Canad. J. Math. Vol. **57** (3), 2005 pp. 535–597

On Local L-Functions and Normalized Intertwining Operators

Henry H. Kim

Abstract. In this paper we make explicit all L-functions in the Langlands–Shahidi method which appear as normalizing factors of global intertwining operators in the constant term of the Eisenstein series. We prove, in many cases, the conjecture of Shahidi regarding the holomorphy of the local L-functions. We also prove that the normalized local intertwining operators are holomorphic and non-vaninishing for $\mathrm{Re}(s) \geq 1/2$ in many cases. These local results are essential in global applications such as Langlands functoriality, residual spectrum and determining poles of automorphic L-functions.

Introduction

The purpose of this paper is threefold; first, to make explicit all L-functions which appear in the constant term of the Eisenstein series by combining the list in [La] and [Sh3]; second, to prove Conjecture 7.1 in [Sh1], regarding the holomorphy of the local L-functions in many cases; third, to prove Assumption (A) in [Ki3], regarding the holomorphy of the normalized local intertwining operators.

More precisely, let $\mathbf{G}$ be a simply connected, split, simple group. Let $\mathbf{M}$ be a maximal parabolic subgroup of $\mathbf{G}$. We explicitly calculate $\mathbf{M}$. Since $\mathbf{G}$ is simply connected, the derived group of $\mathbf{M}$ is simply connected, and hence it is well-known. However, determining the exact structure is a delicate matter and is crucial for the study of L-functions. For example, let us take $\mathbf{G}$ to be the exceptional group of type F_4. One of the maximal Levi subgroup $\mathbf{M}$ has Sp_6 as a derived group. The L-group of Sp_6 is $SO_7(\mathbb{C})$. However, the L-function which appears in the constant term of the Eisenstein series attached to $(\mathbf{G}, \mathbf{M})$ is the degree 8 spin L-function, which exists for $\mathrm{Spin}(7, \mathbb{C})$, but not for $SO_7(\mathbb{C})$. We will see that $\mathbf{M} = GSp_6$, whose L-group is $G\,\mathrm{Spin}(7, \mathbb{C})$. This has been pointed out to us by F. Shahidi. One byproduct of these explicit calculations is that we obtain new L-functions. For example, by considering split spin groups $\mathrm{Spin}(2n)$, $\mathrm{Spin}(2n+1)$, and a maximal Levi subgroup whose derived group is SL_n, we obtain the twisted symmetric square and twisted exterior square L-functions of cuspidal representations of GL_n. Note that Shahidi [Sh4] obtained those L-functions as normalizing factors in the Eisenstein series only when n is even.

We prove Conjecture 7.1 in [Sh1] for E-type groups, except possibly for the following four cases: $E_7 - 3$, $E_8 - 3$, $E_8 - 4$, (xxviii) ($D_7 \subset E_8$). In these four exceptional cases, the Levi subgroups involve either a group of type D_n (spin group) or an exceptional group of type E_6. Due to the lack of a classification of generic discrete series for

Received by the editors March 18, 2002; revised May 1, 2003.

Partially supported by NSF grant DMS9988672, NSF grant DMS9729992 (at the IAS), by an NSERC grant and by the Clay Mathematics Institute, where the author was a recipient of a fellowship.

AMS subject classification: 11F70, 22E55.

the groups of type D_n and E_6, we are unable to prove the conjecture. However, we may only need a partial classification. Indeed, in [Ca-Sh], Casselman and Shahidi proved the conjecture in the case of quasi-split classical groups, using a partial classification of generic discrete series of quasi-split classical groups. In Proposition 3.15, we calculate explicitly the Rankin–Selberg L-function for $GL_k \times$ (quasi-split classical group) for generic discrete series. However, the proof does not extend to spin groups, due to the complicated nature of Levi subgroups. Asgari [As] was able to extend the result to spin groups using G-type groups (also the exceptional group of type F_4) for the following reason: besides the problem of partial classification of discrete series for the Levi factor, one needs to see how the poles of corresponding γ-factors cancel. In order to see this, we have to use multiplicativity of γ-factors [Sh1, Theorem 3.5]. For that, one has to express the intertwining operator as a product of rank-one operators. For G-type groups, the Levi subgroups are very simple. For example, the Levi subgroups of $G\,\mathrm{Spin}(2n)$ are of the form $GL_{n_1} \times \cdots \times GL_{n_k} \times G\,\mathrm{Spin}(2m)$. Hence one can see the cancellation easily. If we use GE-type groups, one might be able to prove the conjecture in the above cases which were excluded.

We should remark that if we can prove that Shahidi's L-functions for supercuspidal representations are Artin L-functions, then the conjecture is immediate: Shahidi's L-functions then are Artin L-functions for discrete series by multiplicativity of γ and L-factors and the conjecture is known for Artin L-functions. However, it is not known that Shahidi's L-functions are Artin L-functions, except for certain cases. Shahidi [Sh5] has shown that for Rankin–Selberg L-functions for $GL_k \times GL_l$, his L-functions are Artin L-functions.

Note that cases $D_5 - 2$, $E_6 - 1$ and $E_7 - 1$ are essential in studying Rankin triple L-functions which in turn give the functorial product $GL_2 \times GL_3 \to GL_6$ [Ki-Sh], and the symmetric cube $GL_2 \to GL_4$. Also the $D_n - 3$ case was used in obtaining the functoriality of the exterior square $GL_4 \to GL_6$, and the symmetric fourth $GL_2 \to GL_5$. The case $E_8 - 2$ has an important application to Ramanujan and Selberg's bounds. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let $\mathrm{diag}(\alpha_v, \beta_v)$ be the Satake parameter for an unramified π_v. Let $\pi_1 = A^3(\pi) = \mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$, constructed in [Ki-Sh], and $\pi_2 = \mathrm{Sym}^4(\pi)$, constructed in [Ki5]. Then we obtain the L-function $L(s, \pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5)$ in the $E_8 - 2$ case. Let S be a finite set of finite places such that π_v is unramified for $v \notin S$, $v < \infty$. By a standard calculation, we have

$$L_S(s, \pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5) = L_S(s, \pi, \mathrm{Sym}^9) L_S(s, \pi, \mathrm{Sym}^7 \otimes \omega_\pi) L_S(s, \pi, \mathrm{Sym}^5 \otimes \omega_\pi^2)^2 \\ \times L_S(s, \mathrm{Sym}^3(\pi) \otimes \omega_\pi^3)^2 L_S(s, \pi \otimes \omega_\pi^4).$$

In [Ki-Sh2], we applied the machinery of [Sh3] and showed that

$$q_v^{-\frac{1}{9}} < |\alpha_v|, |\beta_v| < q_v^{\frac{1}{9}}$$

if π_v is unramified, using the fact that the local L-function $L(s, \pi_v, r_1)$ is holomorphic for $\mathrm{Re}(s) \geq 1$ for π_v unramified [Sh3, Lemma 5.8]. Now our explicit calculations of the L-functions enable us to extend the result to the archimedean places, thanks to Proposition 4.9. Let π_v be a local (finite or infinite) spherical component, given

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

by $\pi_v = \mathrm{Ind}(|\cdot|_v^{s_{1v}} \otimes |\cdot|_v^{s_{2v}})$. Then $|\,\mathrm{Re}(s_{iv})| < \frac{1}{9}$. If $F = \mathbb{Q}$, $v = \infty$, this means $\lambda_1 = \frac{1}{4}(1 - s^2) > \frac{77}{324} \approx 0.238$, where $s = 2s_{1v} = -2s_{2v}$ and λ_1 is the first eigenvalue of the Laplace operator on the corresponding hyperbolic space.

We prove Assumption (A), except possibly for the following 12 cases:

- Cases where standard module conjecture is not available.
  $B_n - 1(\mathrm{Spin}(2n+1))$; $D_n - 1(\mathrm{Spin}(2n))$; **(xxx)** in [La] ($E_6 \subset E_7$); $E_8 - 4$; **(xxxii)** in [La] ($E_7 \subset E_8$).
- Cases where the Levi subgroup contains a group of type B_3, C_3, D_n.
  (xviii) in [La] ($B_3 \subset F_4$); **(xxii)** in [La] ($C_3 \subset F_4$); **(xxiv)** in [La] ($D_5 \subset E_6$); $E_7 - 3$; **(xxvi)** in [La] ($D_6 \subset E_7$); $E_8 - 3$; **(xxviii)** in [La] ($D_7 \subset E_8$).

It seems that for the last 7 cases, we might not be able to prove Assumption (A) purely by local means. We need global information on bounds of Fourier coefficients. (See case **(xxiv)** in [La] for the details.)

1 Preliminaries

Recall several facts and notations from [Ki3]: let $\mathbf{G}$ be a split group over a local field F and $\mathbf{P} = \mathbf{MN}$ is a maximal parabolic subgroup and let α be the unique simple root in $\mathbf{N}$. As in [Sh1], let $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \cdot \rho$, where ρ is half the sum of roots in $\mathbf{N}$. We identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}_{\mathbb{C}}^*$ and denote $I(s, \pi) = I(s\tilde{\alpha}, \pi) = \mathrm{Ind}_P^G \pi \otimes \exp\big(\langle s\tilde{\alpha}, H_P(\,\cdot\,)\rangle\big)$.

Let $A(s, \pi, w_0)$ be the standard intertwining operator from $I(s\tilde{\alpha}, \pi)$ into

$$I(w_0(s\widetilde{\alpha}), w_0(\pi)).$$

Denote by ${}^L M$, the L-group of $\mathbf{M}$ and let ${}^L\mathfrak{n}$ be the Lie algebra of the L-group of $\mathbf{N}$. Let r be the adjoint action of ${}^L M$ on ${}^L\mathfrak{n}$ and decompose $r = \bigoplus_{i=1}^m r_i$, with ordering as in [Sh1]. For each i, $1 \le i \le m$, let $L(s, \pi, r_i)$ be the local L-function defined in [Sh1].

To be more precise, the numbers $\langle \tilde{\alpha}, \beta \rangle$, where $\beta^\vee$ ranges over those dual roots for which $X_{\beta^\vee} \in {}^L\mathfrak{n}$, take a string of integers from 1 through m, where m is a positive integer. Given i, $1 \le i \le m$, let

$$V_i = \{X_{\beta^\vee} \in {}^L\mathfrak{n} \mid \langle \tilde{\alpha}, \beta \rangle = i\}.$$

Then for each i, the adjoint action of ${}^L M$ leaves V_i stable. Let r_i be its restriction to V_i. Each r_i is irreducible [Sh3] and the weights of r_i are the roots $\beta^\vee$ in ${}^L\mathfrak{n}$ which restrict to $i\alpha^\vee$ on ${}^L A^0$.

Let π be an unramified representation of $\mathbf{M}(F)$ and χ the inducing character of the torus. Namely, $\pi \hookrightarrow \mathrm{Ind}_{B(F)}^{G(F)} \chi$, where $B = TU$ is a Borel subgroup and χ is a character of $T(F)$. Let $\hat{t}$ be the semi-simple conjugacy class in ${}^L M^0$ corresponding to π. Then note the relationship

$$\chi \circ \beta^\vee(\varpi) = \beta^\vee(\hat{t}),$$

where $\beta^\vee$ on the right is considered as a root of ${}^L M^0$. Then we have

$$L(s, \pi, r_i) = \prod_{\substack{\beta > 0 \\ \langle \tilde{\alpha}, \beta \rangle = i}} L(s, \chi \circ \beta^\vee) = \prod_{\substack{\beta > 0 \\ \langle \tilde{\alpha}, \beta \rangle = i}} (1 - \chi \circ \beta^\vee(\varpi)\, q^{-s})^{-1}.$$

For an arbitrary generic representation π, the local L-function is defined, using local coefficients. We normalize the intertwining operator $A(s,\pi,w_0)$ as follows:

$$
\begin{aligned}
&A(s,\pi,w_0) = r(s,\pi,w_0)N(s,\pi,w_0),\\
(1.1)\qquad &r(s,\pi,w_0) = \prod_{i=1}^{m} \frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)\epsilon(is,\pi,r_i,\psi)}.
\end{aligned}
$$

2 Local L-Functions Made Explicit

In this section, we make explicit the L-functions which appear in the constant term of Eisenstein series. We look at them case by case from [Sh3, La]. Let F be a number field and $\mathbb{A}$ its ring of adeles. We give a simple, simply connected, split group $\mathbf{G}$, a maximal Levi subgroup $\mathbf{M}$, a cuspidal representation π of $\mathbf{M}(\mathbb{A})$, and $L(s,\pi,r_i)$, $i=1,\dots,m$.

Let η be a character of $\mathbf{M}$. We let $\pi_\eta = \pi\otimes\eta$ be the representation of $\mathbf{M}(\mathbb{A})$ such that

$$(\pi\otimes\eta)(m) = \pi(m)\eta(m).$$

The following is well-known, *cf.* [Ko, p. 616].

Lemma 2.1 *Under the correspondence $\mathbf{M}\to {}^LM^0$, the cocharacters of $\mathbf{M}$ correspond to characters of ${}^LM^0$. Hence if $a=a(t)$, $t\in GL_1$, is in the connected component of the center of $\mathbf{M}$, which is a generator of the cocharacters of $\mathbf{M}$, then it corresponds to the character of ${}^LM^0$ which generates the character group of ${}^LM^0$. Denote it by $\hat{a}$. Let π_v be an unramified representation of $\mathbf{M}(F_v)$ with the central character ω_{π_v}. Let $\hat{t}$ be the semi-simple conjugacy class in ${}^LM^0$ corresponding to π_v. Then*

$$\pi_v(a(\varpi)) = \omega_{\pi_v}(\varpi) = \hat{a}(\hat{t}).$$

In the following, we will consider the twisted L-function only when it gives rise to a new L-function.

2.1 $B_n - 1$ Case ($A_{n-1}\subset B_n$)

Let $\mathbf{G}=\mathrm{Spin}(2n+1)$ be a split spin group. Let $\theta=\Delta-\{e_n\}$ (This is a standard notation for root system. See, for example, [Bou]). Then $\tilde{\alpha}=\frac{1}{2}(e_1+\cdots+e_n)$. Let $\mathbf{M}=\mathbf{M}_\theta(\supset \mathbf{T})$ be the Levi subgroup of $\mathbf{G}$ generated by θ and let $\mathbf{P}=\mathbf{MN}$ be the corresponding standard parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$: $\mathbf{A}=(\bigcap_{\alpha\in\theta}\ker\alpha)^0=\{a(t) : t\in \overline{F}^*\}$, where

$$
a(t) = \begin{cases} H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-1}}(t^{n-1})H_{\alpha_n}(t^{\frac{n}{2}}) & \text{if } n \text{ is even},\\ H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)\cdots H_{\alpha_{n-1}}(t^{2(n-1)})H_{\alpha_n}(t^n) & \text{if } n \text{ is odd}.\end{cases}
$$

Notice t^2 instead of t when n is odd. Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D\simeq SL_n$. We identify $\mathbf{A}$ with GL_1. We fix an identification of $\mathbf{M}_D$ and SL_n under which the element

$$H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-1}}(t^{n-1})$$

goes to the diagonal element $\operatorname{diag}(t, t, \ldots, t, t^{-(n-1)})$. We define a map $\bar{f}\colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_n$ by

$$\bar{f}\colon (a(t), x) \mapsto \begin{cases} (t^{\frac{n}{2}}, t, x) & \text{if } n \text{ is even,} \\ (t^n, t^2, x) & \text{if } n \text{ is odd.} \end{cases}$$

We define a map $GL_1 \times GL_1 \times SL_n \to GL_1 \times GL_n$ by $(a, b, x) \mapsto (a, bx)$. The composition of the above maps is trivial on the set S, where

$$S = \begin{cases} \{(a(t), tI_n) : t^{\frac{n}{2}} = 1\} & \text{if } n \text{ is even,} \\ \{(a(t), t^2 I_n) : t^n = 1\} & \text{if } n \text{ is odd,} \end{cases}$$

where I_n is the identity matrix in SL_n. Now, $\mathbf{M} \simeq (GL_1 \times SL_n)/S$ via the well-defined map which sends $m = a(t)x$ to $(a(t), x)$ and we obtain a map $f\colon \mathbf{M} \to GL_1 \times GL_n$ so that

$$f(H_{\alpha_n}(t)) = (t, \operatorname{diag}(1, \ldots, 1, t^2)).$$

We can easily see it using the equation $a(t) = H_{\alpha_1}(t) \cdots H_{\alpha_{n-1}}(t^{n-1}) H_{\alpha_n}(t^{\frac{n}{2}})$ if n is even. Under the above identification,

$$H_{\alpha_n}(t) = a(t^{\frac{2}{n}}) \operatorname{diag}(t^{-\frac{2}{n}}, t^{-\frac{2}{n}}, \ldots, t^{-\frac{2}{n}}, t^{\frac{2(n-1)}{n}}).$$

When n is odd, it is similar. We remark that it is independent of the choice of roots of unity which show up.

Let σ be a cuspidal representation of $GL_n(\mathbb{A})$ with the central character ω. Let η be a grössencharacter of F. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map f and σ, η. (More precisely[1], we need to proceed in the following way: $\mathbf{M}(\mathbb{A}_F)$ is co-compact in $GL_n(\mathbb{A}_F)$. Consequently $\sigma \otimes \eta|_{f(M)}$, $M = \mathbf{M}(\mathbb{A}_F)$, decomposes to a direct sum of irreducible representations of M. Let π be any irreducible cuspidal constituent of this direct sum. As we shall see, its choice is irrelevant. In what follows, we will omit this argument.) The central character of π is

$$\omega_\pi = \begin{cases} \omega \eta^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \omega^2 \eta^n & \text{if } n \text{ is odd.} \end{cases}$$

Now suppose σ_v is an unramified representation, given by $\sigma_v = \pi(\mu_1, \ldots, \mu_n)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1 \mu_2^{-1}(t), \ldots, \chi \circ H_{\alpha_{n-1}}(t) = \mu_{n-1} \mu_n^{-1}(t), \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_n}(t)) = (t, \operatorname{diag}(1, \ldots, 1, t^2))$, $\chi \circ H_{\alpha_n}(t) = \mu_n^2 \eta_v$.

Hence, the positive roots $\{e_i + e_j, e_i \text{ for all } i, j\}$ contribute to $L(s, \pi_v, r_1)$ and

$$m = 1; \quad L(s, \pi_v, r_1) = L(s, \sigma_v, \operatorname{Sym}^2 \otimes \eta_v).$$

We obtain the twisted symmetric square L-functions of GL_n.

Remark In [Sh4], Shahidi obtained these twisted symmetric square L-functions only when n is even.

[1] Thanks are due to Prof. Shahidi who pointed this out

2.2 $C_n - 1$ Case

Let $\mathbf{G} = Sp_{2n}$ and $\mathbf{M} = GL_{n-1} \times SL_2$. This is the case when $\theta = \Delta - \{e_{n-1} - e_n\}$. In this case, $\tilde{\alpha} = e_1 + \cdots + e_{n-1}$. It is worth remarking this case because the second L-function in the $F_4 - 1$ case appears as the first L-function in the $C_4 - 1$ case. (This is the case which was excluded in [Sh1, p. 298] and [Sh3, Lemma 4.2]. I thank Prof. F. Shahidi who pointed this out.) Let σ_1 (σ_2, resp.) be a cuspidal representation of $GL_{n-1}(\mathbb{A})$ ($GL_2(\mathbb{A})$, resp.) with a central character ω_1 (ω_2, resp.). Let σ_{20} be any irreducible constituent of $\sigma_2|_{SL_2(\mathbb{A})}$. Then $\pi = \sigma_1 \otimes \sigma_{20}$ is a cuspidal representation of $\mathbf{M}(\mathbb{A})$. Now suppose σ_{iv} is an unramified representation, given by

$$\sigma_{1v} = \pi(\mu_1, \ldots, \mu_{n-1}), \quad \sigma_{2v} = \pi(\nu_1, \nu_2).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$, given by σ_{iv}'s. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \ldots, \chi \circ H_{\alpha_{n-2}}(t) = \mu_{n-2}\mu_{n-1}^{-1}(t),$$
$$\chi \circ H_{\alpha_n}(t) = \nu_1\nu_2^{-1}(t), \chi(a(t)) = \omega_1(t).$$

From this, we can see $\chi \circ H_{\alpha_{n-1}} = \mu_{n-1}\nu_1^{-1}\nu_2$. Hence, we can compute that

$$m = 2, \quad L(s, \pi, r_1) = L(s, \sigma_1 \times Ad(\sigma_2)), \quad L(s, \pi, r_2) = L(s, \sigma_1, \wedge^2\rho_{n-1}),$$

where $Ad(\sigma_2)$ is the Gelbart–Jacquet lift of σ_2, which is an automorphic representation of $GL_3(\mathbb{A})$.

2.3 D_n Cases

2.3.1 $D_n - 1$ ($A_{n-1} \subset D_n$)

Let $\mathbf{G} = \mathrm{Spin}(2n)$ be a split spin group. It is a simply connected group of type D_n. There is a two-to-one map $\mathrm{Spin}(2n) \to SO_{2n}$. Let $\theta = \Delta - \{\alpha_n\}$, where $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-3} = e_{n-3} - e_{n-2}, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n$. Then $\tilde{\alpha} = \frac{1}{2}(e_1 + \cdots + e_n)$. Let $\mathbf{M} = \mathbf{M}_\theta(\supset \mathbf{T})$ be the Levi subgroup of $\mathbf{G}$ generated by θ and let $\mathbf{P} = \mathbf{MN}$ be the corresponding standard parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$: $\mathbf{A} = (\bigcap_{\alpha\in\theta} \ker\,\alpha)^0 = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = \begin{cases} H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-2}}(t^{n-2})H_{\alpha_{n-1}}(t^{\frac{n}{2}-1})H_{\alpha_n}(t^{\frac{n}{2}}) & \text{if } n \text{ is even,} \\ H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)\cdots H_{\alpha_{n-2}}(t^{2(n-2)})H_{\alpha_{n-1}}(t^{n-2})H_{\alpha_n}(t^n) & \text{if } n \text{ is odd.} \end{cases}$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_n$. Now we proceed exactly the same way as in the $B_n - 1$ case; under the identifications, $\mathbf{A}$ with GL_1 and $\mathbf{M}_D$ with SL_n, $\mathbf{M} \simeq (GL_1 \times SL_n)/S$, where

$$S = \begin{cases} \{(a(t), tI_n) : t^{\frac{n}{2}} = 1\} & \text{if } n \text{ is even,} \\ \{(a(t), t^2I_n) : t^n = 1\} & \text{if } n \text{ is odd.} \end{cases}$$

We also construct a map $f\colon \mathbf{M} \to GL_1 \times GL_n$ so that

$$f(H_{\alpha_n}(t)) = (t,\ \mathrm{diag}(1, \dots, 1, t, t)).$$

Let σ be a cuspidal representation of $GL_n(\mathbb{A})$ with the central character ω. Let η be a grössencharacter of F. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map f and σ, η. The central character of π is

$$\omega_\pi = \begin{cases} \omega\eta^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \omega^2\eta^n & \text{if } n \text{ is odd.} \end{cases}$$

Now suppose σ_v is an unramified representation, given by $\sigma_v = \pi(\mu_1, \dots, \mu_n)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \dots, \chi \circ H_{\alpha_{n-1}}(t) = \mu_{n-1}\mu_n^{-1}(t), \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_n}(t)) = (t,\ \mathrm{diag}(1, \dots, 1, t, t))$, $\chi \circ H_{\alpha_n}(t) = \mu_{n-1}\mu_n\eta_v$. Hence, we can compute that

$$m = 1, \quad L(s, \pi_v, r_1) = L(s, \sigma_v, \wedge^2 \otimes \eta_v).$$

We obtain the twisted exterior square L-functions of GL_n.

Remark In [Sh4], Shahidi obtained these twisted exterior square L-functions only when n is even.

2.3.2 $D_n - 2$

Let $\mathbf{G} = \mathrm{Spin}(2n)$ be a split spin group and $\theta = \Delta - \{\alpha_{n-2}\}$. Then $\tilde{\alpha} = e_1 + \cdots + e_{n-2}$. Let $\mathbf{M} = \mathbf{M}_\theta (\supset \mathbf{T})$ be the Levi subgroup of $\mathbf{G}$ generated by θ and let $\mathbf{P} = \mathbf{MN}$ be the corresponding standard parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$: $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = \begin{cases} H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-2}}(t^{n-2})H_{\alpha_{n-1}}(t^{\frac{n-2}{2}})H_{\alpha_n}(t^{\frac{n-2}{2}}) & \text{if } n \text{ is even,} \\ H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)\cdots H_{\alpha_{n-2}}(t^{2(n-2)})H_{\alpha_{n-1}}(t^{n-2})H_{\alpha_n}(t^{n-2}) & \text{if } n \text{ is odd.} \end{cases}$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_{n-2} \times SL_2 \times SL_2$. We identify $\mathbf{A}$ with GL_1. We fix an identification of $\mathbf{M}_D$ and $SL_{n-2} \times SL_2 \times SL_2$ under which the element $H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-3}}(t^{n-3})$ goes to the diagonal element $\mathrm{diag}(t, t, \dots, t, t^{-(n-3)})$ of SL_{n-2}, and $H_{\alpha_{n-1}}(t)$, $H_{\alpha_n}(t)$ go to $\mathrm{diag}(t, t^{-1})$ of SL_2. We define a map $\bar{f}\colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times GL_1 \times SL_{n-2} \times SL_2 \times SL_2$ by

$$\bar{f}\colon (a(t), x, y, z) \mapsto \begin{cases} (t, t^{\frac{n-2}{2}}, t^{\frac{n-2}{2}}, x, y, z) & \text{if } n \text{ is even,} \\ (t^2, t^{n-2}, t^{n-2}, x, y, z) & \text{if } n \text{ is odd.} \end{cases}$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Now, $\mathbf{M} \simeq (GL_1 \times SL_{n-2} \times SL_2 \times SL_2)/S$, where

$$S = \begin{cases} \{(a(t), tI_{n-2}, t^{\frac{n-2}{2}} I_2, t^{\frac{n-2}{2}} I_2) : t^{n-2} = 1\} & \text{if } n \text{ is even,} \\ \{(a(t), t^2 I_{n-2}, t^{n-2} I_2, t^{n-2} I_2) : t^{2(n-2)} = 1\} & \text{if } n \text{ is odd.} \end{cases}$$

We obtain a map $f \colon \mathbf{M} \to GL_{n-2} \times GL_2 \times GL_2$ so that

$$f(H_{\alpha_{n-2}}(t)) = (\operatorname{diag}(1, \dots, 1, t), \operatorname{diag}(1, t), \operatorname{diag}(1, t)).$$

Let π_2, π_3 be two cuspidal representations of GL_2 with central characters ω_2, ω_3, resp. and π_1 be a cuspidal representation of GL_{n-2} with the central character ω_1. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map f and π_1, π_2, π_3. The central character of π is

$$\omega_\pi = \begin{cases} \omega_1 \omega_2^{\frac{n-2}{2}} \omega_3^{\frac{n-2}{2}} & \text{if } n \text{ is even,} \\ \omega_1^2 \omega_2^{n-2} \omega_3^{n-2} & \text{if } n \text{ is odd.} \end{cases}$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \dots, \mu_{n-2}), \quad \pi_{2v} = \pi(\nu_1, \nu_2), \quad \pi_{3v} = \pi(\eta_1, \eta_2).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1 \mu_2^{-1}(t), \dots, \chi \circ H_{\alpha_{n-3}}(t) = \mu_{n-3} \mu_{n-2}^{-1}(t),$$
$$\chi \circ H_{\alpha_{n-1}}(t) = \nu_1 \nu_2^{-1}(t), \quad \chi \circ H_{\alpha_n}(t) = \eta_1 \eta_2^{-1}(t), \quad \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_{n-2}}(t)) = (\operatorname{diag}(1, \dots, 1, t), \operatorname{diag}(1, t), \operatorname{diag}(1, t))$,

$$\chi \circ H_{\alpha_{n-2}}(t) = \mu_{n-2} \nu_2 \eta_2.$$

Hence, we can compute that

$$m = 2, \quad L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}), \quad L(s, \pi_v, r_2) = L(s, \pi_{1v}, \wedge^2 \otimes \omega_2 \omega_3).$$

