# Q-ANALOGUE OF A TWO VARIABLE INVERSE PAIR OF SERIES WITH APPLICATIONS TO BASIC DOUBLE HYPERGEOMETRIC SERIES

CHRISTIAN KRATTENTHALER

**1. Introduction.** Let

$$f(z_1, z_2) = \big(f_1(z_1, z_2), f_2(z_1, z_2)\big)$$

be a pair of a formal series (fps) in $z_1$ and $z_2$ of the form

(1.1)     $f_i(z_1, z_2) = z_i / \psi_i(z_1, z_2)$   for $i = 1, 2,$

where $\psi_i(z_1, z_2)$ is an fps with $\psi_i(0, 0) \neq 0$ for $i = 1, 2$. Then there exists a unique pair of fps

$$F(z_1, z_2) = \big(F_1(z_1, z_2), F_2(z_1, z_2)\big),$$

which is also of the form (1.1), with

(1.2)     $f_i\big(F_1(z_1, z_2), F_2(z_1, z_2)\big) = z_i$   for $i = 1, 2.$

This pair is called the *inverse* of $f(z_1, z_2)$.

For $\mathbf{k}, \mathbf{l} \in \mathbf{Z}^2$ (pairs of integers), $\mathbf{k} = (k_1, k_2)$ and $\mathbf{l} = (l_1, l_2)$, we adopt the familiar multidimensional notations, $|\mathbf{k}| = k_1 + k_2$, $\mathbf{k} + \mathbf{l} = (k_1 + l_1, k_2 + l_2)$, $\mathbf{k} \geqq \mathbf{l}$ if and only if $k_1 \geqq l_1$ and $k_2 \geqq l_2$, $\mathbf{0} = (0, 0)$,

$$\mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2},$$

and

$$f^{\mathbf{k}}(z_1, z_2) = f_1^{k_1}(z_1, z_2) f_2^{k_2}(z_1, z_2).$$

By two-variable Lagrange inversion the coefficients of $F_i(z_1, z_2)$ or, even more generally, the coefficients of integral powers of $F_i(z_1, z_2)$, $i = 1, 2$, may be evaluated (see for example [7, (4.5) with $\phi(\mathbf{x}) = \mathbf{x}^{\mathbf{l}}$]):

(1.3)     $\langle \mathbf{z}^{\mathbf{k}} \rangle F^{\mathbf{l}}(z_1, z_2) = \langle \mathbf{z}^{-\mathbf{l}} \rangle f^{-\mathbf{k}}(z_1, z_2) D(f)(z_1, z_2),$

where

$$D(f)(z_1, z_2) = \psi_1(z_1, z_2) \psi_2(z_1, z_2) \frac{\partial f}{\partial z}(z_1, z_2),$$

Received August 14, 1988.

with $\frac{\partial f}{\partial z}(z_1, z_2)$ the Jacobian of $f(z_1, z_2)$. ($\langle \mathbf{z^k} \rangle a(z_1, z_2)$ means the coefficient of $\mathbf{z^k}$ in the formal Laurent series (fLs) $a(z_1, z_2)$.)

An example of such an inverse pair is

$$(1.4)(a) \quad f_0(z_1, z_2) = \left( \frac{z_1}{1 - z_2}, \frac{z_2}{1 - z_1} \right)$$

and

$$(1.4)(b) \quad F_0(z_1, z_2) = \left( \frac{z_1(1 - z_2)}{1 - z_1 z_2}, \frac{z_2(1 - z_1)}{1 - z_1 z_2} \right).$$

With the help of Lagrange inversion the coefficient of $\mathbf{z^k}$ in $F_0^\mathbf{l}(z_1, z_2)$, and, consequently, of

$$(1 - z_2)^{l_1}(1 - z_1)^{l_2} / (1 - z_1 z_2)^{l_1 + l_2},$$

can be evaluated. Essentially, this was done in [6, Theorem 1] and [7, p.190]. In addition, Evans et al. [6, Theorem 9] give the following $q$-analogue of their formula, which may be written as

$$
\begin{aligned}
(1.5) \quad \langle \mathbf{z^k} \rangle & \frac{(qz_1 z_2; q)_{\alpha+\beta}}{(qz_2; q)_\alpha (qz_1; q)_\beta} \\
&= q^{k_1 k_2} \frac{(q^{\beta - k_2 + 1}; q)_{k_1 - 1}(q^{\alpha - k_1 + 1}; q)_{k_2 - 1}}{(q; q)_{k_1}(q; q)_{k_2}} \\
&\quad \times \left( (q^{k_1} - q^\alpha)(q^{k_2} - q^\beta) - (1 - q^{k_1})(1 - q^{k_2}) \right).
\end{aligned}
$$

(For definition of the symbol $(a; q)_\beta$ see (4.2).) They prove it by a basic hypergeometric transformation formula and put the question if a proof by two-variable $q$-Lagrange inversion could be given. The first approach towards multivariable $q$-Lagrange formulas was made by the author in [16]. In the present paper we give a new two-variable $q$-Lagrange formula (Theorem 3), a special case of which helps to establish a proof of (1.5). As a by-product, in Theorem 5, a $q$-analogue of the inverse pair (1.4) is obtained.

The coefficient matrices of the members of the pair which will be given in (4.13) of Theorem 5 are inverses of each other (with respect to matrix multiplication). In [8, 10] Gessel and Stanton, using certain pairs of matrices, which are inverses of each other, systematically derive one-variable basic hypergeometric summations and transformations. Applying the same method to the inverse matrices determined by the pair in (4.13), we are able to deduce a number of two-variable basic hypergeometric summations and transformations, all of which appear to be new. (F. H. Jackson [12, 13] was the first to treat basic double hypergeometric series systematically. While there exists an extensive theory on one-variable basic hypergeometric series, until now there have not appeared many results on basic double series. In particular, the number of summation

theorems is not very large. A collection of papers dealing with basic double series is included in the references, although we do not claim that this list is complete.)

Our paper is organized as follows. Section 2 contains a short outline of [**16**, Section 3] in order to explain how two-variable *q*-Lagrange inversion should be understood. In Section 3 the *q*-Lagrange formula is given which we need to prove the coefficient theorem (1.5). The proof of (1.5) is done in Section 4 by use of the inverse pair (4.13). Finally, Section 5 is devoted to the derivation of basic double hypergeometric summations and transformations.

**2. Preliminaries.** Unless otherwise stated, in this paper we shall always consider fps (fLs) in the indeterminates $z_1$ and $z_2$ of the form

$$\sum_{\mathbf{i} \geq \mathbf{m}} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}, \ \text{ for some } \mathbf{m} \in \mathbf{Z}^2,$$

whose coefficients $a_i$ are rational functions in the indeterminate *q*.

A sequence

$$f = \left( f_{\mathbf{k}}(z_1, z_2) \right)_{\mathbf{k} \in \mathbf{Z}^2}$$

of fLs satisfying

$$(2.1) \qquad f_{\mathbf{k}}(z_1, z_2) = \sum_{\mathbf{n} \geq \mathbf{k}} f_{\mathbf{n}\mathbf{k}} \mathbf{z}^{\mathbf{n}} \quad \text{and} \quad f_{\mathbf{k}\mathbf{k}} \neq 0$$

is called a *diagonal sequence*. The sequence of powers of

$$f(z_1, z_2) = \left( f_1(z_1, z_2), f_2(z_1, z_2) \right)$$

satisfying (1.1), $\left( f^{\mathbf{k}}(z_1, z_2) \right)_{\mathbf{k} \in \mathbf{Z}^2}$, is an example of a diagonal sequence. Given another sequence

$$g = \left( g_{\mathbf{l}}(z_1, z_2) \right)_{\mathbf{l} \in \mathbf{Z}^2},$$

where

$$g_{\mathbf{l}}(z_1, z_2) = \sum_{\mathbf{k} \geq \mathbf{l}} g_{\mathbf{k}\mathbf{l}} \mathbf{z}^{\mathbf{k}},$$

the substitution of *f* into *g* is defined by

$$g(f) = \left( h_{\mathbf{l}}(z_1, z_2) \right)_{\mathbf{l} \in \mathbf{Z}^2}$$

with

$$h_{\mathbf{l}}(z_1, z_2) = \sum_{\mathbf{k} \geq \mathbf{l}} g_{\mathbf{k}\mathbf{l}} f_{\mathbf{k}}(z_1, z_2),$$

or more precisely

$$h_{\mathbf{l}}(z_1, z_2) = \sum_{\mathbf{n}} \left( \sum_{\mathbf{n} \geq \mathbf{k} \geq \mathbf{l}} f_{\mathbf{n}\mathbf{k}} g_{\mathbf{k}\mathbf{l}} \right) \mathbf{z}^{\mathbf{n}}.$$

The sequence

$$\mathcal{F} = \left( F_{\mathbf{l}}(z_1, z_2) \right)_{\mathbf{l} \in \mathbf{Z}^2}$$

is called *inverse* of $f$ if

$$f(\mathcal{F}) = (\mathbf{z}^{\mathbf{k}})_{\mathbf{k} \in \mathbf{Z}^2},$$

i.e., if and only if

(2.2)        $$\sum_{\mathbf{n} \geq \mathbf{k}} f_{\mathbf{n}\mathbf{k}} F_{\mathbf{n}}(z_1, z_2) = \mathbf{z}^{\mathbf{k}} \quad \text{for all } \mathbf{k} \in \mathbf{Z}^2$$

or, equivalently

(2.3)        $$\sum_{\mathbf{m} \geq \mathbf{n} \geq \mathbf{k}} F_{\mathbf{m}\mathbf{n}} f_{\mathbf{n}\mathbf{k}} = \delta_{\mathbf{m}\mathbf{k}} \quad \text{for all } \mathbf{m}, \mathbf{k} \in \mathbf{Z}^2,$$

where we have set

$$F_{\mathbf{l}}(z_1, z_2) = \sum_{k \geq \mathbf{l}} F_{\mathbf{k}\mathbf{l}} \mathbf{z}^{\mathbf{k}}.$$

($\delta_{\mathbf{m}\mathbf{k}}$ is the Kronecker-delta.) That means, $f$ and $\mathcal{F}$ are inverses of each other if and only if the corresponding coefficient matrices $(f_{\mathbf{n}\mathbf{k}})$ and $(F_{\mathbf{k}\mathbf{l}})$ are inverses of each other. This fact will be of importance in Section 5. Equation (2.2) is the analogue for (1.2) in the setting of sequences. Obviously, for any diagonal sequence $f$ there exists a uniquely determined inverse sequence $\mathcal{F}$.

