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ON THE BOUNDARY BEHAVIOUR OF FRIDMAN INVARIANTS

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Abstract

We prove that the Fridman invariant defined using the Carathéodory pseudodistance does not always go to 1 near strongly Levi pseudoconvex boundary points and it always goes to 0 near nonpseudoconvex boundary points. We also discuss whether Fridman invariants can be extended continuously to some boundary points of domains constructed by deleting compact subsets from other domains.

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1. Introduction

Let D be a bounded domain and Ω a bounded homogeneous domain in $\mathbb{C}^n$ and let $z \in D$. Denote by $O(\Omega, D)$ the set of holomorphic maps from Ω into D. Denote by d either the Carathéodory pseudosdistance c or the Kobayashi pseudodistance k on D. Fridman [6, 7] introduced a holomorphic invariant, now called the *Fridman invariant*:

$$e_D^{\Omega^d}(z) = \sup\{\tanh(r) : \ B_D^d(z, r) \subset f(\Omega), f \in O(\Omega, D), f \text{ is injective}\},$$

where $B_D^d(0, r)$ is the d-ball centred at z with radius r. (In [6, 7], $\inf 1/r$ was used instead of $\sup \tanh(r)$.) We denote $e_D^{\Omega^c}(z)$ by $\tilde{e}_D^{\Omega}(z)$ and $e_D^{\Omega^k}(z)$ by $e_D^{\Omega}(z)$ in this paper. When Ω is the unit ball B^n, we denote $\tilde{e}_D^{B^n}(z)$ by $\tilde{e}_D(z)$ and $e_D^{B^n}(z))$ by $e_D(z)$.

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. The Carathéodory pseudodistance on Ω is defined as

$$c_\Omega(z, w) = \sup\{\operatorname{artanh}(|\lambda|) : f \in O(\Omega, \mathbb{D}), f(z) = 0, f(w) = \lambda\}.$$

Let

$$\ell_\Omega(z', z'') = \inf\{\operatorname{artanh}(|\lambda|) : \varphi \in O(\mathbb{D}, \Omega), \varphi(0) = z',\ \varphi(\lambda) = z''\}.$$

The Kobayashi pseudodistance for Ω is

$$k_\Omega(z', z'') = \inf\left\{ \sum_{j=1}^{N} \ell_\Omega(z_{j-1}, z_j) : N \in \mathbb{N}, z_0 = z', z_1, \ldots, z_{N-1} \in \Omega, z_N = z'' \right\}.$$

Another invariant, called the *squeezing function*, was introduced by Deng *et al.* [2]:

$$s_D(z) = \sup\{r : \ rB^n \subset f(D), \ f \in O(D, B^n), \ f(z) = 0, \ f \text{ is injective}\}.$$

From the definitions, it is clear that $e_D^{\Omega^d}$ and s_D are invariant under biholomorphisms. Many properties of $\tilde{e}_D$, e_D and s_D have been explored (see the survey paper [4] and the references therein). For results on the boundary behaviour of e_D, we refer to [7, 11, 12, 14] and for the boundary behaviour of $s_D(z)$ to [3, 5, 9, 13].

Recently, Nikolov and Verma [14, Proposition 4] proved that e_D goes to 1 near strongly pseudoconvex boundary points. Because $\tilde{e}_D(z) \le e_D(z)$, it is of interest to investigate whether the same result holds for $\tilde{e}_D$. Here we give a negative answer. In fact, for any $c \in (0, 1)$, there exist a bounded nonpseudoconvex domain D_c and a strongly pseudoconvex boundary point a such that $\lim_{z\to a} \tilde{e}_{D_c}(z) = c$.

THEOREM 1.1. *Let* $0 < R_1 < R_2 < 2R_1/(1 + R_1^2) < 1$ *and let* $D = B^n \setminus K$, *where* $n \ge 2$ *and* $K = \{z \in \mathbb{C}^n \mid R_1 \le \|z\| \le R_2, \mathrm{Re}\, z_n \ge 0\}$. *Take* $p = (0, 0, \dots, R_1)$ *and* $p_k = (0, 0, \dots, (1 - 1/k)R_1)$, $k \in \mathbb{N}$. *Then,*

$$\lim_{k\to\infty} \tilde{e}_D(p_k) = \frac{R_2 - R_1}{1 - R_1 R_2}.$$

Let Ω be a bounded domain in $\mathbb{C}^n$, $n \ge 2$. Let K be a compact subset of Ω such that $D = \Omega\backslash K$ is connected. Bharali proved that $s_D(z) \le \tanh(k_\Omega(z; \partial D \cap K))$ [1, Theorem 1.8]. From Theorem 1.1, it is clear that there is no such estimation for $\tilde{e}_D$ under the same condition.

