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Example of a non-standard extreme-value law

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Abstract. It has been shown that sufficiently well mixing dynamical systems with positive entropy have extreme-value laws which in the limit converge to one of the three standard distributions known for independently and identically distributed processes, namely Gumbel, Fréchet and Weibull distributions. In this short note, we give an example which has a non-standard limiting distribution for its extreme values. Rotations of the circle by irrational numbers are used and it will be shown that the limiting distribution is a step function where the limit has to be taken along a suitable sequence given by the convergents.

1. *Introduction*

For independently and identically distributed (i.i.d.) random variables, extreme-value theory is a well-established topic. Given a sequence of random variables X_j, one forms the maximum random variable $M_n = \max_{1\le j\le n} X_j$. If there exists a sequence of numbers a_n, b_n, such that $a_n(M_n - b_n)$ converges in distribution to a limit, then one says that the sequence $(X_j)_{j\in\mathbb{N}}$ has the extreme-value property. It is known that for i.i.d. random variables X_j satisfying some weak condition, the limiting distribution is one of three: Gumbel (the distribution function is $e^{-e^{-t}}$), Fréchet (with the distribution function $e^{-t^{-\beta}}$ for $t > 0$ and a parameter $\beta > 0$) or Weibull (with the distribution function $e^{-(-t)^\beta}$ for $t < 0$ and a parameter $\beta > 0$).

In a dynamical system with map T on a space Ω, the random variable chosen is a given function X_0 evaluated along the orbit, that is $X_j = X_0 \circ T^j$. The function chosen is rotational symmetric: a base point x is selected and then $X_0(y) = g(d(x, y))$, where g is a function defined on $\mathbb{R}^+$. Typically, $g(s) = -\log s$ is chosen. The pursuit of extreme values in dynamics is quite recent, with the most notable first paper in this domain being [**2**], where for non-uniformly expanding C^2 maps on the interval, the limiting extreme-value law for $g(s) = -\log s$ was proven to be Gumbel. For the quadratic map on the interval and Benedicks–Carleson parameters, it was shown in [**5**] that the extreme-value statistics for $g(s) = C - s^{\beta}$, for a constant $C > 0$ and a parameter β, tend in the limit to a Weibull distribution. For more general non-uniformly hyperbolic maps, the extreme-value law statistics was addressed in [**10**]. For higher dimensional hyperbolic maps with discontinuities, such as dispersing billiards, Lozi maps and Lorenz-type maps, the limiting extreme-value law was established in [**8**]. Since the functions g are always connected to the metric, the fact that M_n is large for some time geometrically means that a point does not enter the neighborhood of x for this time. The extreme-value law property and the distribution of hitting or return times are therefore intimately connected to each other. In fact, the equivalence was formally established first for absolutely continuous measures in [**6**] and then for more general measures in [**7**].

Here, we provide an example that is contrary to all the quoted results. It yields a non-standard limiting extreme-value law. Since we use circle rotations, we do not obtain the good mixing properties of the systems mentioned, nor a good decay of correlations. In fact, it is known [**11**] that the hitting times distribution for circle maps are 'non-standard'. The limiting distributions turn out to be locally constant and not exponential, as is witnessed by many hyperbolic systems or those that at least display a sufficiently fast decay of correlations (see, e.g., [**1**, **9**]).

2. *General settings*

Let us consider a 'probability' dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, T)$ with invariant measure $\mathbb{P}$ and a measurable function X from Ω to $\mathbb{R}$. Denote the maximum value in the first n trials by M_n, i.e., $M_n = \max_{1\le j\le n} X_j$, where $X_j(x) = X \circ T^j$. The question is whether there exist sequences (a_n) and (b_n) such that the rescaled random variables $a_n(M_n - b_n)$ converge in distribution, and what the limit is.

Let us recall that real random variables Y_n, $n \in \mathbb{N}$, converge to a real random variable Y in distribution if and only if the distribution functions $F_{X_n}(t) = \mathbb{P}(X_n \le t)$ converges to the distribution function $F_X(t) = \mathbb{P}(X \le t)$ at every point t, where the limiting distribution F_X is continuous. It is a simple observation that the convergence in distribution can be expressed in the same way in the terms of functions $\mathbb{P}(X_n > t)$ and $\mathbb{P}(X > t)$.

Hence, we ask if the functions

$$H_n(y) := \mathbb{P}((M_n - b_n)a_n > y), \qquad y \in \mathbb{R}$$

converge to a right-continuous decreasing (not necessarily strictly) function $H(y)$ at every point $y \in \mathbb{R}$, where the function H is continuous. Since any right-continuous function has at most countably many points of discontinuity, the limiting distribution H is uniquely determined (if it exists). It represents a real random variable if the limits of $H(y)$ at plus and minus infinity are 1 and 0, respectively.