By analogy with (1.3), we call an identity of the form

(2.4)        $$\langle \mathbf{z}^{\mathbf{k}} \rangle F_{\mathbf{l}}(z_1, z_2) = \langle \mathbf{z}^{-\mathbf{l}} \rangle \tilde{f}_{\mathbf{k}}(z_1, z_2)$$

a *Lagrange formula*, where the sequence of fLs $\tilde{f} = \left( \tilde{f}_{\mathbf{k}}(z_1, z_2) \right)_{\mathbf{k} \in \mathbf{Z}^2}$ is expressed in terms of $f$. Equation (2.4) immediately implies

(2.5)        $$\tilde{f}_{\mathbf{k}}(z_1, z_2) = \sum_{\mathbf{l} \leq \mathbf{k}} F_{\mathbf{k}\mathbf{l}} \mathbf{z}^{-\mathbf{l}},$$

and therefore

(2.6)        $$\langle \mathbf{z}^{\mathbf{0}} \rangle f_{\mathbf{k}}(z_1, z_2) \tilde{f}_{\mathbf{n}}(z_1, z_2) = \delta_{\mathbf{n}\mathbf{k}}.$$

In [**16**] the author gave a method for finding $\tilde{f}$ for a given sequence $f$. For fLs $a(z_1, z_2)$ and $b(z_1, z_2)$ we introduce a bilinear form $\langle\ ,\ \rangle$ by

$$(2.7) \qquad \langle a(z_1, z_2), b(z_1, z_2)\rangle = \langle \mathbf{z^0}\rangle a(z_1, z_2)b(z_1, z_2).$$

Given any linear operator $L$ mapping fLs into fLs, $L^*$ denotes the adjoint of $L$ with respect to $\langle\ ,\ \rangle$, meaning

$$\langle La(z_1, z_2), b(z_1, z_2)\rangle = \langle a(z_1, z_2), L^*b(z_1, z_2)\rangle$$

for all $a(z_1, z_2)$ and $b(z_1, z_2)$. What we need is the following special case of [**16**, Theorem 1].

LEMMA 1. *Let*

$$f = \bigl(f_{\mathbf{k}}(z_1, z_2)\bigr)_{\mathbf{k}\in\mathbf{Z}^2}$$

*be a diagonal sequence satisfying the system of equations*

$$(2.8) \qquad U_i f_{\mathbf{k}}(z_1, z_2) = c_i(\mathbf{k})V f_{\mathbf{k}}(z_1, z_2) \quad i = 1, 2,$$

*where $U_i$ and $V$ are linear operators (acting on fLs), $V$ being bijective and $c_i(\mathbf{k})$, $i = 1, 2$, are functions from $\mathbf{Z}^2$ into the field of rational functions of $q$ satisfying the property that if $m \neq n$ there exists a $j$ ($j = 1$ or $2$) for which $c_j(\mathbf{m}) \neq c_j(\mathbf{n})$. Let*

$$\bigl(h_{\mathbf{k}}(z_1, z_2)\bigr)_{\mathbf{k}\in\mathbf{Z}^2}$$

*be a non-trivial solution (i.e., $h_k \not\equiv 0$ for all $\mathbf{k} \in \mathbf{Z}^2$) of the dual system*

$$(2.9) \qquad U_i^* h_{\mathbf{k}}(z_1, z_2) = c_i(\mathbf{k})V^* h_{\mathbf{k}}(z_1, z_2) \quad i = 1, 2,$$

*then*

$$(2.10) \quad \tilde{f}_{\mathbf{k}}(z_1, z_2) = \langle f_{\mathbf{k}}(z_1, z_2), V^* h_{\mathbf{k}}(z_1, z_2)\rangle^{-1} V^* h_{\mathbf{k}}(z_1, z_2).$$

**3. The Lagrange formula.** Recall the definition of $q$-powers due to Hofbauer [**11**, **15**].

*Definition* 2. The fps $\varphi_\alpha(t)$ in the indeterminate $t$, $\alpha \in \mathbf{R}$ (real numbers), are called $q$-powers for a fixed fps $\varphi(t)$ if $\varphi_\alpha(0) \neq 0$ for all $\alpha \in \mathbf{R}$ and

$$(3.1) \qquad \varphi_\alpha(qt) = \bigl(1 + (q^\alpha - 1)t\varphi(t)\bigr)\varphi_\alpha(t).$$

It is easy to see that for $q \to 1$ we have

$$\varphi_\alpha(t) \longrightarrow \bar{\varphi}^\alpha(t)$$

where

$$\bar{\varphi}'(t)/\bar{\varphi}(t) = \varphi(t).$$

Let the operators $\epsilon_1, \epsilon_2$ be defined by

$$\epsilon_i \mathbf{z^n} = q^{n_i} \mathbf{z^n} \quad i = 1, 2.$$

The next theorem gives a $q$-analogue of the two-variable Lagrange inversion for that special case of $f(z_1, z_2)$ of Section 1, where in (1.1) $\psi_1$ depends only on $z_2$ and $\psi_2$ only on $z_1$.

THEOREM 3. *Let $\varphi_\alpha(t)$ and $\phi_\alpha(t)$ be $q$- powers for $\varphi(t)$ and $\phi(t)$, respectively. For the inverse sequence $\big(F_\mathbf{l}(z_1, z_2)\big)_{\mathbf{l} \in \mathbf{Z}^2}$ of $\big(f_\mathbf{k}(z_1, z_2)\big)_{\mathbf{k} \in \mathbf{Z}^2}$, where*

$$(3.2) \qquad f_\mathbf{k}(z_1, z_2) = z_1^{k_1} z_2^{k_2} / \varphi_{k_1 + \lambda}(qz_2) \phi_{k_2 + \mu}(qz_1)$$

*with $\lambda, \mu \in \mathbf{R}$, we have that*

$$(3.3) \qquad \langle \mathbf{z^k} \rangle F_\mathbf{l}(z_1, z_2) = \langle \mathbf{z^{-1}} \rangle (1 - q^{\lambda + \mu} z_1 \phi(z_1) z_2 \varphi(z_2) \epsilon_1^{-1} \epsilon_2^{-1})$$
$$\times \mathbf{z^{-k}} \varphi_{k_1 + \lambda}(qz_2) \phi_{k_2 + \mu}(qz_1).$$

*Proof.* By (3.1) we get

$$\begin{cases} \big(1 + (q^{k_2 + \mu} - 1) qz_1 \phi(qz_1)\big) \epsilon_1 f_\mathbf{k} = q^{k_1} f_\mathbf{k} \\ \big(1 + (q^{k_1 + \lambda} - 1) qz_2 \varphi(qz_2)\big) \epsilon_2 f_\mathbf{k} = q^{k_2} f_\mathbf{k}, \end{cases}$$

and after a short calculation

$$\begin{cases} \big[\big(1 - qz_1 \phi(qz_1)\big) \epsilon_1 + q^\mu qz_1 \phi(qz_1) \epsilon_1 \big(1 - qz_2 \varphi(qz_2)\big) \epsilon_2 \big] f_\mathbf{k} \\ \qquad\qquad = q^{k_1}(1 - q^{\lambda + \mu} qz_2 \varphi(qz_2) \epsilon_2 qz_1 \phi(qz_1) \epsilon_1) f_\mathbf{k} \\ \big[q^\lambda qz_2 \varphi(qz_2) \epsilon_2 \big(1 - qz_1 \phi(qz_1)\big) \epsilon_1 + \big(1 - qz_2 \varphi(qz_2)\big) \epsilon_2 \big] f_\mathbf{k} \\ \qquad\qquad = q^{k_2}(1 - q^{\lambda + \mu} qz_2 \varphi(qz_2) \epsilon_2 qz_1 \phi(qz_1) \epsilon_1) f_\mathbf{k} \end{cases}$$

This is a system of "eigenvalue" equations in the sense of (2.8). Thus the dual system for the auxiliary sequence

$$\big(h_\mathbf{k}(z_1, z_2)\big)_{\mathbf{k} \in \mathbf{Z}^2}$$

reads, by use of $\epsilon_i^* = \epsilon_i^{-1} (i = 1, 2)$ and $a(z_1, z_2)^* = a(z_1, z_2)$ for any multiplication operator $a(z_1, z_2)$,

$$\begin{cases} \big[\epsilon_1^{-1} \big(1 - qz_1 \phi(qz_1)\big) \\ \qquad\qquad + q^\mu \epsilon_2^{-1} \big(1 - qz_2 \varphi(qz_2)\big) \epsilon_1^{-1} qz_1 \phi(qz_1)\big] h_\mathbf{k} \\ \qquad = q^{k_1} \big(1 - q^{\lambda + \mu} \epsilon_1^{-1} qz_1 \phi(qz_1) \epsilon_2^{-1} qz_2 \varphi(qz_2)\big) h_\mathbf{k} \\ \big[q^\lambda \epsilon_1^{-1} \big(1 - qz_1 \phi(qz_1)\big) \epsilon_2^{-1} qz_2 \varphi(qz_2) \\ \qquad\qquad + \epsilon_2^{-1} \big(1 - qz_2 \varphi(qz_2)\big)\big] h_\mathbf{k} \\ \qquad = q^{k_2} \big(1 - q^{\lambda + \mu} \epsilon_1^{-1} qz_1 \phi(qz_1) \epsilon_2^{-1} qz_2 \varphi(qz_2)\big) h_\mathbf{k}, \end{cases}$$

which is equivalent to

$$\begin{cases} \left(1 + (q^{k_2+\mu} - 1)z_1\phi(z_1)\right)\epsilon_1^{-1}h_{\mathbf{k}} = q^{k_1}h_{\mathbf{k}} \\ \left(1 + (q^{k_1+\lambda} - 1)z_2\varphi(z_2)\right)\epsilon_2^{-1}h_{\mathbf{k}} = q^{k_2}h_{\mathbf{k}}. \end{cases}$$

A solution of this system is

$$h_{\mathbf{k}}(z_1, z_2) = \mathbf{z}^{-\mathbf{k}}\varphi_{k_1+\lambda}(qz_2)\phi_{k_2+\mu}(qz_1),$$

hence, by (2.10),

$$(3.4) \qquad \tilde{f}_{\mathbf{k}}(z_1, z_2) = (1 - q^{\lambda+\mu}z_1\phi(z_1)z_2\varphi(z_2)\epsilon_1^{-1}\epsilon_2^{-1})$$
$$\times \mathbf{z}^{-\mathbf{k}}\varphi_{k_1+\lambda}(qz_2)\phi_{k_2+\mu}(qz_1),$$

which together with (2.4) proves (3.3).

