $u_k \in C_k$. Start with $C_1$, the lowest colour in the assumed linear order, and $u_1$, the highest vertex of $C_1$. If $t < n$ and if $C_t$ and $u_t$ have been chosen, let $v_t$ be the last element of the sequence (*) for $C_t$. Now if $g(C_t) = C_k$ for some $k$, $1 \leqq k < t$, or if $g(C_t) \notin \mathscr{K}$, choose $C_{t+1}$ to be the lowest colour of $\mathscr{K} - \{C_1, \ldots, C_t\}$ and let $u_{t+1}$ be the highest vertex of that colour. Otherwise, let $C_{t+1} = g(C_t)$, $u_{t+1} = v_t$.

Now for $t = n, n-1, \ldots, 1$, we successively define $f$ on the elements of $C_t$ so that no cycles are introduced. Suppose $f$ has already been defined on $U_t = C_{t+1} \cup \ldots \cup C_n$ and let $f_t$ be this function. Reorder the elements of $C_t$ if necessary so that $u_t$ is the highest element, and apply the construction of the lemma with $U$, $V$, $W$, and $f$ replaced, respectively, by $C_t$, $V_t$, $\bigcup \mathscr{G}(C_t) - V_t$, and $\phi_t$ (as above). The construction has been carried out so that $v_t \notin V_t$, since, if $g(C_t) = C_{t+1}$, we have $f_t^\infty(v_t) = f_t^\infty(u_{t+1}) = f_t^\infty(v_{t+1})$ and, continuing in this manner, it is clear that $f_t^\infty(v_t) \notin C_t$ when $(\mathscr{K}, C, g)$ is a $T$-graph. Therefore, the lemma applies and it determines a function $h$ which may be used to define $f_{t-1} = f_t \cup h$.

The two constructions just described are inverses of each other, so the theorem has been proved. It is possible to give a much simpler proof of this theorem, based directly on the theorem of Tutte (**11**) which expresses the number of subtrees of a directed graph, having a given root, as a determinant. We consider the directed graph having $\bigcup \mathscr{C} \cup \{r_0\}$ as vertices, where $r_0$ is a new symbol; there is an arc in this graph from $v$ to $v'$ if and only if either $v \in C \in \mathscr{K}$ and $v' \in \bigcup \mathscr{G}(C)$, or if $v \in C \in \mathscr{C} - \mathscr{K}$ and $v' = r_0$. The number of $\mathscr{K}$-structures is obviously the number of subtrees of this directed graph having root $r_0$. The corresponding determinant is easily evaluated by using elementary row and column operations; as an example of this evaluation we consider the situation in Figures 1 and 2, where $\mathscr{C} = \mathscr{K} \cup \{R\}$, $\mathscr{K} = \{A, B, C\}$, $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3, c_4\}$. By Tutte's theorem, the number of $\mathscr{K}$-structures is

$$
\det \left(\begin{array}{ccccc|ccc|cccc|c}
 & & A & & & & B & & & & C & & R \\
7 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
0 & 7 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & 7 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 7 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
\hline
-1 & -1 & -1 & -1 & -1 & 9 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 9 & 0 & -1 & -1 & -1 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 9 & -1 & -1 & -1 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -4 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

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$$
= \det \left(\begin{array}{ccccc|ccc|cccc|c}
7 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
-7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & -1 & -1 & -1 & -1 & 9 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -9 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 5 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$
= \det \left(\begin{array}{ccccc|ccc|cccc|c}
7 & 0 & 0 & 0 & 0 & -3 & -1 & -1 & -4 & -1 & -1 & -1 & 0 \\
0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-5 & -1 & -1 & -1 & -1 & 9 & 0 & 0 & -4 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

$$
= (3+4)^{5-1}(5+4)^{3-1}(4+1)^{4-1} \det \begin{pmatrix} 7 & -3 & -4 \\ -5 & 9 & -4 \\ 0 & 0 & 1 \end{pmatrix}.
$$

The formula in the theorem can also be obtained by means of multivariate generating functions and a generalization of Lagrange's inversion formula, as shown by Good (**3**, p. 512). (Several misprints in the formula stated by Good should be corrected.)