The maximum-value statistics are tightly connected with the statistics of the entry times. For a set $B \in \mathcal{B}$, denote by τ_B the entry time function, which is given by

$$\tau_B(x) = \min\{j \ge 1 : T^j(x) \in B\}$$

($\tau_B(x) = \infty$ if x never enters B). The normalized entry times distribution, is then given by

$$F_B(t) = \mathbb{P}\left(\tau_B \le \frac{t}{\mathbb{P}(B)}\right),$$

for $t \in \mathbb{R}$, is increasing, constant on intervals $[j\mathbb{P}(B), (j+1)\mathbb{P}(B))$ for $j = 0, 1, \ldots$ and the jumps at the points $j\mathbb{P}(B)$, $j = 1, 2, \ldots$, are less than or equal to $\mathbb{P}(B)$.

The connection between the maximum value M_n and the entry times can be expressed in terms of level sets:

$$L(y) = \{x \in \Omega, X(x) > y\}, \quad y \in \mathbb{R}.$$

For the distribution of the maximum values we have

$$\mathbb{P}(M_n > y) = \mathbb{P}(\tau_{L(y)} \le n),$$

since the sets on both sides equal.

Consequently, for sequences $a_n, b_n, n = 1, 2, \ldots,$ the rescaled maximum value variables H_n satisfy the following equalities:

$$H_n(y) := \mathbb{P}((M_n - b_n)a_n > y) = \mathbb{P}\left(M_n > \frac{1}{a_n}y + b_n\right) = \mathbb{P}\left(\tau_{L\left(y/a_n + b_n\right)} \le n\right) \quad (1)$$

$$= F_{L(y/a_n+b_n)}\left(n\,\mathbb{P}\left(L\left(\frac{y}{a_n} + b_n\right)\right)\right). \quad (2)$$

We will use this equality in the next section, where we calculate directly the limiting distribution for the sequence H_n in irrational rotation of the interval.

3. *Rotation of the interval*

Let us consider a rotation $T : [0, 1) \to [0, 1)$, $Tx = x + \alpha \bmod 1$, on the unit interval (or circle) by an irrational angle $\alpha \in (0, 1)$. The Lebesgue measure μ is then the only invariant probability measure. We consider the continued fraction expansion

$$\alpha = [c_1, c_2, c_3, \ldots] = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cdots}}}.$$

The convergents of α are then p_k/q_k, where $p_k = c_k p_{k-1} + p_{k-2}$, $p_0 = 0$, $p_1 = 1$ and $q_k = c_k q_{k-1} + q_{k-2}$, $q_0 = 1$, $q_1 = c_1$. The numbers $q_k\alpha - p_k$, $k \in \mathbb{N}$, form an alternating positive and negative sequence and their absolute values $\eta_k = |q_k\alpha - p_k|$ satisfy the implicit formula $\eta_k = \eta_{k-2} - c_k\eta_{k-1}$, $\eta_0 = \alpha$, $\eta_1 = 1 - c_1\alpha$.

Denote the following nested sequence of intervals,

$$B_k = \begin{cases} (-\eta_{k+1}, \eta_k) & \text{for } k \text{ even}, \\ (-\eta_k, \eta_{k+1}) & \text{for } k \text{ odd}. \end{cases}$$

With no danger of ambiguity, the sets B_k, $k \in \mathbb{N}$, are considered as subsets of the state space $[0, 1)$, so we identify the above-mentioned intervals with their images under the projection $\bmod 1 : \mathbb{R} \to [0, 1)$.

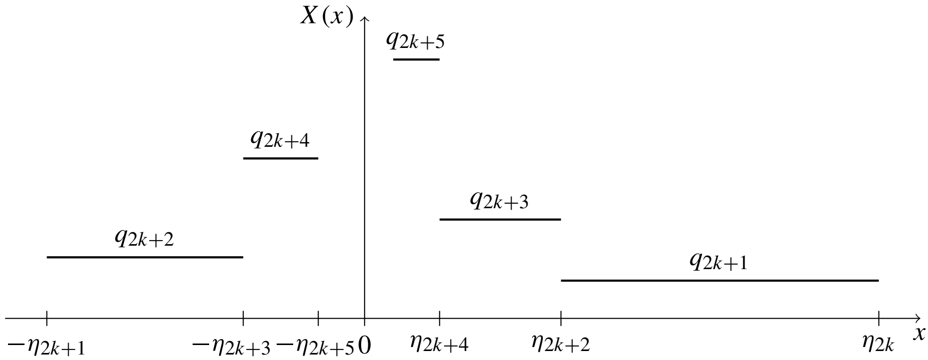

FIGURE 1. Initial distribution X.