**4. Inverse pairs of sequences.** We use the standard notations

$$(4.1) \qquad (a; q)_\infty = \prod_{j=0}^\infty (1 - aq^j),$$

for arbitrary $\beta \in \mathbf{R}$

$$(4.2) \qquad (a; q)_\beta = (a; q)_\infty/(aq^\beta; q)_\infty,$$

$$(4.3) \qquad {}_{r+1}\phi_r \begin{bmatrix} a_1, \ldots, a_{r+1}; \\ b_1, \ldots, b_r; \end{bmatrix} q, z = \sum_{j=0}^\infty \frac{(a_1; q)_j \cdots (a_{r+1}; q)_j}{(b_1; q)_j \cdots (b_r; q)_j} \frac{z^j}{(q; q)_j},$$

and

$$(4.4) \qquad \prod \begin{bmatrix} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{bmatrix} q = \frac{(a_1; q)_\infty \cdots (a_r; q)_\infty}{(b_1; q)_\infty \cdots (b_s; q)_\infty}.$$

We shall frequently write $(a)_\infty$ or $(a)_\beta$ instead of $(a; q)_\infty$ or $(a; q)_\beta$, respectively. That is to say, the base of such an expression is $q$ unless otherwise stated.

By the $q$-binomial theorem [**17**, Appendix (IV. 11)]

$$(4.5) \qquad {}_1\phi_0 \begin{bmatrix} a; \\ -; \end{bmatrix} q, z = \frac{(az; q)_\infty}{(z; q)_\infty}$$

(which we use as an identity of fps in $z$; for an fps-definition of $(z; q)_\infty$ see [**8**, Theorem 3.13]) we have

$$(4.6) \qquad (z; q)_\beta = \sum_{j=0}^\infty \frac{(q^{-\beta}; q)_j}{(q; q)_j} q^{\beta j} z^j.$$

Letting $q \to 1$ we see that $(z; q)_\beta$ is a $q$-analogue for $(1 - z)^\beta$.

We shall freely make use of the relations between expressions of the type $(a; q)_\beta$, contained in [**17**, Appendix II], and the "flip $q$ into $q^{-1}$"-idea, made precise in [**8**, Theorem 3.13]. In particular,

$$(4.7) \qquad (z; q^{-1})_\infty = 1/(qz; q)_\infty$$

and

$$(4.8) \qquad (z; q^{-1})_\beta = 1/(qz; q)_{-\beta}.$$

To obtain the promised inverse pair of sequences, we require the following basic double summation.

LEMMA 4. *For $r_1, r_2 \in \mathbf{R}$ holds*

$$(4.9) \qquad \sum_{\mathbf{j} \geq \mathbf{0}} q^{j_1 j_2} \frac{(q^{r_2})_{j_1} (q^{r_1})_{j_2}}{(q)_{j_1} (q)_{j_2}} \mathbf{z}^{\mathbf{j}} \frac{(z_2)_{r_1 + j_1} (z_1)_{r_2 + j_2}}{(z_1 z_2)_{r_1 + r_2 + j_1 + j_2}} = 1$$

*Proof.* By two-fold use of the $q$-analogue of Gauss's theorem [**17**, Appendix (IV.2)]

$$(4.10) \qquad {}_2\phi_1 \begin{bmatrix} a, b; \\ c; \end{bmatrix} q, c/ab \end{bmatrix} = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty},$$

(which, just as (4.5) we use in a formal sense) we evaluate

$$\sum_{\mathbf{j} \geq \mathbf{0}} q^{j_1 j_2} \frac{(q^{r_2})_{j_1} (q^{r_1})_{j_2}}{(q)_{j_1} (q)_{j_2}} \mathbf{z}^{\mathbf{j}} \frac{(z_2)_{r_1 + j_1} (z_1)_{r_2 + j_2}}{(z_1 z_2)_{r_1 + r_2 + j_1 + j_2}}$$

$$= \sum_{j_2 \geq 0} z_2^{j_2} \frac{(q^{r_1})_{j_2} (z_1)_{r_2 + j_2} (z_2)_{r_1}}{(q)_{j_2} (z_1 z_2)_{r_1 + r_2 + j_2}} {}_2\phi_1 \begin{bmatrix} z_2 q^{r_1}, q^{r_2}; \\ z_1 z_2 q^{r_1 + r_2 + j_2}; \end{bmatrix} q, z_1 q^{j_2} \end{bmatrix}$$

$$= \sum_{j_2 \geq 0} z_2^{j_2} \frac{(q^{r_1})_{j_2} (z_1)_{r_2 + j_2} (z_2)_{r_1}}{(q)_{j_2} (z_1 z_2)_{r_1 + r_2 + j_2}} \frac{(z_1 q^{r_2 + j_2})_\infty (z_1 z_2 q^{r_1 + j_2})_\infty}{(z_1 z_2 q^{r_1 + r_2 + j_2})_\infty (z_1 q^{j_2})_\infty}$$

$$= \frac{(z_2)_{r_1}}{(z_1 z_2)_{r_1}} {}_2\phi_1 \begin{bmatrix} z_1, q^{r_1}; \\ z_1 z_2 q^{r_1}; \end{bmatrix} q, z_2 \end{bmatrix}$$

$$= 1.$$

In (4.9) perform the substitutions $r_1 = k_1 + \lambda$, $r_2 = k_2 + \mu$, $j_1 = n_1 - k_1$ and $j_2 = n_2 - k_2$. Multiplying the resulting identity by $\mathbf{z}^{\mathbf{k}}$ we get

$$(4.11) \qquad \sum_{\mathbf{n} \geq \mathbf{k}} q^{(n_1 - k_1)(n_2 - k_2)} \frac{(q^{k_2 + \mu})_{n_1 - k_1} (q^{k_1 + \lambda})_{n_2 - k_2}}{(q)_{n_1 - k_1} (q)_{n_2 - k_2}}$$

$$\times \mathbf{z}^{\mathbf{n}} \frac{(z_2)_{n_1 + \lambda} (z_1)_{n_2 + \mu}}{(z_1 z_2)_{n_1 + n_2 + \lambda + \mu}} = \mathbf{z}^{\mathbf{k}}.$$

After replacing $z_1$ by $z_1 q^{-\mu}$ and $z_2$ by $z_2 q^{-\lambda}$ in (4.11), turning $q$ into $q^{-1}$ leads to

$$(4.12) \quad \sum_{\mathbf{n} \geq \mathbf{k}} q^{k_1 k_2 - n_1 n_2 + |\mathbf{n} - \mathbf{k}|} \frac{(q^{k_2 + \mu})_{n_1 - k_1} (q^{k_1 + \lambda})_{n_2 - k_2}}{(q)_{n_1 - k_1} (q)_{n_2 - k_2}}$$

$$\times \mathbf{z}^{\mathbf{n}} \frac{(z_2 q^\lambda; q^{-1})_{n_1 + \lambda} (z_1 q^\mu; q^{-1})_{n_2 + \mu}}{(z_1 z_2 q^{\lambda + \mu}; q^{-1})_{n_1 + n_2 + \lambda + \mu}} = \mathbf{z}^{\mathbf{k}},$$

valid for all $\mathbf{k} \in \mathbf{Z}^2$. By comparing this system of equations with (2.2), we obtain

THEOREM 5. *The sequences*

$$\mathcal{g} = \big(g_{\mathbf{k}}(z_1, z_2)\big)_{\mathbf{k} \in \mathbf{Z}^2} \quad and \quad \mathcal{G} = \big(G_{\mathbf{l}}(z_1, z_2)\big)_{\mathbf{l} \in \mathbf{Z}^2}$$

*are inverses of each other where*

$$(4.13)(a) \quad g_{\mathbf{k}}(z_1, z_2) = \epsilon_{12}^{-1} \big(q^{k_1 k_2} \mathbf{z}^{\mathbf{k}} / (q z_2; q)_{k_1 + \lambda} (q z_1; q)_{k_2 + \mu}\big)$$

*and*

$$(4.13)(b) \quad G_{\mathbf{l}}(z_1, z_2) = \mathbf{z}^{\mathbf{l}} \frac{(z_2 q^\lambda; q^{-1})_{l_1 + \lambda} (z_1 q^\mu; q^{-1})_{l_2 + \mu}}{(z_1 z_2 q^{\lambda + \mu}; q^{-1})_{l_1 + l_2 + \lambda + \mu}};$$

$\epsilon_{12}$ *is the linear operator defined by*

$$\epsilon_{12} \mathbf{z}^{\mathbf{n}} = q^{n_1 n_2} \mathbf{z}^{\mathbf{n}}.$$

*Proof.* By the $q$-binomial theorem (4.5) we get

$$\sum_{\mathbf{n} \geq \mathbf{k}} q^{k_1 k_2 - n_1 n_2 + |\mathbf{n} - \mathbf{k}|} \frac{(q^{k_2 + \mu})_{n_1 - k_1} (q^{k_1 + \lambda})_{n_2 - k_2}}{(q)_{n_1 - k_1} (q)_{n_2 - k_2}} \mathbf{z}^{\mathbf{n}}$$

$$= q^{k_1 k_2} \epsilon_{12}^{-1} \big(\mathbf{z}^{\mathbf{k}} / (q z_2)_{k_1 + \lambda} (q z_1)_{k_2 + \mu}\big)$$

$$= g_{\mathbf{k}}(z_1, z_2).$$

Using this, comparison of (2.2) and (4.12) completes the proof.

For $\lambda = \mu = 0$ the sequences $\mathcal{g}$ and $\mathcal{G}$ are the $q$-analogues of the powers of $f_0(z_1, z_2)$ and $F_0(z_1, z_2)$ respectively, given in (1.4). In fact, for $\lambda = \mu = 0$, when $q \to 1$ we have

$$g_{\mathbf{k}}(z_1, z_2) \to \big(z_1 / (1 - z_2)\big)^{k_1} \big(z_2 / (1 - z_1)\big)^{k_2}$$

and

$$G_{\mathbf{l}}(z_1, z_2) \to \left(\frac{z_1(1 - z_2)}{1 - z_1 z_2}\right)^{l_1} \left(\frac{z_2(1 - z_1)}{1 - z_1 z_2}\right)^{l_2}.$$

Another formulation of Theorem 5 is

COROLLARY 6. *The sequences $\bar{g}$ and $\bar{G}$ are inverses of each other where*

$$(4.14)(a) \quad \bar{g}_{\mathbf{k}}(z_1, z_2) = \mathbf{z}^{\mathbf{k}}/(qz_2; q)_{k_1+\lambda}(qz_1; q)_{k_2+\mu}$$

*and*

$$(4.14)(b) \quad \bar{G}_{\mathbf{l}}(z_1, z_2) = \epsilon_{12}\left(q^{-l_1 l_2}\mathbf{z}^{\mathbf{l}}\frac{(z_2 q^{\lambda}; q^{-1})_{l_1+\lambda}(z_1 q^{\mu}; q^{-1})_{l_2+\mu}}{(z_1 z_2 q^{\lambda+\mu}; q^{-1})_{l_1+l_2+\lambda+\mu}}\right).$$

In the proof of Theorem 5 we saw that the coefficients of $\mathbf{z}^{\mathbf{n}}$ in $g_{\mathbf{k}}(z_1, z_2)$ are

$$(4.15) \quad g_{\mathbf{n}\mathbf{k}} = q^{k_1 k_2 - n_1 n_2 + |\mathbf{n}-\mathbf{k}|}\frac{(q^{k_2+\mu}; q)_{n_1-k_1}(q^{k_1+\lambda}; q)_{n_2-k_2}}{(q; q)_{n_1-k_1}(q; q)_{n_2-k_2}}.$$

Using a variation of the $q$-Lagrange formula of Theorem 3, we compute the coefficients of $G_{\mathbf{l}}(z_1, z_2)$.