Let K be a compact subset of B^n, $n \ge 2$, such that $D = B^n\backslash K$ is connected. In [17], we proved that

$$s_D(z) = \min_{w\in\partial K} \tanh[c_{B^n}(z, w)].$$

Moreover, for some special K (for example, a pseudoconvex subdomain of B^n with dense strongly pseudoconvex points in ∂K), we have $s_D(z) = \tilde{e}_D(z)$. It follows from Theorem 1.1 that $\tilde{e}_D = s_D$ does not hold for general compact subsets K. See [15–17] for more results on the comparison of the Fridman invariant and the squeezing function.

However, it is also natural to ask how $\tilde{e}_D(z)$ behaves near nonpseudoconvex boundary points. We show that $\tilde{e}_D(z)$ goes to 0 near such points.

THEOREM 1.2. *Let D be a bounded domain in* $\mathbb{C}^n$, $n \ge 2$, *and assume that* ∂D *is* C^2 *smooth near* $p \in \partial D$. *If p is not pseudoconvex, then*

$$\lim_{z\to p} \tilde{e}_D(z) = 0.$$

Because $s_D(z) \le \tilde{e}_D(z)$ [14, Proposition 1], Theorem 1.2 immediately implies the following result.

COROLLARY 1.3. *Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, and assume that ∂D is C^2 smooth near $p \in \partial D$. If p is not pseudoconvex, then*

$$\lim_{z \to p} s_D(z) = 0.$$

Let $D \subsetneq \mathbb{C}^n, n \geq 2$, be a bounded domain and let $\mathcal{S}$ be a subset of $O(D)$ which contains all the bounded holomorphic functions. Define

$$\partial^{\mathcal{S}} D := \left\{ \xi \in \partial D : \begin{array}{l} \text{there exists } U, \text{ a connected open neighbourhood of } \xi, \\ \text{and } V, \text{ a connected component of } D \cap U, \text{ such that} \\ \text{for all } f \in \mathcal{S}, \text{ there exists } F_f \in O(U) \text{ satisfying } f|_V = F_f|_V \end{array} \right\}$$

Bharali [1, Theorem 1.11] proved that $\lim_{z \to p} s_D(z) = 0$ for each $p \in \partial^{\mathcal{S}} D$. If $p \in \partial D$ is not pseudoconvex, then $p \in \partial^{\mathcal{S}} D$. Thus, the above corollary can also be seen as a special case of [1, Theorem 1.11].

Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, and K a compact subset of Ω such that $D = \Omega \backslash K$ is connected. By Hartog's extension theorem, $\partial K \subset \partial^{\mathcal{S}} D$. Hence Theorem 1.1 shows that, for $p \in \partial^{\mathcal{S}} D$, in general, $\lim_{z \to p} \tilde{e}_D(z) \neq 0$.

We have the following result.

THEOREM 1.4. *Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset of Ω such that $D = \Omega \backslash K$ is connected. Then $\lim_{z \to p} \tilde{e}_D(z)$ exists for any $p \in \partial K$.*

Let D and p be as in Theorem 1.1. Combining Theorem 1.1 with Theorem 1.4 shows that

$$\lim_{z \to p} \tilde{e}_D(z) = \frac{R_2 - R_1}{1 - R_1 R_2}.$$

It is easy to see that for any $c \in (0, 1)$, there exist R_1, R_2 with $0 < R_1 < R_2 < 1$ such that $(R_2 - R_1)/(1 - R_1 R_2) = c$.

It is then natural to ask whether the same result holds for e_D. The answer is negative as the following result shows.

THEOREM 1.5. *Let $0 < R_1 < R_2 < 1$, $K_1 = \{z \mid R_1 \leq \|z\| \leq R_2, \mathrm{Re}\, z_n \geq 0\}$ and $K_2 = \{p_j\}_{j \in \mathbb{N}}$, where $p_j = ((1 - 1/j)R_1, 0, \ldots, 0)$. Take $K = K_1 \cup K_2$ and $D = B^n \setminus K$, $n \geq 2$. Then $e_D(z)$ cannot be extended continuously to ∂K.*

2. Proof of the results

We will use Hartogs's extension theorem (see, for example, [10, Theorem 1.2.6]), which we state as the following lemma.

