Proceedings of the Edinburgh Mathematical Society (2015) **58**, 53–80
DOI:10.1017/S0013091513000631

ON THE EQUIVARIANT IMPLICIT FUNCTION THEOREM WITH LOW REGULARITY AND APPLICATIONS TO GEOMETRIC VARIATIONAL PROBLEMS

RENATO G. BETTIOL[1], PAOLO PICCIONE[2] AND GAETANO SICILIANO[2]

[1] *Department of Mathematics, University of Notre Dame, 255 Hurley Building, Notre Dame, IN 46556-4618, USA* (rbettiol@nd.edu)
[2] *Departamento de Matemática, Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP 05508-090, Brazil* (piccione@ime.usp.br) (sicilian@ime.usp.br)

(Received 18 June 2012)

Abstract We prove an implicit function theorem for functions on infinite-dimensional Banach manifolds, invariant under the (local) action of a finite-dimensional Lie group. Motivated by some geometric variational problems, we consider group actions that are not necessarily differentiable everywhere, but only on some dense subset. Applications are discussed in the context of harmonic maps, closed (pseudo-) Riemannian geodesics and constant mean curvature hypersurfaces.

Keywords: implicit function theorem; geometric variational problems; CMC hypersurfaces; closed geodesics; harmonic maps

2010 *Mathematics subject classification:* Primary 46T05; 47J07; 58C15; 58D19

1. Introduction

The implicit function theorem is a ubiquitous result, from elementary multivariable calculus courses to current pure and applied research problems. Given that the literature on the several formulations of the theorem is so considerable, we do not attempt to give an account of the extensive variety of statements available. In this paper we formulate a version of the theorem for functions on Banach manifolds invariant under the action of a finite-dimensional Lie group, which is not necessarily compact. Previous formulations of the G-equivariant implicit function theorem, most notably by Dancer [**6–8**], considered only *linear* actions of groups on Banach spaces. This paper concerns two crucial improvements, namely, that

(1) the actions may be nonlinear,

(2) the action is only assumed to be by homeomorphisms, and possibly not everywhere differentiable.

Proving an implicit function theorem in such a broad context is not just a matter of abstract generality. Namely, as first noted by Palais [**16**] and others in the 1960s, the natural variational framework of several interesting geometric problems involves functionals on Banach manifolds that are invariant under the continuous (but not differentiable) action of a finite-dimensional Lie group of symmetries. Our result is motivated precisely by this type of problem, which includes constant mean curvature (CMC) embeddings, closed geodesics and harmonic maps, among other potential applications. A rough statement of our main abstract result is as follows.

Theorem 1.1. *Let $\mathfrak{f}\colon \mathfrak{M}\times\Lambda\to\mathbb{R}$ be a map of class $\mathcal{C}^{k+1}$, $k\geqslant 1$, where $\mathfrak{M}$ and Λ are (possibly infinite-dimensional) Banach manifolds, and assume that, for all λ, the functional $f(\cdot,\lambda)$ is invariant under the action of a finite-dimensional Lie group G on $\mathfrak{M}$. Let $(x_0,\lambda_0)\in\mathfrak{M}\times\Lambda$ be such that $(\partial\mathfrak{f}/\partial x)(x_0,\lambda_0)=0$. Assume that the second variation $(\partial^2\mathfrak{f}/\partial x^2)(x_0,\lambda_0)$ is represented by a self-adjoint Fredholm operator, and that the critical point x_0 is equivariantly non-degenerate, i.e.*

$$\ker\left(\frac{\partial^2\mathfrak{f}}{\partial x^2}(x_0,\lambda_0)\right)=T_{x_0}(G\cdot x_0).$$

There then exists a $\mathcal{C}^k$-map $x\colon U\subset\Lambda\to\mathfrak{M}$, defined in a neighbourhood U of λ_0 in Λ, with $x(\lambda_0)=x_0$, such that if $\lambda\in U$ and y is sufficiently close to the orbit $G\cdot x_0$, then $(\partial\mathfrak{f}/\partial x)(y,\lambda)=0$ if and only if y belongs to the G-orbit of $x(\lambda)$.

The precise technical statement of the above result (see Theorem 3.2) with all detailed functional analytical conditions is cast in abstract Banach vector bundle language, and only given later, in § 3, for the sake of exposition. Its formulation involves a set of axioms that describe a rather general setup to which the result applies. Axioms (A1), (A2) and (A3) describe the basic variational setup. Axiom (B) deals with the differentiability of the action, and Axiom (C) with the G-invariance of the set of critical points. Axioms (D1), (D2), (D3) and (D4) give the existence of a *gradient-like* map, and axioms (E1) and (E2) guarantee its equivariance. Finally, Axiom (F) provides a notion of continuity for the tangent space to the group orbits.

Explicit applications to the above-mentioned geometric variational problems are discussed in § 4. More precisely, we quote some of the deformation rigidity results that can be obtained as a direct consequence of Theorem 1.1.

Theorem 1.2. *Let $(\bar{M},\bar{\boldsymbol{g}})$ be a smooth Riemannian manifold, let $\boldsymbol{g}$ be a $\mathcal{C}^k$ Riemannian metric tensor on the manifold M, with $k\geqslant 3$, and let $\phi\colon M\to\bar{M}$ be a $(\boldsymbol{g},\bar{\boldsymbol{g}})$-harmonic map, i.e. $\Delta_{\boldsymbol{g},\bar{\boldsymbol{g}}}(\phi)=\operatorname{tr}(\hat{\nabla}\,\mathrm{d}\phi)=0$. Consider Λ an open subset of a Banach space of symmetric $(0,2)$-tensors of class $\mathcal{C}^k$ on M, with $\boldsymbol{g}\in\Lambda$, such that every tensor in Λ is a Riemannian metric tensor on M. Suppose that ϕ is non-degenerate, i.e. all Jacobi fields along ϕ are of the form $\bar{\boldsymbol{K}}\circ\phi$, where $\bar{\boldsymbol{K}}$ is a Killing vector field of $\bar{M}$. There then exists a neighbourhood $\mathcal{U}$ of $\boldsymbol{g}$ in Λ, a neighbourhood $\mathcal{V}$ of ϕ in $\mathcal{C}^{2,\alpha}(M,\bar{M})$, and a $\mathcal{C}^{k-1}$-function $\mathcal{U}\ni\boldsymbol{h}\mapsto\phi_{\boldsymbol{h}}\in\mathcal{C}^{2,\alpha}(M,\bar{M})$ such that*

(a) *$\phi_{\boldsymbol{h}}$ is an $(\boldsymbol{h},\bar{\boldsymbol{g}})$-harmonic map for all $\boldsymbol{h}\in\mathcal{U}$,*

(b) *if $\boldsymbol{h}\in\mathcal{U}$ and $\varphi\in\mathcal{V}$ is an $(\boldsymbol{h},\bar{\boldsymbol{g}})$-harmonic map, then φ is geometrically equivalent to $\phi_{\boldsymbol{h}}$, i.e. there exists an isometry $\psi\in\operatorname{Iso}(\bar{M},\bar{\boldsymbol{g}})$ such that $\varphi=\psi\circ\phi_{\boldsymbol{h}}$.*

Theorem 1.3. *Let $\boldsymbol{g}$ be a $\mathcal{C}^k$ (pseudo-) Riemannian metric tensor on the manifold M, with $k \geqslant 3$, and let γ be a closed $\boldsymbol{g}$-geodesic on M. Consider Λ an open subset of a Banach space of symmetric $(0,2)$-tensors of class $\mathcal{C}^k$ on M, with $\boldsymbol{g} \in \Lambda$, such that every tensor in Λ is a (pseudo-) Riemannian metric tensor on M. Suppose that γ is non-degenerate, i.e. all periodic Jacobi fields along γ are (constant) multiples of the tangent field γ'. There then exists a neighbourhood $\mathcal{U}$ of $\boldsymbol{g}$ in Λ, a neighbourhood $\mathcal{V}$ of γ in $\mathcal{C}^2(\mathbb{S}^1, M)$, and a $\mathcal{C}^{k-1}$-function $\mathcal{U} \ni \boldsymbol{h} \mapsto \gamma_{\boldsymbol{h}} \in \mathcal{C}^2(\mathbb{S}^1, M)$ such that*

(a) *$\gamma_{\boldsymbol{h}}$ is a closed $\boldsymbol{h}$-geodesic in M for all $\boldsymbol{h} \in \mathcal{U}$,*

(b) *if $\boldsymbol{h} \in \mathcal{U}$ and $\alpha \in \mathcal{V}$ is a closed $\boldsymbol{h}$-geodesic in M, then α is geometrically equivalent to $\gamma_{\boldsymbol{h}}$, i.e. α and $\gamma_{\boldsymbol{h}}$ have the same image (and the same number of turns).*

Theorem 1.4. *Let $x\colon M \hookrightarrow \bar{M}$ be a non-degenerate and transversely oriented CMC embedding, with mean curvature H_0. Assume that there exists an invariant volume functional $\mathcal{V}$ defined in a neighbourhood of x in the set of $\mathcal{C}^1$-embeddings of M into $\bar{M}$. There then exists an open interval $]H_0-\varepsilon, H_0+\varepsilon[$ and a smooth function $]H_0-\varepsilon, H_0+\varepsilon[\, \ni H \mapsto \varphi_H \in \mathcal{C}^{2,\alpha}(M)$, with $\varphi_{H_0}=0$, such that the following hold.*

(a) *For all $H \in\,]H_0-\varepsilon, H_0+\varepsilon[$, the map $x_H\colon M \hookrightarrow \bar{M}$ defined by*

$$x_H(p) = \exp_{x(p)}(\varphi_H(p)\cdot \boldsymbol{n}_x(p)), \quad p \in M,$$

is a CMC embedding having mean curvature equal to H.

(b) *Any given CMC embedding $y\colon M \hookrightarrow \bar{M}$ sufficiently close to x (in the $\mathcal{C}^{2,\alpha}$-topology) is isometrically congruent to some x_H.*

One of the main motivations for the development of our abstract result is Theorem 1.4, which is a generalization of some previous rigidity results for CMC embeddings. These previous results mostly originate from an idea of Kapouleas [**11**,**12**], which was then also employed by Mazzeo *et al.* [**15**], Mazzeo and Pacard [**14**], White [**19**, § 3] and, finally, Pérez and Ros [**17**, Theorem 6.7]. Using a similar idea, we prove a specific formulation of the G-equivariant implicit function theorem for CMC embeddings (see Proposition 2.10), without using Theorem 1.1. The proof is purely geometric, based on a flux argument. This curious proof cannot be extended, for example, to the case where $x(M)$ is not the boundary of an open subset of $\bar{M}$. In particular, this excludes the case of CMC embeddings of manifolds *with boundary*.

Our abstract G-equivariant implicit function theorem applied to this setup covers a much broader situation, culminating in Theorem 1.4, which generalizes Proposition 2.10. This is a practical illustration of the advantages of the generalized equivariant implicit function theorem given by Theorem 1.1. Namely, the hypothesis that $x(M)$ is a boundary is replaced with the more general hypothesis that there exists a generalized volume functional in a $\mathcal{C}^1$-neighbourhood of x that is invariant under (small) isometries of the ambient space. Topological and geometrical conditions that guarantee that this hypothesis is satisfied are discussed in Appendix B. For instance, invariant volume functionals exist when the ambient space is diffeomorphic to a sphere or to $\mathbb{R}^n$, when the ambient space is not compact but it has compact isometry group (in the case of embeddings of manifolds with boundary, compactness of the isometry group is not necessary), or when the image of x is contained in an open subset of $\bar{M}$ that has vanishing de Rham cohomology in dimension $m = \dim(M)$.

Returning to the abstract result, a final technical remark on its proof is in order. The central point is the construction of a sort of *slice** for the group action at a given smooth critical orbit of the variational problem. More precisely, this is a smooth submanifold S, transversal to the given smooth critical orbit, such that every nearby orbit (not necessarily smooth) intercepts S, and with the property that it is a *natural constraint*, i.e. restriction to S of the variational problem has the same critical points as the non-restricted functional. Given the lack of regularity, transversality at an orbit does not imply non-empty intersection with nearby orbits. The transversality argument is replaced by a topological degree argument that uses the finite dimensionality of the group orbits (see Proposition 3.4).

2. An implicit function theorem for CMC hypersurfaces

In this section, we discuss an ad hoc version of the implicit function theorem in the context of CMC embeddings in Riemannian manifolds, which serves as motivation for the abstract formulation given in § 3. The basic setup is given by a CMC hypersurface M of a Riemannian manifold $\bar{M}$. We first recall two elementary applications of Stokes's theorem to the computation of integrals involving Killing fields and mean curvature of submanifolds (see [**9**, Lemma 5.5] for the two-dimensional orientable case).

Lemma 2.1. *Let $(\bar{M}, \bar{\boldsymbol{g}})$ be a Riemannian manifold, let $M \subset \bar{M}$ be a compact submanifold (without boundary), with mean curvature vector field $\boldsymbol{H}$, and let $\boldsymbol{K} \in \mathfrak{X}(M)$ be a Killing field in M. Then,*

$$\int_M \bar{\boldsymbol{g}}(\boldsymbol{K}, \boldsymbol{H}) = 0. \tag{2.1}$$

In addition, if M is the boundary of a (bounded) open subset of $\bar{M}$, then

$$\int_M \bar{\boldsymbol{g}}(\boldsymbol{K}, \boldsymbol{n}) = 0, \tag{2.2}$$

where $\boldsymbol{n}$ is a continuous unit normal field along M.