THEOREM 7. *The coefficient of $\mathbf{z}^{\mathbf{k}}$ in $G_{\mathbf{l}}(z_1, z_2)$ is given by*

$$(4.16) \quad G_{\mathbf{k}\mathbf{l}} = q^{l_1 l_2 - k_1 k_2}\frac{(q^{k_2+\mu}; q^{-1})_{k_1-l_1}(q^{k_1+\lambda}; q^{-1})_{k_2-l_2}}{(q^{-1}; q^{-1})_{k_1-l_1}(q^{-1}; q^{-1})_{k_2-l_2}}$$
$$\times \left(1 - q^{\lambda+\mu+k_1+k_2}\frac{(1-q^{l_1-k_1})(1-q^{l_2-k_2})}{(1-q^{k_2+\mu})(1-q^{k_1+\lambda})}\right).$$

*Proof.* It is a simple fact that the fps $(z; q)_{\alpha}$ are $q$-powers for $-1/(1-z)$. Hence, using (4.14)(a) and (3.4) with $\varphi(z) = \phi(z) = -1/(1-z)$,

$$\tilde{\bar{g}}_{\mathbf{k}}(z_1, z_2) = \left(1 - q^{\lambda+\mu}\frac{z_1 z_2}{(1-z_1)(1-z_2)}\epsilon_1^{-1}\epsilon_2^{-1}\right)\mathbf{z}^{-\mathbf{k}}(qz_2)_{k_1+\lambda}(qz_1)_{k_2+\mu}.$$

Because of (2.6) we get, since $\epsilon_{12}^* = \epsilon_{12}$,

$$\tilde{g}_{\mathbf{k}}(z_1, z_2) = \epsilon_{12}q^{-k_1 k_2}\tilde{\bar{g}}_{\mathbf{k}}(z_1, z_2).$$

Therefore the Lagrange formula (2.4) for $F_{\mathbf{l}} = G_{\mathbf{l}}$, etc., reads

$$\langle\mathbf{z}^{\mathbf{k}}\rangle G_{\mathbf{l}}(z_1, z_2)$$
$$= \langle\mathbf{z}^{-\mathbf{l}}\rangle\epsilon_{12}\left(q^{-k_1 k_2}\left(1 - q^{\lambda+\mu}\frac{z_1 z_2}{(1-z_1)(1-z_2)}\epsilon_1^{-1}\epsilon_2^{-1}\right)\right.$$
$$\left.\times \mathbf{z}^{-\mathbf{k}}(qz_2)_{k_1+\lambda}(qz_1)_{k_2+\mu}\right).$$

By the $q$-binomial theorem (4.5), with $q$ replaced by $q^{-1}$, the right-hand side of the last equation is equal to

$$q^{l_1 l_2 - k_1 k_2} \frac{(q^{k_2+\mu}; q^{-1})_{k_1-l_1}(q^{k_1+\lambda}; q^{-1})_{k_2-l_2}}{(q^{-1}; q^{-1})_{k_1-l_1}(q^{-1}; q^{-1})_{k_2-l_2}}$$
$$\times \left(1 - q^{\lambda+\mu+k_1+k_2}\frac{(1-q^{l_1-k_1})(1-q^{l_2-k_2})}{(1-q^{k_2+\mu})(1-q^{k_1+\lambda})}\right),$$

which establishes (4.16).

The coefficient evaluation (1.5) of Evans et al. [6] is only a reformulation of Theorem 7.

COROLLARY 8. *For $\alpha, \beta \in \mathbf{R}$ we have that*

$$(4.17) \quad \frac{(qz_1z_2)_{\alpha+\beta}}{(qz_2)_\alpha(qz_1)_\beta} = \sum_{\mathbf{k}\geq 0} q^{k_1 k_2} \frac{(q^{\beta-k_2+1})_{k_1-1}(q^{\alpha-k_1+1})_{k_2-1}}{(q)_{k_1}(q)_{k_2}}$$
$$\times \left((q^{k_1}-q^\alpha)(q^{k_2}-q^\beta) - (1-q^{k_1})(1-q^{k_2})\right)\mathbf{z}^{\mathbf{k}}.$$

*Proof.* By (4.13)(b) and (4.16) we have

$$\frac{(z_2 q^\lambda; q^{-1})_{l_1+\lambda}(z_1 q^\mu; q^{-1})_{l_2+\mu}}{(z_1 z_2 q^{\lambda+\mu}; q^{-1})_{l_1+l_2+\lambda+\mu}}$$
$$= \sum_{\mathbf{k}\geq\mathbf{l}} q^{l_1 l_2 - k_1 k_2} \frac{(q^{k_2+\mu}; q^{-1})_{k_1-l_1}(q^{k_1+\lambda}; q^{-1})_{k_2-l_2}}{(q^{-1}; q^{-1})_{k_1-l_1}(q^{-1}; q^{-1})_{k_2-l_2}}$$
$$\times \left(1 - q^{\lambda+\mu+k_1+k_2}\frac{(1-q^{l_1-k_1})(1-q^{l_2-k_2})}{(1-q^{k_2+\mu})(1-q^{k_1+\lambda})}\right)\mathbf{z}^{\mathbf{k}-\mathbf{l}}.$$

In this identity substitute (in order) $\mathbf{k}+\mathbf{l}$ for $\mathbf{k}$, $-\beta-\mu$ for $l_2$, $-\alpha-\lambda$ for $l_1$, $z_2 q^{-\lambda}$ for $z_2$ and $z_1 q^{-\mu}$ for $z_1$ thus obtaining

$$\frac{(z_2; q^{-1})_{-\alpha}(z_1; q^{-1})_{-\beta}}{(z_1 z_2; q^{-1})_{-\alpha-\beta}}$$
$$= \sum_{\mathbf{k}\geq 0} q^{-k_1 k_2 + \alpha k_2 + \beta k_1}\frac{(q^{k_2-\beta}; q^{-1})_{k_1}(q^{k_1-\alpha}; q^{-1})_{k_2}}{(q^{-1}; q^{-1})_{k_1}(q^{-1}; q^{-1})_{k_2}}$$
$$\times \left(1 - q^{k_1+k_2-\alpha-\beta}\frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{k_1-\alpha})(1-q^{k_2-\beta})}\right)\mathbf{z}^{\mathbf{k}}.$$

Regarding (4.8), a short calculation leads to

$$\frac{(qz_1z_2; q)_{\alpha+\beta}}{(qz_2; q)_\alpha(qz_1; q)_\beta} = \sum_{\mathbf{k}\geq 0} q^{k_1 k_2 + k_1 + k_2}\frac{(q^{\beta-k_2}; q)_{k_1}(q^{\alpha-k_1}; q)_{k_2}}{(q; q)_{k_1}(q; q)_{k_2}}$$
$$\times \left(1 - \frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{\alpha-k_1})(1-q^{\beta-k_2})}\right)\mathbf{z}^{\mathbf{k}},$$

which is equivalent to (4.17).

*Remark.* Our approach to the proof of (4.17) relies heavily on Gessel's [**7**, p. 160] $q = 1$-proof. For example, the system of equations (4.12) is the $q$-analogue for the two equations at the top of p. 160 in [**7**] (for $a = d = 0$).

From Corollary 8 we may deduce another identity of [**6**, (6.1)].

COROLLARY 9. *For $\alpha, \beta \in \mathbf{R}$ we have that*

$$(4.18) \quad \frac{(z_1 z_2)_{\alpha+\beta-1}}{(z_2)_\alpha (z_1)_\beta} = \sum_{\mathbf{k} \geq 0} q^{k_1 k_2} \frac{(q^{\beta-k_2})_{k_1} (q^{\alpha-k_1})_{k_2}}{(q)_{k_1} (q)_{k_2}} \mathbf{z}^{\mathbf{k}}.$$

*Proof.* In (4.17) replace $z_1$ by $z_1/q$ and $z_2$ by $z_2/q$. Multiplying the resulting identity by

$$(1 - z_1 z_2/q)^{-1} = \sum_{j=0}^{\infty} (z_1 z_2)^j q^{-j}$$

and collecting the terms on the right-hand side yield (4.18).

Just as the pair (4.13) corresponds to the identity (4.17), there exists an inverse pair of sequences corresponding to (4.18), which we will state without proof.

THEOREM 10. *The sequence of* fLs

$$(4.19)(a) \quad \epsilon_{12}^{-1} \left( q^{k_1 k_2} (1 - q^{k_1+\lambda} z_2 - q^{k_2+\mu} z_1) \frac{\mathbf{z}^{\mathbf{k}}}{(z_2; q)_{k_1+\lambda+1} (z_1; q)_{k_2+\mu+1}} \right)$$

*is the inverse of the sequence*

$$(4.19)(b) \quad \mathbf{z}^{\mathbf{l}} \frac{(z_2 q^{\lambda-1}; q^{-1})_{l_1+\lambda} (z_1 q^{\mu-1}; q^{-1})_{l_2+\mu}}{(z_1 z_2 q^{\lambda+\mu-1}; q^{-1})_{l_1+l_2+\lambda+\mu+1}}.$$