LEMMA 2.1. *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset of Ω such that $\Omega \backslash K$ is connected. If f is holomorphic on $\Omega \backslash K$, then there exists a holomorphic function F on Ω such that $F|_{\Omega \backslash K} = f$.*

PROOF OF THEOREM 1.1. Because B^n is biholomorphic to $B^n(0, R_1)$ and they are both homogeneous, for $p_k = (0, 0, \ldots, (1 - 1/k)R_1)$, there exists a holomorphic embedding

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$f_k : B^n \to B^n(0, R_1)$ such that $f_k(0) = p_k$ and $f_k(B^n) = B^n(0, R_1)$. By Lemma 2.1, $c_D(z_1, z_2) = c_{B^n}(z_1, z_2)$, for all $z_1, z_2 \in D$. From [8, Corollary 2.3.5],

$$\tanh c_{B^n}(a, z) = \left[1 - \frac{(1 - \|a\|^2)(1 - \|z\|^2)}{|1 - \langle z, a\rangle|^2}\right]^{1/2}.$$

Let $w \in \{z \in B^n \mid \operatorname{Re} z_n = 0\}$. It is easy to see that

$$\tanh c_{B^n}(p, w) \geq \tanh c_{B^n}(p, 0) = R_1.$$

Denote

$$d_k = \frac{R_2 - (1 - 1/k)R_1}{1 - (1 - 1/k)R_1R_2}.$$

Because $R_2 < 2R_1/(1 + R_1^2)$, there exists $N > 0$ such that for any $k > N$,

$$B_D^c(p_k, \operatorname{artanh}(d_k)) \subset B^n(0, R_1) = f_k(B^n),$$

and hence $\tilde{e}_D(p_k) \geq d_k$.

We claim that $\tilde{e}_D(p_k) \leq d_k$. For w with $\|w\| = R_2$, it is obvious that

$$\tanh[c_\Omega(p_k, w)] \leq \tanh[c_\Omega(p_k, q)] = d_k,$$

where $q = (0, 0, \ldots, R_2)$. Suppose that $\tilde{e}_D(p_k) > d_k$. Then, there exists $r > \operatorname{artanh}(d_k)$ and a holomorphic embedding $g_k : B^n \to D$ such that $g_k(0) = p_k$ and $B_D^c(p_k, r) \subset g_k(B^n)$. Because the Carathéodory pseudodistance is continuous (see, for example, [8]), we know that $B_D^c(p_k, r)$ and $B_{B^n}^c(p_k, r)$ are open. It follows that there exists $\delta > 0$ such that $B^n(q, \delta) \subset B_{B^n}^c(p_k, r)$. Because $B_D^c(p_k, r) \subset g_k(B^n) \subset D$ and $c_D(z_1, z_2) = c_{B^n}(z_1, z_2)$, we have $B^n(q, \delta) \cap g_k(B^n) \neq \emptyset$ and $B^n(q, \delta) \cap \partial B^n(0, R_2) \subset \partial(g_k(B^n))$.

However, it is clear that $q \in \partial B^n(0, R_2)$ is strongly pseudoconvex for $B^n(0, R_2)$. Thus, there exists a local C^2 defining function ρ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) v_j \bar{v}_k > 0,$$

for all $v \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(q) v_j = 0.$$

However, $g_k(B^n)$ is pseudoconvex and it is clear that $-\rho(z)$ is a local defining function on some neighbourhood of q for $g_k(B^n)$. It follows that

$$\sum_{j,k=1}^{n} \frac{\partial^2 (-\rho)}{\partial z_j \partial \bar{z}_k}(q) v_j \bar{v}_k \geq 0,$$

for all $v \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial (-\rho)}{\partial z_j}(q) v_j = 0,$$

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which is a contradiction. Hence $\tilde{e}_D(p_k) \le d_k$. So we have $\tilde{e}_D(p_k) = d_k$, which implies

$$\lim_{k\to\infty} \tilde{e}_D(p_k) = \frac{R_2 - R_1}{1 - R_1 R_2}. \qquad \square$$

PROOF OF THEOREM 1.2. Because p is not pseudoconvex, we can find a connected neighbourhood U_p of p such that for any holomorphic function f on D, there exists a holomorphic function F on U_p with $F|_{U_p\cap D} = f|_{U_p\cap D}$.