Proof. Denote by $\boldsymbol{K}_M \in \mathfrak{X}(M)$ the vector field on M obtained by orthogonal projection of $\boldsymbol{K}$. We claim that $\operatorname{div}_M(\boldsymbol{K}_M) = \bar{\boldsymbol{g}}(\boldsymbol{K}, \boldsymbol{H})$. Equality (2.1) then follows immediately from Stokes's theorem. In order to compute $\operatorname{div}_M(\boldsymbol{K}_M)$, let $\bar{\nabla}$ denote the Levi–Civita connection of $\bar{\boldsymbol{g}}$, and let ∇ be the Levi–Civita connection of the induced metric on M. If $\mathcal{S}$ is the second fundamental form of M, then, for all pairs $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$, one has

* The terminology here is not standard. Recall that a 'slice' for an action through a point is typically assumed invariant under the action of the isotropy of that point (see [**4**]). This property is not required here.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

that $\bar{\nabla}_{\boldsymbol{X}}\boldsymbol{Y} = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \mathcal{S}(\boldsymbol{X}, \boldsymbol{Y})$. Moreover, differentiating in the direction $\boldsymbol{X}$ the equality $\bar{\boldsymbol{g}}(\boldsymbol{K}_M, \boldsymbol{Y}) = \bar{\boldsymbol{g}}(\boldsymbol{K}, \boldsymbol{Y})$, we get that

$$\bar{\boldsymbol{g}}(\nabla_{\boldsymbol{X}}\boldsymbol{K}_M, \boldsymbol{Y}) + \bar{\boldsymbol{g}}(\boldsymbol{K}_M, \nabla_{\boldsymbol{X}}\boldsymbol{Y}) = \bar{\boldsymbol{g}}(\bar{\nabla}_{\boldsymbol{X}}\boldsymbol{K}, \boldsymbol{Y}) + \bar{\boldsymbol{g}}(\boldsymbol{K}, \bar{\nabla}_{\boldsymbol{X}}\boldsymbol{Y}). \tag{2.3}$$

Substituting $\bar{\boldsymbol{g}}(\boldsymbol{K}, \bar{\nabla}_{\boldsymbol{X}}\boldsymbol{Y}) = \bar{\boldsymbol{g}}(\boldsymbol{K}, \nabla_{\boldsymbol{X}}\boldsymbol{Y}) + \bar{\boldsymbol{g}}(\boldsymbol{K}, \mathcal{S}(\boldsymbol{X}, \boldsymbol{Y}))$ into (2.3),

$$\bar{\boldsymbol{g}}(\nabla_{\boldsymbol{X}}\boldsymbol{K}_M, \boldsymbol{Y}) = \bar{\boldsymbol{g}}(\bar{\nabla}_{\boldsymbol{X}}\boldsymbol{K}, \boldsymbol{Y}) + \bar{\boldsymbol{g}}(\boldsymbol{K}, \mathcal{S}(\boldsymbol{X}, \boldsymbol{Y})). \tag{2.4}$$

Given $x \in M$, an orthonormal frame $e_1, \dots, e_m$ of T_xM, and recalling that, since $\boldsymbol{K}$ is Killing, $\bar{\boldsymbol{g}}(\bar{\nabla}_{e_i}\boldsymbol{K}, e_i) = 0$ for all i, we get that

$$\operatorname{div}_M(\boldsymbol{K}_M) = \sum_i \bar{\boldsymbol{g}}(\nabla_{e_i}\boldsymbol{K}_M, e_i) = \sum_i \bar{\boldsymbol{g}}(\boldsymbol{K}, \mathcal{S}(e_i, e_i)) = \bar{\boldsymbol{g}}(\boldsymbol{K}, \boldsymbol{H}),$$

which proves (2.1). Formula (2.2) is an immediate application of Stokes's theorem, observing that $\operatorname{div}_{\bar{M}} \boldsymbol{K} = 0$, as $\boldsymbol{K}$ is Killing. □

Remark 2.2. It is easy to find counterexamples to (2.2) when M is a hypersurface that is not the boundary of an open subset of $\bar{M}$. If M is the boundary of an open subset of $\bar{M}$, i.e. if the set $\bar{M} \setminus M$ has two connected components, then M is *transversely oriented*. This means that the normal bundle $TM^{\perp}$ is orientable. Conversely, if M is transversely oriented, then the condition that M is the boundary of an open subset of $\bar{M}$ is equivalent to the condition that the homomorphism $H_1(\bar{M}) \to H_1(\bar{M}, \bar{M} \setminus M)$ induced in singular homology by the inclusion $(\bar{M}, \emptyset) \hookrightarrow (\bar{M}, \bar{M} \setminus M)$ is trivial.

Definition 2.3. Given a transversely oriented codimension 1 CMC embedding $x\colon M \hookrightarrow \bar{M}$, the *Jacobi operator* J_x of x is the second-order linear elliptic differential operator

$$J_x(f) = \Delta_x f - (m \operatorname{Ric}_{\bar{M}}(\boldsymbol{n}_x) + \|\mathcal{S}_x\|^2)f, \tag{2.5}$$

defined on the space of $\mathcal{C}^2$-functions $f\colon M \to \mathbb{R}$. In the above formula, $m = \dim(M)$, Δ_x is the (positive) Laplacian of functions on M relative to the pullback metric $x^*(\bar{\boldsymbol{g}})$, $\operatorname{Ric}_{\bar{M}}(\boldsymbol{n}_x)$ is the Ricci curvature of $\bar{M}$ evaluated on the unit normal field $\boldsymbol{n}_x$ of x, and $\mathcal{S}_x$ is the second fundamental form of x.

Definition 2.4. A function f satisfying $J_x(f) = 0$ is called a *Jacobi field* of x.

Remark 2.5. It follows easily from (2.5) that the space $\ker(J_x)$ of Jacobi fields of x is a finite-dimensional space.

Remark 2.6. Given any $\alpha \in \,]0, 1[$, seen as a linear operator from $\mathcal{C}^{2,\alpha}(M)$ to $\mathcal{C}^{0,\alpha}(M)$, J_x is a Fredholm map* of index 0, which is symmetric with respect to the L^2-pairing $\langle\cdot,\cdot\rangle_{L^2}\colon \mathcal{C}^{2,\alpha}(M) \times \mathcal{C}^{0,\alpha}(M) \to \mathbb{R}$, given by $\langle f_1, f_2\rangle_{L^2} = \int_M f_1 \cdot f_2 \,\mathrm{d}M$. In particular, $\ker(J_x) = \operatorname{Im}(J_x)^{\perp}$, relative to the L^2-inner product.

* Second-order self-adjoint elliptic operators acting on sections of Euclidean vector bundles over compact manifolds are Fredholm maps of index 0 from the space of $\mathcal{C}^{j,\alpha}$-sections to the space of $\mathcal{C}^{j-2,\alpha}$-sections, $j \geqslant 2$ (see, for instance, [**19**, § 1.4] and [**20**, Theorem 1.1]). This fact is used throughout the paper.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

Note that if $\bar{\boldsymbol{K}}$ is a Killing field of $(\bar{M}, \bar{\boldsymbol{g}})$, then $f = \bar{\boldsymbol{g}}(\bar{\boldsymbol{K}}, \boldsymbol{n}_x)$ is a Jacobi field of x. The embedding x is equivariantly non-degenerate if every Jacobi field arises in this way.

Definition 2.7. The CMC embedding x is *non-degenerate* if, given any Jacobi field f of x, there exists a Killing field $\bar{\boldsymbol{K}}$ of $(\bar{M}, \bar{\boldsymbol{g}})$ such that $f = \bar{\boldsymbol{g}}(\bar{\boldsymbol{K}}, \boldsymbol{n}_x)$.

Remark 2.8. Non-degeneracy of every CMC embedding of M into $\bar{M}$ is a *generic* property in the set of Riemannian metrics $\bar{\boldsymbol{g}}$ (see [**19**, **20**]).

Definition 2.9. Two embeddings $x_i \colon M \hookrightarrow \bar{M}$, $i = 1, 2$, are said to be *congruent* if there exists a diffeomorphism $\phi \colon M \to M$ such that $x_2 = x_1 \circ \phi$, and *isometrically congruent* if there exists a diffeomorphism $\phi \colon M \to M$ and an isometry $\psi \colon \bar{M} \to \bar{M}$ such that $x_2 = \psi \circ x_1 \circ \phi$. Roughly speaking, congruence classes of embeddings of M into $\bar{M}$ are submanifolds of $\bar{M}$ that are diffeomorphic to M.

We now prove the above-mentioned formulation of the implicit function theorem for CMC embeddings (compare with Theorem 1.4).

Proposition 2.10. *Let $x \colon M \hookrightarrow \bar{M}$ be a non-degenerate codimension 1 CMC embedding of a compact manifold M into a Riemannian manifold $(\bar{M}, \bar{\boldsymbol{g}})$, with mean curvature H_0. Assume also that $x(M)$ is the boundary of an open subset of $\bar{M}$. There then exists an open interval $]H_0 - \varepsilon, H_0 + \varepsilon[$ and a smooth function $]H_0 - \varepsilon, H_0 + \varepsilon[\ni H \mapsto \varphi_H \in \mathcal{C}^{2,\alpha}(M)$, with $\varphi_{H_0} = 0$, such that the following hold.*

(a) *For all $H \in]H_0 - \varepsilon, H_0 + \varepsilon[$, the map $x_H \colon M \hookrightarrow \bar{M}$ defined by*

$$x_H(p) = \exp_{x(p)}(\varphi_H(p) \cdot \boldsymbol{n}_x(p)), \quad p \in M,$$

is a CMC embedding having mean curvature equal to H.

(b) *Any given CMC embedding $y \colon M \hookrightarrow \bar{M}$ sufficiently close to x (in the $\mathcal{C}^{2,\alpha}$-topology) is isometrically congruent to some x_H.*

Proof. By a standard argument in submanifold theory, congruence classes of embeddings $y \colon M \hookrightarrow \bar{M}$ near x are parametrized by functions on M. More precisely, to each function $\varphi \in \mathcal{C}^{2,\alpha}(M)$ one associates the map $x_\varphi \colon M \to \bar{M}$ defined by $x_\varphi(p) = \exp_{x(p)}(\varphi(p) \cdot \boldsymbol{n}_x(p))$, $p \in M$. For φ in a neighbourhood of 0, x_φ is an embedding of M into $\bar{M}$. Conversely, given any embedding $y \colon M \hookrightarrow \bar{M}$ that is sufficiently close to x, there exists $\varphi \in \mathcal{C}^{2,\alpha}(M)$ near 0 such that y is congruent to x_φ. Given a sufficiently small neighbourhood $\mathcal{U}$ of 0 in $\mathcal{C}^{2,\alpha}(M)$, consider the map $\mathcal{H} \colon \mathcal{U} \to \mathcal{C}^{0,\alpha}(M)$ that associates to each φ the mean curvature function of the embedding x_φ. This function is smooth, as it is given by a second-order quasi-linear differential operator having smooth coefficients. The derivative $\mathrm{d}\mathcal{H}(0) \colon \mathcal{C}^{2,\alpha}(M) \to \mathcal{C}^{0,\alpha}(M)$ coincides with the Jacobi operator J_x.

By the non-degeneracy assumption on x, there exist $d = \dim \ker(J_x) \geqslant 0$ Killing vector fields $\bar{\boldsymbol{K}}_1, \dots, \bar{\boldsymbol{K}}_d$ of $(\bar{M}, \bar{\boldsymbol{g}})$ such that the functions $f_i = \bar{\boldsymbol{g}}(\bar{\boldsymbol{K}}_i, \boldsymbol{n}_x)$, $i = 1, \dots, d$, form a basis of $\ker(J_x)$. Consider now the auxiliary map $\tilde{\mathcal{H}} \colon \mathcal{U} \times \mathbb{R}^d \to \mathcal{C}^{0,\alpha}(M)$ defined by

$$\tilde{\mathcal{H}}(f, a_1, \dots, a_d) = \mathcal{H}(f) + \sum_{i=1}^{d} a_i f_i.$$

Clearly, $\tilde{\mathcal{H}}$ is smooth, and

$$\mathrm{d}\tilde{\mathcal{H}}(0)(f, b_1, \dots, b_d) = J_x(f) + \sum_{i=1}^{d} b_i f_i.$$

Now, $\mathrm{d}\tilde{\mathcal{H}}(0)$ is surjective; namely, the f_i span the orthogonal complement of $\mathrm{Im}(J_x)$. Moreover, the kernel of $\mathrm{d}\tilde{\mathcal{H}}(0)$ coincides with $\ker(J_x) \oplus \{0\}$, which is finite dimensional and, therefore, complemented in $\mathcal{C}^{2,\alpha}(M) \oplus \mathbb{R}^d$. In other words, $\tilde{\mathcal{H}}$ is a smooth submersion at 0.

Using the local form of submersions, we get that, for H near H_0, there exists an open neighbourhood $\mathcal{V}$ of 0 in $\mathcal{C}^{2,\alpha}(M) \times \mathbb{R}^d$ such that the set $\tilde{\mathcal{H}}^{-1}(H) \cap \mathcal{V}$ is a smooth embedded submanifold of dimension d. Moreover, using the fact that submersions admit smooth local sections, one has that there exists a smooth function $]H_0 - \varepsilon, H_0 + \varepsilon[\ni H \mapsto \tilde{\varphi}_H \in \mathcal{V}$ such that $\tilde{\mathcal{H}}(\tilde{\varphi}_H) = H$ for all H, and with $\tilde{\varphi}_{H_0} = 0$. We now claim, for all $H \in \mathbb{R}$, given $\tilde{\varphi} = (\varphi, a_1, \dots, a_d) \in \tilde{\mathcal{H}}^{-1}(H)$, that $a_1 = \dots = a_d = 0$ and $\mathcal{H}(\varphi) = H$; in other words, $\tilde{\mathcal{H}}^{-1}(H) = \mathcal{H}^{-1}(H) \times \{0\}$. In order to prove the claim, assume that

$$\mathcal{H}(\varphi) + \sum_{i=1}^{d} a_i f_i = H.$$

Multiplying both sides of this equality by $\sum_i a_i f_i$ and integrating on M, keeping in mind that

$$\int_M \mathcal{H}(\varphi) \sum_{i=1}^{d} a_i f_i \overset{(2.1)}{=} 0 \quad \text{and} \quad H \cdot \int_M \sum_{i=1}^{d} a_i f_i \overset{(2.2)}{=} 0,$$

we get that

$$\int_M \left[\sum_{i=1}^{d} a_i f_i \right]^2 = 0.$$

This implies that $a_1 = \dots = a_d = 0$, and proves the claim. Hence, we have $\tilde{\varphi}_H = (\varphi_H, 0, \dots, 0)$, with $H \mapsto \varphi_H$ satisfying Proposition 2.10 (a).

Item (b) also follows easily. Namely, the action by isometries of $(\bar{M}, \bar{\boldsymbol{g}})$ on each CMC embedding x_{φ_H} produces an orbit that is a d-dimensional submanifold of the Banach space $\mathcal{C}^{2,\alpha}(M)$.* Such an orbit is contained in $\mathcal{H}^{-1}(H)$, which is also a d-dimensional submanifold around x_H. Hence, a neighbourhood of x_H in the orbit of x_H coincides with a neighbourhood of x_H in $\mathcal{H}^{-1}(H)$. This implies that CMC embeddings $\mathcal{C}^{2,\alpha}$-close to x must be isometrically congruent to some x_H. □

Remark 2.11. Observe that the assumption that $x(M)$ is the boundary of a bounded open subset of $\bar{M}$ cannot be omitted in Proposition 2.10, as (2.2) is used in the proof (see Remark 2.2). In particular, Proposition 2.10 does not cover the case of CMC embeddings of manifolds with boundary (compare with Theorem 1.4).