Let X be a random variable on $[0, 1)$, defined as follows:

$$X(x) = q_{k+1} \quad \text{where } k = \min\{\ell \in \mathbb{N} \mid x \notin B_\ell\}.$$

The sets B_k were chosen to have nice return times, namely

$$\tau_{B_k}(x) = \begin{cases} q_{k+1} & \text{if } x \cdot \operatorname{sgn}(q_k\alpha - p_k) \in (0, \eta_k), \\ q_k & \text{if } x \cdot \operatorname{sgn}(q_k\alpha - p_k) \in (-\eta_{k+1}, 0). \end{cases}$$

Consequently, we obtain that the distribution of the entry time satisfies the following conditions (for the detailed proof see Proposition 3 in [**3**]):

$$\mu(\tau_{B_k} \le s) = \begin{cases} 1 & \text{if } s \ge q_{k+1}, \\ q_k\eta_{k+1} + \eta_k[s] & \text{if } s \in [q_k, q_{k+1}), \\ (\eta_k + \eta_{k+1})[s] & \text{if } s \in [0, q_k) \end{cases} \tag{3}$$

(note that $q_{k+1}\eta_k + \eta_{k+1}q_k = 1$), since $q_k(\eta_k + \eta_{k+1}) + \eta_k[s - q_k] = q_k\eta_{k+1} + \eta_{k+1}[s]$.

For an infinite set of integers $K \subset \mathbb{N}$, denote the following limits (if they exist):

$$\lim_{k\in K} \frac{q_{k+j}}{q_k} = \gamma_j, \quad \lim_{k\in K} \frac{\eta_{k+j}}{\eta_k} = \delta_j \quad \text{for all } j \in \mathbb{N}.$$

It is a standard fact that $q_{k+2}/q_k > 2$ for every k. Hence, by definition, the sequence γ_j is increasing (not necessarily strictly) and goes to infinity, whenever the limits γ_j exist. Moreover,

$$\lim_{k\in K} q_k\eta_k = \frac{1}{\gamma_1 + \delta_1}, \tag{4}$$

whenever γ_1 and δ_1 exist. Indeed,

$$\begin{aligned} \lim_{k\in K} q_k\eta_k &= \lim_{k\in K}\left(\frac{1}{q_k\eta_k}\right)^{-1} = \lim_{k\in K}\left(\frac{q_{k+1}\eta_k + q_k\eta_{k+1}}{q_k\eta_k}\right)^{-1} \\ &= \lim_{k\in K}\left(\frac{q_{k+1}}{q_k} + \frac{\eta_{k+1}}{\eta_k}\right)^{-1} = (\gamma_1 + \delta_1)^{-1}. \end{aligned}$$

We express our main theorem in terms of these limits.

THEOREM 1. *Let K be an infinite set of natural numbers and the limits γ_j and δ_j exist (along K) for every $j \geq 0$. Then the random variables M_{q_k}/q_k, $k \in K$, converge in distribution to a random variable M with the following distribution:*

$$H(y) = \mathbb{P}(M > y) = \begin{cases} 1 & \text{if } y < 1, \\ \dfrac{\delta_j + \delta_{j+1}}{\gamma_1 + \delta_1} & \text{if } y \in [\gamma_j, \gamma_{j+1}),\ j = 0, 1, 2, \ldots. \end{cases}$$

Proof. Let us recall a standard fact, that convergence in distribution of the random variables M_{q_k}/q_k to a random variable M agrees with the pointwise convergence of distribution functions H_{q_k} to H at every point, except the points of discontinuities of the limiting function H. By the formula for H, its points of discontinuities are the points γ_j, $j \in \mathbb{N}$. In addition, γ_j goes to infinity (or reaches infinity in a finite step). Thus, we need to prove that $H_{q_k}(y)$ tends to $H(y)$ for every y from the intervals $(-\infty, \gamma_1)$ and (γ_j, γ_{j+1}), $j \in \mathbb{N}_0$. We treat the two cases separately, as follows:

(a) Let $y \leq \gamma_0 = 1$. Then, for every $k \in K$, $yq_k < q_k$, $B_{k-1} \subset L(yq_k)$ and

$$H_{q_k}(y) = \mu(M_{q_k}/q_k > y) = \mu(\tau_{L(q_k y)} \leq q_k) \geq \mu(\tau_{B_{k-1}} \leq q_k) = 1.$$

The last equality follows from (3). Hence, the limit $\lim_{k\in K} H_{q_k}(y)$ is one.