2.3.3 $D_n - 3$

Let $\mathbf{G} = \operatorname{Spin}(2n)$ be a split spin group and $\theta = \Delta - \{\alpha_{n-3}\}$. Then $\tilde{\alpha} = e_1 + \cdots + e_{n-3}$. Let $\mathbf{M} = \mathbf{M}_\theta (\supset \mathbf{T})$ be the Levi subgroup of $\mathbf{G}$ generated by θ and let $\mathbf{P} = \mathbf{MN}$ be the corresponding standard parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$: $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = \begin{cases} H_{\alpha_1}(t^2) H_{\alpha_2}(t^4) \cdots H_{\alpha_{n-3}}(t^{2(n-3)}) H_{\alpha_{n-2}}(t^{2(n-3)}) H_{\alpha_{n-1}}(t^{n-3}) H_{\alpha_n}(t^{n-3}) \\ \qquad \text{if } n \text{ is even,} \\ H_{\alpha_1}(t) H_{\alpha_2}(t^2) \cdots H_{\alpha_{n-3}}(t^{n-3}) H_{\alpha_{n-2}}(t^{n-3}) H_{\alpha_{n-1}}(t^{\frac{n-3}{2}}) H_{\alpha_n}(t^{\frac{n-3}{2}}) \\ \qquad \text{if } n \text{ is odd.} \end{cases}$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_{n-3} \times SL_4$. We identify $\mathbf{A}$ with GL_1. We fix an identification of $\mathbf{M}_D$ and $SL_{n-3} \times SL_4$ under which the element $H_{\alpha_1}(t)H_{\alpha_2}(t^2)\cdots H_{\alpha_{n-4}}(t^{n-4})$ goes to the diagonal element $\mathrm{diag}(t,t,\ldots,t,t^{-(n-4)})$ of SL_{n-3}, and $H_{\alpha_{n-1}}(t)H_{\alpha_{n-2}}(t^2)H_{\alpha_n}(t)$ goes to $\mathrm{diag}(t,t,t^{-1},t^{-1})$ of SL_4. We define a map $\bar{f}\colon \mathbf{A}\times\mathbf{M}_D \to GL_1\times GL_1\times SL_{n-3}\times SL_4$ by

$$\bar{f}\colon (a(t),x,y) \mapsto \begin{cases} (t^2, t^{n-3}, x, y) & \text{if } n \text{ is even,} \\ (t, t^{\frac{n-3}{2}}, x, y) & \text{if } n \text{ is odd.} \end{cases}$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_{n-3} \times SL_4)/S$, where

$$S = \begin{cases} \{(a(t), t^2 I_{n-3}, t^{n-3} I_4) : t^{2(n-3)} = 1\} & \text{if } n \text{ is even,} \\ \{(a(t), tI_{n-3}, t^{\frac{n-3}{2}} I_4) : t^{n-3} = 1\} & \text{if } n \text{ is odd.} \end{cases}$$

We obtain a map $f\colon \mathbf{M} \to GL_{n-3} \times GL_4$ so that

$$f(H_{\alpha_{n-3}}(t)) = (\mathrm{diag}(1,\ldots,1,t), \mathrm{diag}(1,1,t,t)).$$

Let π_1, π_2 be cuspidal representations of $GL_{n-3}(\mathbb{A})$, $GL_4(\mathbb{A})$ with the central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. The central character of π is

$$\omega_\pi = \begin{cases} \omega_1 \omega_2^{\frac{n-3}{2}} & \text{if } n \text{ is odd,} \\ \omega_1^2 \omega_2^{n-3} & \text{if } n \text{ is even.} \end{cases}$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \ldots, \mu_{n-3}), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3, \nu_4).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \ldots, \chi \circ H_{\alpha_{n-4}}(t) = \mu_{n-4}\mu_{n-3}^{-1}(t),$$
$$\chi \circ H_{\alpha_{n-1}}(t) = \nu_1\nu_2^{-1}(t), \quad \chi \circ H_{\alpha_{n-2}}(t) = \nu_2\nu_3^{-1}(t), \quad \chi \circ H_{\alpha_n}(t) = \nu_3\nu_4^{-1}(t),$$
$$\chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_{n-3}}(t)) = (\mathrm{diag}(1,\ldots,1,t),\ \mathrm{diag}(1,1,t,t))$, $\chi \circ H_{\alpha_{n-3}}(t) = \mu_{n-3}\nu_3\nu_4$. Hence, we can compute that

$$m = 2,\ L(s,\pi_v,r_1) = L(s, \pi_{1v} \times \pi_{2v}, \rho_{n-3} \otimes \wedge^2 \rho_4),\ L(s,\pi_v,r_2) = L(s, \pi_{1v}, \wedge^2 \otimes \omega_2).$$

2.3.4 $D_n - 1$ Case

Dealing with $\mathrm{Spin}(2n)$ in the general case is like dealing with SL_n. The Levi subgroups of $\mathrm{Spin}(2n)$ are very complicated just as in SL_n. The idea of Asgari [As] is to deal with $G\,\mathrm{Spin}(2n)$ and use the restriction technique just as we do for GL_n to SL_n.

We define $G\,\mathrm{Spin}(2n)$ to be the maximal Levi subgroup of $\mathrm{Spin}(2(n+1))$ which has $\mathrm{Spin}(2n)$ as the derived group. More precisely, we add $\alpha_0 = e_0 - e_1$ in the root system and consider the Levi subgroup attached to $\theta = \Delta - \{\alpha_0\}$. Then

$$\mathbf{A} = \left\{ H_{\alpha_0}(t^2)H_{\alpha_1}(t^2)\cdots H_{\alpha_{n-2}}(t^2)H_{\alpha_{n-1}}(t)H_{\alpha_n}(t) : t \in \overline{F}^* \right\},$$

and

$$\mathbf{M}_D = \mathrm{Spin}(2n), \quad \mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_{n-1}}(t)H_{\alpha_n}(t) : t^2 = 1\}.$$

We define

$$G\,\mathrm{Spin}(2n) = (GL_1 \times \mathrm{Spin}(2n))/(\mathbf{A} \cap \mathbf{M}_D).$$

Let $\mathbf{G} = \mathrm{Spin}(2n)$ be a split spin group. Note that the center of $\mathbf{G}$ is

$$Z(G) = \begin{cases} \{\prod_{i=1}^{n-2} H_{\alpha_i}((-1)^i)H_{\alpha_{n-1}}(-t)H_{\alpha_n}(t), \text{ and } H_{\alpha_{n-1}}(t)H_{\alpha_n}(t) : t^2 = 1\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is even}, \\ \{H_{\alpha_1}(t^2)\cdots H_{\alpha_{n-2}}(t^{2(n-2)})H_{\alpha_{n-1}}(t)H_{\alpha_n}(t^3) : t^4 = 1\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is odd}. \end{cases}$$

We set $c = H_{\alpha_{n-1}}(-1)H_{\alpha_n}(-1)$, and

$$z = \begin{cases} \prod_{i=1}^{n-2} H_{\alpha_i}((-1)^i)H_{\alpha_{n-1}}(-1) & \text{if } n \text{ is even}, \\ \prod_{i=1}^{n-2} H_{\alpha_i}((-1)^i)H_{\alpha_{n-1}}(\sqrt{-1})H_{\alpha_n}(\sqrt{-1}) & \text{if } n \text{ is odd}. \end{cases}$$

Note that $c = z^2$ if n is odd. Hence $Z(G) \simeq \mathbb{Z}/4\mathbb{Z}$ if n is odd, and $Z(G) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if n is even. This fact implies that when n is odd, there is, up to isomorphism, a unique non simply-connected, non-adjoint group of type D_n, namely, SO_{2n}. However, when n is even, there are two non-isomorphic, non simply-connected, non-adjoint group of type D_n: one is $SO_{2n} \simeq \mathrm{Spin}(2n)/\{1, c\}$; the other is $HS(2n) \simeq \mathrm{Spin}(2n)/\{1, z\}$, the so-called half-spin group.

Then ${}^L G\,\mathrm{Spin}(2n) = GSO_{2n}(\mathbb{C})$, where

$$GO_{2n} = \{g \in GL_{2n} \,|\, {}^t g J_{2n} g = \lambda(g) J_{2n}, \lambda(g) \in GL_1\}, \quad J_{2n} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}.$$

Let $GSO_{2n} = \{g \in GO_{2n} \mid \det(g)\lambda(g)^{-n} = 1\}$. This fact agrees with Borel's observation [Bo, p. 30] that the derived group of ${}^L G$ is simply connected if and only if the center of G is connected. Note that the center of $G\,\mathrm{Spin}(2n)$ is not connected. It is $\mathbf{A} \cup \mathbf{A}z$.

Let $\theta = \Delta - \{\alpha_k\}$. Let $n = k + l$, $k \geq 2$ and $l \geq 4$. Let $\mathbf{M} = \mathbf{M}_\theta (\supset \mathbf{T})$ be the Levi subgroup of $\mathbf{G}$ generated by θ and let $\mathbf{P} = \mathbf{MN}$ be the corresponding standard parabolic subgroup of $\mathbf{G}$. Let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$: $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = \begin{cases} H_{\alpha_1}(t) \cdots H_{\alpha_k}(t^k) H_{\alpha_{k+1}}(t^k) \cdots H_{\alpha_{n-2}}(t^k) H_{\alpha_{n-1}}(t^{\frac{k}{2}}) H_{\alpha_n}(t^{\frac{k}{2}}) & \text{if } k \text{ is even,} \\ H_{\alpha_1}(t^2) \cdots H_{\alpha_k}(t^{2k}) H_{\alpha_{k+1}}(t^{2k}) \cdots H_{\alpha_{n-2}}(t^{2k}) H_{\alpha_{n-1}}(t^k) H_{\alpha_n}(t^k) & \text{if } k \text{ is odd.} \end{cases}$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_k \times \mathrm{Spin}(2l)$. We identify $\mathbf{A}$ with GL_1. We fix an identification of $\mathbf{M}_D$ and $SL_k \times \mathrm{Spin}(2l)$ under which the element $H_{\alpha_1}(t) H_{\alpha_2}(t^2) \cdots H_{\alpha_{k-1}}(t^{k-1})$ goes to the diagonal element $\mathrm{diag}(t, t, \ldots, t, t^{-(k-1)})$ of SL_k, and

$$b(t) = H_{\alpha_{k+1}}(t^2) \cdots H_{\alpha_{n-2}}(t^2) H_{\alpha_{n-1}}(t) H_{\alpha_n}(t)$$

is the toral element in $\mathrm{Spin}(2l)$. We define a map $\bar{f} \colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_k \times \mathrm{Spin}(2l)$ by

$$\bar{f} \colon (a(t), x, y) \mapsto \begin{cases} (t, t^{\frac{k}{2}}, x, y) & \text{if } k \text{ is even,} \\ (t^2, t^k, x, y), & \text{if } k \text{ is odd.} \end{cases}$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_k \times \mathrm{Spin}(2l))/S$, where

$$S = \begin{cases} \{(a(t), tI_k, b(t^{\frac{k}{2}})) : t^k = 1\} & \text{if } n \text{ is even,} \\ \{(a(t), t^2 I_k, b(t^k)) : t^{2k} = 1\} & \text{if } n \text{ is odd.} \end{cases}$$

We obtain a map $f \colon \mathbf{M} \to GL_k \times G\,\mathrm{Spin}(2l)$ so that

$$f(H_{\alpha_k}(t)) = (\mathrm{diag}(1, \ldots, 1, t),\ c(t)),$$

where $c(t)$ is an element in $G\,\mathrm{Spin}(2l)$.

Let π_1 (π_2, resp.) be a cuspidal representation of GL_k ($G\,\mathrm{Spin}(2l)$, resp.) with the central character ω_1 (ω_2, resp.). Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \begin{cases} \omega_1 \omega_2^{k/2} & \text{if } k \text{ is even,} \\ \omega_1^2 \omega_2^k & \text{if } k \text{ is odd.} \end{cases}$$

Let $\hat{t}_1 = \mathrm{diag}(a_1, \ldots, a_k) \in GL_k(\mathbb{C}) = {}^L GL_k$ and

$$\hat{t}_2 = \mathrm{diag}(b_1, \ldots, b_l, b_l^{-1} b_0, \ldots, b_1^{-1} b_0) \in GSO_{2l}(\mathbb{C}) = {}^L G\,\mathrm{Spin}(2l)$$

be the Satake parameters attached to π_{1v}, π_{2v}, resp. Here we note that

$$\mathrm{diag}(b_1, \ldots, b_l, b_l^{-1} b_0, \ldots, b_1^{-1} b_0) \mapsto b_0$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

generates the character group of GSO_{2n} and hence by Lemma 2.1, $b_0 = \omega_2(\varpi)$. Then

$$\chi \circ H_{\alpha_1} = a_1 a_2^{-1}, \dots, \chi \circ H_{\alpha_{k-1}} = a_{k-1} a_k^{-1},$$
$$\chi \circ H_{\alpha_{k+1}} = b_1 b_2^{-1}, \dots, \chi \circ H_{\alpha_{n-1}} = b_{l-1} b_l^{-1}, \chi \circ H_{\alpha_n} = b_{l-1} b_l b_0^{-1},$$
$$\chi(a(t)) = \omega_{\pi_v} = \begin{cases} (a_1 \cdots a_k)(b_0)^{k/2} & \text{if } k \text{ is even,} \\ (a_1 \cdots a_k)^2 (b_0)^k & \text{if } k \text{ is odd.} \end{cases}$$

Since $f(H_{\alpha_k}(t)) = (\mathrm{diag}(1, \dots, 1, t),\ c(t))$, we can see $\chi \circ H_{\alpha_k} = a_k b_1^{-1} b_0$. Hence, we can compute that $m = 2$,

$$L(s, \pi, r_1) = L(s, \pi_1 \times \pi_2),$$
$$L(s, \pi, r_2) = L(s, \pi_1, \wedge^2 \otimes \omega_2).$$

2.4 F_4 Cases

We use a root system as in [G-O-V]. We take simple roots $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_2 = e_4$, $\alpha_3 = e_3 - e_4$, and $\alpha_4 = e_2 - e_3$. Here $(e_i, e_j) = \delta_{ij}$. The positive roots are $e_i \pm e_j$, $1 \le i < j \le 4$, e_i, $i = 1, 2, 3, 4$ and $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$. There are 24 of them. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

The Dynkin diagram is

$$o_1 \;\text{———}\; o_2 \Longleftarrow o_3 \;\text{———}\; o_4.$$

2.4.1 $F_4 - 1$

Let $\mathbf{G}$ be a split simply connected group of type F_4, and $\theta = \{\alpha_1, \alpha_2, \alpha_4\}$. Let $\mathbf{M} = \mathbf{M}_\theta$ be the Levi subgroup of $\mathbf{G}$ generated by θ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$: $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^2) H_{\alpha_2}(t^4) H_{\alpha_3}(t^6) H_{\alpha_4}(t^3).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_3 \times SL_2$. We identify $\mathbf{A}$ with GL_1. We fix an identification of $\mathbf{M}_D$ and $SL_3 \times SL_2$ under which the element $H_{\alpha_1}(t) H_{\alpha_2}(t^2)$ goes to the diagonal element $\mathrm{diag}(t, t, t^{-2})$ of SL_3, and $H_{\alpha_4}(t)$ goes to $\mathrm{diag}(t, t^{-1})$ of SL_2. We define a map $\bar{f} \colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_3 \times SL_2$ by

$$\bar{f} \colon (a(t), x, y) \mapsto (t^2, t^3, x, y).$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_3 \times SL_2)/S$, where

$$S = \{(a(t), t^2 I_3, t^3 I_2) : t^6 = 1\}.$$

We obtain a map $f \colon \mathbf{M} \to GL_3 \times GL_2$ so that

$$f(H_{\alpha_3}(t)) = (\mathrm{diag}(1, 1, t), \mathrm{diag}(1, t)).$$

We remark that it is independent of the choice of 6th root of unity which shows up.

Let π_1, π_2 be cuspidal representations of GL_3, GL_2 with the central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map f and π_1, π_2. The central character of π is

$$\omega_\pi = \omega_1^2 \omega_2^3.$$

Let π_{iv} be an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \mu_2, \mu_3), \quad \pi_{2v} = \pi(\nu_1, \nu_2).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$ and χ the inducing character of the torus. Then

$$\chi \circ H_{\alpha_1}(t) = \mu_1 \mu_2^{-1}(t); \quad \chi \circ H_{\alpha_2}(t) = \mu_2 \mu_3^{-1}(t),$$
$$\chi \circ H_{\alpha_4}(t) = \nu_1 \nu_2^{-1}(t); \quad \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_3}(t)) = (\mathrm{diag}(1, 1, t), \mathrm{diag}(1, t))$, we have $\chi \circ H_{\alpha_3} = \mu_3 \nu_2$. In this case, $\tilde{\alpha} = 2e_1 + e_2 + e_3$ and the positive roots $\{e_1 - e_2, e_1 - e_3, e_2 \pm e_4, e_3 \pm e_4\}$ contribute to $L(s, \pi_v, r_1)$, and so on. Hence we can compute that $m = 4$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v}), \quad L(s, \pi_v, r_2) = L(s, (\tilde{\pi}_{1v} \otimes \omega_1) \otimes \pi_{2v}, \rho_3 \otimes \mathrm{Sym}^2 \rho_2),$$
$$L(s, \pi_v, r_3) = L(s, \pi_{2v} \otimes \omega_1 \omega_2), \quad L(s, \pi_v, r_4) = L(s, \pi_{1v} \otimes \omega_1 \omega_2^2).$$

2.4.2 $F_4 - 2$

Let $\theta = \{\alpha_1, \alpha_3, \alpha_4\}$, and

$$\mathbf{A} = \{H_{\alpha_1}(t^3) H_{\alpha_2}(t^6) H_{\alpha_3}(t^8) H_{\alpha_4}(t^4) : t \in \bar{F}^*\}.$$

Also $\mathbf{M}_D = SL_2 \times SL_3$, and

$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^3) H_{\alpha_3}(t^2) H_{\alpha_4}(t^4) : t^6 = 1\}.$$

By identifying $\mathbf{A}$ with GL_1, we have

$$\mathbf{M} = (GL_1 \times SL_2 \times SL_3)/(\mathbf{A} \cap \mathbf{M}_D).$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

We do exactly the same as in the the $F_4 - 1$ case: we construct a map $f\colon \mathbf{M} \to GL_2 \times GL_3$ such that $f(H_{\alpha_2}(t)) = (\mathrm{diag}(1,t),\ \mathrm{diag}(1,1,t^2))$. Under the identification, $H_{\alpha_3}(t^2)H_{\alpha_4}(t)$ is the diagonal element $\mathrm{diag}(t,t,t^{-2})$ of SL_3.

Let π_1, π_2 be cuspidal representations of GL_2, GL_3 with the central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^3 \omega_2^4.$$

Let π_{iv} be an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \mu_2), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$ and χ the inducing character of the torus. Then

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_4}(t) = \nu_1\nu_2^{-1}(t),$$
$$\chi \circ H_{\alpha_3}(t) = \nu_2\nu_3^{-1}(t), \quad \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_2}(t)) = (\mathrm{diag}(1,t),\ \mathrm{diag}(1,1,t^2))$, we have $\chi \circ H_{\alpha_2} = \mu_2\nu_3^2$. In this case, $\tilde{\alpha} = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$. Hence we can compute that $m = 3$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \rho_2 \otimes \mathrm{Sym}^2 \rho_3);$$
$$L(s, \pi_v, r_2) = L(s, \tilde{\pi}_{2v}, \mathrm{Sym}^2 \rho_3 \otimes \omega_1\omega_2^2);$$
$$L(s, \pi_v, r_3) = L(s, \pi_{1v} \otimes \omega_1\omega_2^2).$$

2.4.3 (xviii) in [La]

Let $\theta = \{\alpha_2, \alpha_3, \alpha_4\}$, and

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^2)H_{\alpha_2}(t^3)H_{\alpha_3}(t^4)H_{\alpha_4}(t^2) : t \in \bar{F}^*\}.$$

Also $\mathbf{M}_D = \mathrm{Spin}(7)$, and

$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_2}(t) : t^2 = 1\}.$$

By identifying A with GL_1, we have

$$\mathbf{M} = (GL_1 \times \mathrm{Spin}(7))/(\mathbf{A} \cap \mathbf{M}_D) \simeq G\,\mathrm{Spin}(7).$$

Let π be a cuspidal representation of $G\,\mathrm{Spin}(7, \mathbb{A})$ with the central character ω. Let η be a grössencharacter of F. Then we can think of η as a character of $\mathbf{M}(\mathbb{A})$ by setting $\eta(a(t)) = \eta(t^2)$. Since $\eta|_{\mathbf{A}\cap\mathbf{M}_D} = 1$, it is well-defined. We consider $\pi_\eta = \pi \otimes \eta$. Let π_v be the unramified representation of $\mathbf{M}(F_v)$ with the corresponding semisimple conjugacy class $\hat{t}$ in $\hat{T}$, the torus in ${}^LM = GSp_6(\mathbb{C})$. Let

$$\hat{t} = \mathrm{diag}(b_1, b_2, b_3, b_0b_3^{-1}, b_0b_2^{-1}, b_0b_1^{-1}).$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Note that $\hat{t} \mapsto b_0$ generates the character group of GSp_6, and hence by Lemma 2.1, $b_0 = \omega_v(\varpi)$.

Let χ be the inducing character of the torus given by $\pi_{\eta,v}$. We have the relationship

$$\chi \circ \alpha^\vee(\varpi) = \alpha^\vee(\hat{t}),$$

where $\alpha^\vee$ on the right is considered as a root of ${}^LM$. Hence

$$\chi \circ H_{\alpha_2} = b_3^2 b_0^{-1}, \quad \chi \circ H_{\alpha_3} = b_2 b_3^{-1},$$
$$\chi \circ H_{\alpha_4} = b_1 b_2^{-1}, \quad \chi(a(t)) = \eta_v^2 \omega_v = \eta_v^2 b_0.$$

From this, we have $\chi \circ H_{\alpha_1} = \eta_v b_1^{-1} b_2^{-1} b_3^{-1} b_0^2$. In this case, $\tilde{\alpha} = e_1$. Hence we can compute that $m = 2$, and

$$\begin{aligned} L(s, \pi_{\eta,v}, r_1)^{-1} = {} & (1 - \eta_v b_1 b_2 b_3 b_0^{-1} q_v^{-s})\big(1 - \eta_v (b_1 b_2 b_3)^{-1} b_0^2 q_v^{-s}\big) \\ & \times \prod_{i=1}^{3} (1 - \eta_v b_i^{-1} b_0 q_v^{-s}) \prod_{i=1}^{3} (1 - \eta_v (b_1 b_2 b_3)^{-1} b_i^2 b_0 q_v^{-s}) \\ & \times \prod_{i=1}^{3} (1 - \eta_v b_1 b_2 b_3 b_i^{-2} q_v^{-s}) \prod_{i=1}^{3} (1 - \eta_v b_i q_v^{-s}) \\ L(s, \pi_{\eta,v}, r_2) = {} & L(s, \eta_v^2 \omega_v). \end{aligned}$$

Here $L(s, \pi_\eta, r_1)$ is called the spherical harmonic of $Sp_6(\mathbb{C})$ and it has degree 14.

2.4.4 (xxii) in [La]

In this case, it is more convenient to use the dual root system: we take simple roots $\alpha_1 = e_1 - e_2 - e_3 - e_4, \alpha_2 = 2e_4, \alpha_3 = e_3 - e_4, \alpha_4 = e_2 - e_3$. Let $\theta = \{\alpha_2, \alpha_3, \alpha_4\}$. Then

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^2) H_{\alpha_2}(t^3) H_{\alpha_3}(t^2) H_{\alpha_4}(t) : t \in \bar{F}^*\}.$$

Also $\mathbf{M}_D = Sp_6$, and

$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_2}(t) H_{\alpha_4}(t) : t^2 = 1\}.$$

By identifying $\mathbf{A}$ with GL_1, we have

$$\mathbf{M} = (GL_1 \times Sp_6)/(\mathbf{A} \cap \mathbf{M}_D) \simeq GSp_6.$$

We can easily see that under this identification, $H_{\alpha_1}(t)$ becomes $\mathrm{diag}(1, 1, 1, t, t, t)$ in GSp_6.

Let π be a cuspidal representation of $GSp_6(\mathbb{A})$ with the central character ω. Suppose π_v is an unramified representation, given by $\mathrm{Ind}_B^{GSp_6}\ \mu_1 \otimes \mu_2 \otimes \mu_3 \otimes \lambda$, where

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

μ_i's and λ are unramified quasi-characters of $F_v^\times$ and $\mu_1 \otimes \mu_2 \otimes \mu_3 \otimes \lambda$ is the character of the torus which assigns to $\mathrm{diag}(x, y, z, tz^{-1}, ty^{-1}, tx^{-1})$ the value

$$\mu_1(x)\mu_2(y)\mu_3(z)\lambda(t).$$

Note that the central character $\omega = \mu_1\mu_2\mu_3\lambda^2$.

Let η be a grössencharacter of F. Then we can think of η as a character of $\mathbf{M}(\mathbb{A})$ by setting $\eta(a(t)) = \eta(t^2)$. Since $\eta|_{A\cap M_D} = 1$, it is well-defined. We consider $\pi_\eta = \pi \otimes \eta$. Let χ be the inducing character of the torus given by $\pi_{\eta,v}$. Then

$$\chi \circ H_{\alpha_2}(t) = \mu_3(t), \quad \chi \circ H_{\alpha_3}(t) = \mu_2\mu_3^{-1}(t),$$
$$\chi \circ H_{\alpha_4}(t) = \mu_1\mu_2^{-1}(t), \quad \chi(a(t)) = \eta_v^2\omega_v(t).$$

From this, we have $\chi \circ H_{\alpha_1} = \eta_v\lambda$. In this case, $\tilde{\alpha} = 2e_1$. Hence we can compute that $m = 2$, and

$$\begin{aligned}
L(s, \pi_{\eta,v}, r_1)^{-1} &= (1 - \eta_v\lambda q_v^{-s})(1 - \eta_v\lambda\mu_1 q_v^{-s})(1 - \eta_v\lambda\mu_2 q_v^{-s})(1 - \eta_v\lambda\mu_3 q_v^{-s}) \\
&\quad \times (1 - \eta_v\lambda\mu_1\mu_2 q_v^{-s})(1 - \eta_v\lambda\mu_1\mu_3 q_v^{-s}) \\
&\quad \times (1 - \eta_v\lambda\mu_2\mu_3 q_v^{-s})(1 - \eta_v\lambda\mu_1\mu_2\mu_3 q_v^{-s}), \\
L(s, \pi_{\eta,v}, r_2)^{-1} &= (1 - \eta_v^2\omega_v q_v^{-s})(1 - \eta_v^2\omega_v\mu_1 q_v^{-s})(1 - \eta_v^2\omega_v\mu_2 q_v^{-s})(1 - \eta_v^2\omega_v\mu_3 q_v^{-s}) \\
&\quad \times (1 - \eta_v^2\omega_v\mu_1^{-1} q_v^{-s})(1 - \eta_v^2\omega_v\mu_2^{-1} q_v^{-s})(1 - \eta_v^2\omega_v\mu_3^{-1} q_v^{-s}).
\end{aligned}$$

$L(s, \pi_\eta, r_1)$ is called the spin L-function and it has degree 8; $L(s, \pi_\eta, r_2)$ is called the standard L-function of symplectic groups and it has degree 7. It appears as the only L-function in the constant term of the Eisenstein series attached to $\omega\eta^2 \otimes \pi'$ of $GL_1 \times Sp_6 \subset Sp_8$, where π' is any irreducible constituent of $\pi|_{Sp_6(\mathbb{A})}$.