**5. Transformations and summations.** Suppose that the infinite lower-triangular matrices $(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbf{Z}^2}$ and $(F_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbf{Z}^2}$ are inverses of each other. Then the "inverse relations"

$$(5.1) \quad b_{\mathbf{n}} = \sum_{\mathbf{n} \geq \mathbf{k} \geq 0} f_{\mathbf{n}\mathbf{k}} a_{\mathbf{k}}, \quad \mathbf{n} \in \mathbf{Z}^2$$

and

$$(5.2) \quad a_{\mathbf{k}} = \sum_{\mathbf{k} \geq \mathbf{l} \geq 0} F_{\mathbf{k}\mathbf{l}} b_{\mathbf{l}}, \quad \mathbf{k} \in \mathbf{Z}^2$$

are equivalent, where $(a_{\mathbf{k}}) = (a_{\mathbf{k}}^{(q)})$ and $(b_{\mathbf{n}}) = (b_{\mathbf{n}}^{(q)})$ are sequences of rational functions in $q$. For the choice

$$f_{\mathbf{nk}} = q^{-k_1 k_2 + n_1 n_2 - |\mathbf{n} - \mathbf{k}|} g_{\mathbf{nk}} \quad \text{and}$$

$$F_{\mathbf{kl}} = q^{k_1 k_2 - l_1 l_2 - |\mathbf{k} - \mathbf{l}|} G_{\mathbf{kl}},$$

by (4.15) and (4.16) the equations (5.1) and (5.2) may be written in the form

$$(5.3) \qquad b_{\mathbf{n}} = \frac{(B)_{n_1}(A)_{n_2}}{(q)_{n_1}(q)_{n_2}} \sum_{\mathbf{k} \geq 0} (-1)^{|\mathbf{k}|} q^{n_1 k_1 + n_2 k_2 - \binom{k_1}{2} - \binom{k_2}{2}}$$

$$\times \frac{(Bq^{n_1})_{k_2 - k_1}(Aq^{n_2})_{k_1 - k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(A)_{k_1}(B)_{k_2}} a_{\mathbf{k}}$$

and

$$(5.4) \qquad a_{\mathbf{k}} = (-1)^{|\mathbf{k}|} q^{\binom{k_1}{2} + \binom{k_2}{2}} \frac{(Bq^{k_2 - k_1})_{k_1}(Aq^{k_1 - k_2})_{k_2}}{(q)_{k_1}(q)_{k_2}}$$

$$\times \left[ \frac{(1 - Aq^{k_1})(1 - Bq^{k_2})}{(1 - Aq^{k_1 - k_2})(1 - Bq^{k_2 - k_1})} \right.$$

$$\times \sum_{\mathbf{l} \geq 0} \frac{(q^{-k_1})_{l_1}(q^{-k_2})_{l_2}}{(q^{k_2 - k_1 + 1}B)_{l_1}(q^{k_1 - k_2 + 1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}}$$

$$- \frac{AB(1 - q^{k_1})(1 - q^{k_2})}{(1 - Aq^{k_1 - k_2})(1 - Bq^{k_2 - k_1})}$$

$$\times \left. \sum_{\mathbf{l} \geq 0} \frac{(q^{-k_1 + 1})_{l_1}(q^{-k_2 + 1})_{l_2}}{(q^{k_2 - k_1 + 1}B)_{l_1}(q^{k_1 - k_2 + 1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}} \right],$$

where we have set $A = q^{\lambda}$ and $B = q^{\mu}$. Setting

$$X_{\mathbf{k}} = \sum_{\mathbf{l} \geq 0} \frac{(q^{-k_1})_{l_1}(q^{-k_2})_{l_2}}{(q^{k_2 - k_1 + 1}B)_{l_1}(q^{k_1 - k_2 + 1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}},$$

(5.4) becomes a recursion for $X_{\mathbf{k}}$, namely,

$$(-1)^{|\mathbf{k}|} q^{-\binom{k_1}{2} - \binom{k_2}{2}} \frac{(q)_{k_1}(q)_{k_2}}{(q^{k_2 - k_1 + 1}B)_{k_1}(q^{k_1 - k_2 + 1}A)_{k_2}} a_{\mathbf{k}}$$

$$= X_{\mathbf{k}} - \frac{AB(1 - q^{k_1})(1 - q^{k_2})}{(1 - Aq^{k_1})(1 - Bq^{k_2})} X_{\mathbf{k} - \mathbf{e}},$$

756 CHRISTIAN KRATTENTHALER

where $\mathbf{e} = (1,1)$. The solution of this recursion is

$$X_{\mathbf{k}} = (-1)^{|\mathbf{k}|} \frac{(q)_{k_1}(q)_{k_2}}{(q^{k_2-k_1+1}B)_{k_1}(q^{k_1-k_2+1}A)_{k_2}}$$
$$\times \sum_{j \geq 0} q^{-\binom{k_1-j}{2}-\binom{k_2-j}{2}}(AB)^j a_{\mathbf{k}-j\mathbf{e}},$$

consequently,

$$(5.5) \qquad \sum_{\mathbf{l} \geq 0} \frac{(q^{-k_1})_{l_1}(q^{-k_2})_{l_2}}{(q^{k_2-k_1+1}B)_{l_1}(q^{k_1-k_2+1}A)_{l_2}} q^{|\mathbf{l}|} b_{\mathbf{l}}$$
$$= (-1)^{|\mathbf{k}|} \frac{(q)_{k_1}(q)_{k_2}}{(q^{k_2-k_1+1}B)_{k_1}(q^{k_1-k_2+1}A)_{k_2}}$$
$$\times \sum_{j \geq 0} q^{-\binom{k_1-j}{2}-\binom{k_2-j}{2}}(AB)^j a_{\mathbf{k}-j\mathbf{e}}.$$

Multiplication of (5.1) (where $f_{\mathbf{nk}} = q^{-k_1k_2+n_1+n_2-|\mathbf{n}-\mathbf{k}|}g_{\mathbf{nk}}$) by $\mathbf{z^n}$ and summing up both sides with respect to $\mathbf{n}$, by use of (4.13)(a) and (4.15), yield the transformation

$$(5.6) \qquad \sum_{\mathbf{n} \geq 0} b_{\mathbf{n}} z_1^{n_1} z_2^{n_2} = \frac{(Bz_1)_\infty (Az_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{\mathbf{k} \geq 0} a_{\mathbf{k}} \frac{z_1^{k_1} z_2^{k_2}}{(Az_2)_{k_1}(Bz_1)_{k_2}}.$$

Equation (5.6) holds if (5.3) or (5.4) is satisfied. Similarly, multiplying (5.2) (where $F_{\mathbf{kl}} = q^{k_1k_2-l_1l_2-|\mathbf{k}-\mathbf{l}|}G_{\mathbf{kl}}$) by $q^{-k_1k_2}\mathbf{z^k}$ and summing with respect to $\mathbf{k}$, we get by (4.13)(b) and (4.16)

$$\sum_{\mathbf{k} \geq 0} q^{-k_1k_2} a_{\mathbf{k}} \mathbf{z^k}$$
$$= \frac{(Bz_1/q; q^{-1})_\infty (Az_2/q; q^{-1})_\infty (z_1z_2/q^2; q^{-1})_\infty}{(z_1/q; q^{-1})_\infty (z_2/q; q^{-1})_\infty (ABz_1z_2/q^2; q^{-1})_\infty}$$
$$\times \sum_{\mathbf{l} \geq 0} q^{-l_1l_2} b_{\mathbf{l}} \mathbf{z^l} \frac{(z_2/q; q^{-1})_{l_1}(z_1/q; q^{-1})_{l_2}}{(z_1z_2/q^2; q^{-1})_{l_1+l_2}}.$$

After replacing $z_1$ by $qz_1$, $z_2$ by $qz_2$ and $q$ by $q^{-1}$, we obtain the transformation

$$(5.7) \qquad \sum_{\mathbf{k} \geq 0} q^{k_1k_2-k_1-k_2} a_{\mathbf{k}}^{(1/q)} z_1^{k_1} z_2^{k_2}$$
$$= \frac{(Bz_1)_\infty (Az_2)_\infty (z_1z_2)_\infty}{(z_1)_\infty (z_2)_\infty (ABz_1z_2)_\infty}$$
$$\times \sum_{\mathbf{l} \geq 0} q^{l_1l_2-l_1-l_2} b_{\mathbf{l}}^{(1/q)} z_1^{l_1} z_2^{l_2} \frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}}.$$

Again, (5.7) holds if (5.3) or (5.4) is satisfied. Different choices of $(a_\mathbf{k})$ and $(b_\mathbf{n})$ which satisfy (5.3) or (5.4) yield basic double hypergeometric summations and transformations when substituted into (5.5), (5.6) or (5.7).

First take

$$(5.8) \qquad a_\mathbf{k} = q^{k_1 k_2} B^{k_1} A^{k_2} \frac{(A)_{k_1}(C)_{k_2}}{(q)_{k_1}(q)_{k_2}}.$$

By two-fold use of the $q$-Vandermonde summation [**17**, Appendix (IV.1)]

$$(5.9) \qquad {}_2\phi_1 \begin{bmatrix} a, q^{-n}; \\ b; \end{bmatrix} q, q = \frac{(b/a; q)_n\, a^n}{(b; q)_n},$$

we have by (5.3) and a bit of manipulation

$$
\begin{aligned}
b_\mathbf{n} &= \frac{(B)_{n_1}(A)_{n_2}}{(q)_{n_1}(q)_{n_2}} \sum_{k_2 \geqq 0} \frac{(q^{-n_2})_{k_2}(q^{n_1}B)_{k_2}(C)_{k_2}}{(q^{1-n_2}/A)_{k_2}(B)_{k_2}(q)_{k_2}} \\
&\quad \times q^{k_2} {}_2\phi_1 \begin{bmatrix} q^{n_2-k_2}A, q^{-n_1}; \\ q^{1-n_1-k_2}/B; \end{bmatrix} q, q \\
&= \frac{(B)_{n_1}(A)_{n_2}}{(q)_{n_1}(q)_{n_2}} \sum_{k_2 \geqq 0} \frac{(q^{-n_2})_{k_2}(q^{n_1}B)_{k_2}(C)_{k_2}}{(q^{1-n_2}/A)_{k_2}(B)_{k_2}(q)_{k_2}} q^{k_2} \\
&\quad \times \frac{(q^{1-n_1-n_2}/AB)_{n_1}}{(q^{1-n_1-k_2}/B)_{n_1}} q^{n_1(n_2-k_2)} A^{n_1} \\
&= (-AB)^{n_1} q^{n_1 n_2 + \binom{n_1}{2}} \frac{(A)_{n_2}(q^{1-n_1-n_2}/AB)_{n_1}}{(q)_{n_1}(q)_{n_2}} {}_2\phi_1 \begin{bmatrix} C, q^{-n_2}; \\ q^{1-n_2}/A; \end{bmatrix} q, q \\
&= (-AB)^{n_1} q^{n_1 n_2 + \binom{n_1}{2}} \frac{(A)_{n_2}(q^{1-n_1-n_2}/AB)_{n_1}}{(q)_{n_1}(q)_{n_2}} \\
&\quad \times \frac{(q^{1-n_2}/AC)_{n_2}}{(q^{1-n_2}/A)_{n_2}} C^{n_2},
\end{aligned}
$$

and finally

$$(5.10) \qquad b_\mathbf{n} = \frac{(q^{n_2}AB)_{n_1}(AC)_{n_2}}{(q)_{n_1}(q)_{n_2}}.$$

