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OVERRINGS OF BEZOUT DOMAINS

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In [2] Brungs shows that every ring T between a principal (right and left) ideal domain R and its quotient field is a quotient ring of R. In this note we obtain similar results without assuming the ascending chain conditions. For a (right and left) Bezout domain R we show that T is a quotient ring of R which is again a Bezout domain; furthermore T is a valuation domain if and only if T is a local ring.

All rings considered are (not-necessarily commutative) integral domains with unity. A weak Bezout domain (also known as a 2-fir) is a ring in which the sum and intersection of any two principal right ideals with nonzero intersection are again principal. In the definition if we omit the phrase "with nonzero intersection" we obtain the definition of a right Bezout domain. It is shown in [3] that the definition of a weak Bezout domain is left-right symmetric; in contrast, a right Bezout domain need not be a left Bezout domain. By a local ring we mean a ring in which the set of nonunits is an ideal. Let us call R a *weak valuation domain* if $aR \cap bR \neq 0$ implies either $aR \subset bR$ or $bR \subset aR$. The left-right symmetry of this definition follows from the following.

PROPOSITION 1. *A ring R is a weak valuation domain if and only if R is a local weak Bezout domain.*

Proof. Assume R is a local weak Bezout domain and let $aR \cap bR \neq 0$. Writing $dR = aR + bR$ we have $d = ax + by$, $a = da_1$, $b = db_1$ and $1 = a_1x + b_1y$. Therefore either a_1x or b_1y must be a unit. This yields, respectively, either $bR \subset aR$ or $aR \subset bR$. If R is a weak valuation domain then R is obviously a weak Bezout domain. To show that R is local it suffices to show that for $a, b \in R$, if $a+b$ is a unit then either a or b is a unit. Accordingly, let

$$(a+b)u = 1. \tag{1}$$

Multiplying (1) on the right by a we obtain

$$bua = a(1-ua). \tag{2}$$

If either a or b is zero then, respectively, b or a is a unit. If both a and b are nonzero then (2) shows that $aR \cap bR \neq 0$. Hence either $aR \subset bR$ or $bR \subset aR$. Assume $aR \subset bR$ (the other case is similar). Then $a = bc$ for some c; substituting this into (1) we find that b has a right inverse and is therefore a unit.

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We remark that a weak valuation ring can be characterized as a ring in which any two factorizations of the same element have common refinements. This is proved in [4, Proposition 1] where large classes of such rings are constructed. We say that R is a *right valuation domain* if its poset of principal right ideals is a chain. If we assume in Proposition 1 that R is a right Ore domain ($aR \cap bR \neq 0$ for all nonzero $a, b \in R$) then we have the following.

COROLLARY. *A ring R is a right valuation domain if and only if R is a local right Bezout domain.*

Each right Bezout domain is a right Ore domain and therefore has a right quotient field $K=\{ab^{-1} \mid a, b \in R, b\neq 0\}$. The construction of K is a particular case of the following which is valid in any integral domain R. Let R^* be the monoid of nonzero elements of R. A submonoid S of R^* is a right Ore system in R if for each $a \in R$, $b \in S$ we have

$$aS \cap bR \neq 0.$$

In this case $R_S=\{ab^{-1} \mid a \in R, b \in S\}$ is a ring under operations that extend those of R (see the references given in [1] or [2] for details). If R is a right Bezout domain (or just a right Ore domain) then R_{R^*} is the right quotient field of R and contains each right quotient ring R_S. It is easy to check that if R is a right Bezout domain then so is each R_S. For right and left Bezout domains the same is true according to the following.

PROPOSITION 2. *Let R be a (right and left) Bezout domain and let S be a right Ore system in R. Then $T=R_S$ is again a (right and left) Bezout domain.*

Proof. Since T is a right Bezout domain (and therefore a weak Bezout domain) it suffices to show that T is a left Ore domain. Let $x=ab^{-1}$, $y=cd^{-1} \in T^*$. Choose b', $d' \in R^*$ such that $db'=bd'$. Then $Rad' \cap Rcb' \neq 0$ and consequently $Rx \cap Ry\neq 0$.

If a right Ore system S in R is saturated, that is, if $ab \in S$ implies both a and b are in S then S is called a *right quotient monoid* in R. It is shown in [1, Lemma 3] that in a right Bezout domain R a saturated submonoid S of R^* is a right quotient monoid if and only if elements similar to members of S are also in S. Recall that two elements a and a' in a ring R are similar ($a\sim_R a'$) if $R/aR\cong R/a'R$ as R-modules. It is shown in [3] that $a\sim_R a'$ iff there is a relation $ab'=ba'$ in R which $aR+bR=R$ and $Ra'+Rb'=R$. Thus if R is a subring of T and if $a\sim_R a'$ then $a\sim_T a'$.

THEOREM. *Let R be a (right and left) Bezout domain with quotient field K. Any subring T between R and K is a quotient ring $T=R_S$ for a suitable right quotient monoid S in R. The ring T is a (right and left) Bezout domain, and is a (right and left) valuation domain if and only if it is a local ring.*

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Proof. Let $S=\{s \in R \mid s^{-1} \in T\}$. Clearly S is a saturated submonoid of R^*. If $a, a' \in R$ and $a \sim_R a'$ then $a \sim_T a'$ from which it follows that $a \in S$ iff $a' \in S$. Thus S is a right quotient monoid in R. Clearly $R_S \subset T$. To show the reverse containment let $z=ab^{-1} \in T$, let $Rd=Ra+Rb$, with $a=a_1d$, $b=b_1d$. Then $Ra_1+Rb_1=R$ so we may find $x, y \in R$ such that

$$1 = xa_1+yb_1. \tag{3}$$

If we multiply (3) on the right side by b_1^{-1} we obtain $b_1^{-1}=xz+y \in T$ so that $R_S=T$. The last statement of the theorem follows from Propositions 1 and 2.

The example given in [2] shows that our results are not valid for rings satisfying only a onesided condition.

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