2.5 E_6 Cases

We take the root system as in [G-O-V]. (We decided not to use the root systems for exceptional groups in [Bou] because the root systems in [G-O-V] may be easier for computations.) We take simple roots, $\alpha_i = e_i - e_{i+1}$, $i = 1, 2, 3, 4, 5$, $\alpha_6 = e_4 + e_5 + e_6 + \epsilon$. Here $(e_i, e_i) = \frac{5}{6}$, $(e_i, e_j) = -\frac{1}{6}$ for $i \neq j$, $\sum e_i = 0$, and ϵ is orthogonal to e_i's and $(\epsilon, \epsilon) = \frac{1}{2}$. The positive roots are $e_i - e_j$, $1 \leq i < j \leq 6$, 2ϵ and $e_i + e_j + e_k + \epsilon$. There are 36 of them. Note that

$$(a_1e_1 + a_2e_2 + \cdots + a_6e_6 + a_0\epsilon, e_i - e_j) = a_i - a_j, \quad 1 \leq i < j \leq 6,$$
$$(a_1e_1 + a_2e_2 + \cdots + a_6e_6 + a_0\epsilon, e_i + e_j + e_k + \epsilon) = (a_i + a_j + a_k) - \frac{1}{2}(a_1 + \cdots + a_6) + \frac{1}{2}a_0.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

The Dynkin diagram is

$$\begin{array}{ccccccccc} o_1 & \text{——} & o_2 & \text{——} & o_3 & \text{——} & o_4 & \text{——} & o_5. \\ &&&& \Big| &&&& \\ &&&& o_6 &&&& \end{array}$$

2.5.1 $E_6 - 1$

Let $\mathbf{G}$ be a simply connected group of type E_6. Let $\theta = \Delta - \{\alpha_3\}$. Then $\tilde{\alpha}_3 = e_1 + e_2 + e_3 + 3\epsilon$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = (\bigcap_{\alpha\in\theta} \ker\, \alpha)^0 = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)H_{\alpha_3}(t^6)H_{\alpha_4}(t^4)H_{\alpha_5}(t^2)H_{\alpha_6}(t^3).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_3 \times SL_3 \times SL_2$. We identify $\mathbf{A}$ with GL_1. We fix an identification of $\mathbf{M}_D$ and $SL_3 \times SL_3 \times SL_2$ under which the element $H_{\alpha_1}(t)H_{\alpha_2}(t^2)$ goes to the diagonal element $\operatorname{diag}(t, t, t^{-2})$ of SL_3, $H_{\alpha_4}(t^2)H_{\alpha_5}(t)$ to $\operatorname{diag}(t, t, t^{-2})$ of SL_3, and $H_{\alpha_6}(t)$ to $\operatorname{diag}(t, t^{-1})$ of SL_2. We define a map $\bar{f}\colon \mathbf{A}\times\mathbf{M}_D \to GL_1 \times GL_1 \times GL_1 \times SL_3 \times SL_3 \times SL_2$ by

$$\bar{f}\colon (a(t), x, y, z) \mapsto (t^2, t^2, t^3, x, y, z).$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_3 \times SL_3 \times SL_2)/S$, where

$$S = \{(a(t), t^2 I_3, t^2 I_3, t^3 I_2) : t^6 = 1\}.$$

We obtain a map $f\colon \mathbf{M} \to GL_3 \times GL_3 \times GL_2$ so that

$$f(H_{\alpha_3}(t)) = (\operatorname{diag}(1, 1, t), \operatorname{diag}(1, 1, t), \operatorname{diag}(1, t)).$$

Let π_1, π_2 be cuspidal representations of $GL_3(\mathbb{A})$ with central characters ω_1, ω_2, resp. Let π_3 be a cuspidal representation of $GL_2(\mathbb{A})$ with the central character ω_3. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2, π_3. Then the central character of π is

$$\omega_\pi = \omega_1^2\omega_2^2\omega_3^3.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \mu_2, \mu_3), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3), \quad \pi_{3v} = \pi(\eta_1, \eta_2).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t), \quad \chi \circ H_{\alpha_4}(t) = \nu_2\nu_3^{-1}(t),$$
$$\chi \circ H_{\alpha_5}(t) = \nu_1\nu_2^{-1}(t), \quad \chi \circ H_{\alpha_6}(t) = \eta_1\eta_2^{-1}(t), \quad \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_3}(t)) = (\mathrm{diag}(1,1,t),\ \mathrm{diag}(1,1,t),\ \mathrm{diag}(1,t))$, we have $\chi \circ H_{\alpha_3}(t) = \mu_3\nu_3\eta_2$. Hence, we can compute that $m = 3$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}),$$
$$L(s, \pi_v, r_2) = L(s, (\tilde{\pi}_{1v} \otimes \omega) \times \tilde{\pi}_{2v}),$$
$$L(s, \pi_v, r_3) = L(s, \pi_{3v} \otimes \omega),$$

where $\omega = \omega_1\omega_2\omega_3$.

2.5.2 $E_6 - 2$

Let $\theta = \Delta - \{\alpha_2\}$. Then $\tilde{\alpha}_2 = e_1 + e_2 + 2\epsilon$.

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^5)H_{\alpha_2}(t^{10})H_{\alpha_3}(t^{12})H_{\alpha_4}(t^8)H_{\alpha_5}(t^4)H_{\alpha_6}(t^6) : t \in \overline{F}^*\},$$

and $\mathbf{M}_D \simeq SL_2 \times SL_5$. We fix an identification of $\mathbf{M}_D$ and $SL_2 \times SL_5$ under which the element $H_{\alpha_1}(t)$ goes to $\mathrm{diag}(t, t^{-1})$ of SL_2, and $H_{\alpha_5}(t^4)H_{\alpha_4}(t^8)H_{\alpha_3}(t^{12})H_{\alpha_6}(t^6)$ goes to $\mathrm{diag}(t^4, t^4, t^4, t^{-6}, t^{-6})$ in SL_5. We define a map $\bar{f} \colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_2 \times SL_5$ by

$$\bar{f} \colon (a(t), x, y) \mapsto (t^5, t^4, x, y).$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_2 \times SL_5)/S$, where

$$S = \{(a(t), tI_2, t^4I_5) : t^{10} = 1\}.$$

We obtain a map $f \colon \mathbf{M} \to GL_2 \times GL_5$ so that

$$f(H_{\alpha_2}(t)) = (\mathrm{diag}(1,t),\ \mathrm{diag}(1,1,1,t,t)).$$

Let π_1, π_2 be cuspidal representations of GL_2, GL_5 with central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^5\omega_2^4.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\eta_1, \eta_2), \quad \pi_{2v} = \pi(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1} = \eta_1\eta_2^{-1}, \quad \chi \circ H_{\alpha_5} = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_4} = \mu_2\mu_3^{-1},$$
$$\chi \circ H_{\alpha_3}(t) = \mu_3\mu_4^{-1}, \quad \chi \circ H_{\alpha_6}(t) = \mu_4\mu_5^{-1}, \quad \chi(a(t)) = \omega_1^5\omega_2^4(t).$$

Since $f(H_{\alpha_2}(t)) = (\mathrm{diag}(1,t), \mathrm{diag}(1,1,1,t,t))$, we can see

$$\chi \circ H_{\alpha_2} = \eta_2\mu_4\mu_5.$$

Hence, we can compute that $m = 2$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \rho_2 \otimes \wedge^2\rho_5),$$
$$L(s, \pi_v, r_2) = L(s, \omega_1\omega_2 \otimes \tilde{\pi}_{2v}).$$

2.5.3 (x) in [La]

Let $\theta = \Delta - \{\alpha_6\}$. Then $\tilde{\alpha}_6 = 2\epsilon$. $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t)H_{\alpha_2}(t^2)H_{\alpha_3}(t^3)H_{\alpha_4}(t^2)H_{\alpha_5}(t)H_{\alpha_6}(t^2),$$

and $\mathbf{M}_D \simeq SL_6$. We fix an identification of $\mathbf{M}_D$ and SL_6 under which the element $H_{\alpha_1}(t)H_{\alpha_2}(t^2)H_{\alpha_3}(t^3)H_{\alpha_4}(t^2)H_{\alpha_5}(t)$ goes to the diagonal element

$$\mathrm{diag}(t,t,t,t^{-1},t^{-1},t^{-1})$$

of SL_6. We define a map $\bar{f}\colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_6$ by

$$\bar{f}\colon (a(t), x) \mapsto (t^2, t, x).$$

Now, $\mathbf{M} \simeq (GL_1 \times SL_6)/S$, where

$$S = \{(a(t), tI_6) : t^2 = 1\}.$$

We obtain a map $f\colon \mathbf{M} \to GL_1 \times GL_6$ so that

$$f(H_{\alpha_6}(t)) = (t, \mathrm{diag}(1,1,1,t,t,t)).$$

Let σ be cuspidal representations of $GL_6(\mathbb{A})$ with central character ω, and η be a grössencharacter of F. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and σ, η. Then the central character of π is

$$\omega_\pi = \omega\eta^2.$$

Now suppose σ_v is an unramified representation, given by

$$\sigma_v = \pi(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$ and let χ be the inducing character of the torus. We have

$$\chi \circ H_{\alpha_1} = \mu_1\mu_2^{-1}, \quad \chi \circ H_{\alpha_2} = \mu_2\mu_3^{-1}(t), \quad \chi \circ H_{\alpha_3} = \mu_3\mu_4^{-1},$$
$$\chi \circ H_{\alpha_4}(t) = \mu_4\mu_5^{-1}, \quad \chi \circ H_{\alpha_5}(t) = \mu_5\mu_6^{-1}, \quad \chi(a(t)) = \omega_v\eta_v^2(t).$$

Since $f(H_{\alpha_6}(t)) = (t, \operatorname{diag}(1,1,1,t,t,t))$, we have $\chi \circ H_{\alpha_6} = \mu_4\mu_5\mu_6\eta_v$. Hence, we can compute that $m = 2$, and

$$L(s,\pi_v,r_1) = L(s,\sigma_v,\wedge^3\rho_6 \otimes \eta_v) = \prod_{1\le i<j<k\le 6} (1-\mu_i\mu_j\mu_k\eta_v q_v^{-s})^{-1},$$
$$L(s,\pi_v,r_2) = L(s,\omega_v\eta_v^2).$$

Here $L(s,\pi,r_1)$ is the exterior cube L-function of GL_6 and it has degree 20.

2.5.4 (xxiv) in [La]

Let $\theta = \Delta - \{\alpha_1\}$. Then $\tilde{\alpha}_1 = e_1 + \epsilon$, and $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^4)H_{\alpha_2}(t^5)H_{\alpha_3}(t^6)H_{\alpha_4}(t^4)H_{\alpha_5}(t^2)H_{\alpha_6}(t^3).$$

Also $\mathbf{M}_D \simeq \operatorname{Spin}(10)$ and

$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_2}(t)H_{\alpha_3}(t^2)H_{\alpha_5}(t^2)H_{\alpha_6}(t^3) : t^4 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times \operatorname{Spin}(10))/(\mathbf{A} \cap \mathbf{M}_D).$$

Since $G\operatorname{Spin}(10) = (GL_1 \times \operatorname{Spin}(10))/\{1,c\}$ (see Section 2.3.4), there is a surjective map $G\operatorname{Spin}(10) \to \mathbf{M}$. Hence we have a dual map ${}^LM \to GSO_{10}(\mathbb{C}) = {}^LG\operatorname{Spin}(10)$. Since the center of $\mathbf{M}$ is connected, the derived group of ${}^LM$ is simply connected, (see [Bo, p. 30]). Hence it is $\operatorname{Spin}(10,\mathbb{C})$. Therefore ${}^LM = G\operatorname{Spin}(10,\mathbb{C})$.

Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$ with central character ω. Let π_v be the unramified representation of $\mathbf{M}(F_v)$ with the corresponding semisimple conjugacy class $\hat{t}$ in $\hat{T}$, the torus in ${}^LM$. We have a 2-to-1 map $\phi\colon {}^LM \to GSO_{10}(\mathbb{C})$. Let $\phi(\hat{t})$ be given by

$$\phi(\hat{t}) = \operatorname{diag}(b_1^2,\dots,b_5^2, b_0^2b_5^{-2},\dots,b_0^2b_1^{-2}).$$

Note that it is the Satake parameter for the representation π_v' of $G\operatorname{Spin}(10,F_v)$, where $\pi' = \bigotimes_v \pi_v'$ is the cuspidal representation of $G\operatorname{Spin}(10,\mathbb{A})$, induced by π and the map $G\operatorname{Spin}(10) \to \mathbf{M}$. Note also that $\omega_\pi = \omega_{\pi'}$, and hence $\omega_{\pi_v} = b_0^2$.

Let η be a grössencharacter of F. Then we can think of η as a character of $\mathbf{M}(\mathbb{A})$ by setting $\eta(a(t)) = \eta(t^4)$. Since $\eta|_{\mathbf{A}\cap\mathbf{M}_D} = 1$, it is well-defined. We consider $\pi_\eta = \pi \otimes \eta$. Let χ be the inducing character of the torus attached to $\pi_{\eta,v}$. Then we have the relationship

$$\chi \circ \alpha^\vee(\varpi) = \alpha^\vee(\hat{t}),$$

where $\alpha^\vee$ on the right is considered as a root of ${}^LM$. Hence

$$\chi\circ H_{\alpha_2}=b_4^2b_5^2b_0^{-2},\quad \chi\circ H_{\alpha_3}=b_3^2b_4^{-2},\quad \chi\circ H_{\alpha_4}=b_2^2b_3^{-2},$$
$$\chi\circ H_{\alpha_5}=b_1^2b_2^{-2},\quad \chi\circ H_{\alpha_6}=b_4^2b_5^{-2},\quad \chi(a(t))=\eta_v^4\omega_v=\eta_v^4b_0^2.$$

From this, we have $\chi\circ H_{\alpha_1}=\eta_v(b_1b_2b_3b_4b_5)^{-1}b_0^3$.

Hence, we can compute that $m=1$, and

$$L(s,\pi_{\eta,v},r_1)^{-1}=(1-\eta_v(b_1b_2b_3b_4b_5)^{-1}b_0^3q_v^{-s})\prod_{i=1}^{5}(1-\eta_vb_1b_2b_3b_4b_5b_0^{-1}b_i^{-2}q_v^{-s})$$
$$\times\prod_{1\le i<j\le 5}(1-\eta_v(b_1b_2b_3b_4b_5)^{-1}b_0(b_ib_j)^2q_v^{-s}).$$

Here r_1 is called the half-spin representation and it has degree 16. We denote it by Spin^{16}. For a future reference, we denote $\mathbf{M}=H\,\mathrm{Spin}(10)$.

2.6 E_7 Cases

We take the root system as in [G-O-V]. We take simple roots, $\alpha_i=e_i-e_{i+1}$, $i=1,2,3,4,5,6$, $\alpha_7=e_5+e_6+e_7+e_8$. Here $(e_i,e_i)=\frac{7}{8}$, $(e_i,e_j)=-\frac{1}{8}$ for $1\le i\ne j\le 8$ and $\sum e_i=0$. The positive roots are e_i-e_j, $1\le i<j\le 7$, $-e_i+e_8$, $i=1,\dots,7$, and $e_i+e_j+e_k+e_8$. There are 63 of them. Note that

$$(a_1e_1+a_2e_2+\cdots+a_8e_8,e_i-e_j)=a_i-a_j,$$
$$(a_1e_1+a_2e_2+\cdots+a_8e_8,e_i+e_j+e_k+e_8)=(a_i+a_j+a_k+a_8)-\frac{1}{2}(a_1+\cdots+a_8).$$

The Cartan matrix is

$$\begin{pmatrix} 2&-1&0&0&0&0&0\\ -1&2&-1&0&0&0&0\\ 0&-1&2&-1&0&0&0\\ 0&0&-1&2&-1&0&-1\\ 0&0&0&-1&2&-1&0\\ 0&0&0&0&-1&2&0\\ 0&0&0&-1&0&0&2 \end{pmatrix}.$$

The Dynkin diagram is

$$\begin{array}{ccccccccccc} o_1 & \text{———} & o_2 & \text{———} & o_3 & \text{———} & o_4 & \text{———} & o_5 & \text{———} & o_6. \\ &&&&&& | &&&& \\ &&&&&& o_7 &&&& \end{array}$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

2.6.1 $E_7 - 1$

Let $\mathbf{G}$ be a simply connected group of type E_7. Let $\theta = \Delta - \{\alpha_4\}$. Then $\tilde{\alpha}_4 = e_1 + e_2 + e_3 + e_4 + 4e_8$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = (\bigcap_{\alpha\in\theta} \ker\alpha)^0 = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^3)H_{\alpha_2}(t^6)H_{\alpha_3}(t^9)H_{\alpha_4}(t^{12})H_{\alpha_5}(t^8)H_{\alpha_6}(t^4)H_{\alpha_7}(t^6).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D \simeq SL_2 \times SL_3 \times SL_4$.

Now we proceed exactly the same way as in the $E_6 - 1$ case; under the identification of $\mathbf{M}_D$ with $SL_2 \times SL_3 \times SL_4$, $\mathbf{M} \simeq (GL_1 \times SL_2 \times SL_3 \times SL_4)/S$, where

$$S = \{(a(t), t^6 I_2, t^4 I_3, t^3 I_4) : t^{12} = 1\}.$$

We also construct a map $f\colon \mathbf{M} \to GL_2 \times GL_3 \times GL_4$ so that

$$f(H_{\alpha_4}(t)) = (\mathrm{diag}(1,t),\ \mathrm{diag}(1,1,t),\ \mathrm{diag}(1,1,1,t)).$$

Let π_i be cuspidal representations of GL_{1+i} with central characters ω_i, $i = 1,2,3$, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by the map f and π_1, π_2, π_3. The central character of π is

$$\omega_\pi = \omega_1^6\omega_2^4\omega_3^3.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\eta_1, \eta_2), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3), \quad \pi_{3v} = \pi(\mu_1, \mu_2, \mu_3, \mu_4).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi\circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi\circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t), \quad \chi\circ H_{\alpha_3}(t) = \mu_3\mu_4^{-1}(t),$$
$$\chi\circ H_{\alpha_5}(t) = \nu_2\nu_3^{-1}(t), \quad \chi\circ H_{\alpha_6}(t) = \nu_1\nu_2^{-1}(t),$$
$$\chi\circ H_{\alpha_7}(t) = \eta_1\eta_2^{-1}(t), \quad \chi(a(t)) = \omega_1^6\omega_2^4\omega_3^3(t).$$

Since $f(H_{\alpha_4}(t)) = (\mathrm{diag}(1,t), \mathrm{diag}(1,1,t), \mathrm{diag}(1,1,1,t))$, we have $\chi\circ H_{\alpha_4}(t) = \mu_4\nu_3\eta_2$. Hence, we can compute that $m = 4$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}),$$
$$L(s, \pi_v, r_2) = L(s, \tilde{\pi}_{2v} \otimes \pi_{3v}, (\rho_3 \otimes \omega_1\omega_2) \otimes \wedge^2\rho_4),$$
$$L(s, \pi_v, r_3) = L(s, (\pi_{1v} \otimes \omega_1\omega_2\omega_3) \times \tilde{\pi}_{3v}),$$
$$L(s, \pi_v, r_4) = L(s, \pi_{2v} \otimes \omega_1^2\omega_2\omega_3).$$

2.6.2 $E_7 - 2$

Let $\theta = \Delta - \{\alpha_3\}$. Then $\tilde{\alpha}_3 = e_1 + e_2 + e_3 + 3e_8$, $\mathbf{M}_D = SL_3 \times SL_5$,

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^5)H_{\alpha_2}(t^{10})H_{\alpha_3}(t^{15})H_{\alpha_4}(t^{18})H_{\alpha_5}(t^{12})H_{\alpha_6}(t^6)H_{\alpha_7}(t^9) : t \in \overline{F}^*\},$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^5)H_{\alpha_2}(t^{10})H_{\alpha_4}(t^3)H_{\alpha_5}(t^{12})H_{\alpha_6}(t^6)H_{\alpha_7}(t^9) : \ t^{15} = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_3 \times SL_5)/(\mathbf{A} \cap \mathbf{M}_D).$$

We proceed exactly in the same way as in the E_6-2 case, and construct a map $f\colon \mathbf{M} \to GL_3 \times GL_5$ such that

$$f(H_{\alpha_3}(t)) = (\operatorname{diag}(1, 1, t), \operatorname{diag}(1, 1, 1, t, t)).$$

Let π_1, π_2 be cuspidal representations of $GL_3(\mathbb{A}), GL_5(\mathbb{A})$ with central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. The central character of π is

$$\omega_\pi = \omega_1^5\omega_2^6.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \mu_2, \mu_3), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t),$$
$$\chi \circ H_{\alpha_6}(t) = \nu_1\nu_2^{-1}(t), \quad \chi \circ H_{\alpha_5}(t) = \nu_2\nu_3^{-1}(t), \quad \chi \circ H_{\alpha_4}(t) = \nu_3\nu_4^{-1}(t),$$
$$\chi \circ H_{\alpha_7}(t) = \nu_4\nu_5^{-1}(t), \quad \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_3}(t)) = (\operatorname{diag}(1, 1, t), \operatorname{diag}(1, 1, 1, t, t))$, we have $\chi \circ H_{\alpha_3}(t) = \mu_3\nu_4\nu_5$. Hence, we can compute that $m = 3$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \rho_3 \otimes \wedge^2\rho_5),$$
$$L(s, \pi_v, r_2) = L(s, (\tilde{\pi}_{1v} \otimes \omega_{1v}) \times (\tilde{\pi}_{2v} \otimes \omega_{2v})),$$
$$L(s, \pi_v, r_3) = L(s, \pi_{2v} \otimes (\omega_1\omega_2)).$$

2.6.3 $E_7 - 3$

Let $\theta = \Delta - \{\alpha_2\}$, $\tilde{\alpha}_2 = e_1 + e_2 + 2e_8$. $\mathbf{M}_D = SL_2 \times \operatorname{Spin}(10)$.

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)H_{\alpha_3}(t^5)H_{\alpha_4}(t^6)H_{\alpha_5}(t^4)H_{\alpha_6}(t^2)H_{\alpha_7}(t^3) : t \in \overline{F}^*\},$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^2)H_{\alpha_3}(t)H_{\alpha_4}(t^2)H_{\alpha_6}(t^2)H_{\alpha_7}(t^3) : \ t^4 = 1\}.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_2 \times \mathrm{Spin}(10))/(\mathbf{A} \cap \mathbf{M}_D).$$

Here we note that $H_{\alpha_2}(t^4)H_{\alpha_3}(t^5)H_{\alpha_4}(t^6)H_{\alpha_5}(t^4)H_{\alpha_6}(t^2)H_{\alpha_7}(t^3)$ is exactly the same as the center of $\mathbf{M} = H\,\mathrm{Spin}(10)$ in Section 2.5.4.

We define a map $\bar{f}\colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_2 \times \mathrm{Spin}(10)$ by

$$\bar{f}\colon (a(t), x, y) \mapsto (t^2, t, x, y).$$

It induces a map $f\colon \mathbf{M} \to GL_2 \times H\,\mathrm{Spin}(10)$. Under the identification, $H_{\alpha_1}(t)$ is the diagonal element $\mathrm{diag}(t, t^{-1})$ in SL_2, $H_{\alpha_3}(t^5)H_{\alpha_4}(t^6)H_{\alpha_5}(t^4)H_{\alpha_6}(t^2)$ is in $\mathrm{Spin}(10)$. From this, we see that $f(H_{\alpha_2}(t)) = (\mathrm{diag}(1, t), b(t))$, where $b(t)$ is an element in $H\,\mathrm{Spin}(10)$.

Let π_1, π_2 be cuspidal representations of $GL_2, H\,\mathrm{Spin}(10)$ with central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^2 \omega_2.$$

Let $\hat{t}_1 = \mathrm{diag}(a_1, a_2) \in GL_2(\mathbb{C})$ be the Satake parameter attached to π_{1v}. Let $\hat{t}_2 \in G\,\mathrm{Spin}(10, \mathbb{C})$ be the Satake parameter attached to π_{2v}. Using the 2-to-1 map $\phi\colon G\,\mathrm{Spin}(10, \mathbb{C}) \to GSO(10, \mathbb{C})$, we can write it as

$$\phi(\hat{t}_2) = \mathrm{diag}(b_1^2, \dots, b_5^2, b_5^{-2}b_0^2, \dots, b_1^{-2}b_0^2) \in GSO_{10}(\mathbb{C}).$$

Then

$$\chi \circ H_{\alpha_1} = a_1 a_2^{-1}, \quad \chi \circ H_{\alpha_6} = b_1^2 b_2^{-2}, \quad \chi \circ H_{\alpha_5} = b_2^2 b_3^{-2},$$
$$\chi \circ H_{\alpha_4} = b_3^2 b_4^{-2}, \quad \chi \circ H_{\alpha_3} = b_4^2 b_5^2 b_0^{-2}, \quad \chi \circ H_{\alpha_7} = b_4^2 b_5^{-2},$$
$$\chi(a(t)) = \omega_1^2 \omega_2 = (a_1 a_2)^2 b_0^2.$$

Since $f(H_{\alpha_2}(t)) = (\mathrm{diag}(1, t),\, b(t))$, we can see $\chi \circ H_{\alpha_2} = a_2 (b_1 b_2 b_3 b_4 b_5)^{-1} b_0^3$. Hence, we can compute that $m = 2$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \rho_2 \otimes \mathrm{Spin}^{16}),$$

$$L(s, \pi_v, r_2) = L(s, \pi_{2v}' \otimes \omega_1) = \prod_{i=1}^{5} (1 - b_i^2 \omega_1 q_v^{-s})^{-1} (1 - b_i^{-2} b_0^2 \omega_1 q_v^{-s})^{-1},$$

where Spin^{16} is the degree 16 half-spin representation (see Section 2.5.4). Here π_2' is the cuspidal representation of $G\,\mathrm{Spin}(10)$, induced by π_2 and the 2-to-1 map $G\,\mathrm{Spin}(10) \to H\,\mathrm{Spin}(10)$. Hence the Satake parameter of π_{2v}' is

$$\phi(\hat{t}_2) = \mathrm{diag}(b_1^2, \dots, b_5^2, b_5^{-2}b_0^2, \dots, b_1^{-2}b_0^2) \in GSO_{10}(\mathbb{C}) = {}^L G\,\mathrm{Spin}(10).$$

Note that the second L-function is the standard L-function for $G\,\mathrm{Spin}(10)$.