(b) Let $y \in (\gamma_j, \gamma_{j+1})$ for some $j \geq 0$. By the definition of γ_j, $q_{k+j} < q_k y < q_{k+j+1}$ eventually for every $k \in \mathbb{N}$. Hence, $L(q_k y) = B_{k+j}$ for k big enough. This implies that

$$H_{q_k}(y) = \mu(\tau_{B_{k+j}} \leq q_k) = q_k(\eta_{k+j} + \eta_{k+j+1}).$$

The last equality follows from (3) and the fact that $q_k < q_{k+j}$. By (4),

$$H_{q_k}(y) = q_k \eta_k \left(\frac{\eta_{k+j}}{\eta_k} + \frac{\eta_{k+j+1}}{\eta_k} \right) \to \frac{\delta_j + \delta_{j+1}}{\gamma_1 + \delta_1}. \qquad \square$$

Remark 1. As we mentioned above (Theorem 1), the sequence $(\gamma_j)_{j\in\mathbb{N}}$ is increasing (not necessarily strictly) and goes to infinity. Hence, the description of $H(y)$ from Theorem 1 characterizes the distribution function in a correct and unique way. Note that $(\gamma_j)_{j\in\mathbb{N}}$ can eventually be equal to $+\infty$. We discuss this case in §4.2.

4. *The limits γ_j and δ_j*

A natural question is, for a given irrational rotation, what is the set K and what are the limits γ_j and δ_j? Let the angle α be given in the form of a continued fraction $\alpha = [c_1, c_2, \ldots]$, and assume $K \subset \mathbb{N}$ is infinite. Put

$$\nu_j = \lim_{k\in K} \frac{q_{k+j-1}}{q_{k+j}}, \quad \theta_j = \lim_{k\in K} \frac{\eta_{k+j}}{\eta_{k+j-1}}, \quad j \geq 1.$$

It is standard that

$$\frac{q_{k-1}}{q_k} = [c_k, c_{k-1}, \ldots, c_1], \quad \frac{\eta_k}{\eta_{k-1}} = [c_{k+1}, c_{k+2}, \ldots], \quad k \in \mathbb{N}.$$

Hence, the limits ν_j and θ_j can be expressed in another way:

$$\nu_j = \lim_{k\in K} [c_{k+j}, c_{k+j-1}, \ldots, c_1], \quad \theta_j = \lim_{k\in K} [c_{k+j+1}, c_{k+j+2}, \ldots], \quad j \geq 0.$$

It follows from the definition that for all $j \geq 1$,

$$\gamma_j = \frac{\gamma_{j-1}}{\nu_j} = \prod_{i=1}^{j} (\nu_i)^{-1}, \quad \delta_j = \theta_j \delta_{j-1} = \prod_{i=1}^{j} \theta_i,$$

whenever the limits used in the equalities exist (where $0^{-1} = \infty$).

4.1. *Finite limits γ_j.* First, let us suppose that the limits $\gamma_j = \lim_{k \in K} (q_{k+j}/q_k)$ exist and are finite for every $j \geq 1$. This condition ensures that

(i) the limits ν_j, $j \geq 1$, exist and are non-zero;
(ii) the limits $\lim_{k \in K} c_{k+j}$, $j \geq 1$, exist and are finite, i.e., for every $j \geq 1$, the sequence $(c_{k+j})_{k \in K}$ is eventually constant;
(iii) the limits θ_j, $j \geq 1$, exist and are non-zero; and
(iv) the limits δ_j, $j \geq 1$, exist and are non-zero.

In such settings, the limit distribution function for extremes $H(y)$, described in the main theorem, has *countably* many jumps (down) in the points γ_j, $j \geq 0$. Letting y go to $+\infty$, the function converges to 0, but never reaches this value.

4.2. *Infinite limits γ_j.* Let us suppose that the limits $\gamma_j = \lim_{k \in K} (q_{k+j}/q_k)$ exist for every $j \geq 1$, and some of them are infinite. In this case, the situation is more complex. We look separately at the two cases when γ_1 is finite and infinite, as follows.

(a) $\gamma_1 < \infty$: Let N be such an index, such that γ_N is the first infinite member of the sequence $(\gamma_j)_{j \geq 1}$. Then $N \geq 2$ and the following conditions hold.

(i) The limits ν_j, $1 \leq j < N$, exist and are non-zero. The limit ν_N exists and is zero.
(ii) The limits $\lim_{k \in K} c_{k+j}$, $1 \leq j < N$, exist and are finite, i.e., for every $1 \leq j < N$, the sequence $(c_{k+j})_{k \in K}$ is eventually constant. The limit $\lim_{k \in K} c_{k+N}$ is infinite.
(iii) The limits θ_j, $1 \leq j < N - 1$, exist and are non-zero. The limit ν_{N-1} exists and is zero.
(iv) The limits δ_j, $1 \leq j < N - 1$, exist and are non-zero. The limit δ_{N-1} exists and is zero.