* This is not a trivial fact, taking into account that the left action of the isometry group of $(\bar{M}, \boldsymbol{g})$ on the space $\mathcal{C}^{2,\alpha}(M)$ obtained via the exponential map of the normal bundle of x is only continuous, and not differentiable. However, it is proved in [**1**] that the orbit of any smooth embedding is a smooth submanifold.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

3. Statement of the G-equivariant implicit function theorem

The usual formulations of the implicit function theorem give a local result, so its statement can be given using open subsets of Banach spaces as domains and codomains of the functions involved. For the equivariant version of the theorem discussed in this section the situation is somewhat different. Namely, we consider group actions on Banach manifolds whose orbits are not necessarily contained in the domain of some local chart, or in the domain of a local trivialization of a vector bundle. In fact, we do not even assume boundedness of the orbits. This suggests that, in spite of the local character of the result and its proof, the equivariant formulation of our theorem is better cast in an abstract Banach manifolds/Banach vector bundles setup.

The basic setup is given by a manifold $\mathfrak{M}$ acted upon by a Lie group G, another manifold Λ and a differentiable function $\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathbb{R}$ that is G-invariant in the first variable. More precisely, our framework is described by the following set of axioms.

Axiom (A1). *$\mathfrak{M}$ and Λ are differentiable manifolds, modelled on a (possibly infinite-dimensional) Banach space.*

Axiom (A2). *G is a finite-dimensional Lie group, acting continuously on $\mathfrak{M}$ (on the left) by homeomorphisms, and $\mathfrak{g}$ denotes its Lie algebra.*

Axiom (A3). *$\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathbb{R}$ is a function of class $\mathcal{C}^{k+1}$, $k \geqslant 1$, satisfying $\mathfrak{f}(g \cdot x, \lambda) = \mathfrak{f}(x, \lambda)$ for all $g \in G$, $x \in \mathfrak{M}$ and $\lambda \in \Lambda$.*

For all $x \in \mathfrak{M}$, denote by

$$\beta_x\colon G \to \mathfrak{M} \quad \text{and} \quad \gamma_g\colon \mathfrak{M} \to \mathfrak{M} \tag{3.1}$$

the map $\beta_x(g) = g \cdot x$ and the homeomorphism $\gamma_g(x) = g \cdot x$, respectively.

As for the regularity of the group action, we make the following assumptions.

Axiom (B). *There exists a dense subset $\mathfrak{M}' \subset \mathfrak{M}$ such that for all $x \in \mathfrak{M}'$ the map $\beta_x\colon G \to \mathfrak{M}$ is differentiable at $1 \in G$.*

We denote by $\partial_1 \mathfrak{f}\colon \mathfrak{M} \times \Lambda \to T\mathfrak{M}^*$ the derivative of $\mathfrak{f}$ with respect to the first variable; our aim is to study the equation $\partial_1 \mathfrak{f}(x, \lambda) = 0$. Observe that, with the weak regularity assumptions on the group action (we do not assume in principle the differentiability of the map γ_g), it does not follow that if $\partial_1 \mathfrak{f}(x, \lambda) = 0$, then $\partial_1 \mathfrak{f}(g \cdot x, \lambda) = 0$ as well for all $g \in G$. We, therefore, explicitly assume that the following holds.

Axiom (C). *For all $\lambda \in \Lambda$, the set $\{x \in \mathfrak{M}\colon \partial_1 \mathfrak{f}(x, \lambda) = 0\}$ is G-invariant.*

We now look at the question of the lack of a *gradient* for the function $\mathfrak{f}$; we define a *gradient-like* map by introducing a suitable vector bundle on the manifold $\mathfrak{M}$, defined by the following axioms.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

Axiom (D1). *$\mathcal{E} \to \mathfrak{M}$ is a $\mathcal{C}^k$-Banach vector bundle.*

Axiom (D2). *There exist $\mathcal{C}^k$-vector bundle morphisms*

$$\mathfrak{i}\colon T\mathfrak{M} \to \mathcal{E} \quad \text{and} \quad \mathfrak{j}\colon \mathcal{E} \to T\mathfrak{M}^*,$$

with $\mathfrak{j}$ injective.

Axiom (D3). *For all $x \in \mathfrak{M}$, the bilinear form $\langle \cdot, \cdot \rangle_x\colon T_x\mathfrak{M} \times T_x\mathfrak{M} \to \mathbb{R}$ defined by $\langle u, v\rangle_x = \mathfrak{j}_x(\mathfrak{i}_x(u))v$ is a (not necessarily complete) positive definite inner product (this implies that $\mathfrak{i}$ is also injective).*

Axiom (D4). *There exists a $\mathcal{C}^k$-map $\delta\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathcal{E}$ such that*

$$\mathfrak{j} \circ \delta\mathfrak{f} = \partial_1\mathfrak{f}.$$

Since $\mathfrak{j}$ is injective, we get that $\partial_1\mathfrak{f}(x, \lambda) = 0$ if and only if $\delta\mathfrak{f}(x, \lambda) = 0$.

We now return to the question of G-invariance of the set of critical points of the functions $\mathfrak{f}(\cdot, \lambda)$. Assuming that the G-action is by diffeomorphisms (i.e. that the maps γ_g are diffeomorphisms), given x_0 such that $\partial_1\mathfrak{f}(x_0, \lambda) = 0$, obviously $\partial_1\mathfrak{f}(g \cdot x_0, \lambda) = 0$ for all $g \in G$. For this conclusion it is necessary to differentiate γ_g; when the action of G is only by homeomorphisms, the G-invariance of the critical set is obtained under a suitable assumption of G-equivariance for the map $\delta\mathfrak{f}$. Given $x \in \mathfrak{M}$, the fibre of $\mathcal{E}$ over x is denoted by $\mathcal{E}_x$.

Axiom (E1). *There exists a continuous left G-action by linear isomorphisms on the fibres of $\mathcal{E}$ compatible with the action on $\mathfrak{M}$, i.e. such that the projection $\mathcal{E} \to \mathfrak{M}$ is equivariant (this means that, for each g, it is given a family of linear isomorphisms $\varphi_{g,x}\colon \mathcal{E}_x \to \mathcal{E}_{g\cdot x}$ depending continuously on $x \in \mathfrak{M}$ and on $g \in G$, such that $\varphi_{gh,x} = \varphi_{g,h\cdot x} \circ \varphi_{h,x}$ for all $g, h \in G$ and all $x \in \mathfrak{M}$).*

Axiom (E2). *The map $\delta\mathfrak{f}(\cdot, \lambda)\colon \mathfrak{M} \to \mathcal{E}$ is equivariant for all $\lambda \in \Lambda$.*

Lemma 3.1. *Axioms* (E1) *and* (E2) *imply* (C).

Proof. Assume that $\partial_1\mathfrak{f}(x_0, \lambda) = 0$; then, $\delta\mathfrak{f}(x_0, \lambda) = 0$. The equivariance property gives that $\delta\mathfrak{f}(g \cdot x_0, \lambda) = 0$ for all $g \in G$, i.e. $\partial_1\mathfrak{f}(g \cdot x_0, \lambda) = 0$ for all $g \in G$. □

Finally, another set of assumptions is needed in order to deal with the lack of the map $x \mapsto \mathrm{d}\beta_x(1) \in \mathrm{Lin}(\mathfrak{g}, T\mathfrak{M})$ for all $x \in \mathfrak{M}$. Our next set of hypotheses gives the existence of a continuous extension to $\mathfrak{M}$ of this map, provided that its codomain is enlarged and endowed with a weaker topology. As above, this set of assumptions is better cast in terms of vector bundles and injective morphisms.

Axiom (F). *There exist a $\mathcal{C}^k$-vector bundle $\mathcal{Y} \to \mathfrak{M}$ and $\mathcal{C}^k$-vector bundle morphisms*

$$\tilde{\jmath}\colon \mathcal{E} \to \mathcal{Y}^* \quad \text{and} \quad \kappa\colon T\mathfrak{M} \to \mathcal{Y},$$

with κ injective, such that

(F1) *$\kappa^* \circ \tilde{\jmath} = \mathfrak{j}$ (from which it follows that $\tilde{\jmath}$ is also injective),*

(F2) *the map $\mathfrak{M}' \ni x \mapsto \kappa_x \circ \mathrm{d}\beta_x(1) \in \mathrm{Lin}(\mathfrak{g}, \mathcal{Y}_x)$ has a continuous extension to a section of the vector bundle $\mathrm{Lin}(\mathfrak{g}, \mathcal{Y}) \to \mathfrak{M}$.*

From the density of $\mathfrak{M}'$, the extension in (F2) is, therefore, unique. We are now ready for the detailed technical statement of Theorem 1.1 in § 1 and its proof.

Theorem 3.2. *In the above setup of Axioms* (A1)–(F), *let* $(x_0, \lambda_0) \in \mathfrak{M}' \times \Lambda$ *be a point such that* $\partial_1 \mathfrak{f}(x_0, \lambda_0) = 0$. *Denote by* $L\colon T_{x_0}\mathfrak{M} \to \mathcal{E}_{x_0}$ *the linear map*

$$L := \pi_{\mathrm{ver}} \circ \partial_1(\delta \mathfrak{f})(x_0, \lambda_0),$$

where $\pi_{\mathrm{ver}}\colon T_{0_{x_0}}\mathcal{E} \to \mathcal{E}_{x_0}$ *is the canonical vertical projection. If*

(G1) L *is Fredholm of index* 0,

(G2) $\ker L = \operatorname{Im} \mathrm{d}\beta_{x_0}(1)$,

then there exists a G*-invariant neighbourhood* $V \subset \mathfrak{M} \times \Lambda$ *of* $(G \cdot x_0, \lambda_0)$ *and a* $\mathcal{C}^k$*-function* $\sigma\colon \Lambda_0 \to \mathfrak{M}$ *defined in a neighbourhood* Λ_0 *of* λ_0 *in* Λ *such that* $(x, \lambda) \in V$ *and* $\partial_1 \mathfrak{f}(x, \lambda) = 0$ *hold if and only if* $x \in G \cdot \sigma(\lambda)$.

Remark 3.3. Condition (G2) is an equivariant *non-degeneracy condition* on the critical orbit $G \cdot x_0$.

Proof. We study a local problem first, and we then use the group action for the proof of the global statement. After suitable local charts and local trivialization of vector bundles around the point (x_0, λ_0) have been chosen, one can assume that the following hold.

- $\mathfrak{M}$ is an open subset of a Banach space X, $\mathfrak{M}'$ is a dense subset of $\mathfrak{M}$ that is endowed with a topology finer than the induced topology from $\mathfrak{M}$, and Λ is an open subset of another Banach space.
- The group action on $\mathfrak{M}$ is described by a map $\mathcal{U} \ni (g, x) \mapsto g \cdot x \in \mathfrak{M}$, with $\mathcal{U}$ an open neighbourhood of $\{1\} \times \mathfrak{M}$ in $G \times \mathfrak{M}$. Such a map satisfies the obvious equalities given by group operations whenever* such equalities make sense in the open set $\mathcal{U}$.
- The $\mathcal{C}^{k+1}$-function $\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathbb{R}$ satisfies $\mathfrak{f}(g \cdot x, \lambda) = \mathfrak{f}(x, \lambda)$ wherever such equality makes sense (as above).
- The vector bundle $\mathcal{E}$ is replaced with the product $\mathfrak{M} \times \mathcal{E}_0$, where $\mathcal{E}_0$ is a fixed Banach space (isometric to the typical fibre of $\mathcal{E}$).
- $\mathfrak{j}\colon \mathfrak{M} \to \operatorname{Lin}(\mathcal{E}_0, X^*)$ is a $\mathcal{C}^k$-map such that $\mathfrak{j}_x$ is injective for all $x \in \mathfrak{M}$.
- $\mathfrak{i}\colon \mathfrak{M} \to \operatorname{Lin}(X, \mathcal{E}_0)$ is a $\mathcal{C}^k$-map such that $\mathfrak{j}_x \circ \mathfrak{i}_x\colon X \to X^*$ is a (not necessarily complete) positive definite inner product on X (which implies, in particular, that $\mathfrak{i}_x$ is injective for all x).

* For instance, the equality $g \cdot (h \cdot x) = (gh) \cdot x$ holds for all $g, h \in G$ and $x \in \mathfrak{M}$ such that $(h, x) \in \mathcal{U}$ and $(g, h \cdot x) \in \mathcal{U}$. In particular, given $x \in \mathfrak{M}$, the equality must hold when g and h belong to some neighbourhood of 1 in G. This could be formalized in terms of *partial actions* of groups (or groupoids) on topological spaces, but this is not relevant in the context of the present paper.

- The vector bundle $\mathcal{Y}$ is replaced with the product $\mathfrak{M} \times \mathcal{Y}_0$, where $\mathcal{Y}_0$ is a fixed Banach space (isometric to the typical fibre of $\mathcal{Y}$).

- $\tilde{\jmath}\colon \mathfrak{M} \to \mathrm{Lin}(\mathcal{E}_0, \mathcal{Y}_0^*)$ and $\kappa\colon \mathcal{M} \to \mathrm{Lin}(X, \mathcal{Y}_0)$ are $\mathcal{C}^k$-maps taking values in the set of injective linear maps, and such that $\kappa_x^* \circ \tilde{\jmath}_x = \mathfrak{j}_x$ for all $x \in \mathfrak{M}$.

- For all $x \in \mathfrak{M}$, the map β_x is only defined on an open neighbourhood of 1 in G. For $x \in \mathfrak{M}'$, its derivative at 1 is a linear map $\mathrm{d}\beta_x(1)\colon \mathfrak{g} \to X$ that depends continuously on x, relative to the finer topology of $\mathfrak{M}'$.

- The map $\mathfrak{M}' \ni x \mapsto \kappa_x \circ [\mathrm{d}\beta_x(1)] \in \mathrm{Lin}(\mathfrak{g}, \mathcal{Y}_0)$ has a continuous extension to $\mathfrak{M}$.

- $\partial_1 \mathfrak{f}\colon \mathfrak{M} \times \Lambda \to X^*$ and $\delta\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathcal{E}_0$ are maps of class $\mathcal{C}^k$ such that $\mathfrak{j}_x(\delta\mathfrak{f}(x,\lambda)) = \partial_1 \mathfrak{f}(x,\lambda)$ for all (x,λ).

- The linear operator $L\colon X \to \mathcal{E}_0$ is given by the partial derivative $\partial_1(\delta\mathfrak{f})(x_0,\lambda_0)$. It is a Fredholm operator of index 0, and $\ker L$ is given by the image of the linear map $\mathrm{d}\beta_{x_0}(1)$.