2.6.4 $E_7 - 4$

Let $\theta = \Delta - \{\alpha_5\}$, $\tilde{\alpha}_5 = e_1 + e_2 + e_3 + e_4 + e_5 + 3e_8$, $\mathbf{M}_D = SL_6 \times SL_2$,

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)H_{\alpha_3}(t^6)H_{\alpha_4}(t^8)H_{\alpha_5}(t^6)H_{\alpha_6}(t^3)H_{\alpha_7}(t^4) : t \in \overline{F}^*\},$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)H_{\alpha_4}(t^2)H_{\alpha_6}(t^3)H_{\alpha_7}(t^4) : t^6 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_6 \times SL_2)/(\mathbf{A} \cap \mathbf{M}_D).$$

As in the $E_7 - 2$ case, we construct a map $f\colon M \to GL_6 \times GL_2$ such that

$$f(H_{\alpha_5}(t)) = (\operatorname{diag}(1,1,1,1,t,t), \operatorname{diag}(1,t)).$$

Let π_1, π_2 be cuspidal representations of $GL_6(\mathbb{A}), GL_2(\mathbb{A})$ with central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^2\omega_2^3.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6), \quad \pi_{2v} = \pi(\nu_1, \nu_2).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t), \quad \chi \circ H_{\alpha_3}(t) = \mu_3\mu_4^{-1}(t),$$
$$\chi \circ H_{\alpha_4}(t) = \mu_4\mu_5^{-1}(t), \quad \chi \circ H_{\alpha_7}(t) = \mu_5\mu_6^{-1}(t), \quad \chi \circ H_{\alpha_6}(t) = \nu_1\nu_2^{-1}(t),$$
$$\chi(a(t)) = \omega_1^2\omega_2^3(t).$$

Since $f(H_{\alpha_5}(t)) = (\operatorname{diag}(1,1,1,1,t,t), \operatorname{diag}(1,t))$, $\chi \circ H_{\alpha_5}(t) = \mu_5\mu_6\nu_2$. Hence, we can compute that $m = 3$, and

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \wedge^2\rho_6 \otimes \rho_2),$$
$$L(s, \pi_v, r_2) = L(s, \tilde{\pi}_{1v}, \wedge^2\rho_6 \otimes (\omega_1\omega_2)),$$
$$L(s, \pi_v, r_3) = L(s, \pi_{2v} \otimes (\omega_1\omega_2)).$$

2.6.5 (xi) in [La]

Let $\theta = \Delta - \{\alpha_7\}$, $\tilde{\alpha}_3 = 2e_8$, $\mathbf{M}_D = SL_7$,

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t^3)H_{\alpha_2}(t^6)H_{\alpha_3}(t^9)H_{\alpha_4}(t^{12})H_{\alpha_5}(t^8)H_{\alpha_6}(t^4)H_{\alpha_7}(t^7) : t \in \overline{F}^*\},$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^3)H_{\alpha_2}(t^6)H_{\alpha_3}(t^2)H_{\alpha_4}(t^5)H_{\alpha_5}(t)H_{\alpha_6}(t^4) : \, t^7 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_7)/(\mathbf{A} \cap \mathbf{M}_D).$$

As in the **(x)** case (Section 2.5.3), we construct a map $f \colon \mathbf{M} \to GL_1 \times GL_7$ such that

$$f(H_{\alpha_7}(t)) = (t, \operatorname{diag}(1, 1, 1, 1, t, t, t)).$$

Let σ be a cuspidal representation of $GL_7(\mathbb{A})$ with the central character ω. Let η be a grössencharacter of F. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and σ, η. Then the central character of π is

$$\omega_\pi = \omega^2\eta^7.$$

Now suppose σ_v is an unramified representation, given by

$$\sigma_v = \pi(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7).$$

Let χ be the character of the torus, given by π_v. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t), \quad \chi \circ H_{\alpha_3}(t) = \mu_3\mu_4^{-1},$$
$$\chi \circ H_{\alpha_4}(t) = \mu_4\mu_5^{-1}, \quad \chi \circ H_{\alpha_5}(t) = \mu_5\mu_6^{-1}, \quad \chi \circ H_{\alpha_6}(t) = \mu_6\mu_7^{-1},$$
$$\chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_7}(t)) = (t, \operatorname{diag}(1, 1, 1, 1, t, t, t))$, $\chi \circ H_{\alpha_7}(t) = \mu_5\mu_6\mu_7\eta_v$. Hence, we can compute that $m = 2$, and

$$L(s, \pi_v, r_1) = L(s, \sigma_v, \wedge^3\rho_7 \otimes \eta_v) = \prod_{1 \le i < j < k \le 7} (1 - \mu_i\mu_j\mu_k\eta_v q_v^{-s})^{-1},$$
$$L(s, \pi_v, r_2) = L(s, \tilde{\sigma}_v \otimes (\omega_v\eta_v^2)).$$

Here $L(s, \pi, r_1)$ is the exterior cube L-function of GL_7 and it has degree 35.

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

2.6.6 (xxvi) in [La]

Let $\theta = \Delta - \{\alpha_6\}, \tilde{\alpha}_6 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + 2e_8, \mathbf{M}_D = \mathrm{Spin}(12)$,

$$\mathbf{A} = \{a(t) = H_{\alpha_1}(t)H_{\alpha_2}(t^2)H_{\alpha_3}(t^3)H_{\alpha_4}(t^4)H_{\alpha_5}(t^3)H_{\alpha_6}(t^2)H_{\alpha_7}(t^2) : t \in \overline{F}^*\},$$

$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t)H_{\alpha_3}(t)H_{\alpha_5}(t)) : t^2 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times \mathrm{Spin}(12))/(\mathbf{A} \cap \mathbf{M}_D).$$

Here note that $\mathbf{M}$ is *not* isomorphic to $G\,\mathrm{Spin}(12)$. In the notation of Section 2.3.4, $\mathbf{A} \cap \mathbf{M}_D = \{1, z\}$. On the other hand, $G\,\mathrm{Spin}(12) = GL_1 \times \mathrm{Spin}(12)/\{1, c\}$. Hence ${}^L M$ is *not* $GSO_{12}(\mathbb{C})$. The derived group of ${}^L M$ is the half-spin group $HS(12, \mathbb{C})$ (the other non simply-connected, non-adjoint group in the notation of Section 2.3.4).

Let $H\,\mathrm{Spin}(12) = (GL_1 \times \mathrm{Spin}(12))/\{1, c, z, cz\}$. Then there are 2-to-1 maps $f\colon \mathbf{M} \to H\,\mathrm{Spin}(12)$ and $G\,\mathrm{Spin}(12) \to H\,\mathrm{Spin}(12)$. Since the center of $H\,\mathrm{Spin}(12)$ is connected, the derived group of ${}^L H\,\mathrm{Spin}(12)$ is simply connected, namely, $\mathrm{Spin}(12, \mathbb{C})$ [Bo, p. 30]. Therefore, ${}^L H\,\mathrm{Spin}(12) = G\,\mathrm{Spin}(12, \mathbb{C})$. We have 2-to-1 maps ${}^L f\colon {}^L H\,\mathrm{Spin}(12) \to {}^L M$ and $\phi\colon {}^L H\,\mathrm{Spin}(12) \to {}^L G\,\mathrm{Spin}(12) = GSO_{12}\mathbb{C}$.

Let π' be a generic cuspidal representation of $H\,\mathrm{Spin}(12, \mathbb{A})$ with central character ω. Let π'_v be the unramified representation of $H\,\mathrm{Spin}(12, F_v)$ with the corresponding semi-simple conjugacy class $\hat{t}$ in $\hat{T}$, the torus in ${}^L H\,\mathrm{Spin}(12)$. Using the 2-to-1 map $\phi\colon G\,\mathrm{Spin}(12, \mathbb{C}) \to GSO_{12}(\mathbb{C})$, we can write

$$\phi(\hat{t}) = \mathrm{diag}(b_1^2, \dots, b_6^2, b_0^2 b_6^{-2}, \dots, b_0^2 b_1^{-2}).$$

Now let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by π' and f. The Satake parameter of π_v is ${}^L f(\hat{t})$ in ${}^L M$. Note that the central character of π is $\omega_\pi = \omega$. Let η be a grössencharacter of F. Then we can think of η as a character of $\mathbf{M}(\mathbb{A})$ by setting $\eta(a(t)) = \eta(t^2)$. Since $\eta|_{A \cap M_D} = 1$, it is well-defined. We consider $\pi_\eta = \pi \otimes \eta$. Let χ be the inducing character of the torus attached to $\pi_{\eta,v}$. Then

$$\chi \circ H_{\alpha_1} = b_1^2 b_2^{-2}, \quad \chi \circ H_{\alpha_2} = b_2^2 b_3^{-2}, \quad \chi \circ H_{\alpha_3} = b_3^2 b_4^{-2},$$

$$\chi \circ H_{\alpha_4} = b_4^2 b_5^{-2}, \quad \chi \circ H_{\alpha_7} = b_5^2 b_6^{-2}, \quad \chi \circ H_{\alpha_5} = b_5^2 b_6^2 b_0^{-2},$$

$$\chi(a(t)) = \eta_v^2 \omega_v = \eta_v^2 b_0^2.$$

From this, we have $\chi \circ H_{\alpha_6} = \eta_v (b_1 \cdots b_6)^{-1} b_0^4$. Hence, we can compute that $m = 2$, and

$$\begin{aligned} L(s, \pi_{\eta,v}, r_1)^{-1} &= (1 - \eta_v b_1^{-1} \cdots b_6^{-1} b_0^4 q_v^{-s})(1 - \eta_v b_1 \cdots b_6 b_0^{-2} q_v^{-s}) \\ &\quad \times \prod_{1 \le i < j \le 6} (1 - \eta_v b_1^{-1} \cdots b_6^{-1} b_0^2 (b_i b_j)^2 q_v^{-s}) \\ &\quad \times \prod_{1 \le i < j \le 6} (1 - \eta_v b_1 \cdots b_6 (b_i b_j)^{-2} q_v^{-s}), \end{aligned}$$

$$L(s, \pi_{\eta,v}, r_2) = L(s, \eta_v^2 \omega_v^2).$$

Here r_1 is called the half-spin representation and it has degree 32.

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Remark Because of the complicated nature of the half-spin group $HS(12,\mathbb{C})$, we were not able to write the explicit formula for the degree 32 half-spin representation of cuspidal representations of $\mathbf{M}(\mathbb{A})$ which do not come from $H\,\mathrm{Spin}(12,\mathbb{A})$.

2.6.7 (xxx) in [La]

Let $\theta = \Delta - \{\alpha_1\}, \tilde{\alpha}_1 = e_1 + e_8, \mathbf{M}_D = E_6$,

$$\mathbf{A} = \{H_{\alpha_1}(t^3)H_{\alpha_2}(t^4)H_{\alpha_3}(t^5)H_{\alpha_4}(t^6)H_{\alpha_5}(t^4)H_{\alpha_6}(t^2)H_{\alpha_7}(t^3) : t \in \overline{F}^*\},$$

$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_2}(t)H_{\alpha_3}(t^2)H_{\alpha_5}(t)H_{\alpha_6}(t^2) :\ t^3 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times E_6)/(\mathbf{A} \cap \mathbf{M}_D) = GE_6.$$

Let π be cuspidal representations of $GE_6(\mathbb{A})$ with central character ω. Then ${}^LM = GE_6(\mathbb{C})$, and we see that $m = 1$, $L(s,\pi_v,r_1)$ is the standard L-function of E_6. It has degree 27.

2.7 E_8 Cases

We take the root system as in [G-O-V]. We take simple roots, $\alpha_i = e_i - e_{i+1}$, $i = 1,\dots,7$, $\alpha_8 = e_6 + e_7 + e_8$. Here $(e_i,e_i) = \frac{8}{9}$, $(e_i,e_j) = -\frac{1}{9}$ for $1 \leq i \neq j \leq 9$ and $\sum e_i = 0$. The positive roots are $e_i - e_j$, $1 \leq i < j \leq 9$, and $e_i + e_j + e_k$, $1 \leq i < j < k \leq 8$, and $-(e_i + e_j + e_9)$, $1 \leq i < j \leq 8$. There are 120 of them. Note that

$$(a_1e_1 + a_2e_2 + \cdots + a_9e_9, e_i - e_j) = a_i - a_j,$$

$$(a_1e_1 + a_2e_2 + \cdots + a_9e_9, e_i + e_j + e_k) = (a_i + a_j + a_k) - \frac{1}{3}(a_1 + \cdots + a_9).$$

The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

The Dynkin diagram is

$$\begin{array}{ccccccccccccc} \circ_1 & \text{———} & \circ_2 & \text{———} & \circ_3 & \text{———} & \circ_4 & \text{———} & \circ_5 & \text{———} & \circ_6 & \text{———} & \circ_7. \\ &&&&&&&& | &&&& \\ &&&&&&&& \circ_8 &&&& \end{array}$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

2.7.1 $E_8 - 1$

Let **G** be a simply-connected exceptional group of type E_8. Let $\theta = \Delta - \{\alpha_5\}$. Then $\tilde{\alpha}_5 = e_1+e_2+e_3+e_4+e_5-5e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and **A** be the connected component of the center of **M**. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^6)H_{\alpha_2}(t^{12})H_{\alpha_3}(t^{18})H_{\alpha_4}(t^{24})H_{\alpha_5}(t^{30})H_{\alpha_6}(t^{20})H_{\alpha_7}(t^{10})H_{\alpha_8}(t^{15}).$$

Since **G** is simply connected, the derived group $\mathbf{M}_D$ of **M** is simply connected, and hence $\mathbf{M}_D = SL_2 \times SL_3 \times SL_5$. As in the $E_7 - 1$ case, we construct a map $f\colon \mathbf{M} \to GL_2 \times GL_3 \times GL_5$ such that

$$f(H_{\alpha_5}(t)) = (\mathrm{diag}(1,t), \mathrm{diag}(1,1,t), \mathrm{diag}(1,1,1,1,t)).$$

Let π_i be cuspidal representations of $GL_2(\mathbb{A}), GL_3(\mathbb{A}), GL_5(\mathbb{A})$ with central characters ω_i, $i = 1,2,3$, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2, π_3. Then the central character of π is

$$\omega_\pi = \omega_1^{15}\omega_2^{10}\omega_3^6.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\eta_1,\eta_2), \quad \pi_{2v} = \pi(\nu_1,\nu_2,\nu_3), \quad \pi_{3v} = \pi(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi\circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi\circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t), \quad \chi\circ H_{\alpha_3}(t) = \mu_3\mu_4^{-1}(t),$$
$$\chi\circ H_{\alpha_4}(t) = \mu_4\mu_5^{-1}(t), \quad \chi\circ H_{\alpha_6}(t) = \nu_2\nu_3^{-1}(t), \quad \chi\circ H_{\alpha_7}(t) = \nu_1\nu_2^{-1}(t),$$
$$\chi\circ H_{\alpha_8}(t) = \eta_1\eta_2^{-1}(t), \quad \chi(a(t)) = \omega_1^{15}\omega_2^{10}\omega_3^6(t).$$

Since $f(H_{\alpha_5}(t)) = (\mathrm{diag}(1,t), \mathrm{diag}(1,1,t), \mathrm{diag}(1,1,1,1,t))$, we have $\chi\circ H_{\alpha_5}(t) = \mu_5\nu_3\eta_2$. Hence, we can compute that $m = 6$, and

$$L(s,\pi_v,r_1) = L(s,\pi_{1v}\times\pi_{2v}\times\pi_{3v}),$$
$$L(s,\pi_v,r_2) = L(s,(\tilde{\pi}_{2v}\otimes\omega_1\omega_2)\otimes\pi_{3v}, \rho_3\otimes\wedge^2\rho_5),$$
$$L(s,\pi_v,r_3) = L(s,(\pi_{1v}\otimes\omega_1\omega_2\omega_3)\otimes\tilde{\pi}_{3v}, \rho_2\otimes\wedge^2\rho_5),$$
$$L(s,\pi_v,r_4) = L(s,(\pi_{2v}\otimes\omega_1^2\omega_2\omega_3)\times\tilde{\pi}_{3v}),$$
$$L(s,\pi_v,r_5) = L(s,(\pi_{1v}\otimes\omega_1^2\omega_2^2\omega_3)\times\tilde{\pi}_{2v}),$$
$$L(s,\pi_v,r_6) = L(s,\pi_{3v}\otimes\omega_1^3\omega_2^2\omega_3).$$

2.7.2 $E_8 - 2$

Let $\theta = \Delta - \{\alpha_4\}$. Then $\tilde{\alpha}_5 = e_1 + e_2 + e_3 + e_4 - 4e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^5)H_{\alpha_2}(t^{10})H_{\alpha_3}(t^{15})H_{\alpha_4}(t^{20})H_{\alpha_5}(t^{24})H_{\alpha_6}(t^{16})H_{\alpha_7}(t^8)H_{\alpha_8}(t^{12}).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D = SL_4 \times SL_5$. As in the $E_7 - 2$ case, we construct a map $f\colon \mathbf{M} \to GL_4 \times GL_5$ such that

$$f(H_{\alpha_4}(t)) = (\operatorname{diag}(1, 1, 1, t), \operatorname{diag}(1, 1, 1, t, t)).$$

Let π_i be cuspidal representations of $GL_4(\mathbb{A}), GL_5(\mathbb{A})$ with central characters ω_i, $i = 1, 2$, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^5 \omega_2^8.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \mu_2, \mu_3, \mu_4), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \quad \chi \circ H_{\alpha_2}(t) = \mu_2\mu_3^{-1}(t), \quad \chi \circ H_{\alpha_3}(t) = \mu_3\mu_4^{-1}(t),$$
$$\chi \circ H_{\alpha_5}(t) = \nu_3\nu_4^{-1}(t), \quad \chi \circ H_{\alpha_6}(t) = \nu_2\nu_3^{-1}(t), \quad \chi \circ H_{\alpha_7}(t) = \nu_1\nu_2^{-1}(t),$$
$$\chi \circ H_{\alpha_8}(t) = \nu_4\nu_5^{-1}(t), \quad \chi(a(t)) = \omega_1^5\omega_2^8(t).$$

Since $f(H_{\alpha_4}(t)) = (\operatorname{diag}(1, 1, 1, t), \operatorname{diag}(1, 1, 1, t, t))$, we have $\chi \circ H_{\alpha_4}(t) = \mu_4\nu_4\nu_5$. Hence, we can compute that $m = 5$, and

$$\begin{aligned}
L(s, \pi_v, r_1) &= L(s, \pi_{1v} \otimes \pi_{2v}, \rho_4 \otimes \wedge^2 \rho_5),\\
L(s, \pi_v, r_2) &= L(s, \pi_{1v} \otimes (\tilde{\pi}_{2v} \otimes \omega_2), \wedge^2 \rho_4 \otimes \rho_5),\\
L(s, \pi_v, r_3) &= L(s, \tilde{\pi}_{1v} \times (\pi_{2v} \otimes \omega_1\omega_2)),\\
L(s, \pi_v, r_4) &= L(s, \tilde{\pi}_{2v}, \wedge^2 \rho_5 \otimes \omega_1\omega_2^2),\\
L(s, \pi_v, r_5) &= L(s, \pi_{1v} \otimes \omega_1\omega_2^2).
\end{aligned}$$

2.7.3 $E_8 - 3$

Let $\theta = \Delta - \{\alpha_3\}$. Then $\tilde{\alpha}_5 = e_1 + e_2 + e_3 - 3e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^4)H_{\alpha_2}(t^8)H_{\alpha_3}(t^{12})H_{\alpha_4}(t^{15})H_{\alpha_5}(t^{18})H_{\alpha_6}(t^{12})H_{\alpha_7}(t^6)H_{\alpha_8}(t^9).$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence $\mathbf{M}_D = SL_3 \times \mathrm{Spin}(10)$. Here we note that

$$H_{\alpha_3}(t^4)H_{\alpha_4}(t^5)H_{\alpha_5}(t^6)H_{\alpha_6}(t^4)H_{\alpha_7}(t^2)H_{\alpha_8}(t^3)$$

is exactly the same as the center of $\mathbf{M} = H\,\mathrm{Spin}(10)$ in Section 2.5.4. We define a map $\bar{f}\colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_3 \times \mathrm{Spin}(10)$ by

$$\bar{f}\colon (a(t), x, y) \mapsto (t^4, t^3, x, y).$$

It induces a map $f\colon \mathbf{M} \to GL_3 \times H\,\mathrm{Spin}(10)$. Under the identification, $H_{\alpha_1}(t)H_{\alpha_2}(t^2)$ is the diagonal element $\mathrm{diag}(t, t, t^{-2})$ in SL_3, $H_{\alpha_4}(t^5)H_{\alpha_5}(t^6)H_{\alpha_6}(t^4)H_{\alpha_7}(t^2)H_{\alpha_8}(t^3)$ is in $\mathrm{Spin}(10)$. From this, we see that $f(H_{\alpha_3}(t)) = (\mathrm{diag}(1, 1, t), b(t))$, where $b(t)$ is an element in $H\,\mathrm{Spin}(10)$.

Let π_1, π_2 be cuspidal representations of $GL_3, H\,\mathrm{Spin}(10)$ with central characters ω_1, ω_2, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^4\omega_2^3.$$

Let $\hat{t}_1 = \mathrm{diag}(a_1, a_2, a_3) \in GL_3(\mathbb{C})$ be the Satake parameter attached to π_{1v}. Let $\hat{t}_2 \in G\,\mathrm{Spin}(10, \mathbb{C})$ be the Satake parameter attached to π_{2v}. Using the 2-to-1 map $\phi\colon G\,\mathrm{Spin}(10, \mathbb{C}) \to GSO_{10}(\mathbb{C})$, we can write it as

$$\phi(\hat{t}_2) = \mathrm{diag}(b_1^2, \dots, b_5^2, b_5^{-2}b_0^2, \dots, b_1^{-2}b_0^2) \in GSO_{10}(\mathbb{C}).$$

Note that $\omega_2 = b_0^2$. Then

$$\chi \circ H_{\alpha_1} = a_1a_2^{-1}, \quad \chi \circ H_{\alpha_2} = a_2a_3^{-1}, \quad \chi \circ H_{\alpha_7} = b_1^2b_2^{-2},$$
$$\chi \circ H_{\alpha_6} = b_2^2b_3^{-2}, \quad \chi \circ H_{\alpha_5} = b_3^2b_4^{-2}, \quad \chi \circ H_{\alpha_8} = b_4^2b_5^{-2},$$
$$\chi \circ H_{\alpha_4} = b_4^2b_5^2b_0^{-2}, \quad \chi(a(t)) = \omega_1^4\omega_2^3 = (a_1a_2a_3)^4b_0^6.$$

From this, we can see $\chi \circ H_{\alpha_3} = a_3(b_1b_2b_3b_4b_5)^{-1}b_0^3$. Hence, we can compute that $m = 4$,

$$L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \rho_3 \otimes \mathrm{Spin}^{16}),$$
$$L(s, \pi_v, r_2) = L(s, (\tilde{\pi}_{1v} \otimes \omega_1) \times \pi'_{2v}),$$
$$L(s, \pi_v, r_3) = L(s, \tilde{\pi}_{2v}, \mathrm{Spin}^{16} \otimes (\omega_1\omega_2)),$$
$$L(s, \pi_v, r_4) = L(s, \pi_{1v} \otimes (\omega_1\omega_2)).$$

Here π'_2 is the cuspidal representation of $G\,\mathrm{Spin}(10)$, induced by π_2 and the 2-to-1 map $G\,\mathrm{Spin}(10) \to H\,\mathrm{Spin}(10)$. Hence the Satake parameter of π'_{2v} is $\phi(\hat{t}_2) \in GSO_{10}(\mathbb{C}) = {}^LG\,\mathrm{Spin}(10)$. Note that the second L-function is the Rankin–Selberg L-function for $GL_3 \times G\,\mathrm{Spin}(10)$.

2.7.4 $E_8 - 4$

Let $\theta = \Delta - \{\alpha_2\}$. Then $\tilde{\alpha}_5 = e_1 + e_2 - 2e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then

$$\mathbf{A} = \left\{ a(t) = H_{\alpha_1}(t^3)H_{\alpha_2}(t^6)H_{\alpha_3}(t^8)H_{\alpha_4}(t^{10})H_{\alpha_5}(t^{12})H_{\alpha_6}(t^8)H_{\alpha_7}(t^4)H_{\alpha_8}(t^6) : t \in \overline{F}^* \right\}.$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence

$$\mathbf{M}_D = SL_2 \times E_6,$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^3)H_{\alpha_3}(t^2)H_{\alpha_4}(t^4)H_{\alpha_6}(t^2)H_{\alpha_7}(t^4) : t^6 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_2 \times E_6)/(\mathbf{A} \cap \mathbf{M}_D).$$

Note that $H_{\alpha_2}(t^3)H_{\alpha_3}(t^4)H_{\alpha_4}(t^5)H_{\alpha_5}(t^6)H_{\alpha_6}(t^4)H_{\alpha_7}(t^2)H_{\alpha_8}(t^3)$ is exactly the same as the center of GE_6 in Section 2.6.7.

We define a map $\bar{f} \colon \mathbf{A} \times \mathbf{M}_D \to GL_1 \times GL_1 \times SL_2 \times E_6$ by

$$\bar{f} \colon (a(t), x, y) \mapsto (t^3, t^2, x, y).$$

It induces a map $f \colon \mathbf{M} \to GL_2 \times GE_6$. Let π_i be cuspidal representations of GL_2, GE_6 with central characters ω_i, $i = 1, 2$, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character of π is

$$\omega_\pi = \omega_1^3 \omega_2^2.$$

In this case, $m = 3$, and $L(s, \pi_v, r_1) = L(s, \pi_{1v} \otimes \pi_{2v}, \rho_2 \otimes \rho_{E_6})$, where ρ_{E_6} is the standard L-function of $GE_6(\mathbb{C})$. The second L-function $L(s, \pi_v, r_2)$ is the standard L-function of GE_6 attached to π_2 ((**xxx**) case; see Section 2.6.7). The third L-function $L(s, \pi_v, r_3)$ is the standard L-function of GL_2 attached to π_1.

2.7.5 $E_8 - 5$

Let $\theta = \Delta - \{\alpha_6\}$. Then $\tilde{\alpha}_6 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^4)H_{\alpha_2}(t^8)H_{\alpha_3}(t^{12})H_{\alpha_4}(t^{16})H_{\alpha_5}(t^{20})H_{\alpha_6}(t^{14})H_{\alpha_7}(t^7)H_{\alpha_8}(t^{10}).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence

$$\mathbf{M}_D = SL_7 \times SL_2,$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^4)H_{\alpha_2}(t^8)H_{\alpha_3}(t^{12})H_{\alpha_4}(t^2)H_{\alpha_5}(t^6)H_{\alpha_7}(t^7)H_{\alpha_8}(t^{10}) : t^{14} = 1\}.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_7 \times SL_2)/(\mathbf{A} \cap \mathbf{M}_D).$$

As in the $E_7 - 4$ case, we construct a map $f \colon \mathbf{M} \to GL_7 \times GL_2$ such that

$$f(H_{\alpha_6}(t)) = (\mathrm{diag}(1,1,1,1,1,t,t), \mathrm{diag}(1,t)).$$

Let π_i be cuspidal representations of GL_7, GL_2 with central characters ω_i, $i = 1, 2$, resp. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and π_1, π_2. Then the central character is

$$\omega_\pi = \omega_1^4 \omega_2^7.$$

Now suppose π_{iv} is an unramified representation, given by

$$\pi_{1v} = \pi(\mu_1, \dots, \mu_7), \quad \pi_{2v} = \pi(\nu_1, \nu_2).$$

Let π_v be the unramified representation of $\mathbf{M}(F_v)$. Then π_v is induced from the character χ of the torus. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t), \dots, \chi \circ H_{\alpha_5}(t) = \mu_5\mu_6^{-1}(t),$$
$$\chi \circ H_{\alpha_8}(t) = \mu_6\mu_7^{-1}(t), \quad \chi \circ H_{\alpha_7}(t) = \nu_1\nu_2^{-1}(t), \quad \chi(a(t)) = \omega_1^4\omega_2^7(t).$$

Since $f(H_{\alpha_6}(t)) = (\mathrm{diag}(1,1,1,1,1,t,t), \mathrm{diag}(1,t))$, we have $\chi \circ H_{\alpha_6}(t) = \mu_6\mu_7\nu_2$. Hence, we can compute that $m = 4$, and

$$\begin{aligned}
L(s, \pi_v, r_1) &= L(s, \pi_{1v} \otimes \pi_{2v}, \wedge^2\rho_7 \otimes \rho_2), \\
L(s, \pi_v, r_2) &= L(s, \tilde{\pi}_{1v}, \wedge^3\rho_7 \otimes \omega_1\omega_2), \\
L(s, \pi_v, r_3) &= L(s, \tilde{\pi}_{1v} \times (\pi_{2v} \otimes \omega_1\omega_2)), \\
L(s, \pi_v, r_4) &= L(s, \pi_{1v} \otimes \omega_1\omega_2^2).
\end{aligned}$$

2.7.6 (xiii) in [La]

Let $\theta = \Delta - \{\alpha_8\}$. Then $\tilde{\alpha}_8 = -3e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$, where

$$a(t) = H_{\alpha_1}(t^3)H_{\alpha_2}(t^6)H_{\alpha_3}(t^9)H_{\alpha_4}(t^{12})H_{\alpha_5}(t^{15})H_{\alpha_6}(t^{10})H_{\alpha_7}(t^5)H_{\alpha_8}(t^8).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence

$$\mathbf{M}_D = SL_8,$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^3)H_{\alpha_2}(t^6)H_{\alpha_3}(t)H_{\alpha_4}(t^4)H_{\alpha_5}(t^7)H_{\alpha_6}(t^2)H_{\alpha_7}(t^5) : t^8 = 1\}.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times SL_8)/(\mathbf{A} \cap \mathbf{M}_D).$$

As in the **(x)** case (Section 2.5.3), we construct a map $f\colon \mathbf{M} \to GL_1 \times GL_8$ such that

$$f(H_{\alpha_8}(t)) = (t, \operatorname{diag}(1,1,1,1,1,t,t,t)).$$

Let σ be a cuspidal representation of GL_8 with the central character ω. Let η be a grössencharacter of F. Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$, induced by f and σ, η. Then the central character of π is

$$\omega_\pi = \omega^3\eta^8.$$

Now suppose σ_v is an unramified representation, given by $\sigma_v = \pi(\mu_1,\dots,\mu_8)$. Let χ be the character of the torus, given by π_v. We have

$$\chi \circ H_{\alpha_1}(t) = \mu_1\mu_2^{-1}(t),\dots,\chi\circ H_{\alpha_7}(t) = \mu_7\mu_8^{-1}(t), \quad \chi(a(t)) = \omega_{\pi_v}(t).$$

Since $f(H_{\alpha_8}(t)) = (t, \operatorname{diag}(1,1,1,1,1,t,t,t))$, we have $\chi \circ H_{\alpha_8}(t) = \mu_6\mu_7\mu_8\eta_v$. Hence, we can compute that $m = 3$, and

$$L(s,\pi_v,r_1) = L(s,\sigma_v,\wedge^3\rho_8 \otimes \eta_v),$$
$$L(s,\pi_v,r_2) = L(s,\tilde{\sigma}_v,\wedge^2\rho_8 \otimes \omega_v\eta_v^2),$$
$$L(s,\pi_v,r_3) = L(s,\sigma_v \otimes \omega_v\eta_v^3).$$

Here $L(s,\pi,r_1)$ is the exterior cube L-function of GL_8 and it has degree 56.