Even though there are alternate means for proving the theorem, the proof given here has several advantages since it establishes a useful correspondence with sequences. It is now possible to enumerate such oriented trees with a given number of vertices of in-degree 2, etc., as in Riordan (**9**), since the in-degree of each vertex is the number of times it appears in the sequences (*).

As an example of the construction in the above proof, consider the tree in Figure 1 and suppose we order the colours $A < B < C < R$. Figure 1 is an $\{A, B, C\}$-structure. The construction selects $C_1 = A$, and we may take $a_1 < a_2 < a_3 < a_4 < a_5$ as an ordering of the elements of $A$, therefore $u_1 = a_5$. Since $f^2(a_5) \in A$ but $f^{3+k}(a_5)$ is not, we set $v_1 = f^3(a_5) = c_2$, $g(A) = C$, and $C_2 = C$. The elements of $C$ must be ordered so that $c_2$ is highest, therefore let $c_1 < c_3 < c_4 < c_2$. Now $f(c_2) = r$ and thus we let $v_2 = r$ and $g(C) = R$. Finally, we take $C_3 = B$ and $b_1 < b_2 < b_3 = u_3$. In this case, $f^2(b_3) = b_1 \in B$ but since $f(b_3)$ is in $C_1 = A$ we take $v_3 = a_3$ not $v_3 = c_2$; hence $g(B) = A$. The construction of the lemma is now used, starting with a sequence for $C_3 = B$. Here, all vertices are free since $V_1$ is vacuous, and the sequence is simply $f(b_1), f(b_2), f(b_3)$:

$$B\colon c_2,\ a_4,\ a_3.$$

The $C_2$ $(=C)$ sequence is constructed next (remembering that $c_1 < c_3 < c_4 < c_2$):

$$C\colon r,\ c_1,\ c_2,\ r.$$

Finally, the $C_1$ $(=A)$ sequence is constructed:

$$A\colon c_4,\ b_1,\ b_1,\ b_2,\ c_2.$$

The original tree is reconstructible from these three sequences. Conversely, from any sequences of this type (i.e., the $A$ sequence contains five elements of $B$ and $C$; the $B$ sequence contains three elements of $A$ and $C$, and if the last element is in $A$, the last element of $A$ is not in $B$; and the $C$ sequence contains four elements of $C$ and $R$, the last in $R$) we can construct an oriented tree which will lead to these sequences.

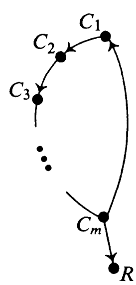


FIGURE 3. Cyclic case

**3. Examples and applications.** Consider a cyclic directed graph like that in Figure 3; for oriented trees, suppose $|R| = 1$. The number of oriented trees in which all arcs go from colour $C_i$ to $C_{i+1}$ or from $C_m$ to $C_1$ or from $C_m$ to $R$ is

$$n_2^{n_1} n_3^{n_2} \ldots n_m^{n_{m-1}} (n_1 + 1)^{n_{m-1}}; \qquad n_j = |C_j|.$$

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If we like, we may merge together colours $R$ and $C_1$; then we find

$$n_2^{n_1-1} n_3^{n_2} \dots n_m^{n_{m-1}} n_1^{n_m}$$

*is the number of oriented trees on* $C_1 \cup C_2 \cup \dots \cup C_m$ *in which all arcs go from colour* $C_i$ *to* $C_{(i+1)(\mathrm{mod}\ m)}$ *and the root is in* $C_1$. The case $m=1$ is Cayley's theorem; the case $m = 2$ was proved by Scoins (**10**) and it also follows from a more general result due to Austin (**1**).