It is clear that $D_1 = U_p \cup D$ is a connected open set. We claim that $c_D(z_1, z_2) = c_{D_1}(z_1, z_2)$, for all $z_1, z_2 \in D$.

Let $f \in O(D, \mathbb{D})$. Then there exists a holomorphic function F on D_1 such that $F|_D = f$. Moreover $F(D_1) = f(D)$. Indeed, if there exists $w \in D_1$ such that $F(w) \notin f(D)$, then $h(z) = 1/(f(z) - F(w))$ is holomorphic on D, but with no holomorphic function $H(z)$ on D_1 such that $H|_D = h$, a contradiction. By the definition of Carathéodory pseudodistance, we have $c_D(z_1, z_2) = c_{D_1}(z_1, z_2)$, for all $z_1, z_2 \in D$.

Assume that $\lim_{z\to p} \tilde{e}_D(z) = 0$ does not hold. Then there exists $p_k \to p$ such that $\lim_{k\to\infty} \tilde{e}_D(p_k) = A > 0$. Because $\lim_{k\to\infty} c_{D_1}(p_k, p) = 0$, for $0 < \epsilon < A/2$, we can find $N > 0$ such that for any $k > N$, there exist $r_k > \operatorname{artanh}(A - \epsilon)$ and a holomorphic embedding $f_k : B^n \to D$ such that $f_k(0) = p_k$, $B^c_D(p_k, r_k) \subset f_k(B^n)$ and $p \in B^c_{D_1}(p_k, r_k)$. Because the Carathéodory pseudodistance is continuous, there exists $\delta_k > 0$ such that $B^n(p, \delta_k) \subset B^c_{D_1}(p_k, r_k)$. Because $c_D(z_1, z_2) = c_{D_1}(z_1, z_2)$, we have $D_1(p, \delta_k) \cap f_k(B^n) \neq \emptyset$ and $B^n(p, \delta_k) \cap \partial D \subset \partial(f_k(B^n))$.

Because $p \in \partial D$ is not pseudoconvex, there is a local C^2 defining function ρ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k < 0,$$

for some $v \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) v_j = 0.$$

However, $f_k(B^n)$ is pseudoconvex and it is clear that $\rho(z)$ is a local defining function on some neighbourhood of p for $f_k(B^n)$. It follows that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k \ge 0,$$

for all $v \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) v_j = 0,$$

which is a contradiction. This implies that $\lim_{z\to p} \tilde{e}_D(z) = 0$. $\square$

To prove Theorem 1.4, the following lemma is needed.

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LEMMA 2.2. *Let D be a bounded domain in $\mathbb{C}^n$. Then,*

$$|\tilde{e}_D(z_1) - \tilde{e}_D(z_2)| \le \tanh[c_D(z_1, z_2)], \quad \textit{for all } z_1, z_2 \in D.$$

PROOF. If $\tilde{e}_D(z_1) = \tilde{e}_D(z_2) = 0$, then we have the conclusion. Thus, without loss of generality, assume that $\tilde{e}_D(z_1) > 0$.

Let $0 < \epsilon < \tilde{e}_D(z_1)$. By definition, there is a holomorphic embedding $f : B^n \to D$ such that $B^c_D(z_1, \operatorname{artanh}[\tilde{e}_D(z_1) - \epsilon]) \subset f(B^n)$.

If $z_2 \notin B^c_D(z_1, \operatorname{artanh}[\tilde{e}_D(z_1) - \epsilon])$, then clearly

$$\tilde{e}_D(z_2) \ge \tilde{e}_D(z_1) - \epsilon - \tanh[c_D(z_1, z_2)].$$

Assume that $z_2 \in B^c_D(z_1, \operatorname{artanh}[\tilde{e}_D(z_1) - \epsilon])$. It is easy to check that $\tanh(t_3) \le \tanh(t_1) + \tanh(t_2)$ for all $t_i \ge 0$, $i = 1, 2, 3$, with $t_3 \le t_1 + t_2$. Then for all z with $\tanh[c_D(z_2, z)] < e^{\Omega}_D(z_1) - \epsilon - \tanh[c_D(z_1, z_2)]\}$,

$$\tanh[c_D(z_1, z)] \le \tanh[c_D(z_2, z)] + \tanh[c_D(z_1, z_2)] < \tilde{e}_D(z_1) - \epsilon.$$