Let $S = \mathrm{Im}(\mathrm{d}\beta_{x_0}(1))^\perp$ be the closed subspace of X given by the orthogonal complement of the subspace $\mathrm{Im}(\mathrm{d}\beta_{x_0}(1))$ relative to the inner product $\langle\cdot,\cdot\rangle = \mathfrak{j}_{x_0} \circ \mathfrak{i}_{x_0}$. Since $\mathrm{Im}(\mathrm{d}\beta_{x_0}(1))$ is finite dimensional, we have a direct sum decomposition

$$X = \mathrm{Im}(\mathrm{d}\beta_{x_0}(1)) \oplus S. \tag{3.2}$$

We now introduce a finite-dimensional subspace $Y \subset \mathcal{E}_0$ as

$$Y = \mathfrak{i}_{x_0}(\ker L);$$

we claim that Y is complementary to the closed subspace $\mathrm{Im}\, L$ in $\mathcal{E}_0$. In order to prove the claim, we first observe that, using the fact that $\mathfrak{i}_{x_0}$ is injective and L has index 0, the dimension of Y equals the codimension of $\mathrm{Im}\, L$. Thus, our claim is proved if we show that $Y \cap \mathrm{Im}\, L = \{0\}$. We have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ L\ } & \mathcal{E}_0 \\ & \searrow^{\partial_1(\partial_1\mathfrak{f})(x_0,\lambda_0)} & \big\downarrow \mathfrak{j}_{x_0} \\ & & X^* \end{array} \tag{3.3}$$

which is easily obtained by differentiating the equality $\mathfrak{j}_x(\delta\mathfrak{f}(x,\lambda_0)) = \partial_1\mathfrak{f}(x,\lambda_0)$ with respect to x at $x = x_0$, keeping in mind that $\delta\mathfrak{f}(x_0,\lambda_0) = 0$. Observe that the second line in (3.3) is a symmetric operator, and, therefore, we obtain that

$$\mathfrak{j}_{x_0}(\mathrm{Im}\, L) \subset [\ker(\mathfrak{j}_{x_0} \circ L)]^\circ = (\ker L)^\circ, \tag{3.4}$$

where W° denotes the annihilator of the subspace $W \subset X$ in X^*. Now, if $v \in \ker L$ is such that $\mathfrak{i}_{x_0}(v) \in \mathrm{Im}\, L$, then, by (3.4), $\mathfrak{j}_{x_0} \circ \mathfrak{i}_{x_0}(v) \in (\ker L)^\circ$, and, in particular, $\mathfrak{j}_{x_0}(\mathfrak{i}_{x_0}(v))v = 0$. By (D3), it follows that $v = 0$, i.e. $Y \cap \mathrm{Im}\, L = \{0\}$, and, therefore,

$$\mathcal{E}_0 = Y \oplus \mathrm{Im}\, L.$$

Let $P\colon \mathcal{E}_0 \to \operatorname{Im} L$ be the projection relative to this direct sum decomposition of $\mathcal{E}_0$. We define the function

$$H\colon (\mathfrak{M}\cap S)\times\Lambda \to \operatorname{Im} L$$
$$H(x,\lambda) = P(\delta\mathfrak{f}(x,\lambda));$$

observe that $H(x_0,\lambda_0)=0$. Such a map has the same regularity as $\delta\mathfrak{f}$. The derivative $\partial_1 H(x_0,\lambda_0)$ is $P\circ L|_S = L|_S\colon S\to \operatorname{Im} L$, and this is an isomorphism by (G2) and by (3.2). We can, therefore, apply the standard implicit function theorem to $H(x,\lambda)=0$ around (x_0,λ_0), obtaining a neighbourhood Λ_0 of λ_0 in Λ and a $\mathcal{C}^k$-function $\sigma\colon \Lambda_0\to(\mathfrak{M}\cap S)$ with $\sigma(\lambda_0)=x_0$ and such that, given (x,λ) in a neighbourhood of (x_0,λ_0) in $(\mathfrak{M}\cap S)\times\Lambda$, the equality $H(x,\lambda)=0$ holds if and only if $x=\sigma(\lambda)$.

In order to complete the proof of our theorem, we show that the following hold.

(1) There exists a neighbourhood W of (x_0,λ_0) in $\mathfrak{M}\times\Lambda$ such that, given $(x,\lambda)\in W$, $H(x,\lambda)=0$ if and only if $\partial_1\mathfrak{f}(x,\lambda)=0$.

(2) If $x\in\mathfrak{M}$ is sufficiently close to x_0, then the orbit $G\cdot x$ has a non-empty intersection with $\mathfrak{M}\cap S$.

By possibly reducing the domain of the function σ, we can assume that its graph is contained in W. The first claim implies that, given (x,λ) sufficiently close to (x_0,λ_0) in $(\mathfrak{M}\cap S)\times\Lambda$, the equality $\partial_1\mathfrak{f}(x,\lambda)=0$ holds if and only if $x=\sigma(\lambda)$. The second claim and assumption (C) imply that, given (x,λ) sufficiently close to $G\cdot x_0\times\{\lambda_0\}$ in $\mathfrak{M}\times\Lambda$, $\partial_1\mathfrak{f}(x,\lambda)=0$ if and only if $x\in G\cdot\sigma(\lambda)$.

In order to prove (1), we first observe that if $(x,\lambda)\in(\mathfrak{M}\cap S)\times\Lambda$ and $\partial_1\mathfrak{f}(x,\lambda)=0$, then $\delta\mathfrak{f}(x,\lambda)=0$ and, therefore, $H(x,\lambda)=0$. Conversely, we show that if $x\in\mathfrak{M}\cap S$ is near x_0, and $H(x,\lambda)=0$, then $\delta\mathfrak{f}(x,\lambda)=0$ (and, thus, $\partial_1\mathfrak{f}(x,\lambda)=0$ as well). We observe that if $H(x,\lambda)=0$, then $\delta\mathfrak{f}(x,\lambda)\in\mathfrak{i}_{x_0}(\ker L)$, and, thus, $\tilde{\jmath}_x(\delta\mathfrak{f}(x,\lambda))$ *annihilates* $[\tilde{\jmath}_x(\mathfrak{i}_{x_0}(\ker L))]_\circ$. Here, given a subspace $Z\subset X^*$, the symbol $Z_\circ$ denotes the subspace of X annihilated by Z. Denote by $B\colon\mathfrak{M}\to\operatorname{Lin}(\mathfrak{g},\mathcal{Y}_0)$ the continuous extension of the map $x\mapsto\kappa_x\circ[\mathrm{d}\beta_x(1)]$ defined in $\mathfrak{M}'$ (by (F2)); we claim that $\tilde{\jmath}_x(\delta\mathfrak{f}(x,\lambda))$ also annihilates the image of $B(x)$. This follows from the fact that, for $x\in\mathfrak{M}'$, $\partial_1\mathfrak{f}(x,\lambda)$ annihilates $\operatorname{Im}(\mathrm{d}\beta_x(1))$, which is easily seen by differentiating at $g=1$ the (constant) map $g\mapsto\mathfrak{f}(\beta_x(g),\lambda)$ (use (B1)) and a continuity argument. Namely, observe that, for $x\in\mathfrak{M}'$,

$$\begin{aligned}0&=\partial_1\mathfrak{f}(x,\lambda)\circ\mathrm{d}\beta_x(1)\\&=\mathfrak{j}_x(\delta\mathfrak{f}(x,\lambda))\circ[\mathrm{d}\beta_x(1)]\\&=\kappa_x^*(\tilde{\jmath}_x(\delta\mathfrak{f}(x,\lambda)))\circ[\mathrm{d}\beta_x(1)]\\&=\tilde{\jmath}_x(\delta\mathfrak{f}(x,\lambda))\circ\kappa_x\circ[\mathrm{d}\beta_x(1)].\end{aligned}$$

This states that the map

$$\mathfrak{M}\ni x\quad\mapsto\quad\tilde{\jmath}_x(\delta\mathfrak{f}(x,\lambda))\circ B(x)\in\mathfrak{g}^*$$

vanishes for $x\in\mathfrak{M}'$. Thus, by continuity, it vanishes identically, i.e. $\tilde{\jmath}_x(\delta\mathfrak{f}(x,\lambda))$ annihilates the image of $B(x)$.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

To conclude the proof of (1), it suffices to show that, for $x \in \mathfrak{M}$ near x_0, one has that

$$\mathrm{Im}(B(x)) + [\tilde{\jmath}_x(\mathfrak{i}_{x_0}(\ker L))]_\circ = \mathcal{Y}_0. \tag{3.5}$$

Using the continuity of B and the fact that the subspace $[\tilde{\jmath}_x(\mathfrak{i}_{x_0}(\ker L))]_\circ$ has *fixed* codimension in $\mathcal{Y}_0$ (i.e. it does not depend on x), it follows that (3.5) is open* in $\mathfrak{M}$. Thus, it suffices to show that it holds at $x = x_0$. We check (3.5) at $x = x_0$. The dimension of $\mathrm{Im}(B(x_0))$ equals the dimension of $\mathrm{Im}(\mathrm{d}\beta_{x_0}(1))$, and this is equal to the dimension of $\ker L$, by (G2). Since $\tilde{\jmath}_{x_0}$ and $\mathfrak{i}_{x_0}$ are injective, the codimension of $[\tilde{\jmath}_x(\mathfrak{i}_{x_0}(\ker L))]_\circ$ is equal to the dimension of $\ker L$. Thus, it suffices to show that

$$\kappa_{x_0}(\mathrm{Im}(\mathrm{d}\beta_{x_0}(1))) \cap [\tilde{\jmath}_{x_0}(\mathfrak{i}_{x_0}(\ker L))]_\circ = \{0\},$$

i.e.

$$\begin{aligned}\mathrm{Im}(\mathrm{d}\beta_{x_0}(1)) \cap \kappa_{x_0}^{-1}[\tilde{\jmath}_{x_0}(\mathfrak{i}_{x_0}(\ker L))]_\circ &= \ker L \cap [\kappa_{x_0}^* \circ \tilde{\jmath}_{x_0}(\mathfrak{i}_{x_0}(\ker L))]_\circ \\ &= \ker L \cap [\mathfrak{j}_{x_0}(\mathfrak{i}_{x_0}(\ker L))]_0 \\ &= \ker L \cap (\ker L)^\perp \\ &= \{0\}.\end{aligned}$$

It remains to show (2), i.e., equivalently, that the set $G \cdot S$ contains an open neighbourhood of x_0. Since we are not assuming differentiability of the group action, this does not follow from a transversality argument. The correct argument in the continuous case uses the notion of topological degree of a map, and it is given separately in Proposition 3.4. In our case, this result is used setting $A = \mathfrak{M}$, $M = G$, $N = \mathfrak{M}$, $P = S$, $m_0 = 1$, $a_0 = x_0$ and χ as the action. □

Proposition 3.4. *Let N be a (possibly infinite-dimensional) Banach manifold, let $P \subset N$ be a Banach submanifold, let M be a finite-dimensional manifold and let A be a topological space. Assume that $\chi\colon A \times M \to N$ is a continuous function such that there exists $a_0 \in A$ and $m_0 \in M$ with*

(a) $\chi(a_0, m_0) \in P$,

(b) $\chi(a_0, \cdot)\colon M \to N$ *of class* $\mathcal{C}^1$,

(c) $\partial_2\chi(a_0, m_0)(T_{m_0}M) + T_{\chi(a_0,m_0)}P = T_{\chi(a_0,m_0)}N$.

Then, for $a \in A$ near a_0, $\chi(a, M) \cap P \neq \emptyset$.

* Details on the proof of openness of condition (3.5) are as follows. Let $e_1, \dots, e_r$ be a basis of $\ker L$; the covectors $\omega_i = \tilde{\jmath}_x(\mathfrak{i}_{x_0}(e_i))$, $i = 1, \dots, r$, are linearly independent in $\mathcal{Y}_0^*$. Consider the surjective linear map $\tau_x\colon \mathcal{Y}_0 \to \mathbb{R}^r$ defined by $\tau_x(v) = (\omega_1(v), \dots, \omega_r(v))$. The map $\mathfrak{M} \ni x \mapsto \tau_x \in \mathrm{Lin}(\mathcal{Y}_0, \mathbb{R}^r)$ is continuous. Condition (3.5) is equivalent to $\mathrm{Im}(B(x)) + \ker \tau_x = \mathcal{Y}_0$, i.e. that the linear map $\tau_x \circ B(x)\colon \mathfrak{g} \to \mathbb{R}^r$ is surjective. This is clearly an open condition.

Proof. Given a function $f\colon U \subset \mathbb{R}^d \to \mathbb{R}^d$ of class $\mathcal{C}^1$, where U is an open neighbourhood of 0, such that $f(0) = 0$ and $\mathrm{d}f(0)$ is an isomorphism, the induced map $\tilde{f}\colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ has topological degree equal to ± 1. Here, $\tilde{f}$ is defined by $\tilde{f}(x) = \|f(xr)\|^{-1} f(xr)$, where $r > 0$ is such that 0 is the unique zero of f in the closed ball $B[0; r]$ of $\mathbb{R}^d$.

Now, if A is any topological space, $f\colon A \times U \to \mathbb{R}^d$ is continuous, and $a_0 \in A$ is such that $f(a_0, \cdot)$ is of class $\mathcal{C}^1$, $f(a_0, 0) = 0$ and $\partial_2 f(a_0, 0)$ is an isomorphism, for a near a_0, and $r > 0$ sufficiently small, $0 \in f(a, B[0; r])$. This follows from the continuity of the topological degree. The same holds for a function $f\colon A \times U \to \mathbb{R}^d$, where now U is an open neighbourhood of 0 in $\mathbb{R}^s$, with $s \geqslant d$, under the assumption that $f(a_0, \cdot)$ is of class $\mathcal{C}^1$, $f(a_0, 0) = 0$, and $\partial_2 f(a_0, 0)$ is surjective. Namely, it suffices to apply the argument above to the function obtained by restricting f to a d-dimensional subspace where $\partial_2 f(a_0, 0)$ is an isomorphism.

To prove the result, use local coordinates adapted to P in N, and assume that M, P and N are Banach spaces, with $N = P \oplus \mathbb{R}^d$, $d \leqslant s = \dim(M)$ is the codimension of P, and $m_0 = 0$. In this situation, the result is obtained by applying the argument above to the function $f\colon A \times M \to \mathbb{R}^d$ given by $f(a, m) = \pi(\chi(a, m))$, where $\pi\colon N \to \mathbb{R}^d$ is the projection relative to the decomposition $N = P \oplus \mathbb{R}^d$. Clearly, $f(a, m) = 0$ if and only if $\chi(a, m) \in P$. Assumption (a) implies that $f(a_0, 0) = 0$, and assumption (c) implies that $\partial_2 f(a_0, 0)$ is surjective. □

Remark 3.5. In Theorem 3.2, some assumptions on the group action can be weakened. For instance, the result also holds for *local* group actions (see Appendix A). This version of the equivariant implicit function theorem is used in the constant mean curvature problem (see § 4.3). Versions of the result for the so-called *partial actions* of groups, or even for actions of groupoids, semigroups, monoids, etc., are also possible.