In such a setting, the limit distribution function for extremes $H(y)$, described in the main theorem, has *finitely* many jumps (down) at the points γ_j, $0 \leq j \leq N - 1$. The function reaches zero at the point γ_{N-1}; indeed,

$$H(y) = \frac{\delta_{N-1} + \delta_N}{\gamma_1 + \delta_1} = 0 \quad \text{for } y \in [\gamma_{N-1}, \gamma_N) = [\gamma_{N-1}, \infty).$$

(b) $\gamma_1 = \infty$: In this case, the limiting distribution H is the characteristic function $1_{[-\infty,1)}$ which can easily be seen following the steps in the proof of Theorem 1, even for the case when the limits δ_j, $j \geq 1$, do not exist.

Theorem 1 ensures that H is non-trivial in all cases, except two: when γ_1 is infinite, or when $\gamma_1 = 1$ and γ_2 is infinite. The former case has already been discussed above. In the latter case, the points of discontinuities, γ_0 and γ_1, are equal. Thus, there is only one point of discontinuity, at $\gamma_0 = 1$. As we already mentioned above, if γ_2 is infinite, then δ_1 is zero, and so is δ_2. We get that the value of the function H on the interval $[1, \infty) = [\gamma_1, \gamma_2)$ is zero.

Example. The two cases when the limiting law for the extreme values is trivial, as well as the non-trivial case with a finite number of jumps, are illustrated by the rotation numbers

$$\alpha = [\overbrace{1, 1, \ldots, 1}^{N-1}, 2, \overbrace{1, 1, \ldots, 1}^{N-1}, 3, \overbrace{1, 1, \ldots, 1}^{N-1}, 4, \ldots],$$

where $N \geq 1$. Here, we choose $K = \{kN : k \in \mathbb{N}\}$.

If $N \geq 2$,

$$\frac{q_{kN+j-1}}{q_{kN+j}} = [\overbrace{1, \ldots, 1}^{j}, k+1, \overbrace{1, \ldots, 1}^{N-1}, k, \overbrace{1, \ldots, 1}^{N-1}, k-1, \ldots]$$

for every $j = 0, \ldots, N-1$. In particular, we immediately get $\nu_0 = 0$, since

$$\nu_0 = \lim_{k\in\mathbb{N}} [k+1, \overbrace{1, \ldots, 1}^{N-1}, k, \overbrace{1, \ldots, 1}^{N-1}, k-1, \ldots] \leq \lim \frac{1}{k+1} = 0.$$

For $j = 1, \ldots, N-1$, we obtain

$$\begin{aligned}\nu_j &= \lim_{k\in\mathbb{N}} [\overbrace{1, \ldots, 1}^{j}, k+1, \overbrace{1, \ldots, 1}^{N-1}, k, \overbrace{1, \ldots, 1}^{N-1}, k-1, \ldots] \\ &= \lim_{k\in\mathbb{N}} \frac{1}{1 + [\underbrace{1, \ldots, 1}_{j-1}, k+1, \underbrace{1, \ldots, 1}_{N-1}, k, \underbrace{1, \ldots, 1}_{N-1}, k-1, \ldots]} = \frac{1}{1+\nu_{j-1}}.\end{aligned}$$

This recursive formula ensures that $\nu_j = s_j/s_{j+1}$ for every $j = 0, \ldots, N-1$, where s_j is the Fibonacci sequence given recursively by $s_{j+1} = s_j + s_{j-1}$ and $s_0 = 0$, $s_1 = 1$. For $j = N$, we obtain $\nu_N = \nu_0 = 0$. Hence, $\gamma_j = s_{j+1}$ for $j = 1, \ldots, N-1$ and $\gamma_N = \infty$. On the other hand,

$$\frac{\eta_{kN+j}}{\eta_{kN+j-1}} = [\overbrace{1, \ldots, 1}^{N-1-j}, k+2, \overbrace{1, \ldots, 1}^{N-1}, k+3, \overbrace{1, \ldots, 1}^{N-1}, k+4, \ldots],$$

for every $j = 0, \ldots, N-1$. For the value $j = N-1$, we immediately get $\theta_{N-1} = \nu_0 = 0$. For $j = 2, \ldots, N-1$, we obtain $\theta_j = \nu_{N-1-j} = s_{N-1-j}/s_{N-j}$, and therefore $\delta_j = s_{N-j-1}/s_{N-1}$ for every $j = 1, \ldots, N-1$. In particular, $\delta_{N-1} = 0$. This implies that function H vanishes at $[s_N, \infty)$.