This implies that

$$B^c_D(z_2, \operatorname{artanh}[\tilde{e}_D(z_1) - \epsilon - \tanh[c_D(z_1, z_2)]]) \subset B^c_D(z_1, \operatorname{artanh}[\tilde{e}_D(z_1) - \epsilon]) \subset f(B^n).$$

Hence

$$\tilde{e}_D(z_2) \ge \tilde{e}_D(z_1) - \epsilon - \tanh[c_D(z_2, z_1)].$$

Because ϵ is arbitrary,

$$\tilde{e}_D(z_2) \ge \tilde{e}_D(z_1) - \tanh[c_D(z_1, z_2)].$$

If $\tilde{e}_D(z_2) = 0$, then $\tilde{e}_D(z_1) \le \tanh[c_D(z_1, z_2)]$ and hence

$$|\tilde{e}_D(z_1) - \tilde{e}_D(z_2)| \le \tanh[c_D(z_1, z_2)].$$

If $\tilde{e}_D(z_2) > 0$, then following the same discussion as for $\tilde{e}_D(z_1) > 0$,

$$\tilde{e}_D(z_1) \ge \tilde{e}_D(z_2) - \tanh[c_D(z_2, z_1)].$$

This completes the proof. □

PROOF OF THEOREM 1.4. By Lemma 2.1, $c_D(z_1, z_2) = c_\Omega(z_1, z_2)$, for all $z_1, z_2 \in D$. Let $p \in \partial K$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\tanh c_D(z_1, z_2) \le \epsilon$ for all $z_1, z_2 \in B^n(p, \delta) \cap D$. By Lemma 2.2, $|\tilde{e}_D(z_1) - \tilde{e}_D(z_2)| \le \tanh[c_D(z_1, z_2)] \le \epsilon$. Hence $\lim_{z \to p} \tilde{e}_D(z)$ exists for any $p \in \partial K$. □

For the proof of Theorem 1.5, we need the following two results.

LEMMA 2.3 [8, Corollary 3.4.3]. *Let D be a bounded domain and A an analytic subset of D of codimension at least two. Then,*

$$k_{D \setminus A} = k_D|_{(D \setminus A) \times (D \setminus A)}.$$

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LEMMA 2.4 [14, Proposition 4]. *Let D be a bounded domain and p_0 a strongly pseudoconvex boundary point. Then,*

$$\lim_{z\to p_0} e_D(z) = 1.$$

PROOF OF THEOREM 1.5. Let $p = (R_1, 0, \ldots, 0)$. It is clear that $p \in \partial K$ and $p_k \to p$. Set $D_j = D \cup \{p_j\}$.

We will first prove that $\lim_{z\to p_j} e_D(z) = 0$. Fix j and suppose that there exist $z_i \to p_j$ such that $\lim_{i\to\infty} e_D(z_i) = A > 0$. By Lemma 2.3, $k_D(z_1, z_2) = k_{D_j}(z_1, z_2)$, for all $z_1, z_2 \in D$. For $0 < \epsilon < A/2$, we can find $N > 0$ such that for any $i > N$, there are $r_i > \operatorname{artanh}(A - \epsilon)$ and a holomorphic embedding $f_i : B^n \to D$ such that $f_i(0) = z_i$, $B^k_D(z_i, r_i) \subset f_i(B^n)$ and $p_j \in B^k_{D_j}(z_i, r_i)$. Because the Kobayashi pseudodistance is continuous (see, for example, [8]), there exists $\delta_i > 0$ such that $B^n(p_j, \delta_i) \subset B^k_{D_j}(z_i, r_i)$. Because $B^k_D(z_i, r_i) \subset f_i(B^n)$, we have $\{z \mid 0 < \|z - p_j\| < \delta_i\} \subset f_i(B^n)$ but $p_j \notin f_i(B^n)$, which contradicts the fact that $f_i(B^n)$ is pseudoconvex.

Denote $S = \{z \mid \|z\| = R_1, \operatorname{Re} z_n > 0\}$. It is clear that S is a smooth subset of ∂D and each point of S is strongly pseudoconvex. Assume that $e_D(z)$ can be extended continuously to ∂K. Because $\lim_{z\to p_j} e_D(z) = 0$ and $p_j \to p$, we have $\lim_{z\to p} e_D(z) = 0$. However, there exist $w_j \in S \to p$. By Lemma 2.4, $\lim_{z\to w_j} e_D(z) = 1$. Hence $\lim_{z\to p} e_D(z) = 1$, which is a contradiction. □

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