4. Applications to geometric variational problems

We now describe concrete applications of our abstract result to three classic geometric variational problems: harmonic maps, closed geodesics and constant mean curvature hypersurfaces, corresponding to Theorems 1.2, 1.3 and 1.4 in § 1, respectively.

4.1. Harmonic maps

Let $(M, \boldsymbol{g})$ and $(\bar{M}, \bar{\boldsymbol{g}})$ be Riemannian manifolds.

Definition 4.1. A $\mathcal{C}^2$-map $\phi\colon M \to \bar{M}$ is said to be $(\boldsymbol{g}, \bar{\boldsymbol{g}})$-*harmonic* if

$$\Delta_{\boldsymbol{g},\bar{\boldsymbol{g}}}(\phi) := \operatorname{tr}(\hat{\nabla}\, \mathrm{d}\phi) = 0, \tag{4.1}$$

where $\hat{\nabla}$ is the connection on the vector bundle $TM^* \otimes \phi^*(T\bar{M})$ over M induced by the Levi–Civita connections ∇ of $\boldsymbol{g}$ and $\bar{\nabla}$ of $\bar{\boldsymbol{g}}$.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

Remark 4.2. Harmonic maps form a class that contains several geometrically important objects (see [**10**]). For instance, if $\dim M = 1$, harmonic maps $\phi\colon M \to \bar{M}$ are the geodesics of $\bar{M}$. In particular, setting $M = \mathbb{S}^1$, the previous statements in § 4.2 regarding closed geodesics of $\bar{M}$ can be reobtained (in the Riemannian case). The harmonic variational problem is also related to the CMC problem described in § 4.3. Namely, an isometric immersion $\phi\colon M \to \bar{M}$ is minimal if and only if it is harmonic. In addition, setting $\bar{M} = \mathbb{R}$, harmonic maps are simply harmonic functions on M; and, setting $\bar{M} = \mathbb{S}^1$, harmonic maps are canonically identified with the harmonic 1-forms on M with integral periods.

Henceforth, we assume compactness of the source manifold M to use the classic variational characterization of harmonic maps. Let $\mathfrak{M}$ be the Banach manifold $\mathcal{C}^{2,\alpha}(M, \bar{M})$ consisting of all maps $\phi\colon M \to \bar{M}$ that satisfy the $\mathcal{C}^{2,\alpha}$-Hölder condition. Let Λ be the open subset of the Banach space of symmetric $(0,2)$-tensors of class $\mathcal{C}^k$ on M, with $k \geqslant 3$, consisting of all positive definite tensors, i.e. elements of Λ are Riemannian metric tensors of class $\mathcal{C}^k$ on M. Set $\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathbb{R}$,

$$\mathfrak{f}(\phi, \boldsymbol{g}) = \frac{1}{2} \int_M \|\mathrm{d}\phi(x)\|^2_{\mathrm{HS}} \operatorname{vol}_{\boldsymbol{g}},$$

where $\operatorname{vol}_{\boldsymbol{g}}$ is the volume form (or density) of $\boldsymbol{g}$ and $\|\mathrm{d}\phi(x)\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm of the linear map $\mathrm{d}\phi(x)$. For a given $\boldsymbol{g}_0 \in \Lambda$, critical points of the map $\phi \mapsto \mathfrak{f}(\phi, \boldsymbol{g}_0)$ are precisely the $(\boldsymbol{g}_0, \bar{\boldsymbol{g}})$-harmonic maps $\phi\colon M \to \bar{M}$. For $\phi \in \mathfrak{M}$, the tangent space $T_\phi\mathfrak{M}$ is identified with the Banach space $\mathcal{C}^{2,\alpha}(\phi^* T\bar{M})$ of all $\mathcal{C}^{2,\alpha}$-Hölder vector fields along ϕ. Given such a $\boldsymbol{V} \in T_\phi\mathfrak{M}$, the derivative $\partial_1\mathfrak{f}(\phi, \boldsymbol{g})\boldsymbol{V}$ is given by

$$\begin{aligned}
\partial_1\mathfrak{f}(\phi, \boldsymbol{g})\boldsymbol{V} &= \int_M \operatorname{tr}(\mathrm{d}\phi^* \bar{\nabla}\boldsymbol{V}) \operatorname{vol}_{\boldsymbol{g}} \\
&= \int_M [\operatorname{div}(\mathrm{d}\phi^*(\boldsymbol{V})) - \bar{\boldsymbol{g}}(\Delta_{\boldsymbol{g},\bar{\boldsymbol{g}}}(\phi), \boldsymbol{V})] \operatorname{vol}_{\boldsymbol{g}} \\
&\overset{\text{Stokes}}{=} -\int_M \bar{\boldsymbol{g}}(\Delta_{\boldsymbol{g},\bar{\boldsymbol{g}}}(\phi), \boldsymbol{V}) \operatorname{vol}_{\boldsymbol{g}},
\end{aligned} \tag{4.2}$$

where the trace is meant on the entries $\mathrm{d}\phi^*(\cdot)\bar{\nabla}_{(\cdot)}\boldsymbol{V}$.

Definition 4.3. The corresponding *Jacobi operator* J along a $(\boldsymbol{g}, \bar{\boldsymbol{g}})$-harmonic map ϕ is the linear differential operator

$$J_\phi(\boldsymbol{V}) = -\Delta\boldsymbol{V} + \operatorname{tr}(\bar{R}(\mathrm{d}\phi(\cdot), \boldsymbol{V})\,\mathrm{d}\phi(\cdot)), \tag{4.3}$$

defined in $\mathcal{C}^{2,\alpha}(\phi^* T\bar{M})$. Here, $\bar{R}$ is the curvature tensor of $\bar{\boldsymbol{g}}$, and $\Delta\boldsymbol{V}$ is a vector field along ϕ uniquely defined by

$$\bar{\boldsymbol{g}}(\Delta\boldsymbol{V}, \boldsymbol{W}) = \operatorname{div}(\bar{\nabla}\boldsymbol{V}^*)\boldsymbol{W} - \bar{\boldsymbol{g}}(\bar{\nabla}\boldsymbol{V}, \bar{\nabla}\boldsymbol{W}), \quad \boldsymbol{W} \in \mathcal{C}^{2,\alpha}(\phi^* T\bar{M}), \tag{4.4}$$

i.e. $\Delta\boldsymbol{V}(x) = \sum_i (\bar{\nabla}_{e_i}\bar{\nabla}\boldsymbol{V})e_i$, where $(e_i)_i$ is an orthonormal basis of T_xM.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

Definition 4.4. A vector field $\boldsymbol{V}$ that satisfies $J_\phi(\boldsymbol{V}) = 0$ is called a *Jacobi field*.

Observe that if $\boldsymbol{K}$ is a Killing vector field, then $J_\phi(\boldsymbol{K} \circ \phi) = 0$.

Definition 4.5. A $(\boldsymbol{g}, \bar{\boldsymbol{g}})$-harmonic map $\phi\colon M \to \bar{M}$ is said to be *non-degenerate* if all Jacobi fields along ϕ are of the form $\boldsymbol{K} \circ \phi$, where $\boldsymbol{K}$ is Killing.

Let G be the isometry group $\mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$ of the target manifold, acting on $\mathfrak{M}$ by left composition. Clearly, the functional $\mathfrak{f}$ is invariant in the first variable under this action.* Using results from [**16**], it is possible to prove that this action is smooth, since it is given by left composition with smooth maps (see also [**18**] for the non-compact case). As a consequence, part of the technical arguments in Theorem 3.2 to deal with low regularity assumptions is not necessary in this context.

Definition 4.6. Two harmonic maps ϕ_1 and ϕ_2 are called *geometrically equivalent* if they are in the same $\mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$-orbit, i.e. if there exists an isometry $\psi\colon \bar{M} \to \bar{M}$ such that $\phi_2 = \psi \circ \phi_1$.

We are now ready for the following proof.

Proof of Theorem 1.2. All assumptions of Theorem 3.2 are satisfied by the harmonic maps problem, using the following objects.

- $\mathfrak{M}'$ coincides with $\mathfrak{M} = \mathcal{C}^{2,\alpha}(M, \bar{M})$.
- $\mathcal{E}$ is the mixed vector bundle whose fibre $\mathcal{E}_\phi$ is $\mathcal{C}^{0,\alpha}(\phi^* T\bar{M})$, the Banach space of all $\mathcal{C}^{0,\alpha}$-Hölder vector fields along ϕ, endowed with the topology $\mathcal{C}^{2,\alpha}$ on the base and $\mathcal{C}^{0,\alpha}$ on the fibres.
- $\mathfrak{i}$ is the inclusion, and $\mathfrak{j}$ is induced by the L^2-pairing that uses the inner product induced by $\bar{\boldsymbol{g}}$, and integrals taken with respect to the volume form (or density) of some fixed auxiliary† Riemannian metric $\boldsymbol{g}_*$ on M.
- Given $\phi\colon M \to \bar{M}$ of class $\mathcal{C}^{2,\alpha}$ and a Riemannian metric tensor $\boldsymbol{g}$ on M, $\delta\mathfrak{f}(\phi, \boldsymbol{g})$ is given by $-\zeta_{\boldsymbol{g}} \cdot \Delta_{\boldsymbol{g},\bar{\boldsymbol{g}}}(\phi)$, where $\zeta_{\boldsymbol{g}}\colon M \to \mathbb{R}^+$ is the positive $\mathcal{C}^k$-function satisfying $\zeta_{\boldsymbol{g}} \cdot \mathrm{vol}_{\boldsymbol{g}_*} = \mathrm{vol}_{\boldsymbol{g}}$ (see (4.1) and (4.2)).

* One should observe that the harmonic map functional is also invariant under the action of the isometry group $\mathrm{Iso}(M, \boldsymbol{g})$ of the source manifold $(M, \boldsymbol{g})$, which acts by right composition in the space of maps from M to $\bar{M}$. However, equivariance with respect to such action will not be considered here. Namely, observe that, as the metric $\boldsymbol{g}$ varies, clearly the group $\mathrm{Iso}(M, \boldsymbol{g})$ also varies; thus, in order to deal with such equivariance, a formulation of the equivariant implicit function theorem for varying groups is needed. The assumption of equivariant non-degeneracy in Theorem 1.2 restricts the result to the case where $(M, \boldsymbol{g})$ has discrete isometry group or, more generally, when, given any Killing field $\boldsymbol{K}$ of $(M, \boldsymbol{g})$, the field $\mathrm{d}\phi(\boldsymbol{K})$ along ϕ is the restriction to $\phi(M)$ of some Killing field $\bar{\boldsymbol{K}}$ of $(\bar{M}, \bar{\boldsymbol{g}})$.

† Note that one cannot use the volume form of $\boldsymbol{g}$ in order to define $\mathfrak{j}$, because this metric is *variable* in the problem that we are considering.

- $\mathcal{Y} = T\mathfrak{M}$, κ is the identity map and $\tilde{\jmath}$ is induced by the L^2-pairing, as above.

- Identifying the Lie algebra $\mathfrak{g}$ with the space of (complete) Killing vector fields on $(\bar{M}, \bar{\boldsymbol{g}})$, for $\phi \in \mathfrak{M}$, the map $\mathrm{d}\beta_\phi(1)\colon \mathfrak{g} \to T_\phi\mathfrak{M}$ associates to a Killing vector field $\bar{\boldsymbol{K}}$ the vector field $\bar{\boldsymbol{K}} \circ \phi$ along ϕ.

- Given a $(\boldsymbol{g}, \bar{\boldsymbol{g}})$-harmonic map $\phi\colon M \to \bar{M}$, the vertical projection of the derivative $\partial_1(\delta\mathfrak{f})(\phi, \boldsymbol{g})$ is identified with $\zeta_{\boldsymbol{g}} \cdot J_\phi$, where J_ϕ is the Jacobi operator in (4.3). This is an elliptic second-order partial differential operator, and $\zeta_{\boldsymbol{g}} \cdot J_\phi\colon \mathcal{C}^{2,\alpha}(\phi^*T\bar{M}) \to \mathcal{C}^{0,\alpha}(\phi^*T\bar{M})$ is a Fredholm operator of index 0 (see [**20**, Theorem 1.1, (2)]).

□

4.2. Closed geodesics

Let M be an arbitrary manifold, let $\mathfrak{M}$ be the Banach manifold $\mathcal{C}^2(\mathbb{S}^1, M)$ consisting of all closed curves of class $\mathcal{C}^2$ in M, let $\mathcal{B}$ be a Banach space of symmetric $(0,2)$-tensors of class $\mathcal{C}^k$ on M, with $k \geqslant 3$, and let Λ denote an open subset of $\mathcal{B}$ consisting of tensors that are everywhere non-degenerate on M. Thus, elements of Λ are (pseudo-) Riemannian metric tensors on M. We also fix an auxiliary Riemannian metric $\boldsymbol{g}_R$ on M; this metric induces a positive definite inner product and a norm $\|\cdot\|_R$ on each tangent and cotangent space to M, and on all tensor products of these spaces. This is used implicitly throughout, whenever our constructions require the use of a norm or of an inner product of tensors.* Given a (pseudo-) Riemannian metric tensor $\boldsymbol{g}$ on M, we denote by $\mathcal{T}_{\boldsymbol{g}}$ the $\boldsymbol{g}_R$-symmetric $(1,1)$-tensor on M defined by

$$\boldsymbol{g} = \boldsymbol{g}_R(\mathcal{T}_{\boldsymbol{g}}\cdot, \cdot). \tag{4.5}$$

Consider the smooth function $\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathbb{R}$ given by

$$\mathfrak{f}(\gamma, \boldsymbol{g}) = \frac{1}{2}\int_{\mathbb{S}^1} \boldsymbol{g}(\gamma', \gamma')\,\mathrm{d}\theta;$$

for a given $\boldsymbol{g}_0 \in \Lambda$, the critical points of the map $\gamma \mapsto \mathfrak{f}(\gamma, \boldsymbol{g}_0)$ are precisely the periodic $\boldsymbol{g}_0$-geodesics on M. For $\gamma \in \mathfrak{M}$, the tangent space $T_\gamma\mathfrak{M}$ is identified with the Banach space of all periodic vector fields $\boldsymbol{V}$ of class $\mathcal{C}^2$ along γ. Given such a $\boldsymbol{V} \in T_\gamma\mathfrak{M}$, recall that the derivative $\partial_1\mathfrak{f}(\gamma, \boldsymbol{g})\boldsymbol{V}$ is given by

$$\partial_1\mathfrak{f}(\gamma, \boldsymbol{g})\boldsymbol{V} = \int_{\mathbb{S}^1} \boldsymbol{g}\left(\gamma', \frac{\mathrm{D}^{\boldsymbol{g}}}{\mathrm{d}\theta}\boldsymbol{V}\right)\mathrm{d}\theta, \tag{4.6}$$

where $\mathrm{D}^{\boldsymbol{g}}/\mathrm{d}\theta$ is the covariant derivative operator along γ relative to the Levi–Civita connection $\nabla^{\boldsymbol{g}}$ of $\boldsymbol{g}$.