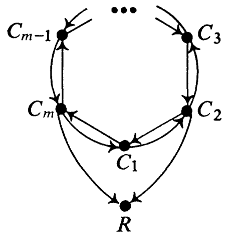



FIGURE 4. Symmetric cycle

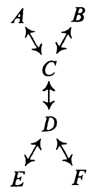



FIGURE 5. A free tree

If the arcs are allowed to go in either direction between colour $C_i$ and colour $C_{i+1}$, we get a situation like Figure 4. In Figure 4, consider $R$ as essentially a specified element of $C_1$ which has been temporarily given a new name. Since the arcs in the remaining graph are symmetric, we may consider free trees instead of oriented trees, namely, connected graphs without cycles; *the number of free trees on* $C_1 \cup C_2 \cup \dots \cup C_m$, *with a vertex of colour* $C_i$ *adjacent to a vertex of colour* $C_j$ *only if* $i \equiv (j \pm 1)(\mathrm{mod}\ m)$, *is*

$$(n_m + n_2)^{n_1-1}(n_1 + n_3)^{n_2-1}(n_2 + n_4)^{n_3-1} \dots (n_{m-1} + n_1)^{n_m-1} n_1 n_2 \dots n_m \times \left(\frac{1}{n_m n_1} + \frac{1}{n_1 n_2} + \dots + \frac{1}{n_{m-1} n_m}\right), \quad n_j = |C_j|, \quad m \geqq 3.$$

In general, enumeration formulas for free trees can be obtained in this way when the graph of "chromatic constraints" has a symmetric incidence matrix. Another interesting case occurs when the directed graph $\mathscr{G}$ is itself a free tree with symmetric arcs. Thus, for example, *the number of free trees on* $A \cup B \cup C \cup D \cup E \cup F$, *with adjacent vertices having adjacent colours in the diagram of Figure* 5, *is*

$$|C|^{|A|-1}|C|^{|B|-1}|A \cup B \cup D|^{|C|-1}|C \cup E \cup F|^{|D|-1}|D|^{|E|-1}|D|^{|F|-1}|C|^2|D|^2.$$

In general, the number of such free trees is

$$\prod_{C \in \mathscr{C}} |\cup\{C' \mid C' \text{ adjacent to } C\}|^{|C|-1}|C|^{\mathrm{degree}(C)-1}$$

when the chromatic constraints themselves form a free tree.

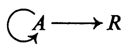
The above formulas can also be used to derive non-obvious summation identities. Let $\mathscr{G}$ be a directed graph on $\{C_1, C_2, \dots, C_m, R\}$ and let

$p(n_1, n_2, \ldots, n_m, x)$ be the formula for the number of $\{C_1, C_2, \ldots, C_m\}$-structures according to the theorem in § 2, where $n_k = |C_k|$ and $x = |R|$. Then we have the convolution formula

$$\sum_{k_1, k_2, \ldots, k_m} \binom{n_1}{k_1}\binom{n_2}{k_2} \ldots \binom{n_m}{k_m} p(k_1, k_2, \ldots, k_m, x) p(n_1 - k_1, n_2 - k_2, \ldots, n_m - k_m, y) = p(n_1, n_2, \ldots, n_m, x + y).$$

For in every $T$-graph $(C_1 \cup \ldots \cup C_m, C_1 \cup \ldots \cup C_m \cup R, f)$ we can partition the vertices $v$ of $C_1 \cup \ldots \cup C_m$ according to the values of $f^\infty(v) \in R$; the above formula expresses the number of ways colours $C_1, \ldots, C_m$ can be split into $k_1, \ldots, k_m$ and $n_1 - k_1, \ldots, n_1 - k_m$ respective elements so that the first group falls into $x$ specified elements of $R$ and the second group falls into the other $y$ elements of $R$.

As an example, the simple graph




yields the identity

$$\sum_k \binom{n}{k} x(x + k)^{k-1} y(y + n - k)^{n-k-1} = (x + y)(x + y + n)^{n-1},$$

which is directly related to Abel's generalization of the binomial theorem (see **4**).

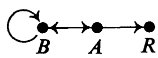



FIGURE 6

From the graph of Figure 6 we get the following identity in integers $m$, $n$, $x$, $y$:

$$\sum_{j,k} \binom{m}{j}\binom{n}{k} xj(x + k)^{j-1}(j + k)^{k-1} y(m - j)(y + n - k)^{m-j-1} \times (m + n - j - k)^{n-k-1} = (x + y)m(x + y + n)^{m-1}(m + n)^{n-1}.$$

(Suitable conventions are assumed when 0/0 appears.) This identity appears to be very difficult to derive by any other means, and more complicated graphs will give still more intricate formulas of this type.

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