If $N \geq 3$, the function H is non-trivial and has $N-1$ jumps at the points $\gamma_j = s_{j+1}$, $j = 1, \ldots, N-1$. By Theorem 1, for every $j = 1, \ldots, N-2$, the value of H on the interval $[s_{j+1}, s_{j+2})$ is equal to s_{N-j}/s_N. The function is zero on $[s_N, \infty)$.

If $N = 2$, then $\gamma_1 = s_2 = 1$, and γ_2 is infinite. Here, we get the trivial limit distribution H.

If $N = 1$, then

$$\alpha = [2, 3, 4, \ldots],$$

and we get $\nu_1 = 0$, which implies $\gamma_1 = \infty$. Here, $K = \mathbb{N}$. This is the second case when H is the trivial limiting distribution.

If the sequence of entries $2, 3, 4, \ldots$ in the continued fraction expansion are replaced by a sequence of numbers $n_1, n_2, n_3, \ldots$ that converges to infinity, then we will get the same values for γ_j and δ_j.

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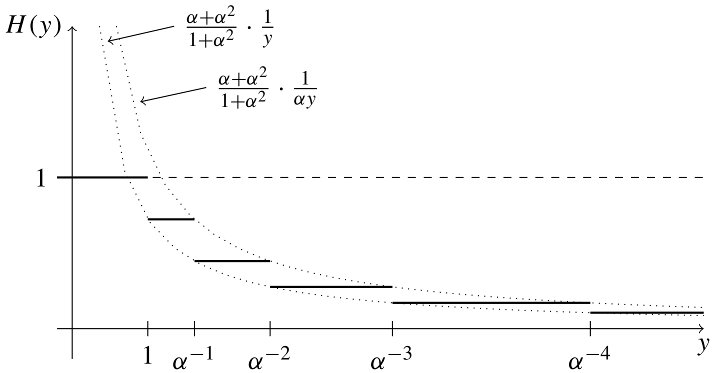

FIGURE 2. Limiting extreme-value law for rotation numbers of constant type.

Below, we apply Theorem 1 to two classical situations: rotation numbers of constant type and diverging rotation numbers.

4.3. *Rotation numbers of constant type.* In this case, one has $\alpha = [c, c, c, \ldots]$, i.e., $\alpha = \frac{1}{2}(\sqrt{c^2+4} - c)$. The next corollary and Figure 2 describe the limiting extreme-value law for these special numbers.

In this case, the sequences $(\nu_j)_{j\in\mathbb{N}}$ and $(\theta_j)_{j\in\mathbb{N}}$ are constant, namely

$$\nu_j = \lim_{k\in\mathbb{N}} \frac{q_{k+j-1}}{q_{k+j}} = \lim_{k\in\mathbb{N}} [\overbrace{c, c, \ldots, c}^{k+j}] = [c, c, \ldots] = \alpha,$$
$$\theta_j = \lim_{k\in\mathbb{N}} [c, c, \ldots] = [c, c, \ldots] = \alpha.$$

Thus,

$$\gamma_j = \alpha^{-j}, \quad \delta_j = \alpha^j, \quad j \in \mathbb{N}.$$

Applying Theorem 1, we get that the random variables M_{q_k}/q_k, $k \in \mathbb{N}$, converge in distribution to a random variable M with the distribution

$$H(y) = \mathbb{P}(M > y) = \begin{cases} 1 & \text{if } y < 1, \\ \alpha^j \dfrac{\alpha+\alpha^2}{1+\alpha^2} & \text{if } y \in [\alpha^{-j}, \alpha^{-j-1}),\ j = 0, 1, 2, \ldots. \end{cases}$$

Note that in the case when $c = 1$, we obtain the golden mean $\alpha = \frac{1}{2}(\sqrt{5} - 1) = [1, 1, 1, \ldots]$, and for the limiting distribution (as $\alpha + \alpha^2 = 1$),

$$H(y) = \mathbb{P}(M > y) = \begin{cases} 1 & \text{if } y < 1, \\ \dfrac{\alpha^j}{2-\alpha} & \text{if } y \in [\alpha^{-j}, \alpha^{-j-1}),\ j = 0, 1, 2, \ldots. \end{cases}$$

4.4. *Divergent rotation number.* Let $\alpha = [c_1, c_2, c_3, \ldots]$ be such that c_n converges to infinity. Let $K = \mathbb{N}$. In this case, the coefficient γ_1 is equal to $+\infty$ and the random variables M_{q_k}/q_k converge in distribution to the constant random variable $M = 1$ (see §4.2).

5. *Limiting behavior for rotations*

In this section, we show another way to determine the limiting distribution of extreme values, using the limiting distribution for the entry time.