* These norms can be used, for example, to give a simple construction of the Banach space $\mathcal{B}$. Consider ∇^R the Levi–Civita connection of $\boldsymbol{g}_R$. Then, $\mathcal{B}$ may be taken as the space of $(0,2)$-tensors s of class $\mathcal{C}^k$ on M that are $\boldsymbol{g}_R$-*bounded*, i.e. such that $\|s\|_{\mathcal{B}} = \max_{1\leqslant i\leqslant k}\{\sup_{x\in M}\|(\nabla^R)^i s(x)\|_R\} < +\infty$.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

Definition 4.7. If γ is a $\boldsymbol{g}$-geodesic, the *Jacobi operator* J along γ is the linear differential operator

$$J(\boldsymbol{V}) = \left(\frac{\mathrm{D}^{\boldsymbol{g}}}{\mathrm{d}\theta^2}\right)^2 \boldsymbol{V} + R^{\boldsymbol{g}}(\gamma', \boldsymbol{V})\gamma', \tag{4.7}$$

defined in the space of $\mathcal{C}^2$-vector fields $\boldsymbol{V}$ along γ. Here, $R^{\boldsymbol{g}}$ is the curvature tensor of the Levi–Civita connection of $\boldsymbol{g}$. A Jacobi field along γ is a vector field $\boldsymbol{V}$ satisfying $J(\boldsymbol{V}) = 0$.

Definition 4.8. A closed $\boldsymbol{g}$-geodesic γ is said to be *non-degenerate* if the only periodic Jacobi fields along γ are (constant) multiples of the tangent field γ'.

Remark 4.9. Non-degeneracy of all closed geodesics (including iterates) is a *generic* property in the set of (pseudo-) Riemannian metric tensors $\boldsymbol{g}$ (see [**2**, **3**]).

Definition 4.10. Let G be the circle $\mathbb{S}^1$, acting on $\mathfrak{M}$ by rotation, i.e. by right composition. This action is only continuous (and not differentiable), but each $g \in G$ gives a diffeomorphism of $\mathfrak{M}$. The stabilizer of every non-constant closed curve γ in M is a finite cyclic subgroup of $\mathbb{S}^1$. When such a stabilizer is trivial, we say that γ is *prime*, i.e. it is not the iterate of some other closed curve in M. If $n > 1$ is the order of the stabilizer of a curve γ, then γ is the n-fold iterate of some prime closed curve on M. Two closed curves γ_1 and γ_2 on M belong to the same $\mathbb{S}^1$-orbit if and only if

(a) $\gamma_1(\mathbb{S}^1) = \gamma_2(\mathbb{S}^1)$,

(b) γ_1 and γ_2 have stabilizers of the same order.

When (a) and (b) are satisfied, we say that γ_1 and γ_2 are *geometrically equivalent*.

We are now ready for the following proof.

Proof of Theorem 1.3. All assumptions of Theorem 3.2 are satisfied by the harmonic maps problem, using the following objects.

- $\mathfrak{M}'$ is the set $\mathcal{C}^3(\mathbb{S}^1, M)$, endowed with the $\mathcal{C}^3$-topology.
- $\mathcal{E}$ is the mixed vector bundle whose fibre $\mathcal{E}_\gamma$ is the Banach space of all periodic continuous vector fields along γ, endowed with the topology $\mathcal{C}^2$ on the base and $\mathcal{C}^0$ on the fibres.
- $\mathfrak{i}$ is the inclusion, and $\mathfrak{j}$ is induced by the L^2-pairing (this uses the inner product given by $\boldsymbol{g}_R$).
- $\mathcal{Y}$ is the mixed vector bundle whose fibre $\mathcal{Y}_\gamma$ is the Banach space of all periodic $\mathcal{C}^1$-vector fields along γ, endowed with the topology $\mathcal{C}^2$ on the base and $\mathcal{C}^1$ on the fibres.
- $\tilde{\jmath}$ is induced by the L^2-pairing (this uses the inner product given by $\boldsymbol{g}_R$).
- κ is the inclusion.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

- Using the identification $\mathfrak{g} \cong \mathbb{R}$, $\mathrm{Lin}(\mathfrak{g}, T\mathfrak{M}) \cong T\mathfrak{M}$ and $\mathrm{Lin}(\mathfrak{g}, \mathcal{Y}) \cong \mathcal{Y}$, for $\gamma \in \mathfrak{M}'$, the map $\mathrm{d}\beta_\gamma(1)$ is the element $\gamma' \in T\mathfrak{M}$.

- The map $\kappa \circ [\mathrm{d}\beta_\gamma(1)]$ has the same expression of $\mathrm{d}\beta_\gamma(1)$, where now $\gamma \in \mathfrak{M}$ and $\gamma' \in \mathcal{Y}$.

- The map $\delta\mathfrak{f}$ is defined by $\delta\mathfrak{f}(\gamma, \boldsymbol{g}) = -\mathcal{T}_{\boldsymbol{g}}(\mathrm{D}^{\boldsymbol{g}}\gamma'/\mathrm{d}\theta)$, where $\mathcal{T}_{\boldsymbol{g}}$ is defined in (4.5). Note that $\mathrm{D}^{\boldsymbol{g}}\gamma'/\mathrm{d}\theta$ is a continuous vector field along γ, and

$$\begin{aligned}
\mathfrak{j}_\gamma(\delta\mathfrak{f}(\gamma, \boldsymbol{g}))\boldsymbol{V} &= \int_{\mathbb{S}^1} \boldsymbol{g}_R(\delta\mathfrak{f}(\gamma, \boldsymbol{g}), \boldsymbol{V})\,\mathrm{d}\theta \\
&= -\int_{\mathbb{S}^1} \boldsymbol{g}_R\left(\mathcal{T}_{\boldsymbol{g}}\left(\frac{\mathrm{D}^{\boldsymbol{g}}}{\mathrm{d}\theta}\gamma'\right), \boldsymbol{V}\right)\mathrm{d}\theta \\
&= -\int_{\mathbb{S}^1} \boldsymbol{g}\left(\frac{\mathrm{D}^{\boldsymbol{g}}}{\mathrm{d}\theta}\gamma', \boldsymbol{V}\right)\mathrm{d}\theta \\
&= \int_{\mathbb{S}^1} \boldsymbol{g}\left(\gamma', \frac{\mathrm{D}^{\boldsymbol{g}}}{\mathrm{d}\theta}\boldsymbol{V}\right)\mathrm{d}\theta \\
&= \partial_1\mathfrak{f}(\gamma, \boldsymbol{g})\boldsymbol{V}.
\end{aligned}$$

- The derivative $\partial_1(\delta\mathfrak{f})$ is given by

$$\partial_1(\delta\mathfrak{f})(\gamma, \boldsymbol{g})\boldsymbol{V} = -(\nabla^{\boldsymbol{g}}_{\boldsymbol{V}}\mathcal{T}_{\boldsymbol{g}})\left(\frac{\mathrm{D}^{\boldsymbol{g}}}{\mathrm{d}\theta}\gamma'\right) - \mathcal{T}_{\boldsymbol{g}}(J(\boldsymbol{V})),$$

  where $\boldsymbol{V} \in T_\gamma\mathfrak{M}$ and J is the Jacobi operator (4.7).

  The operator J acting on the space of periodic fields of class $\mathcal{C}^2$ along γ and taking values in the space of periodic continuous vector fields along γ is a Fredholm operator of index 0, as it is a compact perturbation of an isomorphism. Since the composition on the left-hand side with $\mathcal{T}_{\boldsymbol{g}}$ is an isomorphism, it follows that the operator $\boldsymbol{V} \mapsto \mathcal{T}_{\boldsymbol{g}}(J(\boldsymbol{V}))$ is a Fredholm operator of index 0 from the space of periodic fields of class $\mathcal{C}^2$ along γ to the space of periodic continuous vector fields along γ. The operator $\boldsymbol{V} \mapsto -(\nabla^{\boldsymbol{g}}_{\boldsymbol{V}}\mathcal{T}_{\boldsymbol{g}})(\mathrm{D}^{\boldsymbol{g}}\gamma'/\mathrm{d}\theta)$ from the space of $\mathcal{C}^2$-vector fields to the space of $\mathcal{C}^0$-vector fields is compact, as it is continuous relative to the $\mathcal{C}^0$-topology, and the inclusion $\mathcal{C}^2 \hookrightarrow \mathcal{C}^0$ is compact. Hence, $\partial_1(\delta\mathfrak{f})(\gamma, \boldsymbol{g})$ is Fredholm of index 0.

- If $\gamma \in \mathfrak{M}'$, then the orbit $\mathbb{S}^1 \cdot \gamma$ is a $\mathcal{C}^1$-submanifold of $\mathfrak{M}$ that is diffeomorphic to $\mathbb{S}^1$. The tangent space $T_\gamma(\mathbb{S}^1 \cdot \gamma) \subset T_\gamma\mathfrak{M}$ is spanned by the tangent field γ'. Non-degeneracy of a critical orbit thus corresponds to the non-degeneracy of the closed geodesic.

□

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

4.3. CMC hypersurfaces

Using our main abstract result, we now prove Theorem 1.4, which is an improvement of Proposition 2.10. More precisely, we employ a version of Theorem 3.2 for local actions (see Theorem A 5). We need a technical assumption concerning the existence of an *invariant volume functional* around a given CMC embedding $x\colon M \hookrightarrow \bar{M}$. This will be a volume functional invariant under left compositions with isometries of the ambient space (see Definition B 1). Examples where this assumption is satisfied are discussed in Appendix B. We stress that this assumption is indeed necessary (see Example 4.11).

Proof of Theorem 1.4. Consider the set $\mathfrak{E}(M, \bar{M})$ of all *unparametrized embeddings* of class $\mathcal{C}^{2,\alpha}$ of M into $\bar{M}$, i.e. the set of congruence classes of $\mathcal{C}^{2,\alpha}$-embeddings $y\colon M \hookrightarrow \bar{M}$. Such a set does not have a natural global differentiable structure, but it admits an atlas of charts that make it an infinite-dimensional topological manifold modelled on the Banach space $\mathcal{C}^{2,\alpha}(M)$ (see [**1**]). Given a smooth embedding $y\colon M \to \bar{M}$, nearby congruence classes of embeddings are parametrized by sections of the normal bundle of y, using the exponential map of $(\bar{M}, \bar{\boldsymbol{g}})$. We identify congruence classes of embeddings near x with functions belonging to a neighbourhood of 0 in the Banach space $\mathcal{C}^{2,\alpha}(M)$; for this identification the transversal orientation of $x(M)$ is used. Let $\mathfrak{M}$ be a sufficiently small neighbourhood of x in $\mathfrak{E}(M, \bar{M})$, identified with a neighbourhood of 0 in the space $\mathcal{C}^{2,\alpha}(M)$.

Consider the isometry group $G = \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$ of the ambient manifold. There is a *local action* (see Appendix A) of G on $\mathfrak{M}$, defined as follows. If $y\colon M \hookrightarrow \bar{M}$ is an embedding near x and ϕ is an isometry of $(\bar{M}, \bar{\boldsymbol{g}})$, then the action of ϕ on (the congruence class of) y is given by the (congruence class of the) left composition $\phi \circ y$. The domain of this action consists of pairs (ϕ, y) such that $\phi \circ y$ belongs to $\mathfrak{M}$; the axioms of local actions are readily verified for this map. The local action of G on the set of unparametrized embeddings is continuous (see [**1**]), but the action is differentiable only on the dense subset $\mathfrak{M}'$ of $\mathfrak{M}$ consisting of congruence classes of embeddings of class $\mathcal{C}^{3,\alpha}$. The orbit of each of these elements is a $\mathcal{C}^1$-submanifold of $\mathcal{C}^{2,\alpha}(M)$.

Given an embedding $y\colon M \hookrightarrow \bar{M}$, denote by $\mathcal{A}(y)$ the volume of M relative to the volume form of the pullback metric $y^*(\bar{\boldsymbol{g}})$, and consider an invariant volume functional $\mathcal{V}$ defined in a neighbourhood of x in $\mathcal{C}^1(M, \bar{M})$. The values $\mathcal{A}(y)$ and $\mathcal{V}(y)$ do not depend on the parametrization of y, and $\mathcal{A}$ and $\mathcal{V}$ define functions on $\mathfrak{M}$ that are smooth in every local chart (see [**1**] for details). Finally, let Λ be an open interval of $\mathbb{R}$ containing H_0, and define

$$\mathfrak{f}\colon \mathfrak{M} \times \Lambda \to \mathbb{R}$$
$$\mathfrak{f}(y, \lambda) := \mathcal{A}(y) + m\lambda\mathcal{V}(y).$$

It is well known that $\partial_1 \mathfrak{f}(y, \lambda) = 0$ if and only if y is a CMC embedding with mean curvature equal to λ. The second variation of $\mathfrak{f}(\cdot, H_0)$ at x is identified with the Jacobi operator J_x in (2.5). In particular, $J_x\colon \mathcal{C}^{2,\alpha}(M) \to \mathcal{C}^{0,\alpha}(M)$ is a Fredholm operator of index 0 (see [**20**, Theorem 1.1, (2)]). Since $\mathcal{A}$ and $\mathcal{V}$ are invariant under the local action of left composition with elements of the isometry group $G = \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$, so is the function $\mathfrak{f}$. Recall that G is a Lie group, and is compact when $\bar{M}$ is compact.

The desired result now follows as a direct application of the equivariant implicit function theorem for local actions (see Theorem A 5); the objects described in Axioms (A), (B), (D) and (F) are defined as follows for the CMC variational problem.