The resulting evaluation (5.3) reads

$$(5.11) \quad \sum_{\mathbf{k} \geq 0} \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(C)_{k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(B)_{k_2}(q)_{k_1}(q)_{k_2}}$$

$$\times q^{k_1 k_2 - \binom{k_1}{2} - \binom{k_2}{2}}(-Bq^{n_1})^{k_1}(-Aq^{n_2})^{k_2}$$

$$= \frac{(q^{n_2}AB)_{n_1}(AC)_{n_2}}{(B)_{n_1}(A)_{n_2}}.$$

For $B = C$ we get the symmetric formula

$$(5.12) \quad \sum_{\mathbf{k} \geq 0} \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(q)_{k_1}(q)_{k_2}}$$

$$\times q^{k_1 k_2 - \binom{k_1}{2} - \binom{k_2}{2}}(-Bq^{n_1})^{k_1}(-Aq^{n_2})^{k_2}$$

$$= \frac{(AB)_{n_1+n_2}}{(B)_{n_1}(A)_{n_2}}.$$

A standard notation for a $q$-Appell type basic double hypergeometric series is

$$(5.13) \quad \phi_{m:s;v}^{p:r;u} \left[ \begin{array}{l} e_1, \ldots, e_p : a_1, \ldots, a_r ; c_1, \ldots, c_u; \\ f_1, \ldots, f_m : b_1, \ldots, b_s ; d_1, \ldots, d_v; \end{array} q; z_1, z_2 \right]$$

$$= \sum_{\mathbf{k} \geq 0} \frac{\prod_{j=1}^{p}(e_j;q)_{k_1+k_2} \prod_{j=1}^{r}(a_j;q)_{k_1} \prod_{j=1}^{u}(c_j;q)_{k_2}}{\prod_{j=1}^{m}(f_j;q)_{k_1+k_2} \prod_{j=1}^{s}(b_j;q)_{k_1} \prod_{j=1}^{v}(d_j;q)_{k_2}} z_1^{k_1} z_2^{k_2}.$$

The dual evaluation (5.5) for the above choice of $(a_{\mathbf{k}})$ and $(b_{\mathbf{n}})$ can be written as

$$(5.14) \quad \phi_{0:1;2}^{1:1;2} \left[ \begin{array}{l} AB : q^{-k_1}; \qquad AC, q^{-k_2}; \\ \quad - : q^{k_2-k_1+1}B; AB, q^{k_1-k_2+1}A; \end{array} q; q, q \right]$$

$$= (-1)^{|\mathbf{k}|} q^{k_1 k_2 - \binom{k_1}{2} - \binom{k_2}{2}} B^{k_1} A^{k_2} \frac{(A)_{k_1}(C)_{k_2}}{(Bq^{k_2-k_1+1})_{k_1}(Aq^{k_1-k_2+1})_{k_2}}$$

$$\times {}_3\phi_2 \left[ \begin{array}{l} q^{k_1}, q^{k_2}, q^{-1}; \\ q^{k_1-1}A, q^{k_2-1}C; \end{array} q^{-1}, q^{-1} \right]$$

$$= (-1)^{|\mathbf{k}|} q^{k_1 k_2 - \binom{k_1}{2} - \binom{k_2}{2}} B^{k_1} A^{k_2} \frac{(A)_{k_1}(C)_{k_2}}{(Bq^{k_2-k_1+1})_{k_1}(Aq^{k_1-k_2+1})_{k_2}}$$

$$\times {}_3\phi_2 \left[ \begin{array}{l} q^{-k_1}, q^{-k_2}, q; \\ q^{1-k_1}/A, q^{1-k_2}/C; \end{array} q, q/AC \right].$$

By (5.6) we get the transformation

$$(5.15) \quad \sum_{\mathbf{n} \geq 0} \frac{(q^{n_2} AB)_{n_1} (AC)_{n_2}}{(q)_{n_1} (q)_{n_2}} z_1^{n_1} z_2^{n_2}$$

$$= \frac{(Bz_1)_\infty (Az_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{\mathbf{k} \geq 0} q^{k_1 k_2} B^{k_1} A^{k_2}$$

$$\times \frac{(A)_{k_1} (C)_{k_2}}{(q)_{k_1} (q)_{k_2}} \frac{z_1^{k_1} z_2^{k_2}}{(Az_2)_{k_1} (Bz_1)_{k_2}}.$$

The special case $B = C$

$$(5.16) \quad \sum_{\mathbf{n} \geq 0} \frac{(AB)_{n_1 + n_2}}{(q)_{n_1} (q)_{n_2}} z_1^{n_1} z_2^{n_2}$$

$$= \frac{(Bz_1)_\infty (Az_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{\mathbf{k} \geq 0} q^{k_1 k_2} B^{k_1} A^{k_2}$$

$$\times \frac{(A)_{k_1} (B)_{k_2}}{(q)_{k_1} (q)_{k_2}} \frac{z_1^{k_1} z_2^{k_2}}{(Az_2)_{k_1} (Bz_1)_{k_2}}$$

turns out to be a $q$-analogue of

$$(1 - z_1 - z_2)^{-a-b} = (1 - z_1)^{-b} (1 - z_2)^{-a} \left(1 - \frac{z_1}{1 - z_2}\right)^{-a} \left(1 - \frac{z_2}{1 - z_1}\right)^{-b},$$

when setting $A = q^a$, $b = q^b$ and letting $q \to 1$.

Finally, the dual transformation (5.7) reads

$$\sum_{\mathbf{k} \geq 0} \frac{(A; q^{-1})_{k_1} (C; q^{-1})_{k_2}}{(q^{-1}; q^{-1})_{k_1} (q^{-1}; q^{-1})_{k_2}} (B/q)^{k_1} (A/q)^{k_2} z_1^{k_1} z_2^{k_2}$$

$$= \frac{(Bz_1)_\infty (Az_2)_\infty (z_1 z_2)_\infty}{(z_1)_\infty (z_2)_\infty (ABz_1 z_2)_\infty} \sum_{\mathbf{l} \geq 0} q^{l_1 l_2 - l_1 - l_2}$$

$$\times \frac{(q^{-l_2} AB; q^{-1})_{l_1} (AC; q^{-1})_{l_2}}{(q^{-1}; q^{-1})_{l_1} (q^{-1}; q^{-1})_{l_2}}$$

$$\times z_1^{l_1} z_2^{l_2} \frac{(z_2)_{l_1} (z_1)_{l_2}}{(z_1 z_2)_{l_1 + l_2}}.$$

The left-hand side can be summed by the $q$-binomial theorem (4.5). In basic hypergeometric notation we obtain after a short calculation

$$\phi_{1:0;1}^{1:1;2}\left[\begin{matrix} 1/AB : z_2; 1/AC, z_1; \\ z_1z_2 : -; 1/AB; \end{matrix}\ q; z_1, z_2\right]$$

$$= \prod\left[\begin{matrix} ABz_1z_2, z_1, z_2; \\ z_1z_2, ABz_1, ACz_2; \end{matrix}\ q\right],$$

or, after setting $F = 1/AB$ and $E = 1/AC$,

$$(5.17)\quad \phi_{1:0;1}^{1:1;2}\left[\begin{matrix} F \quad : z_2; E, z_1; \\ z_1z_2 : -; F; \end{matrix}\ q; z_1, z_2\right]$$

$$= \prod\left[\begin{matrix} z_1z_2/F, z_1, z_2; \\ z_1z_2, z_1/F, z_2/E; \end{matrix}\ q\right],$$

As the cases $z_1 = 0$ and $z_2 = 0$, respectively, show, this is a generalization of the $q$-binomial theorem (4.5).

   Another choice is

$$(5.18)\quad a_{\mathbf{k}} = \delta_{k_1k_2}q^{2\binom{k_1}{2}}\frac{(A)_{k_1}(B)_{k_1}}{(C)_{k_1}(q)_{k_1}}C^{k_1}.$$

By the $q$-Gauss sum (4.10) we obtain from (5.3)

$$(5.19)\quad b_{\mathbf{n}} = \frac{(C)_{n_1+n_2}(B)_{n_1}(A)_{n_2}}{(C)_{n_1}(C)_{n_2}(q)_{n_1}(q)_{n_2}}.$$

By (5.5) we get

$$(5.20)\quad \phi_{0:2;2}^{1:2;2}\left[\begin{matrix} C : B, q^{-k_1}; \qquad A, q^{-k_2}; \\ - : C, q^{k_2-k_1+1}B; C, q^{k_1-k_2+1}A; \end{matrix}\ q; q, q\right]$$

$$= \delta_{k_1k_2}C^{k_1}\frac{(1-A)(1-B)}{(1-q^{k_1}A)(1-q^{k_1}B)}\frac{(q)_{k_1}}{(C)_{k_1}}$$

$$\times {}_3\phi_2\left[\begin{matrix} q^{-k_1}, q^{1-k_1}/C, q; \\ q^{1-k_1}/A, q^{1-k_1}/B; \end{matrix}\ q, q\right].$$

For the special case $C = qAB$ the Pfaff-Saalschütz summation [17, Appendix (IV.4)]

$$(5.21)\quad {}_3\phi_2\left[\begin{matrix} a, b, q^{-n}; \\ c, d; \end{matrix}\ q, q\right] = \frac{(c/a; q)_n(c/b; q)_n}{(c; q)_n(c/ab; q)_n},$$

provided $cd = abq^{1-n}$, can be applied to find

$$(5.22)\quad \phi_{0:2;2}^{1:2;2}\left[\begin{matrix} qAB : B, q^{-k_1}; & A, q^{-k_2}; \\ - : qAB, q^{k_2-k_1+1}B; & qAB, q^{k_1-k_2+1}A; \end{matrix} q; q, q\right]$$

$$= \delta_{k_1 k_2}(AB)^{k_1}\frac{(q)_{k_1}}{(qAB)_{k_1}}.$$

Another example is when $B = C$ and the right-hand side of (5.20) may be summed. This time by the $q$-Vandermonde summation (5.9)

$$(5.23)\quad \phi_{0:1;2}^{1:1;2}\left[\begin{matrix} B : q^{-k_1}; & A, q^{-k_2}; \\ - : q^{k_2-k_1+1}B; & B, q^{k_1-k_2+1}A; \end{matrix} q; q, q\right]$$

$$= \delta_{k_1 k_2}B^{k_1}\frac{(q)_{k_1}}{(qB)_{k_1}}.$$

The transformation (5.6) is

$$(5.24)\quad \phi_{0:1;1}^{1:1;1}\left[\begin{matrix} C : B; A; \\ - : C; C; \end{matrix} q; z_1, z_2\right]$$

$$= \frac{(Bz_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty}\sum_{k\geq 0}q^{k^2-k}C^k\frac{(A)_k(B)_k}{(C)_k(q)_k}\frac{(z_1z_2)^k}{(Az_2)_k(Bz_1)_k}.$$

The special case $A = B = C$

$$(5.25)\quad \sum_{\mathbf{n}\geq \mathbf{0}}\frac{(A)_{n_1+n_2}}{(q)_{n_1}(q)_{n_2}}z_1^{n_1}z_2^{n_2}$$

$$= \frac{(Az_1)_\infty(Az_2)_\infty}{(z_1)_\infty(z_2)_\infty}\sum_{k\geq 0}q^{k^2-k}A^k\frac{(A)_k}{(q)_k}\frac{(z_1z_2)^k}{(Az_2)_k(Az_1)_k}$$

is a $q$-analogue of

$$(1 - z_1 - z_2)^{-a} = (1 - z_1)^{-a}(1 - z_2)^{-a}\left(1 - \frac{z_1z_2}{(1-z_1)(1-z_2)}\right)^{-a}.$$