2.7.7 (xxviii) in [La]

Let $\theta = \Delta - \{\alpha_7\}$. Then $\tilde{\alpha}_7 = e_1 + \cdots + e_7 - e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A} = \{a(t) : t \in \overline{F}^*\}$ where

$$a(t) = H_{\alpha_1}(t^2)H_{\alpha_2}(t^4)H_{\alpha_3}(t^6)H_{\alpha_4}(t^8)H_{\alpha_5}(t^{10})H_{\alpha_6}(t^7)H_{\alpha_7}(t^4)H_{\alpha_8}(t^5).$$

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence

$$\mathbf{M}_D = \operatorname{Spin}(14),$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_1}(t^2)H_{\alpha_3}(t^2)H_{\alpha_5}(t^2)H_{\alpha_6}(t^3)H_{\alpha_8}(t) : t^4 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times \operatorname{Spin}(14))/(\mathbf{A} \cap \mathbf{M}_D).$$

Since $G\,\mathrm{Spin}(14) = (GL_1 \times \mathrm{Spin}(14))/\{1, c\}$ (see Section 2.3.4), there is a surjective map $G\,\mathrm{Spin}(14) \to \mathbf{M}$. Hence we have a dual map ${}^LM \to GSO_{14}(\mathbb{C}) = {}^LG\,\mathrm{Spin}(14)$. Since the center of $\mathbf{M}$ is connected, the derived group of ${}^LM$ is simply connected (See [Bo, p. 30]). Hence it is $\mathrm{Spin}(14, \mathbb{C})$. Therefore ${}^LM = G\,\mathrm{Spin}(14, \mathbb{C})$.

Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$ with central character ω. Let π_v be the unramified representation of $\mathbf{M}(F_v)$ with the corresponding semisimple conjugacy class $\hat{t}$ in $\hat{T}$, the torus in ${}^LM$. We have a 2-to-1 map $\phi\colon {}^LM \to GSO_{14}(\mathbb{C})$. Let $\phi(\hat{t})$ be given by

$$\phi(\hat{t}) = \mathrm{diag}(b_1^2, \dots, b_7^2, b_0^2 b_7^{-2}, \dots, b_0^2 b_1^{-2}).$$

Note that it is the Satake parameter for the representation π'_v of $G\,\mathrm{Spin}(14, F_v)$, where $\pi' = \otimes_v \pi'_v$ is the cuspidal representation of $G\,\mathrm{Spin}(14, \mathbb{A})$, induced by π and the map $G\,\mathrm{Spin}(14) \to \mathbf{M}$. Note also that $\omega_\pi = \omega_{\pi'}$, and hence $\omega_{\pi_v} = b_0^2$.

Let η be a grössencharacter of F. Then we can think of η as a character of $\mathbf{M}(\mathbb{A})$ by setting $\eta(a(t)) = \eta(t^4)$. Since $\eta|_{\mathbf{A}\cap\mathbf{M}_D} = 1$, it is well-defined. We consider $\pi_\eta = \pi \otimes \eta$. Let χ be the inducing character of the torus attached to $\pi_{\eta,v}$. Then we have the relationship

$$\chi \circ \alpha^\vee(\varpi) = \alpha^\vee(\hat{t}),$$

where $\alpha^\vee$ on the right is considered as a root of ${}^LM$. Then

$$\chi \circ H_{\alpha_1} = b_1^2 b_2^{-2}, \dots, \chi \circ H_{\alpha_5} = b_5^2 b_6^{-2},$$

$$\chi \circ H_{\alpha_8} = b_6^2 b_7^{-2}, \quad \chi \circ H_{\alpha_6} = b_6^2 b_7^2 b_0^{-2}, \quad \chi(a(t)) = \eta_v^4 \omega_v = \eta_v^4 b_0^2.$$

From this, we have $\chi \circ H_{\alpha_7} = \eta_v (b_1 \cdots b_7)^{-1} b_0^4$. Hence, we can compute that $m = 2$, and

$$\begin{aligned} L(s, \pi_{\eta,v}, r_1)^{-1} = {} & (1 - \eta_v (b_1 \cdots b_7)^{-1} b_0^4 q_v^{-s}) \\ & \times \prod_{1 \le i < j \le 7} (1 - \eta_v (b_1 \cdots b_7)^{-1} b_0^2 (b_i b_j)^2 q_v^{-s}), \\ & \times \prod_{1 \le i < j < k \le 7} (1 - \eta_v b_1 \cdots b_7 (b_i b_j b_k)^{-2} q_v^{-s}) \\ & \times \prod_{i=1}^{7} (1 - \eta_v b_1 \cdots b_7 b_0^{-2} b_i^{-2} q_v^{-s}), \end{aligned}$$

$$L(s, \pi_{\eta,v}, r_2) = L(s, \pi'_v \otimes \eta_v^2) = \prod_{i=1}^{7} (1 - \eta_v^2 b_i^2 q_v^{-s})^{-1} (1 - \eta_v^2 b_i^{-2} b_0^2 q_v^{-s})^{-1}.$$

Here r_1 is called the half-spin representation and has degree 64, and the second L-function is the standard L-function for $G\,\mathrm{Spin}(14)$.

2.7.8 (xxxii) in [La]

Let $\theta = \Delta - \{\alpha_1\}$. Then $\tilde{\alpha}_1 = e_1 - e_9$. Let $\mathbf{P} = \mathbf{P}_\theta = \mathbf{MN}$ and $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then

$$\mathbf{A} = \{H_{\alpha_1}(t^2) H_{\alpha_2}(t^3) H_{\alpha_3}(t^4) H_{\alpha_4}(t^5) H_{\alpha_5}(t^6) H_{\alpha_6}(t^4) H_{\alpha_7}(t^2) H_{\alpha_8}(t^3) : t \in \overline{F}^*\}.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Since $\mathbf{G}$ is simply connected, the derived group $\mathbf{M}_D$ of $\mathbf{M}$ is simply connected, and hence

$$\mathbf{M}_D = \text{simply-connected } E_7,$$
$$\mathbf{A} \cap \mathbf{M}_D = \{H_{\alpha_2}(t)H_{\alpha_4}(t)H_{\alpha_8}(t) : t^2 = 1\}.$$

If we identify $\mathbf{A}$ with GL_1, then

$$\mathbf{M} = (GL_1 \times E_7)/(\mathbf{A} \cap \mathbf{M}_D) = GE_7.$$

Let π be a cuspidal representation of $GE_7(\mathbb{A})$ with the central character ω. Then ${}^LM = GE_7(\mathbb{C})$, and we see that $m = 2$, $L(s, \pi_v, r_1)$ is the standard L-function of $GE_7(\mathbb{C})$. It has degree 56. Also $L(s, \pi_v, r_2) = L(s, \omega_v)$.

3 Proof of a Conjecture of Shahidi

In this section, let F be a local field of characteristic zero and we omit the subscript v. Here G, M denote the group of F-rational points $\mathbf{G}(F), \mathbf{M}(F)$, resp. Recall Conjecture 7.1 of [Sh1]:

Conjecture *Assume π is tempered and generic. Then each $L(s, \pi, r_i)$ is holomorphic for* $\mathrm{Re}(s) > 0$.

This conjecture is true for archimedean places [A]. In fact, for archimedean places, the L-function $L(s, \pi, r_i)$ and the ϵ-factor are Artin factors [Sh7]. In particular $L(s, \pi, r_i)$ is holomorphic for $\mathrm{Re}(s) > 0$. This conjecture has many important applications. It played a crucial role in proving the functorial product of $GL_2 \times GL_3$ and functoriality of symmetric cube in [Ki-Sh]. First we start with known results.

Proposition 3.1 ([Sh1, p. 309]) *Assume π is tempered and generic.*

(1) *If $m = 1$, $L(s, \pi, r)$ is holomorphic for* $\mathrm{Re}(s) > 0$.
(2) *If $m = 2$ and $L(s, \pi, r_2) = \prod_j (1 - \alpha_j q_v^{-s})^{-1}$, possibly an empty product where each $\alpha_j \in \mathbb{C}$ is of absolute value one (in particular if r_2 is one-dimensional, this holds), then $L(s, \pi, r_1)$ is holomorphic for* $\mathrm{Re}(s) > 0$.

Proposition 3.2 ([Ca-Sh, p. 573]) *If G is a quasi-split classical group, then the conjecture holds.*

Proposition 3.3 (Asgari [As]) *Let G be a simply connected split group of type D_n and F_4. Then the conjecture holds.*

Lemma 3.4 ([Sh1, Proposition 7.3 and Corollary 7.6]) *Let ρ be a generic supercuspidal representation of M. Then*

(1) *For $i = 1, 2$, $L(s, \rho, r_i)$ is a product of $(1 - uq^{-s})^{-1}$, where u is a complex number of absolute value* 1.
(2) *If $i \geq 3$, $L(s, \rho, r_i) = 1$.*

(3) *If $L(s,\rho,r_1)L(s,\rho,r_2)$ has pole at $s=0$, then it is simple. Namely, only one of $L(s,\rho,r_1)$ and $L(s,\rho,r_2)$ has a simple pole at $s=0$.*

Recall the following induction step, which we can see immediately through our explicit calculations in Section 2.

Proposition 3.5 ([Sh1, Proposition 4.1]) *Let F be a number field. Let $\mathbf{G}$ be a quasi-split connected reductive group over F. Let $\mathbf{P}=\mathbf{MN}$ be a standard maximal parabolic subgroup of $\mathbf{G}$ with respect to an F-Borel subgroup $\mathbf{B}$. Let π be a globally generic cuspidal representation of $\mathbf{M}(\mathbb{A})$. Let $r=\bigoplus_{i=1}^m r_i$ be the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$ as before. Then for each i, $2\le i\le m$, there exists a quasi-split connected reductive F-group $\mathbf{G}_i$, a maximal F-parabolic subgroup $\mathbf{P}_i=\mathbf{M}_i\mathbf{N}_i$ of $\mathbf{G}_i$, a globally generic cuspidal representation π' of $\mathbf{M}_i(\mathbb{A})$, such that, if the adjoint action r' of ${}^LM_i$ on ${}^L\mathfrak{n}_i$ decomposes as $r'=\bigoplus_{j=1}^{m'} r'_j$, then*

$$L(s,\pi,r_i)=L(s,\pi',r'_1).$$

Lemma 3.6 *Let π be a generic, tempered representation. Then for $i\ge 3$, $L(s,\pi,r_i)$ is holomorphic for* $\mathrm{Re}(s)>0$.

Proof Except for r_3 in the case of E_8-1, all r_i, $i\ge 3$, come from non-self conjugate parabolic subgroups with $m=1$. Hence Proposition 3.1 applies.

Suppose we are in the E_8-1 case. Then r_3 comes from the E_6-2 case. In that case, we calculate directly to see our assertion. We postpone the proof until Section 3.2.2. ■

Proposition 3.7 *Let π be tempered and generic. Then $L(s,\pi,r_i)$ is holomorphic at* $\mathrm{Re}(s)=\frac{1}{2}$.

Proof Note that

$$\gamma(s,\pi,r_i,\psi)=\epsilon(s,\pi,r_i,\psi)\frac{L(1-s,\pi,\tilde{r}_i)}{L(s,\pi,r_i)},$$

and $L(s,\pi,r_i)$ is defined to be

$$L(s,\pi,r_i)=P_{\pi,i}(q^{-s})^{-1},$$

where $P_{\pi,i}$ is the unique polynomial satisfying $P_{\pi,i}(0)=1$ such that $P_{\pi,i}(q^{-s})$ is the numerator of $\gamma(s,\pi,r_i,\psi)$.

Suppose $L(s,\pi,r_i)$ has a pole at $\mathrm{Re}(s)=\frac{1}{2}$. Then it contains the inverse of a factor $1-uq^{1/2-s}$, where u is a complex number with $|u|=1$. Then by unitarity of π and [Sh1, Proposition 7.8], we see $L(1-s,\pi,\tilde{r}_i)$ contains the inverse of a factor $1-\overline{uq^{1/2}}q^{-(1-s)}=1-u^{-1}q^{s-\frac{1}{2}}=u^{-1}q^{s-\frac{1}{2}}(1-uq^{\frac{1}{2}-s})$. Hence there is a cancellation. This contradicts the definition of $L(s,\pi,r_i)$. ■

The following is a slight generalization of Proposition 3.1.

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Proposition 3.8 ([Sh1, Theorem 3.5] and [Sh2, Proposition 3.3.1]) *Let π be tempered and generic, and let $C_\chi(s,\pi,w_0)$ be the local coefficient attached to (M,π)* [Sh2]. *Then we have*

$$C_\chi(s,\pi,w_0)=\prod_{i=1}^m \gamma(is,\pi,r_i,\psi).$$

In particular, $\prod_{i=1}^m \gamma(is,\pi,r_i,\psi)$ has no zeros for $\mathrm{Re}(s)>0$, *and $L(s,\pi,r_1)$ is holomorphic for* $\mathrm{Re}(s)>0$ *if $\prod_{i=2}^m L(1-is,\pi,r_i)$ has poles only at* $\mathrm{Re}(s)=\frac{1}{2}$ *in the region* $\mathrm{Re}(s)>0$.

Proof Note that since we are only dealing with split groups, there is no λ-function in the formula. Also we can make $a=1$ in Theorem 3.5 of [Sh1], by making ψ and w_0 compatible. By the definition of $C_\chi(s,\pi,w_0)$, $C_\chi(s,\pi,w_0)A(s,\pi,w_0)$ has no zeros. Since $A(s,\pi,w_0)$ is holomorphic for $\mathrm{Re}(s)>0$, $C_\chi(s,\pi,w_0)$ has no zeros for $\mathrm{Re}(s)>0$. The last statement follows from Proposition 3.7. ■

Recall the multiplicativity of γ-factors. Let π be an irreducible generic admissible representation of M. Suppose $\pi\subset \mathrm{Ind}_{M_\theta N_\theta}^M\,\sigma\otimes 1$, where $M_\theta N_\theta$, $\theta\subset\Delta$, is a parabolic subgroup of M and σ is an irreducible generic admissible representation of M_θ. Let $\theta'=w(\theta)\subset\Delta$ and fix a reduced decomposition $w=w_{n-1}\cdots w_1$ of w as in Lemma 2.1.1 of [Sh2]. Then for each j, there exists a unique root $\alpha_j\in\Delta$ such that $w_j(\alpha_j)<0$. For each j, $2\le j\le n-1$, let $\bar w_j=w_{j-1}\cdots w_1$. Set $\bar w_1=1$. Also let $\Omega_j=\theta_j\cup\{\alpha_j\}$, where $\theta_1=\theta$, $\theta_n=\theta'$, and $\theta_{j+1}=w_j(\theta_j)$, $1\le j\le n-1$. Then the group M_{Ω_j} contains $M_{\theta_j}N_{\theta_j}$ as a maximal parabolic subgroup and $\bar w_j(\sigma)$ is a representation of M_{θ_j}. The L-group ${}^LM_\theta$ acts on V_i. Given an irreducible constituent of this action, there exists a unique j, $1\le j\le n-1$, which under w_j is equivalent to an irreducible constituent of the action of ${}^LM_{\theta_j}$ on the Lie algebra of ${}^LN_{\theta_j}$. We denote by $i(j)$ the index of this subspace of the Lie algebra of ${}^LN_{\theta_j}$. Finally, let S_i denote the set of all such i's where S_i, in general, is a proper subset of $1\le i\le n-1$.

Proposition 3.9 ([Sh1, 3.13]) *For each $j\in S_i$, let $\gamma(s,\bar w_j(\sigma),r_{i(j)},\psi)$ denote the corresponding factor. Then*

$$\gamma(s,\pi,r_i,\psi)=\prod_{j\in S_i}\gamma(s,\bar w_j(\sigma),r_{i(j)},\psi).$$

We follow the exposition in [Sh6, p. 280]. Let $\phi\colon W_F\times SL_2(\mathbb{C})\to {}^LM$ be the parametrization of π. Then ϕ factors through ${}^LM_\theta$, *i.e.*, there exists

$$\phi'\colon W_F\times SL_2(\mathbb{C})\to {}^LM_\theta$$

such that $\phi=i\circ\phi'$, where $i\colon {}^LM_\theta\hookrightarrow {}^LM$. Let $r_i'=r_i|_{{}^LM_\theta}$. Then $r_i'=\oplus_j r_{i(j)}$, and

$$\gamma(s,\phi,r_i,\psi)=\prod_j\gamma(s,\phi',r_{i(j)},\psi).$$

Given an irreducible component of $r_i|_{{}^LM_\theta}$, there exists a unique j, which under w_j, makes this component equivalent to an irreducible constituent of the action of ${}^LM_{\theta_j}$ on the Lie algebra of ${}^LN_{\theta_j}$. Hence we have:

Proposition 3.10 *Suppose π, σ be as in Proposition 3.9. Suppose π is tempered, and $\gamma(s, \overline{w}_j(\sigma), r_{i(j)}, \psi)$ is an Artin factor for each $j \in S_i$, namely, $\gamma(s, \overline{w}_j(\sigma), r_{i(j)}, \psi) = \gamma(s, \phi', r_{i(j)}, \psi)$ for each j. Then $\gamma(s, \pi, r_i, \psi)$ and $L(s, \pi, r_i)$ are also Artin factors. In particular, $L(s, \pi, r_i)$ is holomorphic for* $\mathrm{Re}(s) > 0$.

Proof Clear from the multiplicativity formulas. Since π is tempered, γ-factors determine the L-factors uniquely. Artin L-functions satisfy the holomorphy. ■

Hence once we know that $\gamma(s, \rho, r_i, \psi)$ is an Artin factor for supercuspidal ρ, Conjecture 7.1 of [Sh1] is obvious by Proposition 3.10 and multiplicativity of γ-factors. However, except for a few cases, it is not known that $\gamma(s, \rho, r_i, \psi)$ is an Artin factor. For example, Shahidi [Sh5] has shown that for Rankin–Selberg L-functions for $GL_k \times GL_l$, his L-functions are Artin L-functions. However it is not even known that Shahidi's exterior square L-function, $L(s, \rho, \wedge^2)$, is an Artin L-function, where ρ is a supercuspidal representation of $GL_n(F)$. Later on, in many situations, all the rank-one factors in Proposition 3.10 are the Rankin–Selberg γ and L-factors for $GL_k \times GL_l$.

We have:

Lemma 3.11 *Let ρ_1 be a tempered representation of GL_{n-2} and ρ_2, ρ_3 be tempered representations of GL_2. Then in $D_n - 2$ case, the triple L-function $L(s, \rho_1 \times \rho_2 \times \rho_3) = L(s, \rho_1 \times (\rho_2 \boxtimes \rho_3))$ is an Artin L-function, where $\rho_2 \boxtimes \rho_3$ is the functorial product given by the local Langlands correspondence* [Ra]. *The same is true for the ϵ-factor.*

Proof It is enough to prove it when ρ_i's are supercuspidal representations. Let $\sigma_1, \sigma_2, \sigma_3$ be cuspidal representations of $GL_{n-2}(\mathbb{A}), GL_2(\mathbb{A}), GL_2(\mathbb{A})$, resp. such that $\sigma_{iv} = \rho_i$ and σ_{iw} is unramified for all $w \neq v$ and $w < \infty$. By considering the $D_n - 2$ case, we obtain the triple L-function $L(s, \sigma_1 \times \sigma_2 \times \sigma_3)$. Let $\sigma_2 \boxtimes \sigma_3$ be the functorial product, obtained in [Ra]. It is an automorphic representation of $GL_4(\mathbb{A})$. Now we compare two functional equations:

$$L(s, \sigma_1 \times \sigma_2 \times \sigma_3) = \epsilon(s, \sigma_1 \times \sigma_2 \times \sigma_3) L(1 - s, \tilde{\sigma}_1 \times \tilde{\sigma}_2 \times \tilde{\sigma}_3),$$

$$L(s, \sigma_1 \times (\sigma_2 \boxtimes \sigma_3)) = \epsilon(s, \sigma_1 \times (\sigma_2 \boxtimes \sigma_3)) L(1 - s, \tilde{\sigma}_1 \times (\tilde{\sigma}_2 \boxtimes \tilde{\sigma}_3)).$$

Since $L(s, \sigma_{1w} \times \sigma_{2w} \times \sigma_{3w}) = L(s, \sigma_{1w} \times (\sigma_{2w} \boxtimes \sigma_{3w}))$ for all $w \neq v$, we have (see [Ki5, Proposition 5.1.3] for the details)

$$\gamma(s, \sigma_{1v} \times \sigma_{2v} \times \sigma_{3v}, \psi_v) = \gamma(s, \sigma_{1v} \times (\sigma_{2v} \boxtimes \sigma_{3v}), \psi_v).$$

Note that $\rho_2 \boxtimes \rho_3$ is tempered (see [Ki5, Proposition 5.1.4]). Hence the equality of γ-factors implies the equality of L-factors. ■

Next we have [Sh6, Theorem 5.2]:

Proposition 3.12 (Multiplicativity of L-factors) *Let π, σ be as in Proposition 3.9. Suppose π is tempered, and σ is a discrete series. Suppose Conjecture 7.1 of* [Sh1] *is valid for every $L(s, \overline{w}_j(\sigma), r_{i(j)})$, $j \in S_i$. Then*

$$L(s, \pi, r_i) = \prod_{j \in S_i} L(s, \overline{w}_j(\sigma), r_{i(j)}).$$

Now we show the application of Conjecture 7.1 of [Sh1] to the functorial lift: let G be a reductive group over a local field F, and suppose we have a homomorphism of L-groups $f\colon {}^LG \to GL_N(\mathbb{C})$. Then Langlands' functoriality predicts that, given an irreducible admissible representation π of $G(F)$, there exists a local lift Π of $GL_N(F)$ such that if $\phi\colon W_F \times SL_2(\mathbb{C}) \to {}^LG$ parametrizes π, then $f \circ \phi$ parametrizes Π. If such parametrization is available (namely, the local Langlands correspondence), it is easy to see that if π is tempered, then Π is tempered. Note that π is tempered if and only if the image $\phi(W_F)$ is bounded (see, for example, [Ku, Lemma 5.2.1]). In that case, it is obvious that $f \circ \phi(W_F)$ is bounded. Hence Π is tempered.

In general, the local Langlands correspondence is not available. Hence we introduce the concept of the local lift in the following way:

We say Π is the local lift of π if it satisfies

$$\gamma(s, \sigma \times \pi, \psi) = \gamma(s, \sigma \times \Pi, \psi), \quad L(s, \sigma \times \pi) = L(s, \sigma \times \Pi),$$

where σ is a discrete series representation of $GL_m(F)$.

The left-hand sides are Shahidi's γ and L-factors, and the right-hand sides are Rankin–Selberg γ and L-factors. Hence according to Section 2, this makes sense only when $G = GL_2 \times GL_2, GL_2 \times GL_3$, GL_4, and groups of type B_n, C_n, D_n. The case $GL_2 \times GL_2$ is a subject of [Ra] and [Ki5] (We need the $D_4 - 2$ case in Section 2); The case $GL_2 \times GL_3$ is a subject of [Ki-Sh] (We need the $D_5 - 2, E_6 - 1, E_7 - 1$ cases in Section 2). In those two cases, the L-group homomorphisms are tensor product maps $GL_2(\mathbb{C}) \times GL_k(\mathbb{C}) \to GL_{2k}(\mathbb{C})$, $k = 2, 3$. The case GL_4 is a subject of [Ki5] (We need the $D_n - 3$, $n = 4, 5, 6, 7$ cases in Section 2). It is an exterior square lift, where the L-group homomorphism is the exterior square $GL_4(\mathbb{C}) \to GL_6(\mathbb{C})$. When $G = SO_{2n+1}$, the local lift was obtained in [CKPSS]. Suppose conjecture 7.1 of [Sh1] is valid in all those cases (We will prove it in Theorem 3.16).

Proposition 3.13 *Suppose the local lift exists. If π is tempered (unitary), Π is tempered.*

Proof Suppose Π is not tempered. We write it as a Langlands' quotient of $\Xi =$ Ind $|\det|^{r_1}\sigma_1 \otimes \cdots \otimes |\det|^{r_k}\sigma_k$, where the σ_i's are (unitary) discrete series representations of smaller GL's and $r_1 \geq \cdots \geq r_k$. Since π is unitary, Π has the unitary central character and hence $r_k < 0$. (Since Π is not tempered, not all r_i's are zero.) Consider the equality $L(s, \tilde{\sigma}_k \times \pi) = L(s, \tilde{\sigma}_k \times \Pi)$. The left hand side is holomorphic for $\mathrm{Re}(s) > 0$ by Conjecture 7.1 of [Sh1]. However,

$$L(s, \tilde{\sigma}_k \times \Pi) = \prod_{i=1}^{k} L(s + r_i, \tilde{\sigma}_k \times \sigma_i),$$

has a pole at $s = -r_k > 0$. ■

In the following, we indicate a proof of Proposition 3.2 due to Casselman and Shahidi [Ca-Sh]. The proof requires two ingredients. The first is that due to the fact that the Levi subgroups are simple, namely, of the form $GL_{n_1} \times \cdots \times GL_{n_k} \times G_l$, where G_l is a quasi-split classical group, the multiplicativity of γ-factors (Proposition 3.9) becomes simple. The second is a partial classification of generic discrete series of quasi-split classical groups. We now have a complete classification of discrete series with generic supercuspidal support of quasi-split classical groups due to Moeglin and Tadic [M-Ta] (cf. [Ja1–Ja3]). In [Ca-Sh], due to a lack of classification at the time, the authors first had to give a partial classification of generic discrete series of quasi-split classical groups.