- $\mathcal{E}$ is the Banach space $\mathcal{C}^{0,\alpha}(M)$.
- $\mathfrak{i}$ is the inclusion $\mathcal{C}^{2,\alpha} \hookrightarrow \mathcal{C}^{0,\alpha}$ and $\mathfrak{j}$ is induced by the L^2-pairing $(f, g) \mapsto \int_M f \cdot g \,\mathrm{vol}_{\boldsymbol{g}}$.
- $\mathcal{Y}$ is the Banach space $\mathcal{C}^{1,\alpha}(M)$.
- $\tilde{\jmath}$ is induced by the L^2-pairing.
- κ is the inclusion.
- Identifying the Lie algebra $\mathfrak{g}$ with the space of (complete) Killing vector fields on $(\bar{M}, \bar{\boldsymbol{g}})$, for $y \in \mathfrak{M}'$, the map $\mathrm{d}\beta_y(1) \colon \mathfrak{g} \to T_y\mathfrak{M}$ associates to a Killing vector field $\bar{\boldsymbol{K}}$ the orthogonal component of $\bar{\boldsymbol{K}}$ along y.
- Given a $\mathcal{C}^{2,\alpha}$-embedding y, $\delta\mathfrak{f}(y, \lambda)$ is the mean curvature function of y (which is a $\mathcal{C}^{0,\alpha}$-function on M).
- $\partial_1(\delta\mathfrak{f})(x, H_0)$ is identified with the Jacobi operator J_x.

The assumptions of Theorem A 5 are easily verified, concluding the proof. □

The following examples show that neither the assumption on the existence of an invariant volume functional nor the assumption on the transversal orientation in the case of minimal embeddings can be omitted in Theorem 1.4.

Example 4.11. Consider $M = \mathbb{S}^1$ and let $\bar{M} = \mathbb{S}^1 \times \mathbb{S}^1$ be the two-torus endowed with the flat metric. The embedding $x \colon \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $x(z) = (z, 1)$, $z \in \mathbb{S}^1$, is obviously minimal (i.e. a geodesic). It is also easy to see that such an embedding is nondegenerate, i.e. every periodic Jacobi field along x is the restriction of a Killing vector field. However, near x there exists no embedding of $\mathbb{S}^1$ into $\mathbb{S}^1 \times \mathbb{S}^1$ with constant geodesic curvature different from 0. Namely, every constant geodesic curvature embedding should be the projection on $\mathbb{S}^1 \times \mathbb{S}^1$ of a circle in the plane $\mathbb{R}^2$; such a projection is a curve with trivial homotopy class, hence it cannot be close to x in the $\mathcal{C}^1$-topology. Observe that, in this example, the image of x is not contained in any open subset of $\mathbb{S}^1 \times \mathbb{S}^1$ with trivial first cohomology space, and there exists no volume functional defined in a neighbourhood of x in $\mathcal{C}^1(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^1)$ that is invariant under isometries of $\mathbb{S}^1 \times \mathbb{S}^1$.

Example 4.12. We observe that the transverse orientability is a necessary condition in Theorem 1.4. Namely, such a condition is closed (and also open) relative to the $\mathcal{C}^1$-topology; thus, if $(x_H)_{H \in\,]-\varepsilon,\varepsilon[}$ is a continuous family of CMC embeddings, such that each x_H has mean curvature H, then x_0 must be transversely oriented.

For instance, consider the real projective plane $\mathbb{R}P^2$ with the standard metric, and the minimal (i.e. geodesic) embedding $x_0\colon \mathbb{S}^1 \hookrightarrow \mathbb{R}P^2$ obtained by projecting in $\mathbb{R}P^2$ a minimal geodesic between two antipodal points in $\mathbb{S}^2$. This is a non-degenerate minimal embedding, which is not transversely oriented. The only CMC immersions of $\mathbb{S}^1$ in $\mathbb{S}^2$ are *parallel* to the equator. The corresponding immersions obtained in the quotient $x_H\colon \mathbb{S}^1 \hookrightarrow \mathbb{R}P^2$ are *not* $\mathcal{C}^1$-close to x_0, since, for instance, the length of x_h tends to twice the length of x_0 as H goes to 0.

An alternative form of stating Theorem 1.4 uses the notion of *rigidity* for a path of CMC embeddings.

Definition 4.13. Given a one-parameter family $x_s\colon M \hookrightarrow \bar{M}$, $s \in [a,b]$, of CMC embeddings, we say that the family $X = \{x_s\}_{s\in[a,b]}$ is *rigid* if there exists an open neighbourhood $\mathcal{U}$ of X in $\mathcal{C}^{2,\alpha}(M,\bar{M})$ such that any CMC embedding $x\colon M \hookrightarrow \bar{M}$ in $\mathcal{U}$ is isometrically congruent to some x_s. We say that the family is *locally rigid* at $s_0 \in [a,b]$ if there exists $\varepsilon > 0$ such that, setting $I = [a,b] \cap [s_0-\varepsilon, s_0+\varepsilon]$, the family $\{x_s\}_{s\in I}$ is rigid.

Corollary 4.14. *Let $x_s\colon M \hookrightarrow \bar{M}$, $s \in [a,b]$, be a $\mathcal{C}^1$-family of CMC embeddings, denote by $\mathcal{H}(s)$ the mean curvature of x_s, and let $s_0 \in [a,b]$ be such that*

- *x_{s_0} is non-degenerate,*
- *there exists an invariant volume functional in a $\mathcal{C}^1$-neighbourhood of x_{s_0},*
- *$\mathcal{H}'(s_0) \neq 0$.*

Then, $X = \{x_s\}_{s\in[a,b]}$ is locally rigid at s_0.

Proof. The assumption that $\mathcal{H}'(s_0) \neq 0$ implies the existence of a $\mathcal{C}^1$-function

$$]\mathcal{H}(s_0)-\varepsilon, \mathcal{H}(s_0)+\varepsilon[\ \ni H \quad \mapsto \quad s(H) \in\ ]s_0-\delta, s_0+\delta[,$$

with $\varepsilon, \delta > 0$ small enough, such that $\mathcal{H}(s(H)) = H$ for all H. Apply Theorem 1.4 to $x = x_{s_0}$, obtaining a new path $H \mapsto x_H$ of CMC embeddings. Note that x_{s_0} must be transversely oriented, even in the case $\mathcal{H}(s_0) = 0$; namely, by the assumption that $\mathcal{H}'(s_0) \neq 0$, it follows that $\mathcal{H}$ is not constant in any neighbourhood of s_0, and, thus, x_{s_0} is the limit of transversely oriented embeddings. By Theorem 1.4 (b), $x_{s(H)}$ must be isometrically congruent to x_H for all H near $H(s_0)$, and the local rigidity follows readily. □

CMC embeddings of manifolds with boundary

A result totally analogous to Theorem 1.4 holds in the case of codimension 1 CMC embeddings $x\colon M \hookrightarrow \bar{M}$ of manifolds M with boundary ∂M. In this situation, one is interested in variations of x that fix the boundary, and the corresponding infinitesimal variations are Jacobi fields that vanish on ∂M. We have the corresponding definition.

Definition 4.15. If $\partial M \neq \emptyset$, a CMC embedding $x\colon M \hookrightarrow \bar{M}$ is *non-degenerate* if every Jacobi field f along x that vanishes on ∂M is of the form $f = \bar{\boldsymbol{g}}(\bar{\boldsymbol{K}}, \boldsymbol{n}_x)$ for some Killing field $\bar{\boldsymbol{K}}$ of $(\bar{M}, \bar{\boldsymbol{g}})$ tangent to $x(\partial M)$.

If x is non-degenerate in this sense, then the implicit function theorem gives the existence of a variation $(x_H)_{H \in \,]H_0-\varepsilon, H_0+\varepsilon[}$ of x by CMC embeddings $x_H\colon M \hookrightarrow \bar{M}$ such that $x_H|_{\partial M} = x|_{\partial M}$ for all H. The proof of this fixed-boundary version is totally analogous to that of Theorem 1.4, *mutatis mutandis*. We require the existence of a volume functional defined in a $\mathcal{C}^1$-neighbourhood of x in the set of embeddings $y\colon M \hookrightarrow \bar{M}$ with fixed boundary, i.e. $y(\partial M) = x(\partial M)$, and invariant under isometries of $(\bar{M}, \bar{\boldsymbol{g}})$ that preserve $x(\partial M)$. Note that the group of such isometries is always compact, because the action of the isometry group is proper and ∂M is compact. In this case, invariant volume functionals can be obtained from invariant primitives of the volume form, using an averaging procedure (see Appendix B).

Remark 4.16. As for the variational framework, in the non-empty boundary case $\mathfrak{M}$ is the manifold of fixed boundary unparametrized embeddings of M into $\bar{M}$ of class $\mathcal{C}^{2,\alpha}$, which is modelled on the Banach space $\mathcal{C}^{2,\alpha}_0(M, \mathbb{R}) = \{f \in \mathcal{C}^{2,\alpha}(M, \mathbb{R})\colon f|_{\partial M} \equiv 0\}$. Note that, when M has boundary, the Jacobi operator $J_x\colon \mathcal{C}^{2,\alpha}_0(M, \mathbb{R}) \to \mathcal{C}^0(M, \mathbb{R})$ is Fredholm of index 0.

Natural extensions of the CMC implicit function theorem

Theorem 1.4 extends naturally to more general situations involving hypersurfaces that are stationary for a parametric elliptic functional with a volume constraint, like, for instance, hypersurfaces with constant *anisotropic mean curvature* (see [**13**]). Such extension is quite straightforward, and is not discussed here. It is also interesting to point out that Theorem 1.4 can be extended to the case of CMC *immersions*, rather than embeddings. The procedure here is to endow the set of unparametrized immersions with a local differential structure based on the exponential map of the normal bundle. This is possible in the case of the so-called *free immersions*, i.e. immersions $x\colon M \to \bar{M}$ with the property that the unique diffeomorphism ϕ of M satisfying $x \circ \phi = x$ is the identity. This is the case, for instance, when there exists some point in the image of x whose inverse image consists of a single point of M. Details can be found in [**5**].

Appendix A. Local actions

It is useful to have a version of the equivariant implicit function theorem for local actions of Lie groups on a manifold. Once more, the paradigmatic example to keep in mind is the CMC embedding problem, in which one has a local action of the isometry group of the target manifold on a neighbourhood of 0 of a Banach space (see § 4.3). One observes that, given the local character of the result, the proof of Theorem 3.2 carries over to the case where the action of the Lie group G is only locally defined, in a sense that is clarified in this appendix.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press

Definition A 1. Let G be a Lie group and let $\mathfrak{M}$ be a topological manifold. A *local action* of G on $\mathfrak{M}$ is a continuous map $\rho\colon \operatorname{Dom}(\rho) \subset G \times \mathfrak{M} \to \mathfrak{M}$, defined on an open subset $\operatorname{Dom}(\rho) \subset G \times \mathfrak{M}$ containing $\{1\} \times \mathfrak{M}$ satisfying that

(a) $\rho(1, x) = x$ for all $x \in \mathfrak{M}$,

(b) $\rho(g_1, \rho(g_2, x)) = \rho(g_1 g_2, x)$ whenever both sides of the equality are defined, i.e. for all those $x \in \mathfrak{M}$ and $g_1, g_2 \in G$ such that $(g_2, x) \in \operatorname{Dom}(\rho)$, $(g_1, \rho(g_2, x)) \in \operatorname{Dom}(\rho)$ and $(g_1 g_1, x) \in \operatorname{Dom}(\rho)$.

Remark A 2. The particular case of actions is when the domain $\operatorname{Dom}(\rho)$ coincides with the entire $G \times \mathfrak{M}$.

Remark A 3. Local actions can be restricted, in the sense that, given any open subset $\mathcal{A}$ of $\operatorname{Dom}(\rho)$ containing $\{1\} \times \mathfrak{M}$, the restriction $\rho|_{\mathcal{A}}$ of ρ to $\mathcal{A}$ is again a local action of G on $\mathfrak{M}$.

Given a local action ρ of G on $\mathfrak{M}$, for $g \in G$, let ρ_g denote the map $\rho(g, \cdot)$, defined on an open (possibly empty) set $\operatorname{Dom}(\rho_g) = \operatorname{Dom}(\rho) \cap \{g\} \times \mathfrak{M}$. The following properties follow easily from the definition.

Lemma A 4. *Let $\rho\colon \operatorname{Dom}(\rho) \subset G \times \mathfrak{M} \to \mathfrak{M}$ be a local action of G on M. The following then hold.*

(i) *For all $g \in G$, the map $\rho_g\colon \rho_g^{-1}(\operatorname{Dom}(\rho_{g^{-1}})) \to \rho_{g^{-1}}^{-1}(\operatorname{Dom}(\rho_g))$ is a homeomorphism.*

(ii) *The set $\{(g, x) \in G \times \mathfrak{M}\colon x \in \rho_g^{-1}(\operatorname{Dom}(\rho_{g^{-1}}))\}$ is an open subset that contains $\{1\} \times \mathfrak{M}$.*

In particular,

(iii) *for all $x \in \mathfrak{M}$, there exists an open neighbourhood U_x of 1 in G such that, for all $g \in U_x$, $x \in \rho_g^{-1}(\operatorname{Dom}(\rho_{g^{-1}}))$.*

In view of (iii), one can define a map $\beta_x\colon \operatorname{Dom}(\beta_x) \subset G \to \mathfrak{M}$ on a neighbourhood $\operatorname{Dom}(\beta_x)$ of 1 in G, by $\beta_x(g) = \rho(g, x)$ (see (3.1)). In particular, if $x \in \mathfrak{M}$ is such that β_x is differentiable (at 1), then one has a well-defined linear map $\mathrm{d}\beta_x(1)\colon \mathfrak{g} \to T_x\mathfrak{M}$. A subset $C \subset \mathfrak{M}$ is called G-invariant if $x \in C$ implies that $\rho(g, x) \in C$ for all $g \in \operatorname{Dom}(\beta_x)$.

Theorem A 5. *Theorem 3.2 holds when one replaces (A2) with the assumption that a local action ρ of G on $\mathfrak{M}$ is given, and replaces (A3) with the assumption that $\mathfrak{f}$ satisfies*

$$\mathfrak{f}(\rho(g, x), \lambda) = \mathfrak{f}(x, \lambda) \quad \text{for all } (g, x) \in \operatorname{Dom}(\rho).$$

In this situation, the conclusion is that there exist open subsets $\Lambda_0 \subset \Lambda$ and $\mathfrak{M}_0 \subset \mathfrak{M}$, with $\lambda_0 \in \Lambda_0$ and $x_0 \in \mathfrak{M}_0$ and a $\mathcal{C}^k$-map $\sigma\colon \Lambda_0 \to \mathfrak{M}_0$ such that, for $(x, \lambda) \in \mathfrak{M}_0 \times \Lambda_0$, the identity $\partial_1 \mathfrak{f}(x, \lambda) = 0$ holds if and only if there exists $g \in G$, with $(\phi(\lambda), g) \in \operatorname{Dom}(\rho)$ such that $x = \rho(g, \sigma(\lambda))$.