By (5.7) the dual transformation is

$$\sum_{k\geq 0}q^{-k}\frac{(A; q^{-1})_k(B; q^{-1})_k}{(C; q^{-1})_k(q^{-1}; q^{-1})_k}(Cz_1z_2)^k$$

$$= \frac{(Bz_1)_\infty(Az_2)_\infty(z_1z_2)_\infty}{(z_1)_\infty(z_2)_\infty(ABz_1z_2)_\infty}\sum_{\mathbf{l}\geq \mathbf{0}}q^{l_1l_2-l_1-l_2}$$

$$\times \frac{(C; q^{-1})_{l_1+l_2}(B; q^{-1})_{l_1}(A; q^{-1})_{l_2}}{(C; q^{-1})_{l_1}(C; q^{-1})_{l_2}(q^{-1}; q^{-1})_{l_1}(q^{-1}; q^{-1})_{l_2}}$$

$$\times z_1^{l_1}z_2^{l_2}\frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1z_2)_{l_1+l_2}},$$

which after setting $E = 1/A$, $D = 1/B$, $F = 1/C$ is transformed into

$$(5.26) \quad \phi_{1:1;1}^{1:2;2} \begin{bmatrix} F : D, z_2; E, z_1; \\ z_1 z_2 : F; F; \end{bmatrix} q; z_1/D, z_2/E \end{bmatrix}$$

$$= \prod \begin{bmatrix} z_1, z_2, z_1 z_2/DE; \\ z_1/D, z_2/E, z_1 z_2; \end{bmatrix} q \end{bmatrix} {}_2\phi_1 \begin{bmatrix} D, E; \\ F; \end{bmatrix} q, z_1 z_2/DE \end{bmatrix}.$$

If $D = F$ the ${}_2\phi_1$ reduces to a ${}_1\phi_0$, which is summable by the $q$-binomial theorem (4.5), thus arriving at (5.17) a second time. For $D = E = F$, again by (4.5), we deduce from (5.26)

$$(5.27) \quad \phi_{1:0;0}^{1:1;1} \begin{bmatrix} D : z_2; z_1; \\ z_1 z_2 : -; -; \end{bmatrix} q; z_1/D, z_2/D \end{bmatrix}$$

$$= \prod \begin{bmatrix} z_1, z_2, z_1 z_2/D; \\ z_1/D, z_2/D, z_1 z_2; \end{bmatrix} q \end{bmatrix}.$$

This is a $q$-analogue of

$$\left(1 - \frac{z_2(1 - z_1)}{1 - z_1 z_2} - \frac{z_1(1 - z_2)}{1 - z_1 z_2}\right)^{-d}$$

$$= (1 - z_1)^{-d}(1 - z_2)^{-d}(1 - z_1 z_2)^d.$$

Also of interest is the limiting case $F \to \infty$ in (5.26),

$$(5.28) \quad \sum_{l \geqq 0} q^{l_1 l_2} \frac{(D)_{l_1}(E)_{l_2}}{(q)_{l_1}(q)_{l_2}} (z_1/D)^{l_1}(z_2/E)^{l_2} \frac{(z_2)_{l_1}(z_1)_{l_2}}{(z_1 z_2)_{l_1 + l_2}}$$

$$= \prod \begin{bmatrix} z_1, z_2, z_1 z_2/DE; \\ z_1/D, z_2/E, z_1 z_2; \end{bmatrix} q \end{bmatrix},$$

it is a $q$-analogue of

$$\left(1 - \frac{z_1(1 - z_2)}{1 - z_1 z_2}\right)^{-d} \left(1 - \frac{z_2(1 - z_1)}{1 - z_1 z_2}\right)^{-e}$$

$$= (1 - z_1)^{-d}(1 - z_2)^{-e}(1 - z_1 z_2)^{d+e}.$$

Equations (5.27) and (5.28) could also be obtained as limiting cases of Al-Salam's [1] $q$-Saalschützian double series theorems.

We have only proved (5.26) in a formal sense. But it is easy to deduce that

(5.26) remains valid for $|q| < 1$, $|z_1| < |D|$ and $|z_2| < |E|$, when it is interpreted as an identity for basic hypergeometric functions. Setting $z_1 = q^a$, $z_2 = q^b$, $D = q^d$, $E = q^e$, $F = q^f$ and letting $q \to 1$, the resulting $_2F_1$ on the right-hand side of (5.26) is summable by ordinary Gauss-summation [**17**, Appendix (III.3)], which leads to

$$(5.29) \quad F^{1:1;2}_{1:1;1} \begin{bmatrix} f : d, b; e, a; \\ a + b : f; f; \end{bmatrix} 1, 1 \end{bmatrix}$$

$$= \Gamma \begin{bmatrix} a - d, b - e, a + b, f, f - d - e \\ a, b, a + b - d - e, f - d, f - e \end{bmatrix},$$

subject to suitable restrictions on $a, b, d, e, f$ such that convergence is provided. This identity is due to Carlitz [**5**, (1.7)].

Of course it is possible first to choose $(b_{\mathbf{n}})$ and, by (5.4), compute the $a_{\mathbf{k}}$'s. If we take

$$(5.30) \quad b_{\mathbf{n}} = \frac{(C)_{n_1}(D)_{n_2}}{(q)_{n_1}(q)_{n_2}},$$

two-fold application of $q$-Vandermonde summation (5.9) yields

$$(5.31) \quad a_{\mathbf{k}} = (-1)^{|\mathbf{k}|} q^{\binom{k_1}{2} + \binom{k_2}{2}}$$

$$\times \frac{(q^{k_2 - k_1 + 1} B/C)_{k_1} (q^{k_1 - k_2 + 1} A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}}$$

$$\times C^{k_1} D^{k_2} \left( 1 - \frac{AB}{CD} \frac{(1 - q^{k_1})(1 - q^{k_2})}{(1 - q^{k_2} B/C)(1 - q^{k_1} A/D)} \right).$$

The evaluation (5.3) becomes

$$\frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} = \sum_{\mathbf{k} \geqq \mathbf{0}} \left( q^{n_1 k_1 + n_2 k_2} C^{k_1} D^{k_2} \right.$$

$$\times \frac{(Bq^{n_1})_{k_2 - k_1} (Aq^{n_2})_{k_1 - k_2} (q^{k_2 - k_1 + 1} B/C)_{k_1}}{(B)_{k_2}(A)_{k_1}}$$

$$\times \left. \frac{(q^{k_1 - k_2 + 1} A/D)_{k_2} (q^{-n_1})_{k_1} (q^{-n_2})_{k_2}}{(q)_{k_1}(q)_{k_2}} \right)$$

$$- \frac{AB}{CD} \sum_{\mathbf{k} \geqq \mathbf{0}} \left( q^{n_1 k_1 + n_2 k_2} C^{k_1} D^{k_2} \right.$$

$$\times \frac{(Bq^{n_1})_{k_2 - k_1} (Aq^{n_2})_{k_1 - k_2} (q^{k_2 - k_1 + 1} B/C)_{k_1 - 1} (q^{k_1 - k_2 + 1} A/D)_{k_2 - 1}}{(B)_{k_2}(A)_{k_1}}$$

$$\times \left. \frac{(q^{-n_1})_{k_1} (q^{-n_2})_{k_2}}{(q)_{k_1 - 1}(q)_{k_2 - 1}} \right).$$

Let the first of the two sums on the right-hand side of this identity be abbreviated by $Y_{\mathbf{n}}(A,B,C,D)$. Thus we may rewrite it as

$$\frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} = Y_{\mathbf{n}}(A,B,C,D)$$
$$- \frac{(1-q^{n_1})(1-q^{n_2})}{(1-A)(1-B)} AB Y_{\mathbf{n-e}}(qA,qB,qC,qD).$$

The solution of this recursive relation is

$$Y_{\mathbf{n}}(A,B,C,D) = \frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} \sum_{j \geqq 0} q^{j^2 - j}(AB)^j$$
$$\times \frac{(q^{n_1};q^{-1})_j (q^{n_2};q^{-1})_j}{(C;q)_j (D;q)_j},$$

consequently we obtain the transformation

(5.32)
$$\sum_{\mathbf{k} \geqq 0} \Bigg( q^{n_1 k_1 + n_2 k_2} C^{k_1} D^{k_2}$$
$$\times \frac{(Bq^{n_1})_{k_2-k_1}(Aq^{n_2})_{k_1-k_2}(q^{k_2-k_1+1}B/C)_{k_1}}{(A)_{k_1}(B)_{k_2}}$$
$$\times \frac{(q^{k_1-k_2+1}A/D)_{k_2}(q^{-n_1})_{k_1}(q^{-n_2})_{k_2}}{(q)_{k_1}(q)_{k_2}} \Bigg)$$
$$= \frac{(C)_{n_1}(D)_{n_2}}{(B)_{n_1}(A)_{n_2}} {}_3\phi_2 \left[ \begin{array}{c} q^{-n_1}, q^{-n_2}, q; \\ C, D; \end{array} q, q^{n_1+n_2}AB \right].$$

The transformation (5.6) becomes

$$\sum_{\mathbf{n} \geqq 0} \frac{(C)_{n_1}(D)_{n_2}}{(q)_{n_1}(q)_{n_2}} z_1^{n_1} z_2^{n_2}$$
$$= \frac{(Bz_1)_\infty (Az_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{\mathbf{k} \geqq 0} \Bigg[ (-1)^{|\mathbf{k}|} q^{\binom{k_1}{2}+\binom{k_2}{2}}$$
$$\times \frac{(q^{k_2-k_1+1}B/C)_{k_1}(q^{k_1-k_2+1}A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}}$$
$$\times C^{k_1}D^{k_2}\left( 1 - \frac{AB}{CD}\frac{(1-q^{k_1})(1-q^{k_2})}{(1-q^{k_2}B/C)(1-q^{k_1}A/D)} \right) \frac{z_1^{k_1} z_2^{k_2}}{(Az_2)_{k_1}(Bz_1)_{k_2}} \Bigg].$$

The left-hand side is summable by the $q$-binomial theorem (4.5). Replacing $z_1$ by $z_1/B$, $z_2$ by $z_2/A$, $C/B$ by $E$ and $D/A$ by $F$, we obtain after some manipulation

(5.33)
$$\frac{(Ez_1)_\infty (Fz_2)_\infty}{(z_1)_\infty (z_2)_\infty} = \sum_{\mathbf{k} \geqq 0} \Bigg[ q^{2k_1 k_2} \frac{(q^{-k_2}E)_{k_1}(q^{-k_1}F)_{k_2}}{(q)_{k_1}(q)_{k_2}}$$
$$\times \left( 1 - \frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{-k_2}E)(1-q^{-k_1}F)} \right) \frac{z_1^{k_1} z_2^{k_2}}{(z_2)_{k_1}(z_1)_{k_2}} \Bigg].$$