Let the same parameters be as before, i.e., $n = q_k$, $b_n = 0$, $a_n = 1/q_k$ and the infinite set $K \subset \mathbb{N}$, according to the assumption of Theorem 1. Let us assume that the following two conditions hold for all $y \in \mathbb{R}$.

(i) The following limit exists:

$$g(y) = \lim_{k\in K} q_k \mu(L(q_k y)).$$

(ii) The measure of the level sets $\mu(L(q_k y))$ goes to zero, when k goes to infinity.

(iii) The sequence of distribution functions for entry times $F_{L(q_k y)}$, $k \in K$, converges uniformly to a distribution function ϕ_y.

Note that the last two conditions imply in particular that ϕ_y is continuous.

Under these three conditions, the extreme-value distribution is

$$H(y) = \lim_{k\in K} H_{q_k y}(y) = \lim_{k\in K} F_{L(q_k y)}(q_k \mu(L(q_k y))) = \phi_y(g(y)),$$

where the last equality follows from the fact that ϕ_y need to be continuous (see [**12**]). The next proposition shows that these assumptions are valid for every y for which Theorem 1 ensures the existence of a non-trivial limit of $H_{q_k}(y)$.

PROPOSITION 2. *Let K be an infinite set of integers $K \subset \mathbb{N}$, such that the limits γ_j and δ_j exist for every $j \in \mathbb{N}$. Assume $\gamma_{j+1} < \infty$ for some $j \in \mathbb{N}$. Then, for $y \in (\gamma_j, \gamma_{j+1})$, the following hold.*

(i) *The limit $\lim_{k\in K} \mu(L(q_k y))$ is zero.*

(ii) *The limit $g(y) = \lim_{k\in K} q_k \mu(L(q_k y))$ exists, is finite and satisfies the following equality:*

$$g(y) = \frac{\delta_j + \delta_{j+1}}{\gamma_1 + \delta_1}.$$

(iii) *The distribution functions for entry times $F_{L(q_k y)}$, $k \in K$, converge uniformly to the distribution function ϕ_y that linearly interpolates the points*

$$(0, 0), \quad \left(\frac{(1+\theta_{j+1})\nu_{j+1}}{1+\theta_{j+1}\nu_{j+1}}, \frac{(1+\theta_{j+1})\nu_{j+1}}{1+\theta_{j+1}\nu_{j+1}}\right), \quad \left(\frac{1+\theta_{j+1}}{1+\theta_{j+1}\nu_{j+1}}, 1\right).$$

(iv) *The sequence $H_{q_k}(y)$, $k \in K$, converges to a number $H(y)$, where*

$$H(y) = \phi_y(g(y)) = g(y).$$

Proof. Take y from some finite interval (γ_j, γ_{j+1}). By definition, $q_{k+j} < q_k y < q_{k+j+1}$ for $k \in K$ big enough. This implies that $L(q_k y) = B_{k+j}$.

(i) We immediately get that $\mu(L(q_k y))$ tends to zero.

(ii) Using equation (4), we get

$$g(y) = \lim_{k\in K} q_k \mu(B_{k+j}) = \lim_{k\in K} q_k(\eta_{k+j} + \eta_{k+j+1}) = \lim_{k\in K} q_k \eta_k \left(\frac{\eta_{k+j}}{\eta_k} + \frac{\eta_{k+j+1}}{\eta_k}\right)$$
$$= \frac{\delta_j + \delta_{j+1}}{\gamma_1 + \delta_1} \le \frac{1+1}{1} \le 2.$$

Hence, the limit $g(y)$ exists and is finite.

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(iii) The uniform convergence of distribution functions $F_{L(q_k y)}$, $k \in \mathbb{N}$, is a direct application of a result due to Coelho and de Faria [**3**, Theorem I]. If we translate their result into our settings and notation, we get that if there exist limits ν_{j+1} and θ_{j+1}, the distribution functions $F_{B_{k+j}}$, $k \in K$, uniformly converge to the function ϕ_y described in the statement of the proposition. Since $F_{B_{k+j}} = F_{L(q_k y)}$, eventually, for $k \in K$, we need only to verify that the limits ν_{j+1} and θ_{j+1} exist. Since γ_ℓ exists and is finite for every $\ell \leq j+1$, the limits ν_ℓ must exist and be positive, for every $\ell \leq j+1$, i.e.,

$$\nu_\ell = \lim_{k \in K} [c_{k+\ell}, c_{k+\ell-1}, \ldots, c_1] > 0.$$

In particular, for every $\ell \leq j+1$,

$$\eta_{k+\ell-1}/\eta_{k+\ell} < c_{k+\ell+1} + 1, \quad k \in K, \ell \leq j.$$

As

$$\frac{\eta_{k+j}}{\eta_k} = \prod_{\ell=0}^{j-1} \frac{\eta_{k+j+1}}{\eta_{k+\ell}} \geq \prod_{\ell=0}^{j-1} \frac{1}{c_{k+\ell+1}+1} > 0$$

is a lower bound for the sequence $(\eta_{k+j}/\eta_k)_{k \in K}$, we conclude that δ_j is strictly positive and therefore θ_{j+1} exists and is equal to δ_{j+1}/δ_j.