Recall that a discrete series of GL_n comes from a distinguished unipotent orbit (p), which gives rise to a complex parameter

$$\Big(\frac{p-1}{2}, \frac{p-1}{2}-1, \frac{p-1}{2}-2, \dots, -\frac{p-1}{2}\Big).$$

This gives rise to an induced representation

$$\operatorname{Ind} \rho|\det|^{\frac{p-1}{2}} \otimes \rho|\det|^{\frac{p-1}{2}-1} \otimes \cdots \otimes \rho|\det|^{-\frac{p-1}{2}}$$

where ρ is a supercuspidal representation of GL. Let $St(\rho, p)$ be the discrete series which is the unique subrepresentation of the above induced representation. Then $L(s, St(\rho, p) \times \tilde{\rho})^{-1}$ is obtained as a numerator of $\gamma(s, St(\rho, p) \times \tilde{\rho}, \psi)$ which comes from the induced representation

$$\operatorname{Ind} St(\rho, p)|\det|^{\frac{s}{2}} \otimes \rho|\det|^{-\frac{s}{2}}.$$

It is a subrepresentation of

$$\operatorname{Ind} \rho|\det|^{\frac{s}{2}+\frac{p-1}{2}} \otimes \rho|\det|^{\frac{s}{2}+\frac{p-1}{2}-1} \otimes \cdots \otimes \rho|\det|^{\frac{s}{2}-\frac{p-1}{2}} \otimes \rho|\det|^{-\frac{s}{2}}.$$

By multiplicativity of γ-factors (Proposition 3.9),

$$\gamma(s, St(\rho, p) \times \tilde{\rho}, \psi) = \prod_{i=0}^{p-1} \gamma(s + \frac{p-1}{2} - i, \rho \times \tilde{\rho}, \psi).$$

Note that $\gamma(s, \rho \times \tilde{\rho}, \psi) = \epsilon(s, \rho \times \tilde{\rho}, \psi)\frac{L(1-s, \rho \times \tilde{\rho})}{L(s, \rho \times \tilde{\rho})}$ and $L(s, \rho \times \tilde{\rho}) = (1 - q^{-rs})^{-1} = \prod_{i=1}^{r}(1 - \eta_i(\varpi)q^{-s})^{-1}$, where r is the order of the cyclic group of unramified characters η_i of F^* such that $\rho \simeq \rho \otimes \eta_i(\det)$. Hence

$$L(s, St(\rho, p) \times \tilde{\rho}) = L(s + \frac{p-1}{2}, \rho \times \tilde{\rho}).$$

Notice the cancellation in $\gamma(s, St(\rho, p) \times \tilde{\rho}, \psi)$. Also if $p \geq q$, then

$$L(s, St(\rho, p) \times \widetilde{St(\rho, q)}) = \prod_{i=0}^{q-1} L(s + \frac{p-1}{2} + \frac{q-1}{2} - i, \rho \times \tilde{\rho}).$$

Let $G = G_n$ be a quasi-split classical group of type B_n, C_n, D_n, and let σ be a discrete series of GL_k and τ be a discrete series of G_l with generic supercuspidal support. We describe a partial classification of discrete series for quasi-split classical groups which we need. First, we remark that if τ is a discrete series which is a subrepresentation of $\operatorname{Ind} |\det|^a \rho \otimes \tau_0$, where ρ is a supercuspidal representation of GL_k and τ_0 is a generic supercuspidal representation of a quasi-split group, then $a = \frac{1}{2}$ or 1. This is a deep result of Shahidi [Sh1]. We say that (ρ, τ_0) satisfies (Ci) if $\operatorname{Ind} |\det|^a \rho \otimes \tau_0$ is reducible at $s = i$.

Following the GL_n example, we introduce a concept of chains. Given integers $a > b > 0$ (we assume that a, b have the same parity) and a supercuspidal representation ρ of GL_k, denote by $\delta(a, b, \rho), \delta(a, \rho)$, the representations

$$\delta(a, b, \rho) = |\det|^{\frac{a-1}{2}} \rho \otimes |\det|^{\frac{a-1}{2}-1} \rho \otimes \cdots \otimes |\det|^{-\frac{b-1}{2}} \rho$$

$$\delta(a, \rho) = |\det|^{\frac{a-1}{2}} \rho \otimes |\det|^{\frac{a-1}{2}-1} \rho \otimes \cdots \otimes |\det|^{\frac{a+1}{2}-[\frac{a}{2}]} \rho,$$

where

$$\frac{a+1}{2} - \left[\frac{a}{2}\right] = \begin{cases} \frac{1}{2}, & \text{if } a \text{ is even,} \\ 1, & \text{if } a \text{ is odd.} \end{cases}$$

Note that $\delta(a, b, \rho)$ gives rise to $[\delta(a, b, \rho)] = |\det|^{\frac{a-b}{4}} St(\rho, \frac{a+b}{2})$ as the unique subrepresentation of

$$\operatorname{Ind}_{GL_k \times \cdots \times GL_k}^{GL_{k\frac{a+b}{2}}} \delta(a, b, \rho);$$

$\delta(a, \rho)$ gives rise to $[d(a, \rho)] = |\det|^{\frac{1}{2}[\frac{a+1}{2}]} St(\rho, [\frac{a}{2}])$. Then a partial classification of discrete series shows (*cf.* [M-Ta], [Ja1–Ja3]) that a discrete series τ with generic supercuspidal support is a subrepresentation of

$$\operatorname{Ind} [\delta(a_1, b_1, \rho_1)] \otimes \cdots \otimes [\delta(a_r, b_r, \rho_r)] \otimes [\delta(a_{r+1}, \rho_{r+1})] \otimes \cdots \otimes [\delta(a_{r+l}, \rho_{r+l})] \otimes \tau_0,$$

where

(1) $\rho_1, \ldots, \rho_{r+l}$ are self-contragredient supercuspidal representations of GL and τ_0 is a generic supercuspidal representation of G_{l_0}, and
(2) the chain $\delta(a_{r+j}, \rho_{r+j})$ can be present only when (ρ_{r+j}, τ_0) satisfies $(C\frac{1}{2})$ or $(C1)$. In that case, a_{r+j} is even or odd, depending on (ρ_{r+j}, τ_0) satisfies $(C\frac{1}{2})$ or $(C1)$, resp. Also the ρ_{r+j}'s are pairwise non-equivalent.

Of course, the complete classification of discrete series requires additional conditions on a_i, b_i's, such as a_1, b_1, a_2, b_2 are all distinct when $\rho_1 \simeq \rho_2$.

The necessity of the parity condition in (2) can be seen in the following proposition. First we need

Lemma 3.14 *Suppose* (ρ, τ_0) *satisfies* (C1). *Then* $L(s, \rho \times \rho)^{-1}$ *divides* $L(s, \rho \times \tau_0)^{-1}$ *as polynomials in* q^{-s}*, namely,*

$$L(s, \rho \times \tau_0) = L(s, \rho \times \rho) \prod_j (1 - u_j q^{-s})^{-1},$$

where $u_j \in \mathbb{C}$ *is of absolute value* 1.

Proof Let δ be the square integrable representation, which is the unique subrepresentation of Ind $|\det|\rho \otimes \tau_0$. Then by Proposition 3.1(2), $L(s, \rho \times \delta)$ is holomorphic for $\mathrm{Re}(s) > 0$. (For example, if we consider $GL \times SO(\text{odd})$, the second L-function is $L(s, \rho, \mathrm{Sym}^2)$, which is a form given in Proposition 3.1(2) by Lemma 3.4.)

Consider the induced representation Ind $|\det|^s \rho \otimes \delta$. By multiplicativity of γ-factors,

$$\gamma(s, \rho \times \delta, \psi) = \gamma(s, \rho \times \tau_0, \psi)\gamma(s+1, \rho \times \rho, \psi)\gamma(s-1, \rho \times \rho, \psi).$$

If $1 - uq^{-s}$ divides $L(s, \rho \times \rho)^{-1}$, then $1 - uq^{1-s}$ appears in the numerator of $\gamma(s-1, \rho \times \rho, \psi)$. Since $L(s, \rho \times \delta)$ is holomorphic for $\mathrm{Re}(s) > 0$, it should cancel with a factor in the denominator of $\gamma(s, \rho \times \tau_0, \psi)$. Hence $L(s, \rho \times \tau_0)^{-1}$ contains a factor $1 - u^{-1}q^{-s}$. Note that $L(s, \rho \times \rho) = (1 - q^{-rs})^{-1} = \prod (1 - u_i q^{-s})^{-1}$, where $|u_i| = 1$. Hence if $1 - uq^{-s}$ divides $L(s, \rho \times \rho)^{-1}$, then $1 - u^{-1}q^{-s}$ also divides $L(s, \rho \times \rho)^{-1}$. ■

Let σ be a discrete series of GL_k and τ be a discrete series of G_l with generic supercuspidal support. In [Sh1], the γ-factor $\gamma(s, \sigma \times \tau, \psi)$ and the L-function $L(s, \sigma \times \tau)$ are defined only when τ itself is generic. However, if τ is not generic, we define the γ-factor $\gamma(s, \sigma \times \tau, \psi)$, using the multiplicativity of γ-factors in Proposition 3.9. And then as usual, we define the L-function $L(s, \sigma \times \tau)$ to be

$$L(s, \sigma \times \tau) = P(q^{-s})^{-1},$$

where $P(X)$ is the unique polynomial satisfying $P(0) = 1$ such that $P(q^{-s})$ is the numerator of $\gamma(s, \sigma \times \tau, \psi)$. We define the ϵ-factor $\epsilon(s, \sigma \times \tau, \psi)$ to satisfy the relation

$$\gamma(s, \sigma \times \tau, \psi) = \epsilon(s, \sigma \times \tau, \psi)\frac{L(1-s, \tilde{\sigma} \times \tilde{\tau})}{L(s, \sigma \times \tau)}.$$

Note that if two discrete series are subquotients of the same induced representation, they are in the same L-packet. Hence our definition of L-functions agrees with Shahidi's conjecture [Sh1, Section 9] that two discrete series which are in the same L-packet have the same γ-function.

Proposition 3.15 *Suppose* $\sigma = St(\rho, p)$ *and* τ *is a subrepresentation of*

$$\mathrm{Ind}[\delta(a, b, \rho)] \otimes \tau_0.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

(We assume $\frac{a+b}{2} \geq p > b$. The other cases are similar.) Then

$$L(s,\sigma\times\tau)=L(s+\frac{p-1}{2},\rho\times\tau_0)\prod_{i=0}^{p-1}L(s+\frac{a-1}{2}+\frac{p-1}{2}-i,\rho\times\rho)$$
$$\times\prod_{i=0}^{b-1}L(s+\frac{b-1}{2}+\frac{p-1}{2}-i,\rho\times\rho),$$

If τ is a subrepresentation of $\mathrm{Ind}[\delta(a,\rho)]\otimes\tau_0$*, then (assume $\frac{a}{2}\geq p$ — the other cases are similar)*

$$L(s,\sigma\times\tau)=\begin{cases}L(s+\frac{p-1}{2},\rho\times\tau_0)\prod_{i=0}^{p-1}L(s+\frac{a-1}{2}+\frac{p-1}{2}-i,\rho\times\rho) & \text{if } a \text{ is even,}\\ \dfrac{L(s+\frac{p-1}{2},\rho\times\tau_0)}{L(s+\frac{p-1}{2},\rho\times\rho)}\prod_{i=0}^{p-1}L(s+\frac{a-1}{2}+\frac{p-1}{2}-i,\rho\times\rho) & \text{if } a \text{ is odd.}\end{cases}$$

Proof First we calculate $L(s,\sigma\times\tau_0)$. $I(s,\sigma\otimes\tau_0)$ is a subrepresentation of

$$\mathrm{Ind}\,|\det|^{s+\frac{p-1}{2}}\rho\otimes|\det|^{s+\frac{p-1}{2}-1}\rho\otimes\cdots\otimes|\det|^{s-\frac{p-1}{2}}\rho\otimes\tau_0.$$

By multiplicativity of γ-factors,

$$\gamma(s,\sigma\times\tau_0,\psi)=\prod_{i=0}^{p-1}\gamma(s+\frac{p-1}{2}-i,\rho\times\tau_0,\psi).$$

If $L(s,\rho\times\tau_0)$ has a pole at $s=0$ (*i.e.*, when $I(s,\rho\otimes\tau_0)$ is reducible at $s=1$), then note that there is a cancellation between $\gamma(s+\frac{p-1}{2}-i-1,\rho\times\tau_0,\psi)$ and $\gamma(s-\frac{p-1}{2}+i,\rho\times\tau_0,\psi)$. Hence

$$L(s,\sigma\times\tau_0)=L(s+\frac{p-1}{2},\rho\times\tau_0).$$

If $L(s,\rho\times\tau_0)$ has no pole at $s=0$, then $L(s,\sigma\times\tau_0)=1$.

Next, suppose τ is a subrepresentation of $\mathrm{Ind}[\delta(a,b,\rho)]\otimes\tau_0$. Then $I(s,\sigma\otimes\tau)$ is a subrepresentation of

$$\mathrm{Ind}\,|\det|^{s}St(\rho,p)\otimes|\det|^{\frac{a-b}{4}}St\big(\rho,\frac{a+b}{2}\big)\otimes\tau_0.$$

By multiplicativity of γ-factors,

$$\gamma(s,\sigma\times\tau,\psi)=\gamma(s,St(\rho,p)\times\tau_0,\psi)\gamma\Big(s\pm\frac{a-b}{4},St(\rho,p)\times St\big(\rho,\frac{a+b}{2}\big),\psi\Big).$$

We only do the case $\frac{a+b}{2} \geq p > b$. Then $a > p$, and

$$L(s, \sigma \times \tau) = L(s + \frac{p-1}{2}, \rho \times \tau_0) \prod_{i=0}^{p-1} L(s + \frac{a-1}{2} + \frac{p-1}{2} - i, \rho \times \rho)$$
$$\times \prod_{i=0}^{b-1} L(s + \frac{b-1}{2} + \frac{p-1}{2} - i, \rho \times \rho).$$

Next, suppose τ is a subrepresentation of $\mathrm{Ind}[\delta(a, \rho)] \otimes \tau_0$. Then $I(s, \sigma \otimes \tau)$ is a subrepresentation of

$$\mathrm{Ind}\ |\det|^s St(\rho, p) \otimes |\det|^{\frac{1}{2}[\frac{a+1}{2}]} St\big(\rho, \Big[\frac{a}{2}\Big]\big) \otimes \tau_0.$$

By multiplicativity of γ-factors,

$$\gamma(s, \sigma \times \tau, \psi) = \gamma(s, St(\rho, p) \times \tau_0, \psi)\gamma\big(s \pm \frac{1}{2}\Big[\frac{a+1}{2}\Big], St(\rho, p) \times St\big(\rho, \Big[\frac{a}{2}\Big]\big), \psi).$$

Suppose first a is even and for convenience, $\frac{a}{2} \geq p$. Then in $\gamma(s - \frac{a}{4}, St(\rho, p) \times St(\rho, \frac{a}{2}), \psi)$, there is a cancellation between $\gamma(s - \frac{1}{2} - \frac{p-1}{2} + i, \rho \times \rho, \psi)$ and $\gamma(s - \frac{1}{2} + \frac{p-1}{2} - i, \rho \times \rho, \psi)$ for $i = 0, 1, \dots, [\frac{p-1}{2}]$. Hence if p is odd, there is a middle term $\gamma(s - \frac{1}{2}, \rho \times \rho, \psi)$, which cancels with itself. Therefore,

$$L(s, \sigma \times \tau) = L(s + \frac{p-1}{2}, \rho \times \tau_0) \prod_{i=0}^{p-1} L(s + \frac{a-1}{2} + \frac{p-1}{2} - i, \rho \times \rho).$$

Suppose a is odd and for convenience, $\frac{a-1}{2} \geq p$. Recall that (ρ, τ_0) satisfies $(C1)$. Then in $\gamma(s - \frac{a+1}{4}, St(\rho, p) \times St(\rho, \frac{a-1}{2}), \psi)$, there is a cancellation between $\gamma(s - 1 - \frac{p-1}{2} + i + 1, \rho \times \rho, \psi)$ and $\gamma(s - 1 + \frac{p-1}{2} - i, \rho \times \rho, \psi)$ for $i = 0, \dots, [\frac{p-1}{2}]$. Hence if p is odd, only $\gamma(s - 1 - \frac{p-1}{2}, \rho \times \rho, \psi)$ contributes. If p is even, two terms $\gamma(s - \frac{1}{2}, \rho \times \rho, \psi)$ and $\gamma(s - 1 - \frac{p-1}{2}, \rho \times \rho, \psi)$ contribute. However, $\gamma(s - \frac{1}{2}, \rho \times \rho, \psi)$ cancels with itself. By the above lemma, $\gamma(s - 1 - \frac{p-1}{2}, \rho \times \rho, \psi)$ cancels with $\gamma(s - \frac{p-1}{2}, \rho \times \tau_0, \psi)$. Hence

$$L(s, \sigma \times \tau) = \frac{L(s + \frac{p-1}{2}, \rho \times \tau_0)}{L(s + \frac{p-1}{2}, \rho \times \rho)} \prod_{i=0}^{p-1} L(s + \frac{a-1}{2} + \frac{p-1}{2} - i, \rho \times \rho).$$

This completes the proof of Proposition 3.15. ∎

In general, when a discrete series τ is a subrepresentation of

$$\mathrm{Ind}[\delta(a_1, b_1, \rho_1)] \otimes \cdots \otimes [\delta(a_r, b_r, \rho_r)] \otimes [\delta(a_{r+1}, \rho_{r+1})] \otimes \cdots \otimes [\delta(a_{r+l}, \rho_{r+l})] \otimes \tau_0,$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

then

$$\gamma(s,\sigma\times\tau,\psi)=\gamma(s,St(\rho,p)\times\tau_0,\psi)$$
$$\times\prod_{i=1}^{r}\gamma\Big(s\pm\frac{a_i-b_i}{4},St(\rho,p)\times St\Big(\rho_i,\frac{a_i+b_i}{2}\Big),\psi\Big)$$
$$\times\prod_{j=1}^{l}\gamma\Big(s\pm\frac{1}{2}\Big[\frac{a_{r+j}+1}{2}\Big],St(\rho,p)\times St\Big(\rho_{r+j},\Big[\frac{a_{r+j}}{2}\Big]\Big),\psi\Big).$$

Hence we have a similar formula as in Proposition 3.15 and we can see that $L(s,\sigma\times\tau)$ is holomorphic for $\mathrm{Re}(s)>0$.

Exceptional groups will be treated on a case by case analysis. One of the key arguments is the use of the multiplicativity of γ-factors (Proposition 3.9). In the following, π is a generic tempered representation.

3.1 D_n Cases

3.1.1 D_n-1 Case

See [As, Proposition 3.3]. Due to the complicated nature of the Levi subgroup, it is difficult to apply the multiplicativity of γ-factors with $\mathrm{Spin}(2n)$, especially for Steinberg representations. Asgari's idea is to use $G\,\mathrm{Spin}(2n)$.

3.1.2 D_n-2 Case

See [As, Proposition 3.3] or Lemma 3.11.

3.1.3 D_n-3 Case

See [As, Proposition 3.3].

3.2 E_6 Cases

3.2.1 E_6-1

Case 1: π *is a discrete series.* If one of π_i's is not supercuspidal, then by multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for $GL_k\times GL_l$. Apply Proposition 3.10. If all of π_i's are supercuspidal, then apply Lemma 3.4 and Proposition 3.1.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for D_4-2, D_5-2 and $GL_k\times GL_l$. Apply Proposition 3.10 and Lemma 3.11.

3.2.2 $E_6 - 2$

Case 1: π *is a discrete series.* If π_2 is supercuspidal, apply Lemma 3.4 and Proposition 3.1. If π_2 is a non-cuspidal square integrable representation, it is given as the unique subrepresentation of Ind $\mu|\cdot|^2 \otimes \mu|\cdot| \otimes \mu \otimes \mu|\cdot|^{-1} \otimes \mu|\cdot|^{-2}$. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $GL_1 \times GL_2 \subset GL_3$. Hence it is an Artin factor. Apply Proposition 3.10.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $D_5 - 3$, $D_4 - 2$ and $GL_k \times GL_l$. Similarly for L-factors. Apply Proposition 3.12. (Since π is unitarily induced from discrete series, there are no shifts in the complex parameter s).

3.2.3 (x) Case; (xxiv) Case in [La]

Apply Proposition 3.1.

3.3 E_7 Cases

3.3.1 $E_7 - 1$

Case 1: π *is a discrete series.* If π_1 or π_2 is not supercuspidal, then by multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $GL_k \times GL_l$. Hence it is an Artin factor. Apply Proposition 3.10. Suppose π_1 and π_2 are both supercuspidal. If π_3 is supercuspidal, apply Lemma 3.4 and Proposition 3.1. If π_3 is given as the unique subrepresentation of Ind $\rho|\det|^{\frac{1}{2}} \otimes \rho|\det|^{-\frac{1}{2}}$, where ρ is a supercuspidal representation of GL_2, then the rank-one situation in the multiplicativity of γ-factors, is $D_5 - 2$ and $GL_k \times GL_l$. Apply Proposition 3.10.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $E_6 - 1$, $D_6 - 2$, $D_5 - 2$, $D_4 - 2$, and $GL_k \times GL_l$. Similarly for L-factors. Apply Proposition 3.12.

3.3.2 $E_7 - 2$

Case 1: π *is a discrete series.* It is exactly the same as $E_6 - 2$ case.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $E_6 - 2$, $D_n - 3$, $n = 4, 5, 6$, $D_n - 2$, $n = 4, 5$, and $GL_k \times GL_l$. Similarly for L-factors. Apply Proposition 3.12.

3.3.3 $E_7 - 4$

Case 1: π *is a discrete series.* Suppose π_1 is supercuspidal. If π_2 is supercuspidal, then apply Lemma 3.4 and Proposition 3.1. If π_2 is a Steinberg representation, given as the unique subrepresentation of Ind $\mu|\cdot|^{\frac{1}{2}} \otimes \mu|\cdot|^{-\frac{1}{2}}$, then from Section 2.6.4, $L(1-3s,\pi,r_3)$ can have a pole only at $\mathrm{Re}(s)=\frac{1}{2}$. Apply Proposition 3.8.

Suppose π_1 is given as the unique subrepresentation of Ind $\rho|\det|^{\frac{1}{2}} \otimes \rho|\det|^{-\frac{1}{2}}$, where ρ is a supercuspidal representation of GL_3. If π_2 is supercuspidal, from Section 2.6.4, we see that $L(s,\pi,r_3)=1$ and $L(s,\pi,r_2)$ is of the form $L(s,\tilde\rho\times\tilde\rho\otimes\omega')$, where ω' is a unitary character. Hence we can apply Proposition 3.1. If π_2 is a Steinberg representation, by multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for $GL_k\times GL_l$. Apply Proposition 3.10.

Suppose π_1 is given as the unique subrepresentation of Ind $\rho|\det|^{1}\otimes\rho\otimes\rho|\det|^{-1}$, where ρ is a supercuspidal representation of GL_2. By multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for D_4-2 and $GL_2\times GL_1$. Apply Proposition 3.10 and Lemma 3.11.

If π_1 is given as the unique subrepresentation of Ind $\mu|\cdot|^{\frac{5}{2}}\otimes\mu|\cdot|^{\frac{3}{2}}\otimes\mu|\cdot|^{\frac{1}{2}}\otimes\mu|\cdot|^{-\frac{1}{2}}\otimes\mu|\cdot|^{-\frac{3}{2}}\otimes\mu|\cdot|^{-\frac{5}{2}}$, it is similar.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for E_6-2, D_6-2, D_5-2, D_4-2 and $GL_k\times GL_l$. Same for L-factors. Apply Proposition 3.12.

3.3.4 (xi) in [La]

Case 1: σ *is a discrete series.* If σ is supercuspidal, apply Lemma 3.4 and Proposition 3.1. If σ is a Steinberg representation, given as the unique subrepresentation of Ind, $\mu|\cdot|^{3}\otimes\mu|\cdot|^{2}\otimes\cdots\otimes\mu|\cdot|^{-3}$, then by multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for $GL_2\subset GL_3$. Apply Proposition 3.10.

Case 2: σ *is not a discrete series.* Then σ is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for **(x)** in [La], $A_{n-1}\subset D_n$, $n=4,5,6$, and $GL_k\times GL_l$. Same for L-factors. Apply Proposition 3.12.

3.3.5 (xxvi) and (xxx) in [La]

Apply Proposition 3.1.

3.4 E_8 Cases

3.4.1 E_8-1

Case 1: π *is a discrete series.* If all of π_i's are supercuspidal, apply Lemma 3.4 and Proposition 3.1. If not all of π_i's are supercuspidal, one of them is a Steinberg representation, which is a unique subrepresentation of a principal series. Then by multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-factors for $GL_k\times GL_l$. Apply Proposition 3.10.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from a discrete series. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for E_7-1, E_6-1, D_n-2, $n = 4, 5, 6, 7$, and $GL_k \times GL_l$. Similarly for L-factors. Apply Proposition 3.12.

3.4.2 $E_8 - 2$

Case 1: π *is a discrete series.* If π_2 is a Steinberg representation, which is a subrepresentation of a principal series, then by multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-factors for $GL_k \times GL_l$. Apply Proposition 3.10.

Suppose π_2 is supercuspidal. If π_1 is supercuspidal, apply Lemma 3.4 and Proposition 3.1. If π_1 is a Steinberg representation, which is a subrepresentation of a principal series, then apply Proposition 3.10 through multiplicativity of γ-factors. If π_1 is given as the unique subrepresentation of Ind $\rho|\det|^{\frac{1}{2}} \otimes \rho|\det|^{-\frac{1}{2}}$, where ρ is a supercuspidal representation of GL_2, then by multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $E_6 - 2$, namely,

$$\gamma(s, \pi, r_1, \psi) = \gamma(s + \frac{1}{2}, \sigma_1, \psi)\gamma(s - \frac{1}{2}, \sigma_2, \psi),$$

where σ_1, σ_2 are square integrable representations of M' whose derived group is $SL_2 \times SL_5$. Note that $L(s, \sigma_i)$ is holomorphic for $\mathrm{Re}(s) > 0$ by the $E_6 - 2$ case and hence the only possible pole of $L(s, \pi, r_1)$ is $\mathrm{Re}(s) = \frac{1}{2}$, which is excluded.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for E_7-1, E_7-2, E_6-2, D_n-3, $n = 4, 5, 6, 7$, $D_n - 2$, $n = 4, 5, 6$, and $GL_k \times GL_l$. Similarly for L-factors. Apply Proposition 3.12.

3.4.3 $E_8 - 5$

Case 1: π *is a discrete series.* If π is supercuspidal, apply Lemma 3.4 and Proposition 3.1. If π is not supercuspidal, one of π_i's is a subrepresentation of a principal series, and apply Proposition 3.10.

Case 2: π *is not a discrete series.* Then π is a full induced representation, unitarily induced from a discrete series. By multiplicativity of γ-factors, $\gamma(s, \pi, r_1, \psi)$ is a product of γ-functions for rank-one situations for $E_7 - 4$, $E_6 - 1$, $D_5 - 3$, $D_n - 2$, $n = 4, 5, 6, 7$, and $GL_k \times GL_l$. Similarly for L-factors. Apply Proposition 3.12.

3.4.4 (xiii) in [La]

Case 1: σ *is a discrete series.* If σ is supercuspidal, apply Lemma 3.4 and Proposition 3.1. If σ is given as the unique subrepresentation of Ind $\rho|\det|^{\frac{1}{2}} \otimes \rho|\det|^{-\frac{1}{2}}$, where ρ is a supercuspidal representation of GL_4, then from Section 2.7.6, we see that $L(s, \pi, r_3) = 1$ and $L(s, \pi, r_2)$ is of the form $L(s, \tilde{\rho} \times \tilde{\rho} \otimes \omega')$, where ω' is a unitary character. Hence we can apply Proposition 3.1. If σ is given as the unique subrepresentation of Ind $\rho|\det|^{\frac{3}{2}} \otimes \rho|\det|^{\frac{1}{2}} \otimes \rho|\det|^{-\frac{1}{2}} \otimes \rho|\det|^{-\frac{3}{2}}$, where ρ is a supercuspidal representation of GL_2, then by multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-factors for D_4-2 and $GL_2\times GL_1$. Apply Proposition 3.10 and Lemma 3.11. If σ is a Steinberg representation, which is a subrepresentation of a principal series, then apply Proposition 3.10 through multiplicativity of γ-factors.