In view of (4.17), this identity is a $q$-analogue for

$$(1-z_1)^{-e}(1-z_2)^{-f} = \frac{\left(1 - \dfrac{z_1}{1-z_2}\dfrac{z_2}{1-z_1}\right)^{e+f}}{\left(1 - \dfrac{z_1}{1-z_2}\right)^e \left(1 - \dfrac{z_2}{1-z_1}\right)^f}.$$

Setting

$$R(z_1, z_2) = \sum_{\mathbf{k} \geq \mathbf{0}} q^{2k_1 k_2} \frac{(q^{-k_2}E)_{k_1}(q^{-k_1}F)_{k_2}}{(q)_{k_1}(q)_{k_2}} \frac{z_1^{k_1} z_2^{k_2}}{(z_2)_{k_1}(z_1)_{k_2}}$$

(5.33) may be rewritten as

$$\frac{(Ez_1)_\infty (Fz_2)_\infty}{(z_1)_\infty (z_2)_\infty} = R(z_1, z_2) - \frac{z_1 z_2}{(1-z_1)(1-z_2)} R(qz_1 qz_2).$$

Thus, we have

$$R(z_1, z_2) = \frac{(Ez_1)_\infty (Fz_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{j \geq 0} q^{j^2-j} \frac{(z_1 z_2)^j}{(Ez_1)_j (Fz_2)_j},$$

hence,

$$(5.34) \quad \sum_{\mathbf{k} \geq \mathbf{0}} q^{2k_1 k_2} \frac{(q^{-k_2}E)_{k_1}(q^{-k_1}F)_{k_2}}{(q)_{k_1}(q)_{k_2}} \frac{z_1^{k_1} z_2^{k_2}}{(z_2)_{k_1}(z_1)_{k_2}}$$

$$= \frac{(Ez_1)_\infty (Fz_2)_\infty}{(z_1)_\infty (z_2)_\infty} \sum_{j \geq 0} q^{j^2-j} \frac{(z_1 z_2)^j}{(Ez_1)_j (Fz_2)_j}.$$

From (4.18) we see that this is a $q$-analogue of

$$\frac{\left(1 - \dfrac{z_1}{1-z_2}\dfrac{z_2}{1-z_1}\right)^{e+f-1}}{\left(1 - \dfrac{z_1}{1-z_2}\right)^e \left(1 - \dfrac{z_2}{1-z_1}\right)^f}$$

$$= (1-z_1)^{-e}(1-z_2)^{-f}\left(1 - \frac{z_1 z_2}{(1-z_1)(1-z_2)}\right)^{-1}.$$

Finally, we turn to the dual transformation (5.7),

$$\sum_{\mathbf{k} \geq 0} \Bigg[ (-1)^{|\mathbf{k}|} q^{k_1 k_2 - \binom{k_1+1}{2} - \binom{k_2+1}{2}}$$

$$\times \frac{(q^{-k_2+k_1-1}B/C; q^{-1})_{k_1} (q^{-k_1+k_2-1}A/D; q^{-1})_{k_2}}{(q^{-1}; q^{-1})_{k_1} (q^{-1}; q^{-1})_{k_2}}$$

$$\times C^{k_1} D^{k_2} \left( 1 - \frac{AB}{CD} \frac{(1-q^{-k_2})(1-q^{-k_1})}{(1-q^{-k_2}B/C)(1-q^{-k_1}A/D)} \right) z_1^{k_1} z_2^{k_2} \Bigg]$$

$$= \frac{(Bz_1)_\infty (Az_2)_\infty (z_1 z_2)_\infty}{(z_1)_\infty (z_2)_\infty (ABz_1 z_2)_\infty}$$

$$\times \sum_{\mathbf{l} \geq 0} \Bigg[ q^{l_1 l_2 - l_1 - l_2} \frac{(C; q^{-1})_{l_1} (D; q^{-1})_{l_2}}{(q^{-1}; q^{-1})_{l_1} (q^{-1}; q^{-1})_{l_2}}$$

$$\times z_1^{l_1} z_2^{l_2} \frac{(z_2)_{l_1} (z_1)_{l_2}}{(z_1 z_2)_{l_1+l_2}} \Bigg]$$

or, equivalently,

$$(5.35) \quad \sum_{\mathbf{k} \geq 0} \Bigg[ q^{k_1 k_2} \frac{(q^{-k_2}B/C)_{k_1} (q^{-k_1}A/D)_{k_2}}{(q)_{k_1} (q)_{k_2}}$$

$$\times \left( 1 - \frac{AB}{CD} \frac{(1-q^{-k_2})(1-q^{-k_1})}{(1-q^{-k_2}B/C)(1-q^{-k_1}A/D)} \right)$$

$$\times (Cz_1)^{k_1} (Dz_2)^{k_2} \Bigg] = \frac{(Bz_1)_\infty (Az_2)_\infty (z_1 z_2)_\infty}{(z_1)_\infty (z_2)_\infty (ABz_1 z_2)_\infty}$$

$$\times \sum_{\mathbf{l} \geq 0} q^{l_1 l_2} \frac{(1/C)_{l_1} (1/D)_{l_2}}{(q)_{l_1} (q)_{l_2}}$$

$$\times (Cz_1)^{l_1} (Dz_2)^{l_2} \frac{(z_2)_{l_1} (z_1)_{l_2}}{(z_1 z_2)_{l_1+l_2}}.$$

By use of (4.18), the left-hand side turns out to be equal to

$$(5.36) \quad \frac{(CDz_1 z_2)_\infty (Az_2)_\infty (Bz_1)_\infty}{(q^{-1}ABz_1 z_2)_\infty (Cz_1)_\infty (Dz_2)_\infty} (1 - q^{-1} ABz_1 z_2),$$

therefore from (5.35) we derive

$$(5.37) \quad \sum_{\mathbf{l} \geq 0} q^{l_1 l_2} \frac{(1/C)_{l_1} (1/D)_{l_2}}{(q)_{l_1} (q)_{l_2}} (Cz_1)^{l_1} (Dz_2)^{l_2} \frac{(z_2)_{l_1} (z_1)_{l_2}}{(z_1 z_2)_{l_1+l_2}}$$

$$= \frac{(Cz_1)_\infty (Dz_2)_\infty (z_1 z_2)_\infty}{(z_1)_\infty (z_2)_\infty (CDz_1 z_2)_\infty}.$$

This identity appeared before in (5.28). Another identity, which was proved "en passant" by (5.36) is

$$(5.38) \quad \frac{(CDz_1z_2)_\infty(Az_2)_\infty(Bz_1)_\infty}{(ABz_1z_2)_\infty(Cz_1)_\infty(Dz_2)_\infty} = \sum_{\mathbf{k}\geq 0}\left[ q^{k_1k_2}\frac{(q^{-k_2}B/C)_{k_1}(q^{-k_1}A/D)_{k_2}}{(q)_{k_1}(q)_{k_2}} \right.$$
$$\times \left( 1 - \frac{AB}{CD}\frac{(1-q^{-k_1})(1-q^{-k_2})}{(1-q^{-k_2}B/C)(1-q^{-k_1}A/D)} \right)$$
$$\left. \times (Cz_1)^{k_1}(Dz_2)^{k_2} \right].$$

After replacing $z_1$ by $z_1/C$, $z_2$ by $z_2/D$, $B/C$ by $q^\alpha$ and $A/D$ by $q^\beta$, we get by (4.2)

$$(5.39) \quad \frac{(z_1z_2)_{\alpha+\beta}}{(z_2)_\alpha(z_1)_\beta} = \sum_{\mathbf{k}\geq 0}\left( q^{k_1k_2}\frac{(q^{\alpha-k_2+1})_{k_1-1}(q^{\beta-k_1+1})_{k_2-1}}{(q)_{k_1}(q)_{k_2}} \right.$$
$$\left. \times \left( (1-q^{\alpha-k_2})(1-q^{\beta-k_1}) - q^{\alpha+\beta}(1-q^{-k_1})(1-q^{-k_2}) \right) \right),$$

which also appears in [6, Theorem 10].

## References

1. W. A. Al-Salam, *Saalschützian theorems for basic double series,* J. London. Math. Soc. *40* (1965), 455–458.
2. G. E. Andrews, *Summations and transformations for basic Appell series,* J. London Math. Soc. (2) *4* (1972), 618–622.
3. ——— *Problems and prospects for basic hypergeometric functions,* from *Theory and Application of Special Functions* (Academic Press, New York, 1975).
4. J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions, I and II,* Quart. J. Math. Oxford Ser. *11* (1940), 249–270; ibid *12* (1941), 112–128.
5. L. Carlitz, *A summation theorem for double hypergeometric series,* Rend. Sem. Mat. Univ. Padova *37* (1967), 230-233.
6. R. Evans, M. E. H. Ismail and D. Stanton, *Coefficients in expansions of certain rational functions,* Can. J. Math. *34* (1982), 1011–1024.
7. I. Gessel, *A combinatorial proof of the multivariable Lagrange inversion formula,* J. Combin. Theory A *45* (1987), 178–196.
8. I. Gessel and D. Stanton, *Applications of q-Lagrange inversion to basic hypergeometric series,* Trans. Amer. Math. Soc. *277* (1983), 173–203.
9. ——— *Short proofs of Saalschütz's and Dixon's theorems,* J. Combin. Theory A *38* (1985), 87–90.
10. ——— *Another family of q-Lagrange inversion formulas,* Rocky Mountain J. Math. *16* (1986), 373–384.
11. J. Hofbauer, *A q-analogue of the Lagrange expansion,* Arch. Math. *42* (1984), 536–544.
12. F. H. Jackson, *On basic double hypergeometric functions,* Quart. J. Math. Oxford Ser. *13* (1942), 69–82.
13. ——— *Basic double hypergeometric functions (II),* Quart. J. Math. Oxford Ser. *15* (1944), 49–61.

**14.** V. K. Jain, *Some expansions involving basic hypergeometric functions of two variables*, Pacific J. Math. *91* (1980), 349–361.

**15.** C. Krattenthaler, *A new q-Lagrange formula and some applications,* Proc. Amer. Math. Soc. *90* (1984), 338-344.

**16.** ——— *Operator methods and Lagrange inversion: A unified approach to Lagrange formulas*, Trans. Amer. Math. Soc. *305* (1988), 431–465.

**17.** L. J. Slater, *Generalized hypergeometric functions* (Cambridge University Press, Cambridge, (1966).

**18.** H. M. Srivastava, *A certain class of q-series transformations*, J. Math. Anal. Appl. *107* (1985), 498–508.

**19.** H. M. Srivastava and V. K. Jain, *q-series identities and reducibility of basic double hypergeometric functions,* Can. J. Math. *38* (1986), 215-231.

**20.** H. M. Srivastava, *Sums of certain double q-hypergeometric series,* Rend. Sem. Mat. Univ. Padova *72* (1984), 1–8.

*Universität Wien,*
*Vienna, Austria*