(iv) The equality $H(y) = \phi_y(g(y))$ is a direct consequence of the previous parts (ii) and (iii). Since

$$\begin{aligned}
&\frac{(1+\eta_{k+j+1}/\eta_{k+j})(q_{k+j}/q_{k+j+1})}{1+(\eta_{k+j+1}/\eta_{k+j}) \cdot (q_{k+j}/q_{k+j+1})} \\
&\quad = \frac{(\eta_{k+j}+\eta_{k+j+1})q_{k+j}}{\eta_{k+j}q_{k+j+1}+\eta_{k+j+1}q_{k+j}} = (\eta_{k+j}+\eta_{k+j+1})q_{k+j} \\
&\quad = \frac{(\eta_{k+j}+\eta_{k+j+1})q_{k+j}}{\eta_k q_{k+1}+\eta_{k+1}q_k} = \frac{(\eta_{k+j}/\eta_k+\eta_{k+j+1}/\eta_k)(q_{k+j}/q_k)}{q_{k+1}/q_k+\eta_{k+1}/\eta_k},
\end{aligned}$$

passing to the limit on both sides yields

$$g(y) = \frac{\delta_j+\delta_{j+1}}{\gamma_1+\delta_1} \leq \frac{(\delta_j+\delta_{j+1})\gamma_j}{\gamma_1+\delta_1} = \frac{(1+\theta_{j+1})\nu_{j+1}}{1+\theta_{j+1}\nu_{j+1}}.$$

By the definition of ϕ_y, we conclude that $\phi_y(g(y)) = g(y)$. □

In comparison with Theorem 1, the last proposition does not answer the question of what happens for $y \in (-\infty, 1)$ and for $y \in (\gamma_j, \gamma_{j+1})$ when γ_{j+1} is infinite. To extend the last proposition and use the formula $H(y) = \phi_y(g(y))$ for these cases also is quite complicated, because the limiting function ϕ_y or the limiting value $g(y)$ need not exist for every y from these intervals.

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REFERENCES

[**1**] M. Abadi. Hitting, returning and the short correlation function. *Bull. Braz. Math. Soc. (N.S.)* **37**(4) (2006), 1–17.

[**2**] P. Collet. Statistics of closest return for some non-uniformly hyperbolic systems. *Ergod. Th. & Dynam. Sys.* **21** (2001), 401–420.

[**3**] Z. Coelho and E. de Faria. Limit laws of entrance times for homeomorphisms of the circle. *Israel J. Math.* **93** (1996), 93–112.

[**4**] A. C. M. Freitas and J. M. Freitas. On the link between dependence and independence in Extreme Value Theory for Dynamical Systems. *Statist. Probab. Lett.* **78** (2008), 1088–1093.

[**5**] A. C. M. Freitas and J. M. Freitas. Extreme values for Benedicks Carleson maps. *Ergod. Th. & Dynam. Sys.* **28**(4) (2008), 1117–1133.

[**6**] A. C. M. Freitas, J. M. Freitas and M. Todd. Hitting time statistics and extreme value theory. *Probab. Theory Related Fields* **147**(3) (2010), 675–710.

[**7**] A. C. M. Freitas, J. M. Freitas and M. Todd. Extreme value laws in dynamical systems for non-smooth observations. *J. Stat. Phys.* **142**(1) (2011), 108–126.

[**8**] C. Gupta, M. Holland and M. Nicol. Extreme value theory and return time statistics for dispersing billiard maps and flows, Lozi maps and Lorenz-like maps. *Ergod. Th. & Dynam. Sys.* **31**(5) (2011), 1363–1390.

[**9**] N. Haydn. Entry and return times distribution. *Dyn. Syst.* **28**(3) (2013), 333–353.

[**10**] M. Holland, M. Nicol and A. Török. Extreme value distributions for non-uniformly hyperbolic dynamical systems. *Trans. Amer. Math. Soc.* **364**(2) (2012), 661–688.

[**11**] D. H. Kim and K. K. Park. The first return time properties of an irrational rotation. *Proc. Amer. Math. Soc.* **136**(11) (2008), 3941–3951.

[**12**] M. Kupsa and Y. Lacroix. Asymptotics for hitting times. *Ann. Probab.* **33**(2) (2005), 610–619.