Case 2: *σ is not a discrete series.* Then σ is a full induced representation, unitarily induced from discrete series. By multiplicativity of γ-factors, $\gamma(s,\pi,r_1,\psi)$ is a product of γ-functions for rank-one situations for **(x)**, **(xi)**, $A_{n-1}\subset D_n$, $n=4,5,6,7$, and $GL_k\times GL_l$. Similarly for L-factors. Apply Proposition 3.12.

3.4.5 (xxxii) in [La]

Apply Proposition 3.1.

In conclusion, we have proved:

Theorem 3.16 *Let π be tempered and generic. Then, except possibly for the four cases E_7-3, E_8-3, E_8-4, and* **(xxviii)** *in* [La] *($D_7\subset E_8$), $L(s,\pi,r_1)$ is holomorphic for* $\mathrm{Re}(s)>0$.

Remark In the four exceptional cases above, the Levi subgroups involve either a group of type D_n (spin group) or an exceptional group of type E_6. Due to lack of the classification of generic discrete series for the groups of type D_n and E_6, we are unable to prove the conjecture. However, we may only need a partial classification.

4 Proof of Assumption (A)

Recall the following from [Ki3]:

***Assumption* (A)** *Let $\pi=\bigotimes_v \pi_v$ be a generic cuspidal representation of* $\mathbf{M}(\mathbb{A})$. *Then $N(s,\pi_v,w_0)$ is holomorphic and non-zero for* $\mathrm{Re}(s)\geq\frac{1}{2}$ *for any v.*

This assumption is absolutely necessary in determining poles of automorphic L-functions in Langlands functionality [CKPSS, Ki-Sh, Ki5]. It is also essential in determining the residual spectrum (*cf.* [Ki1]). In fact, we need a stronger asseriton that $N(s,\pi_v,w_0)$ is holomorphic for $\mathrm{Re}(s)\geq 0$. We start with:

Lemma 4.1 *Let ρ be a supercuspidal representation of $\mathbf{M}(F_v)$. Then the normalized intertwining operator $N(s,\rho,w_0)$ is holomorphic and non-zero except possibly at* $\mathrm{Re}(s)=-1$, *unless $m\geq 2$ and the induced representation $I(s,\rho)$ is reducible at $s=\frac{1}{2}$, in which case $N(s,\rho,w_0)$ is holomorphic and non-zero except at* $\mathrm{Re}(s)=-\frac{1}{2}$.

Proof By the general theory in [Sh1], in (1.1), $\prod_{i=1}^m L(is,\rho,r_i)^{-1}A(s,\rho,w_0)$ is entire and non-zero for a supercuspidal representation ρ. Therefore the poles of $N(s,\rho,w_0)$ come from zeros of $\prod_{i=1}^m L(1+is,\rho,r_i)^{-1}$. However, by Lemma 3.4,

$$\prod_{i=1}^m L(1+is,\rho,r_i)^{-1}$$

has a zero at $\mathrm{Re}(s) = -\frac{1}{2}$ or -1, at only one of them. ■

Lemma 4.2 *Let π_v be a tempered, generic representation of $\mathbf{M}(F_v)$. Then $N(s, \pi_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}(s) \geq 0$, except for the four cases excluded in Theorem 3.16.*

Proof In (1.1), $A(s, \pi_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}(s) > 0$. By Theorem 3.16, $L(s, \pi_v, r_i)$ is holomorphic for $\mathrm{Re}(s) > 0$ except for the cases excluded in Theorem 3.16. Hence $N(s, \pi_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}(s) > 0$. For $\mathrm{Re}(s) = 0$, it is well-known by the theory of R-groups. (Or see [Zh, Lemma 2].) ■

Lemma 4.3 *Let π_v be a generic tempered representation which is a subrepresentation of $I(\Lambda, \rho)$, where ρ is a supercuspidal representation and the coordinates of Λ are half-integers,* i.e., *$\langle \Lambda, \beta^\vee \rangle$ is a half-integer for all positive roots. Then $N(s, \pi_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}(s) > -\frac{1}{2m}$, where m is as in* (1.1).

Proof We only have to show for $-\frac{1}{2m} < \mathrm{Re}(s) < 0$. By the assumption, $I(s, \pi_v) \subset I(s\tilde{\alpha} + \Lambda, \rho)$. Then

$$N(s, \pi_v, w_0) = N(s\tilde{\alpha} + \Lambda, \rho, w')|_{I(s,\pi_v)}.$$

Note that $\langle s\tilde{\alpha}+\Lambda, \beta^\vee \rangle = is+$ half-integers, where $i = 1, \ldots, m$. Hence by Lemma 4.1, $N(s\tilde{\alpha} + \Lambda, \rho, w')$ is holomorphic except for $\mathrm{Re}(s) = \frac{n}{i}$ or $\frac{n}{2i}$, where $i = 1, \ldots, m$ and $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$, we have $\frac{n}{i}, \frac{n}{2i} \notin (-\frac{1}{2m}, 0)$, and so $N(s, \pi_v, w_0)$ is holomorphic for $-\frac{1}{2m} < \mathrm{Re}(s) < 0$. Since its inverse is holomorphic in this region, it would have to be non-zero there also. ■

In many cases, such as $\mathbf{M} = GL_k \times SO_{2l}$ or $GL \times SO_{2l+q}$, we have $\langle s\tilde{\alpha}+\Lambda, \beta^\vee \rangle = is+$ integers. In those cases, $N(s, \pi_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}(s) > -\frac{1}{m}$.

Corollary 4.4 *Let π_v be a generic tempered representation.*

(1) *In the case of $D_n - 2$, $N(s, \pi_v, w_0)$ is holomorphic and non-zero for* $\mathrm{Re}(s) > -\frac{1}{4}$.
(2) *In the case of $A_{n-1} \subset D_n$, $N(s, \pi_v, w_0)$ is holomorphic and non-zero for* $\mathrm{Re}(s) > -\frac{1}{2}$.

Proof Just observe that in the case of $D_n - 2$, $A_{n-1} \subset D_n$, π_v is a tempered representation of GL_k and we know that any tempered representation of GL_k is a subrepresentation of $I(\Lambda, \rho)$, where ρ is a supercuspidal representation of GL and the coordinates of Λ are half-integers. ■

In the case of $GL_k \times GL_l \subset GL_{k+l}$, we have (see [Ki4, Lemma 2.10]):

Proposition 4.5 ([M-W2]) *Let σ (τ) be a tempered representation of GL_k (GL_l, resp.). Then the normalized intertwining operator $N(s, \sigma \otimes \tau, w_0)$ is holomorphic and non-zero for* $\mathrm{Re}(s) > -1$.

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

Now let $\pi = \bigotimes_v \pi_v$ be a generic unitary cuspidal representation of $\mathbf{M}(\mathbb{A})$. Then for all v, π_v is generic and unitary. Suppose π_v is non-tempered. The following standard module conjecture is proved for various cases including GL_n. Especially it is true for archimedean places due to Vogan [V]. In [Mu1], it is proved for Sp_{2n} and SO_{2n+1} over non-archimedean places. In [Ca-Sh], it is proved for any quasi-split group when π_0 is supercuspidal.

Standard Module Conjecture *Given a non-tempered, generic π_v, there is a tempered data π_0 and a complex parameter Λ_0 which is in the corresponding positive Weyl chamber so that*

$$\pi_v = I_{M_0}(\Lambda_0, \pi_0) = \mathrm{Ind}_{M_0}^{M}\left(\pi_0 \otimes q_v^{\langle \Lambda_0, H_{P_0}^M(\ )\rangle}\right).$$

Recall the following [Ki3, Lemma 2.4].

Lemma 4.6 *If $s\tilde{\alpha} + \Lambda_0$ is in the corresponding positive Weyl chamber for* $\mathrm{Re}(s) \geq \frac{1}{2}$ *together with standard module conjecture and Conjecture 7.1 of [Sh1], then Assumption (A) holds.*

Lemma 4.7 ([Zh]) *Let π_0 be an irreducible tempered, generic representation and consider the induced representation $I(\Lambda, \pi_0)$. Assume Conjecture 7.1 of [Sh1] for each rank-one situation. If $N(\Lambda, \pi_0, w_0)$ is holomorphic at Λ_0, then it is non-zero at Λ_0.*

Recall:

Proposition 4.8 (Langlands [La2, Lemma 7.5] or [Ki3, Proposition 2.1]) *Let π be a cuspidal representation of $\mathbf{M}(\mathbb{A})$. Unless $\mathbf{P} = \mathbf{MN}$ is self-conjugate and $w_0\pi \simeq \pi$, the global intertwining operator $M(s, \pi, w_0)$ is holomorphic for* $\mathrm{Re}(s) \geq 0$.

The following proposition is an immediate consequence of [Ki4, Proposition 1.8], (see the proof of [Sh3, Theorem 5.2]).

Proposition 4.9 *Let $\pi = \bigotimes_v \pi_v$ be a unitary, generic cuspidal representation of $\mathbf{M}(\mathbb{A})$. Fix a place v. If π_v is non-tempered, assume the standard module conjecture and write π_v as $\pi_v = I_{M_0}(\Lambda_0, \pi_0)$. Assume Conjecture 7.1 of* [Sh1] *for each rank-one situation so that Lemma 4.7 may be applied. Then the normalized operator $N(s, \pi_v, w_0)$ and the local L-function $L(s, \pi_v, r_1)$ are holomorphic for* $\mathrm{Re}(s) \geq 1$.

Proof Fix a place w where π_w is spherical. By checking the L-functions in Section 2 (or use [Ki-Sh, Proposition 2.1]), we can take a grössencharacter χ such that

(1) $\chi_v = 1$ and χ_w is highly ramified;
(2) $w_0(\pi \otimes \chi) \not\simeq \pi \otimes \chi$;
(3) $w_0'(\pi_i' \otimes \chi) \not\simeq \pi_i' \otimes \chi$ for all i, where π_i' is as in Proposition 3.5, namely, $L(s, \pi, r_i) = L(s, \pi_i', r_1')$, and w_0' is the Weyl group element for π_i'.

Then $M(s, \pi \otimes \chi, w_0)$ and $M(s, \pi_i' \otimes \chi, w_0')$ are holomorphic for $\mathrm{Re}(s) \geq 0$ by Proposition 4.8. Hence by omitting χ, we can assume that $M(s, \pi, w_0)$ and $M(s, \pi_i', w_0')$ are holomorphic for $\mathrm{Re}(s) \geq 0$.

Recall (see [Sh3, (2.7)])

$$M(s,\pi,w_0)f=\prod_{i=1}^{m}\frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}\bigotimes_{u\notin S}\tilde{f}_u\otimes\bigotimes_{u\in S}A(s,\pi_u,w_0)f_u, \tag{4.1}$$

where S is a finite set of places including archimedean places such that $v\in S$ and π_u is unramified for $u\notin S$, and $f=\bigotimes_u f_u$ is such that for each $u\notin S$, f_u is the unique K_u-fixed function normalized by $f_u(e_u)=1$ and $\tilde{f}_u$ is the K_u-fixed function in the space of $I(-s,w_0(\pi_u))$, normalized in the same way.

Now, by induction, we show that for all i, $L_S(s,\pi,r_i)$ is holomorphic for $\mathrm{Re}(s)\geq\frac{1}{2}$, and has no zeros for $\mathrm{Re}(s)\geq 1$. For each $u\in S$, $A(s,\pi_u,w_0)$ is not a zero operator. Since $M(s,\pi,w_0)$ is holomorphic for $\mathrm{Re}(s)\geq 0$, the quotient $\prod_{i=1}^m\frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}$ is holomorphic for $\mathrm{Re}(s)\geq 0$. Now starting at $\mathrm{Re}(s)>N_0$, where $\prod_{i=1}^m L_S(is,\pi,r_i)$ is absolutely convergent, and arguing inductively, we can see that $\prod_{i=1}^m L_S(is,\pi,r_i)$ is holomorphic for $\mathrm{Re}(s)\geq 0$.

Next, recall the ψ-Fourier coefficient of the Eisenstein series [Sh2] (see [Ki3, Lemma 2.3]):

$$E_\psi(s,f,e,P)=\frac{\prod_{u\in S}W_{f_u}(s,e_u)}{\prod_{i=1}^m L_S(1+is,\pi,r_i)}.$$

Since $M(s,\pi,w_0)$ is holomorphic for $\mathrm{Re}(s)\geq 0$, the Eisenstein series is holomorphic in the same region, and hence $\prod_{i=1}^m L_S(1+is,\pi,r_i)$ has no zeros for $\mathrm{Re}(s)\geq 0$.

Now we apply the induction on m. First, let $m=1$. It is clear. Suppose our assertion is true for $L_S(s,\pi,r_i)$, $i=2,\dots,m$, *i.e.*, for all $2\leq i\leq m$, $L_S(s,\pi,r_i)$ is holomorphic for $\mathrm{Re}(s)\geq\frac{1}{2}$, and has no zeros for $\mathrm{Re}(s)\geq 1$. Since $\prod_{i=1}^m L_S(is,\pi,r_i)$ is holomorphic for $\mathrm{Re}(s)\geq 0$, $L_S(s,\pi,r_1)$ is holomorphic for $\mathrm{Re}(s)\geq\frac{1}{2}$. Since $\prod_{i=1}^m L_S(1+is,\pi,r_i)$ has no zeros for $\mathrm{Re}(s)\geq 0$, $L_S(s,\pi,r_1)$ has no zeros for $\mathrm{Re}(s)\geq 1$. This finishes the induction step.

Applying the induction again on m, this time for the local L-functions, we can assume that $L(s,\pi_v,r_i)$, $i=2,\dots,m$, is holomorphic for $\mathrm{Re}(s)\geq 1$. Now we normalize $A(s,\pi_v,w_0)$ as in (1.1). Since for each $u\in S$, $u\neq v$, $A(s,\pi_u,w_0)$ is not a zero operator, pick f_u, $u\in S$, $u\neq v$, so that $A(s,\pi_u,w_0)f_u\neq 0$. Then (4.1) is written as

$$\begin{aligned}M(s,\pi,w_0)f=\prod_{i=1}^{m}\frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}\prod_{i=1}^{m}\frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)}\bigotimes_{u\notin S}\tilde{f}_u\\ \otimes\bigotimes_{u\in S,u\neq v}A(s,\pi_u,w_0)f_u\otimes\frac{N(s,\pi_v,w_0)}{\prod_{i=1}^m\epsilon(s,\pi_v,r_i,\psi_v)}.\end{aligned}$$

Now pick $N_0\geq 1$ so large that $L(1+s,\pi_v,r_1)$ has no poles for $\mathrm{Re}(s)\geq N_0$. If $\mathrm{Re}(s)\geq N_0-1$, the left-hand side is holomorphic and each term on the right-hand side except possibly $N(s,\pi_v,w_0)$ is not zero there. Hence the normalized operator $N(s,\pi_v,w_0)$ is holomorphic for $\mathrm{Re}(s)\geq N_0-1$. By Lemma 4.7, $N(s,\pi_v,w_0)$ is non-vanishing for $\mathrm{Re}(s)\geq N_0-1$ (apply it by identifying $N(s,\pi_v,w_0)$ with $N(s\tilde{\alpha}+\Lambda_0,\pi_0,w_0)$). Hence $L(s,\pi_v,r_1)$ has no poles for $\mathrm{Re}(s)\geq N_0-1$. Arguing inductively, we see that $L(s,\pi_v,r_1)$ has no poles for $\mathrm{Re}(s)\geq 1$. ■

The above proposition has a very important application when applied to the E_8-2 case. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let $\mathrm{diag}(\alpha_v, \beta_v)$ be the Satake parameter for an unramified π_v. Let $\pi_1 = A^3(\pi) = \mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$, constructed in [Ki-Sh], and $\pi_2 = \mathrm{Sym}^4(\pi)$, constructed in [Ki5]. Then we obtain the L-function $L(s, \pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5)$ in the $E_8 - 2$ case. In [Ki-Sh2], we applied the machinery of [Sh3] and showed that $q_v^{-1/9} < |\alpha_v|, |\beta_v| < q_v^{1/9}$, using the fact that the local L-function $L(s, \pi_v, r_1)$ is holomorphic for $\mathrm{Re}(s) \geq 1$ for π_v unramified [Sh3, Lemma 5.8]. Now our explicit calculation of the L-functions enable us to extend the result to the archimedean places, thanks to Proposition 4.9. Let S be a finite set of places of finite places such that π_v is unramified for $v \notin S$, $v < \infty$. By standard calculation, we have

$$L_S(s, \pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5) = L_S(s, \pi, \mathrm{Sym}^9) L_S(s, \pi, \mathrm{Sym}^7 \otimes \omega_\pi) \\ \times L_S(s, \pi, \mathrm{Sym}^5 \otimes \omega_\pi^2)^2 L_S(s, \mathrm{Sym}^3(\pi) \otimes \omega_\pi^3)^2 L_S(s, \pi \otimes \omega_\pi^4).$$

This immediately implies meromorphic continuation and the functional equation of the 9th symmetric power L-functions. Now Proposition 4.9 implies that for each v, $L(s, \pi_{1v} \otimes \pi_{2v}, \rho_4 \otimes \wedge^2 \rho_5)$ is holomorphic for $\mathrm{Re}(s) \geq 1$, and so is $L(s, \pi_v, \mathrm{Sym}^9)$. Therefore we have:

Theorem 4.10 *Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let π_v be a local (finite or infinite) spherical component, given by $\pi_v = \mathrm{Ind}(|\cdot|_v^{s_{1v}} \otimes |\cdot|_v^{s_{2v}})$. Then*

$$|\,\mathrm{Re}(s_{iv})| < \frac{1}{9}.$$

If $F = \mathbb{Q}$, $v = \infty$, this means

$$\lambda_1 = \frac{1}{4}(1 - s^2) > \frac{77}{324} \approx 0.238,$$

where $s = 2s_{1v} = -2s_{2v}$ and λ_1 is the first eigenvalue of the Laplace operator on the corresponding hyperbolic space.

Now we prove:

Theorem 4.11 *Assumption* (A) *holds except possibly for the following twelve cases. Five cases where the standard module conjecture is not available: $B_n - 1$ $(\mathrm{Spin}(2n+1))$; $D_n - 1$ $(\mathrm{Spin}(2n))$;* **(xxx)** *in* [La] *($E_6 \subset E_7$); $E_8 - 4$;* **(xxxii)** *in* [La] *($E_7 \subset E_8$). Seven cases where the Levi subgroup contains a group of type B_3, C_3, D_n:* **(xviii)** *in* [La] *($B_3 \subset F_4$);* **(xxii)** *in* [La] *($C_3 \subset F_4$);* **(xxiv)** *in* [La] *($D_5 \subset E_6$); $E_7 - 3$;* **(xxvi)** *in* [La] *($D_6 \subset E_7$); $E_8 - 3$;* **(xxviii)** *in* [La] *($D_7 \subset E_8$).*

By Lemma 4.2, we only have to show for non-tempered π_v. Using standard module conjecture, we denote

$$I(s, \pi_v) = I(s\tilde{\alpha} + \Lambda_0, \pi_0) \subset I(s\tilde{\alpha} + \Lambda_0', \sigma_v),$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

where π_0 is a generic tempered representation and σ_v is a generic discrete series. Hence we can identify $N(s, \pi_v, w_0)$ with $N(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w}_0)$. Also we have

$$N(s, \pi_v, w_0) = N(s\tilde{\alpha} + \Lambda_0', \sigma_v, w')|_{I(s,\pi_v)}.$$

It is enough to show that $N(s\tilde{\alpha} + \Lambda_0', \sigma_v, w')$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$. Then $N(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w}_0)$ is holomorphic there, and by Zhang's lemma (Lemma 4.7), it is non-zero as well. In what follows, we can assume that s is real. All we need to do is that for $\frac{1}{2} \leq s < 1$, rank-one normalized operators are holomorphic. We can see checking case by case, that in the cases under consideration, rank-one operators for the exceptional four cases which were excluded in Theorem 3.16 do not appear. By identifying roots of $\mathbf{G}$ with respect to a parabolic subgroup, with those of $\mathbf{G}$ with respect to the maximal torus, it is enough to check $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle > -1$ if the rank-one operators are for those of $GL_k \times GL_l \subset GL_{k+l}$. If there are rank-one operators for other situation, we need to check $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle > -\frac{1}{2m}$.

We check case by case. First recall the classification of unitary representations of $GL_n(F_v)$ due to Tadic [Ta]: a generic, unitary representation π_v is of the form

$$\begin{aligned}\pi_v &= \mathrm{Ind}\, |\det|^{r_1}\sigma_1 \otimes |\det|^{r_k}\sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes |\det|^{-r_k}\sigma_k \otimes \cdots \otimes |\det|^{-r_1}\sigma_1 \\ &= I(\Lambda_0, \pi_0),\end{aligned}$$

where $0 < r_k \leq \cdots \leq r_1 < \frac{1}{2}$ and $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l$ are discrete series of $GL_{n_i}(F_v)$. Here we can write Λ_0 as $\Lambda_0 = s_1e_1 + s_2e_2 + \cdots + (-s_2)e_{n-1} + (-s_1)e_n$, where $0 \leq s_{[\frac{n}{2}]} \leq \cdots \leq s_2 \leq s_1 < \frac{1}{2}$. In terms of roots, $\Lambda_0 = s_1\alpha_1 + (s_1+s_2)\alpha_2 + \cdots + (s_1 + \cdots + s_{[\frac{n}{2}]})\alpha_{[\frac{n}{2}]} + (s_1 + \cdots + s_{[\frac{n}{2}]-1})\alpha_{[\frac{n}{2}]+1} + \cdots + s_1\alpha_{n-1}$, where $\{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n\}$ is the set of simple roots.

Also let $\tau = \bigotimes_v \tau_v$ be a generic cuspidal representation of

$$G_n(\mathbb{A}) = Sp_{2n}(\mathbb{A}), SO_{2n+1}(\mathbb{A}).$$

We showed in [Ki4, Lemma 3.3] that τ_v is of the form

$$\tau_v = \mathrm{Ind}\, |\det|^{r_1}\tau_1 \otimes \cdots \otimes |\det|^{r_k}\tau_k \otimes \tau_0 = I(\Lambda_0, \pi_0),$$

where $0 < r_k \leq \cdots \leq r_1 < 1$ and $\tau_1, \ldots, \tau_k$ are discrete series of $GL_{n_i}(F_v)$ and τ_0 is a generic tempered representation of $G_l(F_v)$. We can write Λ_0 as $\Lambda_0 = s_1e_1 + \cdots + s_ne_n$, where $0 \leq s_n \leq \cdots \leq s_1 < 1$. We did not treat SO_{2n} in [Ki4] because the standard module conjecture was not available. However, it is now proved for SO_{2n} by Muić [Mu2]. Hence we have the same result, except that Langlands' data are more complicated [Ja1]: τ_v is of the same form as above, or it is induced from the Levi subgroup $M = GL_{n_1} \times \cdots \times GL_{n_k} \times F^\times$. In that case,

$$\tau_v = \mathrm{Ind}\, |\det|^{r_1}\tau_1 \otimes \cdots \otimes |\det|^{r_{k-1}}\tau_{k-1} \otimes |\ |^{r_k}\mu = I(\Lambda_0, \pi_0),$$

where $|r_k| < r_{k-1} < \cdots < r_1 < 1$, and $\tau_1, \ldots, \tau_k$ are tempered representations of GL_{n_i} and μ is a unitary character of $F^\times$. Hence in the case of SO_{2n}, we can write Λ_0 as $\Lambda_0 = s_1e_1 + \cdots + s_{n-1}e_{n-1} + s_ne_n$, where $|s_n| \leq s_{n-1} \leq \cdots \leq s_1 < 1$.

4.1 D_n Cases

$D_n - 1$: We cannot prove Assumption (A) if π_2 is an arbitrary generic cuspidal representation of $G\,\mathrm{Spin}(2l, \mathbb{A})$ since the standard module conjecture is not available. So let π_2 be a generic cuspidal representation of $GSO_{2l}(\mathbb{A})$ and extend it to a generic cuspidal representation of $G\,\mathrm{Spin}(2l, \mathbb{A})$, using the homomorphism $G\,\mathrm{Spin}(2l) \to GSO_{2l}$. Then we can apply the standard module conjecture.

In this case, $\tilde{\alpha} = e_1 + \cdots + e_k$; $\Lambda_0 = r_1e_1 + r_2e_2 + \cdots + (-r_2)e_{k-1} + (-r_1)e_k + r_{k+1}e_{k+1} + \cdots + r_{n-1}e_{n-1} + r_ne_n$, where $\frac{1}{2} > r_1 \geq \cdots \geq r_{[\frac{k}{2}]} \geq 0$, $1 > r_{k+1} \geq \cdots \geq r_{n-1} \geq |r_n|$. Hence

$$s\tilde{\alpha} + \Lambda_0 = (s + r_1)e_1 + \cdots + (s - r_1)e_k + r_{k+1}e_{k+1} + \cdots + r_{n-1}e_{n-1} + r_ne_n.$$

Therefore, we see that if $s \geq \frac{1}{2}$, $s - r_1 - r_{k+1} > -1$ for rank-one situations of $GL_a \times GL_b$. Other rank-one situations appear only when $r_m = 0$ for some $m > k$. In that case, rank-one operators are in the corresponding positive Weyl chamber, and Lemma 4.6 applies.

$D_n - 2$: In this case, $\tilde{\alpha} = e_1 + \cdots + e_{n-2}$; $\Lambda_0 = r_1e_1 + r_2e_2 + \cdots + (-r_2)e_{n-3} + (-r_1)e_{n-2} + s_1(e_{n-1} - e_n) + s_2(e_{n-1} + e_n)$, where $\frac{1}{2} > r_1 \geq r_2 \geq \cdots \geq 0$ and $\frac{1}{2} > s_1, s_2 \geq 0$. Here π_{2v} is tempered if $s_1 = 0$. Hence

$$s\tilde{\alpha} + \Lambda_0 = (s+r_1)e_1 + (s+r_2)e_2 + \cdots + (s-r_2)e_{n-3} + (s-r_1)e_{n-2} + (s_1+s_2)e_{n-1} + (-s_1+s_2)e_n.$$

The rank-one situations are $GL_k \times GL_l$, unless s_1 or s_2 is zero, in which case we can see that the rank-one situations are in the corresponding positive Weyl chamber, and Lemma 4.6 applies. Suppose none of s_1 and s_2 are zero. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle$ is $s - (r_1 + s_1 + s_2) > -1$, if $s \geq \frac{1}{2}$.

$D_n - 3$: In this case, $\Lambda_0 = r_1e_1 + r_2e_2 + \cdots + (-r_2)e_{n-4} + (-r_1)e_{n-3} + (r_1' + r_2')e_{n-2} + (r_1' - r_2')e_{n-1}$, where $\frac{1}{2} > r_1 \geq \cdots \geq r_{[\frac{n-3}{2}]} \geq 0$, $\frac{1}{2} > r_1' \geq r_2' \geq 0$. Here $r_1 = 0$ if π_{1v} is tempered. The same is true for π_{2v}. Hence

$$s\tilde{\alpha} + \Lambda_0 = (s + r_1)e_1 + \cdots + (s - r_1)e_{n-3} + (r_1' + r_2')e_{n-2} + (r_1' - r_2')e_{n-1}.$$

The rank-one situations are $GL_k \times GL_l$, unless $r_1' = r_2' \neq 0$, in which case the rank-one operator is for $D_k - 2$. It is the case when $\pi_{2v} = \mathrm{Ind}\,|\det|^{r'}\rho \otimes |\det|^{-r'}\rho$, where ρ is a tempered representation of GL_2. Then by direct computation, we see that $N(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w}_0)$ is a product of the following three operators; $N(s\tilde{\alpha}' + \Lambda_0', \pi_{1v} \otimes \rho \otimes \rho, w_0')$, $N((s-2r')\tilde{\alpha}' + \Lambda_0', \pi_{1v} \otimes \omega_\rho, w_0')$, $N((s+2r')\tilde{\alpha}' + \Lambda_0', \pi_{1v} \otimes \omega_\rho, w_0')$, where $s\tilde{\alpha}' + \Lambda_0' = (s + r_1)e_1 + \cdots + (s - r_1)e_{n-3}$. The first operator is the operator for the $D_k - 2$ case and it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. The last two operators are the operators for $GL_k \times GL_1$. Since $s - 2r' - r_1 > -1$ if $s \geq \frac{1}{2}$, they are holomorphic.