Proof. The proof of Theorem 3.2 carries over to this case, with minor modifications. One constructs a submanifold $S \subset \mathfrak{M}$ through x_0 with the property that $T_{x_0}S \oplus \mathrm{Im}(\mathrm{d}\beta_{x_0}(1)) = T_{x_0}\mathfrak{M}$, and considers the restriction of $\mathfrak{f}$ to the product $S \times \varLambda$. The only extra consideration here is to show that the set $\rho((G \times S) \cap \mathrm{Dom}(\rho))$ is a neighbourhood of x_0; this is the analogue of (2) in the proof of Theorem 3.2. Once again, this follows as an application of Proposition 3.4, used in the following setup: A is an open neighbourhood of 1 in G, M is an open neighbourhood of x_0 in $\mathfrak{M}$, these open subsets being chosen in such a way that the product $A \times M$ is contained in the open set $\{(g, x) \in G \times \mathfrak{M} \colon x \in \rho_g^{-1}(\mathrm{Dom}(\rho_{g^{-1}}))\}$ (see Lemma A 4 (ii)). Set $N = \mathfrak{M}$, $P = S$, $a_0 = 1$ and $m_0 = x_0$; the function χ is the restriction of ρ to $A \times M$. The conclusion of Proposition 3.4 states that, for all $x \in M$, there exists $g \in A$ such that $x \in \rho_g^{-1}(\mathrm{Dom}(\rho_{g^{-1}}))$ and $\rho(g, x) \in S$; since $\rho(g, x) \in \mathrm{Dom}(\rho_{g^{-1}})$, $x = \rho(g^{-1}, \rho(g, x)) \in \rho((G \times S) \cap \mathrm{Dom}(\rho))$, i.e. $\rho((G \times S) \cap \mathrm{Dom}(\rho))$ contains the open subset $M \ni x_0$. The rest of the proof of Theorem 3.2 can now be repeated *verbatim*. □

Appendix B. Invariant volume functionals

A technical assumption made in Theorem 1.4 concerns the existence of a generalized volume functional $\mathcal{V}$ that is invariant under left composition with isometries of $(\bar{M}, \bar{\boldsymbol{g}})$. We consider a compact differentiable manifold M, possibly with boundary ∂M, and a Riemannian manifold $(\bar{M}, \bar{\boldsymbol{g}})$, with $m = \dim(M) = \dim(\bar{M}) - 1$.

Definition B 1. Let $\mathcal{U}$ be an open subset of embeddings $x \colon M \hookrightarrow \bar{M}$. An *invariant volume functional* on $\mathcal{U}$ is a real-valued function $\mathcal{V} \colon \mathcal{U} \to \mathbb{R}$ satisfying the following.

(a) $\mathcal{V}(x) = \int_M x^*(\eta)$, where η is an m-form defined on an open subset $U \subset \bar{M}$ such that $\mathrm{d}\eta$ is equal to the volume form $\mathrm{vol}_{\bar{\boldsymbol{g}}}$ of $\bar{\boldsymbol{g}}$ in U.

(b) Given $x \in \mathcal{U}$, for all isometry ϕ of $(\bar{M}, \bar{\boldsymbol{g}})$ sufficiently close to the identity, the function is $\mathcal{V}(\phi \circ x) = \mathcal{V}(x)$.

If M has boundary, the invariance property (b) is required to hold only for isometries ϕ near the identity, and preserving $x(\partial M)$, i.e. $\phi(x(\partial M)) = x(\partial M)$.

By (b), the generalized volume $\mathcal{V}(x)$ does not depend on the parametrization of x, i.e. $\mathcal{V}(x \circ \psi) = \mathcal{V}(x)$ for all diffeomorphisms ψ of M. Hence, $\mathcal{V}$ defines a smooth map (in local charts) in a neighbourhood of x in the set of unparametrized embeddings of M into $\bar{M}$ (see [**1**]).

Example B 2. Assume that the image $x(M)$ of x is the boundary of a bounded open subset $\varOmega$ of $\bar{M}$; note that this is an open condition in the set of unparametrized embeddings.* For y sufficiently close to x in the $\mathcal{C}^1$-topology, one has that $y(M) = \partial\varOmega_y$ for some open bounded subset $\varOmega_y$ of $\bar{M}$. Setting $\mathcal{V}(y) = \mathrm{vol}(\varOmega_y)$, i.e. the volume of this

* If x is transversely oriented, then such a condition is equivalent to the fact that x induces the null map in homology $H^1(M) \to H^1(\bar{M}, M \setminus x(M))$. In particular, the condition is stable by $\mathcal{C}^1$-small perturbations of x.

open bounded subset, we have an invariant volume functional $\mathcal{V}$ (use Stokes's theorem). In fact, when $x(M) = \partial\Omega$, any function $\mathcal{V}$ satisfying Definition B 1 (a) coincides with such a volume functional, by Stokes's theorem.

Example B 3. Assume that $\bar{M}$ is non-compact, so its volume form $\mathrm{vol}_{\bar{\boldsymbol{g}}}$ is exact, and assume that $G = \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$ is compact. Let η be a primitive of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$, and set

$$\eta^G = \int_G \phi^*(\eta)\,\mathrm{d}\phi,$$

the integral being taken relative to a Haar measure of volume 1 of G. Then, η^G is a G-invariant primitive of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$. In particular, the functional $\mathcal{V}(x) = \int_M x^*(\eta^G)$ is an invariant volume functional.

Using the compactness argument above, we immediately obtain the following corollary.

Corollary B 4. *Assume that M is a compact manifold with (non-empty) boundary ∂M, and that $\bar{M}$ is non-compact. Given an embedding $x\colon M \hookrightarrow \bar{M}$, there exists a volume functional $\mathcal{V}$ defined in the set*

$$\{y\colon M \hookrightarrow \bar{M} \text{ embedding}\colon y(\partial M) = x(\partial M)\},$$

which is invariant under all isometries ϕ that preserve $x(\partial M)$.

Proof. The action of the isometry group $G = \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$ on $\bar{M}$ is proper; since ∂M is compact, the closed subgroup G_0 of G consisting of isometries ϕ satisfying $\phi(x(\partial M)) = x(\partial M)$ is compact. If η is a primitive of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$ in $\bar{M}$, then one can average η on G_0, obtaining a G_0-invariant primitive η^{G_0} of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$. Clearly, the volume functional defined by η^{G_0} is also G_0-invariant. □

Non-compactness of the ambient manifold $\bar{M}$ and compactness of its isometry group is a rather restrictive assumption; we now determine more general conditions that guarantee the existence of invariant volume functionals.

Lemma B 5. *Let $U \subset \bar{M}$ be an open subset and let η be any primitive of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$ on U. Consider the volume functional $\mathcal{V}(y) = \int_M y^*(\eta)$, defined on the set of embeddings $\mathfrak{M}(U)$ of M into $\bar{M}$ with image in U, and let*

$$\rho\colon\ \mathrm{Dom}(\rho) \subset \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}}) \times \mathfrak{M}(U) \to \mathfrak{M}(U)$$

be the natural local action by left composition with isometries of $(\bar{M}, \bar{\boldsymbol{g}})$.

For all $\phi \in \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$, the map $y \mapsto \mathcal{V}(y) - \mathcal{V}(\phi \circ y)$ is then locally constant on $\mathrm{Dom}(\rho_\phi)$. If $\eta - \phi^*(\eta)$ is exact in U and $x \in \mathfrak{M}(U)$, then, for ϕ sufficiently close to the identity, $\mathcal{V}(\phi \circ x) = \mathcal{V}(x)$.

Proof. For all $\phi \in \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$, $\phi^*(\eta)$ is a primitive of the volume form $\mathrm{vol}_{\bar{\boldsymbol{g}}}$, so $\eta - \phi^*(\eta)$ is closed in its domain. If $x, y \in \mathfrak{M}(U)$ are $\mathcal{C}^0$-close, then x and y are homotopic; hence, using Stokes's theorem, if x and y are in $\mathrm{Dom}(\rho_\phi)$, $\int_M x^*(\eta - \phi^*(\eta)) = \int_M y^*(\eta - \phi^*(\eta))$, i.e. $\mathcal{V}(x) - \mathcal{V}(\phi \circ x) = \mathcal{V}(y) - \mathcal{V}(\phi \circ y)$.

If $\eta - \phi^*(\eta)$ is exact, then so is $x^*(\eta - \phi^*(\eta))$; thus, $\int_M x^*(\eta - \phi^*(\eta)) = 0$, i.e. $\mathcal{V}(x) = \mathcal{V}(\phi \circ x)$. Note that this equality also holds when M has boundary, provided that $\phi(x(\partial M)) = x(\partial M)$. □

Lemma B 5 suggests how to construct invariant volume functionals. The natural setup consists of a pair of open subsets $U_1 \subset U_2 \subset \bar{M}$, an m-form η on U_2 that is a primitive of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$ in U_2, with the following properties.

- $\phi(U_1) \subset U_2$ for ϕ in a neighbourhood of the identity in $\mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$.
- $\eta - \phi^*(\eta)$ is exact in U_1 for $\phi \in \mathrm{Iso}(\bar{M}, \bar{\boldsymbol{g}})$ near the identity.

Corollary B 6. *Given objects U_1, U_2 and η as above, the map $\mathcal{V}(x) = \int_M x^*(\eta)$ is an invariant volume functional in the set of embeddings $x\colon M \hookrightarrow \bar{M}$ with image contained in U_1.*

Proof. This follows immediately from the last statement of Lemma B 5. □

Note that $\eta - \phi^*(\eta)$ is closed; hence, if U_1 has vanishing de Rham cohomology in dimension m, then it is exact. This observation provides a large class of examples of manifolds $(\bar{M}, \bar{\boldsymbol{g}})$ where it is possible to define local invariant volume functionals.

Example B 7. If $\bar{M}$ is a non-compact manifold whose mth de Rham cohomology space is 0, then the volume functional defined by any primitive of $\mathrm{vol}_{\bar{\boldsymbol{g}}}$ is invariant under the (whole) isometry group. More generally, if $x\colon M \hookrightarrow \bar{M}$ is an embedding with image contained in an open subset whose mth de Rham cohomology space is 0, then there exists a volume functional invariant under isometries near the identity, defined in an open neighbourhood $\mathcal{U}$ of x in the set of embeddings of M into $\bar{M}$. In particular, this applies when $\bar{M}$ is $\mathbb{R}^{m+1}$ or $\bar{M} = S^{m+1}$. Manifolds of the form $\bar{M}^{m+1} = \mathbb{R}^k \times N^{m+1-k}$, $k \geqslant 1$, have trivial mth de Rham cohomology space. Manifolds of the form $\bar{M}^{m+1} = S^k \times N^{m+1-k}$, $k \geqslant 1$, have open dense subsets with trivial mth de Rham cohomology space.

Acknowledgements. R.G.B. was partly supported by the NSF, USA (Grants DMS-0941615 and DMS-1209387). P.P. was partly supported by the FAPESP, Brazil (Projeto Temático 2011/21362-2), and by the CNPq, Brazil. G.S. was supported by the FAPESP, Brazil (Grant 2010/00068-6).

References

1. L. J. Alías and P. Piccione, On the manifold structure of the set of unparameterized embeddings with low regularity, *Bull. Braz. Math. Soc.* **42**(2) (2011), 171–183.
2. R. G. Bettiol, Generic properties of semi-Riemannian geodesic flows, MSc Dissertation, University of São Paulo, Brazil (2010).
3. L. Biliotti, M. A. Javaloyes and P. Piccione, On the semi-Riemannian bumpy metric theorem, *J. Lond. Math. Soc. (2)* **84**(1) (2011), 1–18.
4. G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, Volume 46 (Academic, 1972).

5. V. Cervera, F. Mascaró and P. Michor, The action of the diffeomorphism group on the space of immersions, *Diff. Geom. Applic.* **1**(4) (1991), 391–401.
6. E. N. Dancer, An implicit function theorem with symmetries and its application to nonlinear eigenvalue equations, *Bull. Austral. Math. Soc.* **21**(1) (1980), 81–91.
7. E. N. Dancer, The G-invariant implicit function theorem in infinite dimension, *Proc. R. Soc. Edinb.* A **92**(1–2) (1982), 13–30.
8. E. N. Dancer, The G-invariant implicit function theorem in infinite dimension, II, *Proc. R. Soc. Edinb.* A **102**(3–4) (1986), 211–220.
9. B. Daniel and P. Mira, Existence and uniqueness of constant mean curvature spheres in Sol_3, *J. Reine Angew. Math.* **2013**(685) (2013), 1–32.
10. J. Eells and L. Lemaire, *Two reports on harmonic maps* (World Scientific, 1995).
11. N. Kapouleas, Constant mean curvature surfaces in Euclidean three-space, *Bull. Am. Math. Soc.* **17**(2) (1987), 318–320.
12. N. Kapouleas, Complete constant mean curvature surfaces in Euclidean three-space, *Annals Math. (2)* **131**(2) (1990), 239–330.
13. M. Koiso and B. Palmer, Geometry and stability of surfaces with constant anisotropic mean curvature, *Indiana Univ. Math. J.* **54**(6) (2005), 1817–1852.
14. R. Mazzeo and F. Pacard, Constant mean curvature surfaces with Delaunay ends, *Commun. Analysis Geom.* **9**(1) (2001), 169–237.
15. R. Mazzeo, F. Pacard and D. Pollack, Connected sums of constant mean curvature surfaces in Euclidean 3-space, *J. Reine Angew. Math.* **536** (2001), 115–165.
16. R. S. Palais, *Foundations of global nonlinear analysis* (W. A. Benjamin, 1968).
17. J. Pérez and A. Ros, The space of properly embedded minimal surfaces with finite total curvature, *Indiana Univ. Math. J.* **45**(1) (1996), 177–204.
18. P. Piccione and D. V. Tausk, On the Banach differential structure for sets of maps on non-compact domains, *Nonlin. Analysis TMA* **46**(2) (2001), 245–265.
19. B. White, The space of m-dimensional surfaces that are stationary for a parametric elliptic functional, *Indiana Univ. Math. J.* **36** (1987), 567–602.
20. B. White, The space of minimal submanifolds for varying Riemannian metrics, *Indiana Univ. Math. J.* **40** (1991), 161–200.

https://doi.org/10.1017/S0013091513000631 Published online by Cambridge University Press