Suppose we are not in the above case. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle$ is $s - (r_1 + r_1' + r_2') > -1$, if $s \geq \frac{1}{2}$.

4.2 F_4 Cases

$F_4 - 1$: $\tilde{\alpha}_3 = 2e_1 + e_2 + e_3$; $\Lambda_0 = r_1\alpha_1 + r_1\alpha_2 + r_2\alpha_4$, $0 \le r_1, r_2 < \frac{1}{2}$. Here if $r_2 = 0$, π_{2v} is tempered. Then

$$s\tilde{\alpha}_3 + \Lambda_0 = \left(2s + \frac{r_1}{2}\right) e_1 + (s - \frac{r_1}{2} + r_2)e_2 + (s - \frac{r_1}{2} - r_2)e_3 + \frac{r_1}{2}e_4.$$

The rank-one situations are $GL_k \times GL_l$, and the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_2) > -1$ if $s \ge \frac{1}{2}$.

$F_4 - 2$: $\tilde{\alpha}_2 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$; $\Lambda_0 = r_1\alpha_1 + r_2\alpha_3 + r_2\alpha_4$, $0 \le r_1, r_2 < \frac{1}{2}$. Here π_{1v} is tempered if $r_1 = 0$. Then

$$s\tilde{\alpha}_2 + \Lambda_0 = \frac{3s + r_1}{2}e_1 + \left(\frac{s - r_1}{2} + r_2\right) e_2 + \frac{s - r_1}{2}e_3 + \left(\frac{s - r_1}{2} - r_2\right) e_4.$$

The rank-one situations are $GL_k \times GL_l$, and the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_2) > -1$ if $s \ge \frac{1}{2}$.

(xviii) in [La] ($B_3 \subset F_4$): Then $\tilde{\alpha}_1 = e_1$. We cannot prove Assumption (A) if π is an arbitrary generic cuspidal representations of $G\,\mathrm{Spin}(7, \mathbb{A})$ since the standard module conjecture is not available. So let π be a generic cuspidal representation of $SO_7(\mathbb{A})$ and extend it to a generic cuspidal representation of $G\,\mathrm{Spin}(7, \mathbb{A})$, using the homomorphism $G\,\mathrm{Spin}(7) \to SO_7$. Then we can apply the standard module conjecture. However, $\Lambda_0 = a_2e_2 + a_3e_3 + a_4e_4$, where $1 > a_2 \ge a_3 \ge a_4 \ge 0$ and

$$s\tilde{\alpha}_1 + \Lambda_0 = se_1 + a_2e_2 + a_3e_3 + a_4e_4.$$

We can see that the least value of $\langle s\tilde{\alpha}_1 + \Lambda_0, \beta^\vee\rangle$ is $s - (a_2 + a_3 + a_4)$. Hence we need to assume that $a_2 < \frac{1}{2}$, in order to conclude that $s - (a_2 + a_3 + a_4) > -1$ if $s \ge \frac{1}{2}$. In order to obtain $a_2 < \frac{1}{2}$, we need a functorial lift from cuspidal representations of SO_7 to GL_6.

(xxii) in [La] ($C_3 \subset F_4$): Similar to the above case.

4.3 E_6 Cases

$E_6 - 1$: $\tilde{\alpha}_3 = e_1 + e_2 + e_3 + 3\epsilon$; $\Lambda_0 = r_1\alpha_1 + r_1\alpha_2 + r_2\alpha_4 + r_2\alpha_5 + r_3\alpha_6$, where $0 \le r_1, r_2, r_3 < \frac{1}{2}$. Here $r_1 = 0$ if π_{1v} is tempered. Then

$$\begin{aligned} s\tilde{\alpha}_3 + \Lambda_0 = (s + r_1)e_1 + se_2 + (s - r_1)e_3 + (r_2 + r_3)e_4 \\ + r_3e_5 + (r_3 - r_2)e_6 + (3s + r_3)\epsilon. \end{aligned}$$

The rank-one situations $GL_k \times GL_l$ and the least value of $\langle s\tilde{\alpha}_3 + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_2 + r_3)$ when $\beta = \alpha_3$. And $s - (r_1 + r_2 + r_3) > -1$ if $s \ge \frac{1}{2}$.

$E_6 - 2$: $\tilde{\alpha}_2 = e_1 + e_2 + 2\epsilon$; $\Lambda_0 = r_1\alpha_1 + r_2\alpha_5 + (r_2 + r_3)\alpha_4 + (r_2 + r_3)\alpha_3 + r_2\alpha_6$, where $0 \le r_1 < \frac{1}{2}$ and $0 \le r_3 \le r_2 \le r_1 < \frac{1}{2}$. Note that π_{1v} is tempered if $r_1 = 0$. Then

$$\begin{aligned} s\tilde{\alpha}_2 + \Lambda_0 = (s + r_1)e_1 + (s - r_1)e_2 + (r_2 + r_3)e_3 \\ + r_2e_4 + (r_2 - r_3)e_5 + (2s + r_2)\epsilon. \end{aligned}$$

The rank-one situations are $GL_k \times GL_l$, unless π_{2v} is tempered, *i.e.*, $r_2 = r_3 = 0$, in which case the rank-one operator is for $A_4 \subset D_5$ and it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. Suppose π_{2v} is not tempered. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle$ is $s - (r_1 + r_2 + r_3) > -1$, if $s \geq \frac{1}{2}$.

(x) in [La] ($SL_6 \subset E_6$): $\tilde{\alpha}_6 = 2\epsilon$; $\Lambda_0 = r_1 e_1 + r_2 e_2 + r_3 e_3 + (-r_3)e_4 + (-r_2)e_5 + (-r_1)e_6$, where $0 \leq r_3 \leq r_2 \leq r_1 < \frac{1}{2}$. Hence

$$s\tilde{\alpha}_6 + \Lambda_0 = r_1 e_1 + r_2 e_2 + r_3 e_3 + (-r_3)e_4 + (-r_2)e_5 + (-r_1)e_6 + 2s\epsilon.$$

The rank-one situations are $GL_k \times GL_l$, unless $r_2 = r_3 = 0$, in which case the rank-one operator is for $A_3 \subset D_4$. It is the case when $\sigma_v = \mathrm{Ind}_{F^\times \times GL_4 \times F^\times}^{GL_6} |\cdot|^{r_1}\mu \otimes \rho \otimes |\cdot|^{-r_1}\mu$, where ρ is a tempered representation of $GL_4(F_v)$. We can see easily that it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. Suppose we are not in the above case. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle$ is $s - (r_1 + r_2 + r_3) > -1$, if $s \geq \frac{1}{2}$.

(xxiv) in [La] ($D_5 \subset E_6$): $\tilde{\alpha}_1 = e_1 + \epsilon$. We cannot prove Assumption (A) if π is an arbitrary generic cuspidal representations of $G\,\mathrm{Spin}(10, \mathbb{A})$ since the standard module conjecture is not available. So let π be a generic cuspidal representation of $SO_{10}(\mathbb{A})$ and extend it to a generic cuspidal representation of $G\,\mathrm{Spin}(10, \mathbb{A})$, using the homomorphism $G\,\mathrm{Spin}(10) \to SO_{10}$. Then we can apply the standard module conjecture. However,

$$\begin{aligned}\Lambda_0 = r_1\alpha_5 + (r_1 + r_2)\alpha_4 + (r_1 + r_2 + r_3)\alpha_3 \\ + \frac{1}{2}(r_1 + r_2 + r_3 + r_4 - r_5)\alpha_2 + \frac{1}{2}(r_1 + r_2 + r_3 + r_4 + r_5)\alpha_6,\end{aligned}$$

where $1 > r_1 \geq r_2 \geq r_3 \geq r_4 \geq |r_5|$. Then

$$\begin{aligned}s\tilde{\alpha}_1 + \Lambda_0 = se_1 &+ \frac{1}{2}(r_1 + r_2 + r_3 + r_4 - r_5)e_2 + \frac{1}{2}(r_1 + r_2 + r_3 - r_4 + r_5)e_3 \\ &+ \frac{1}{2}(r_1 + r_2 - r_3 + r_4 + r_5)e_4 + \frac{1}{2}(r_1 - r_2 + r_3 + r_4 + r_5)e_5 \\ &+ \frac{1}{2}(-r_1 + r_2 + r_3 + r_4 + r_5)e_6 + (s + \frac{1}{2}(r_1 + r_2 + r_3 + r_4 + r_5))\epsilon.\end{aligned}$$

We see that $(s\tilde{\alpha}_1 + \Lambda_0, e_1 - e_2) = s - \frac{1}{2}(r_1 + r_2 + r_3 + r_4 - r_5)$. We need $r_1 < \frac{1}{2}$ to see that $(s\tilde{\alpha}_1 + \Lambda_0, e_1 - e_2) > -1$ if $s \geq \frac{1}{2}$. In order to obtain $r_1 < \frac{1}{2}$, we need a functorial lift from cuspidal representations of SO_{10} to GL_{10}, which is not yet known.

4.4 E_7 Cases

$E_7 - 1$: $\tilde{\alpha}_4 = e_1 + e_2 + e_3 + e_4 + 4e_8$; $\Lambda_0 = r_1\alpha_1 + (r_1 + r_2)\alpha_2 + r_1\alpha_3 + r_3\alpha_5 + r_3\alpha_6 + r_4\alpha_7$, where $0 \leq r_2 \leq r_1 < \frac{1}{2}, 0 \leq r_3, r_4 < \frac{1}{2}$. Here π_{1v} is tempered when $r_1 = r_2 = 0$.

Then

$$s\tilde{\alpha}_4 + \Lambda_0 = (s+r_1)e_1 + (s+r_2)e_2 + (s-r_2)e_3 + (s-r_1)e_4 + (r_3+r_4)e_5 \\ + r_4e_6 + (r_4-r_3)e_7 + (4s+r_4)e_8.$$

All rank-one situations are $GL_k \times GL_l$, unless $r_1 = r_2$, $r_3 = r_4 = 0$, in which case the rank-one operator is for $D_5 - 2$. It is the case when $\pi_{3v} = \mathrm{Ind}\,|\det|^r\rho \otimes |\det|^{-r}\rho$, where ρ is a tempered representation of $GL_2(F_v)$. We can see easily that it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. Suppose we are not in the above case. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_3 + r_4) > -1$, if $s \geq \frac{1}{2}$.

$E_7 - 2$: $\tilde{\alpha}_3 = e_1+e_2+e_3+3e_8$; $\Lambda_0 = r_1\alpha_1+r_1\alpha_2+r_2\alpha_6+(r_2+r_3)\alpha_5+(r_2+r_3)\alpha_4+r_2\alpha_7$, where $0 \leq r_1 < \frac{1}{2}, 0 \leq r_3 \leq r_2 < \frac{1}{2}$. Here $r_1 = 0$ if π_{1v} is tempered. Then

$$s\tilde{\alpha}_3 + \Lambda_0 = (s+r_1)e_1 + se_2 + (s-r_1)e_3 + (r_2+r_3)e_4 \\ + r_2e_5 + (r_2-r_3)e_6 + (3s+r_2)e_8.$$

The possible rank-one cases are $A_4 \subset D_5$, in which case $r_2 = r_3 = 0$, or $D_5 - 2$, in which case, $r_1 = 0$, $r_2 = r_3$. The remaining cases are $GL_k \times GL_l$. In the first two cases, the rank-one operators are in the corresponding positive Weyl chamber and Lemma 4.6 applies. In the remaining cases, the least value of $\langle s\tilde{\alpha}_3 + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_2 + r_3) > -1$, if $s \geq \frac{1}{2}$.

$E_7 - 4$: $\tilde{\alpha}_5 = e_1 + e_2 + e_3 + e_4 + e_5 + 3e_8$; $\Lambda_0 = r_1\alpha_1 + (r_1 + r_2)\alpha_2 + (r_1 + r_2 + r_3)\alpha_3 + (r_1 + r_2)\alpha_4 + r_1\alpha_7 + r_4\alpha_6$, *where* $r_4 = 0$ if π_{2v} *is tempered and* $0 \leq r_3 \leq r_2 \leq r_1 < \frac{1}{2}$. Then

$$s\tilde{\alpha}_5 + \Lambda_0 = (s+r_1)e_1 + (s+r_2)e_2 + (s+r_3)e_3 + (s-r_3)e_4 \\ + (s-r_2)e_5 + (r_1+r_4)e_6 + (r_1-r_4)e_7 + (3s+r_1)e_8.$$

All rank-one cases are $GL_k \times GL_l$, unless $r_1 = r_2 = r_3 = 0$, in which case the rank-one operator is for $A_5 \subset D_6$. We can easily see that it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. Suppose we are not in the above case. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_2 + r_4) > -1$, if $s \geq \frac{1}{2}$.

(xi) in [La] ($SL_7 \subset E_7$): $\tilde{\alpha}_7 = 2e_8$; $\Lambda_0 = r_1e_1 + r_2e_2 + r_3e_3 - r_3e_5 - r_2e_6 - r_1e_7$, *where* $0 \leq r_3 \leq r_2 \leq r_1 < \frac{1}{2}$. Then

$$s\tilde{\alpha}_7 + \Lambda_0 = r_1e_1 + r_2e_2 + r_3e_3 - r_3e_5 - r_2e_6 - r_1e_7 + 2se_8.$$

All rank-one cases are $GL_k \times GL_l$, unless $r_2 = r_3 = 0$, in which case the rank-one operator is for $A_5 \subset D_6$. It is the case when $\sigma_v = \mathrm{Ind}_{F^\times \times GL_5 \times F^\times}^{GL_7} |\cdot|^r\mu \otimes \rho \otimes |\cdot|^{-r}\mu$, where ρ is a tempered representation of $GL_5(F_v)$. We can easily see that it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. Suppose we are not in the above case. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee\rangle$ is $s - (r_1 + r_2 + r_3) > -1$, if $s \geq \frac{1}{2}$.

4.5 E_8 Cases

E_8-1: $\tilde\alpha_5 = e_1+e_2+e_3+e_4+e_5-5e_9$; $\Lambda_0 = r_1e_1+r_2e_2-r_2e_4-r_1e_5+r_3e_6-r_3e_8+r_4(e_6+e_7+e_8)$, *where* $0\le r_2\le r_1<\frac12$, $0\le r_3<\frac12$, *and* $0\le r_4<\frac12$. Then

$$s\tilde\alpha_5+\Lambda_0 = (6s+r_1)e_1+(6s+r_2)e_2+6se_3+(6s-r_2)e_4 \\ +(6s-r_1)e_5+(5s+r_3+r_4)e_6+(5s+r_4)e_7+(5s+r_4-r_3)e_8.$$

The possible rank-one cases are E_6-1, in which case $r_2=r_3=r_4=0$, or D_5-2, in which case $r_1=r_2$, $r_3=r_4=0$. The remaining cases are $GL_k\times GL_l$. In the first two cases, the rank-one operators are in the corresponding positive Weyl chamber and Lemma 4.6 applies. In the remaining cases, the least value of $\langle s\tilde\alpha_5+\Lambda_0,\beta^\vee\rangle$ is $s-(r_1+r_3+r_4)>-1$, if $s\ge\frac12$.

E_8-2: $\tilde\alpha_4=e_1+e_2+e_3+e_4-4e_9$; $\Lambda_0=r_1e_1+r_2e_2-r_2e_3-r_1e_4+r_3\alpha_7+(r_3+r_4)\alpha_6+(r_3+r_4)\alpha_5+r_3\alpha_8$, *where* $0\le r_2\le r_1<\frac12$ *and* $0\le r_4\le r_3<\frac12$. Then

$$s\tilde\alpha_4+\Lambda_0=(5s+r_1)e_1+(5s+r_2)e_2+(5s-r_2)e_3+(5s-r_1)e_4 \\ +(4s+r_3+r_4)e_5+(4s+r_3)e_6+(4s+r_3-r_4)e_7+4se_8.$$

The possible rank-one cases are E_6-2, in which case $r_1=r_2$, $r_3=r_4=0$, or D_6-2, in which case $r_1=r_2=0$, $r_3=r_4$, or D_4-2, in which case $r_1=r_2$, $r_3=r_4$. The remaining cases are $GL_k\times GL_l$. In the first case, the rank-one operator is in the corresponding positive Weyl chamber and Lemma 4.6 applies. The next two cases occur when $\pi_{2v}=\mathrm{Ind}_{GL_2\times F^\times\times GL_2}^{GL_5}|\det|^r\rho\otimes\mu\otimes|\det|^{-r}\rho$, where ρ is a tempered representation of GL_2. By direct computation, we see that the operators for D_6-2 and D_4-2 are in the corresponding positive Weyl chamber. For example, if π_{1v} is tempered, then we have the operator $N(s,\pi_{1v}\otimes\rho\otimes\rho,w')$. If π_{1v} is of the form $\mathrm{Ind}_{GL_2\times GL_2}^{GL_4}|\det|^{r'}\rho'\otimes|\det|^{-r'}\rho'$, where ρ is a tempered representation of $GL_2(F_v)$, then we have the operator $N(s-r',\rho'\otimes\rho\otimes\rho,w'')$.

In the remaining cases, the least value of $\langle s\tilde\alpha_4+\Lambda_0,\beta^\vee\rangle$ is $s-(r_1+r_3+r_4)>-1$, if $s\ge\frac12$.

E_8-5: $\tilde\alpha_6=e_1+e_2+e_3+e_4+e_5+e_6-3e_9$; $\Lambda_0=r_1\alpha_1+(r_1+r_2)\alpha_2+(r_1+r_2+r_3)\alpha_3+(r_1+r_2+r_3)\alpha_4+(r_1+r_2)\alpha_5+r_1\alpha_8+r_4\alpha_7$, *where* $0\le r_3\le r_2\le r_1<\frac12$ *and* $0\le r_4<\frac12$. Then

$$s\tilde\alpha_6+\Lambda_0=(4s+r_1)e_1+(4s+r_2)e_2+(4s+r_3)e_3+4se_4 \\ +(4s-r_3)e_5+(4s-r_2)e_6+(3s+r_1+r_4)e_7+(3s+r_1-r_4)e_8.$$

All rank-one cases are $GL_k\times GL_l$. The least value of $\langle s\tilde\alpha_6+\Lambda_0,\beta^\vee\rangle$ is $s-(r_1+r_2+r_4)>-1$, if $s\ge\frac12$.

(xiii) in [La] ($SL_8\subset E_8$): $\tilde\alpha_8=-3e_9$; $\Lambda_0=r_1e_1+r_2e_2+r_3e_3+r_4e_4-r_4e_5-r_3e_6-r_2e_7-r_1e_8$, *where* $0\le r_4\le r_3\le r_2\le r_1<\frac12$. Then

$$s\tilde\alpha_8+\Lambda_0=(3s+r_1)e_1+(3s+r_2)e_2+(3s+r_3)e_3+(3s+r_4)e_4 \\ +(3s-r_4)e_5+(3s-r_3)e_6+(3s-r_2)e_7+(3s-r_1)e_8.$$

https://doi.org/10.4153/CJM-2005-023-x Published online by Cambridge University Press

All rank-one cases are $GL_k \times GL_l$, unless $r_2 = r_3 = r_4 = 0$, in which case the rank-one operator is for $A_5 \subset D_6$. It is the case when $\sigma_v = \mathrm{Ind}_{F^\times \times GL_6 \times F^\times}^{GL_8} |\cdot|^r \mu \otimes \rho \otimes |\cdot|^{-r}\mu$, where ρ is a tempered representation of $GL_6(F_v)$. We can easily see that it is in the corresponding positive Weyl chamber and Lemma 4.6 applies. Suppose we are not in the above case. Then the least value of $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle$ is $s - (r_1 + r_2 + r_3) > -1$, if $s \geq \frac{1}{2}$.

Corollary 4.12 (Corollary to the $D_n - 1$ case) *Look at the $D_n - 1$ case ($A_{n-1} \subset D_n$ with n odd). Let π be a cuspidal representation of $GL_n(\mathbb{A})$ with n odd. Then the twisted exterior square L-function $L(s, \pi, \wedge^2 \otimes \chi)$ is entire for any grössencharacter χ. Hence the twisted symmetric square L-function $L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ always has a pole at $s = 0, 1$.*

Proof Apply [Ki3]. ■

Corollary 4.13 (Corollary to the $E_6 - 2$ case) *Look at the $E_6 - 2$ case. Let π_1, π_2 be cuspidal representations of $GL_2(\mathbb{A}), GL_5(\mathbb{A})$, resp. Then the completed L-function $L(s, \pi_1 \otimes \pi_2, \rho_2 \otimes \wedge^2 \rho_5)$ is entire.*

Proof Apply [Ki3, Theorem 3.11]. ■

Remark In the case when $G = Sp_{2n}, SO_{2n+1}, SO_{2n}$, and $M = GL_n$, we have a stronger result that $N(s, \sigma_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}(s) \geq 0$. Just note that a generic, unitary representation σ_v is of the form

$$\sigma_v = \mathrm{Ind}\, |\det|^{r_1}\sigma_1 \otimes |\det|^{r_k}\sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes |\det|^{-r_k}\sigma_k \otimes \cdots \otimes |\det|^{-r_1}\sigma_1,$$

where $0 < r_k \leq \cdots \leq r_1 < \frac{1}{2}$ and $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l$ are discrete series of GL. Then $I(s, \sigma_v)$ is

$$\mathrm{Ind}\, |\det|^{\frac{s}{2}+r_1}\sigma_1 \otimes |\det|^{\frac{s}{2}+r_k}\sigma_k \otimes |\det|^{\frac{s}{2}}\tau_1 \otimes \cdots \otimes |\det|^{\frac{s}{2}}\tau_l \otimes |\det|^{\frac{s}{2}-r_k}\sigma_k \otimes \cdots \otimes |\det|^{\frac{s}{2}-r_1}\sigma_1,$$

if $G = SO_{2n+1}, SO_{2n}$, and

$$\mathrm{Ind}\, |\det|^{s+r_1}\sigma_1 \otimes |\det|^{s+r_k}\sigma_k \otimes |\det|^{s}\tau_1 \otimes \cdots \otimes |\det|^{s}\tau_l \otimes |\det|^{s-r_k}\sigma_k \otimes \cdots \otimes |\det|^{s-r_1}\sigma_1,$$

if $G = Sp_{2n}$.

All rank-one operators are holomorphic and non-zero for $\mathrm{Re}(s) \geq 0$.

Acknowledgments The main result of this paper was obtained at the Institute for Advanced Study in the special year 1999–2000 in the Theory of Automorphic Forms and L-functions. I would like to thank the organizers, E. Bombieri, H. Iwaniec, R. P. Langlands, and P. Sarnak for inviting me to participate in the special year. I would like to thank Prof. F. Shahidi for his constant help in explaining his results and for many discussions, and also Dr. P. Trapa for helpful discussions regarding spin groups. Thanks are due to the referee who read this manuscript thoroughly and gave many comments and suggestions.

References

[A] J. Arthur *Intertwining operators and residues I. Weighted characters.* J. Funct. Anal. **84**(1989), 19–84.

[As] M. Asgari *Local L–functions for split spinor groups.* Canad. J. Math. **54**(2002), 673–693.

[Bo] A. Borel, *Automorphic L–functions.* In: Automorphic Forms and Automorphic Representations, Proc. Sympos. Pure Math. 33, American Mathematical Society, Providence, RI, 1979, pp. 27–61.

[Bou] N. Bourbaki, *Groupes et algébres de Lie.*, Chap. 4-6, Paris: Hermann, 1968.

[Ca-Sh] W. Casselman and F. Shahidi, *On irreducibility of standard modules for generic representations.* Ann. Sci. École Norm. Sup. **31**(1998), 561–589.

[CKPSS] J. Cogdell, H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to GL_N.* Publ. Math. Inst. Hautes Études Sci. **93**(2001), 5–30.

[G-O-V] V. V. Gorbatsevich, A. L. Onishchik and E. B. Vinberg, *Lie Groups and Lie Algebras III.* In: Encyclopedia of Mathematical Sciences, Vol 41, Springer-Verlag, 1990.

[Ja1] C. Jantzen, *Degenerate principal series for orthogonal groups.* J. Reine Angew. Math.**441**(1993), 61–98.

[Ja2] ———, *On supports of induced representations for symplectic and odd-orthogonal groups.* Amer. J. Math. **119**(1997), 1213–1262 .

[Ja3] ———, *On square-integrable representations of classical p-adic groups.* Can. J. Math. **52**(2000), 539–581.

[Ki1] H. Kim, *The residual spectrum of Sp_4.* Comp. Math. **99**(1995), 129–151

[Ki2] ———, *The residual spectrum of G_2.* Can. J. Math. **48**(1996), 1245–1272.

[Ki3] ———, *Langlands-Shahidi method and poles of automorphic L-functions: application to exterior square L-functions.* Can. J. Math. **51**(1999), 835–849.

[Ki4] ———, *Langlands-Shahidi method and poles of automorphic L-functions II.* Israel J. Math. **117**(2000), 261–284.

[Ki5] ———, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2.* J. Amer. Math. Soc. **16**(2003), 139–183.

[Ki-Sh] H. Kim and F. Shahidi, *Functorial products for $GL_2 \times GL_3$ and symmetric cube for GL_2.* Ann. of Math. **155**(2002), 837–893.

[Ki-Sh2] ———, *Cuspidality of symmetric powers with applications.* Duke Math. J. **112**(2002), 177–197.

[Ko] R. E. Kottwitz, *Stable trace formula: cuspidal tempered terms.* Duke Math. J. **51**(1984), 611–650.

[Ku] S. Kudla, *The local Langlands correspondence: the non-archimedean case.* In: Proceedings of Symposia in Pure Mathematics 55, American Mathematical Society, Providence, RI, 1994, pp. 365–391.

[La] R. P. Langlands, *Euler Products.* Yale University Press, New Haven, CN, 1971.

[La2] ———, *On the Functional Equations Satisfied by Eisenstein Series.* Lecture Notes in Mathematics vol. 544, Springer-Verlag, Berlin, 1976.

[M-W1] C. Moeglin and J. L. Waldspurger, *Spectral Decomposition and Eisenstein series, une paraphrase de l'Ecriture.* Cambridge Tracts in Mathematics 113, Cambridge, Cambridge University Press, 1995.

[M-W2] ———, *Le spectre résiduel de $GL(n)$.* Ann. Scient. Éc. Norm. Sup. **22**(1989), 605–674.

[M-Ta] C. Moeglin and M. Tadic, *Construction of discrete series for classical p-adic groups.* J. Amer. Math. Soc. **15**(2002), 715–786 .

[Mu1] G. Muić, *Some results on square integrable representations; irreducibility of standard representations.* Internat. Math. Res. Notices **14**(1998), 705–726.

[Mu2] ———, *A proof of Casselman–Shahidi's conjecture for quasi-split classical groups.* Canad. Math. Bull. **44**(2001), 298–312.

[Ra] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for $SL(2)$.* Ann. of Math. **152**(2000), 45–111.

[Sh1] F. Shahidi, *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups.* Annals of Math. **132**(1990), 273–330.

[Sh2] ———, *On certain L-functions.* Amer. J. Math. **103**(1981), 297–355.

[Sh3] ———, *On the Ramanujan conjecture and finiteness of poles for certain L-functions.* Ann. of Math. **127**(1988), 547–584.

[Sh4] ———, *On non-vanishing of twisted symmetric and exterior square L-functions for $GL(n)$.* Pacific J. Math. Olga Taussky-Todd memorial issue (1997), 311–322.

[Sh5] ———, *Fourier transforms of intertwining operators and Plancherel measures for GL(n).* Amer. J. Math. **106**(1984), 67–111 .

[Sh6] ———, *On multiplicativity of local factors.* In: Festschrift in honor of I. I. Piatetski-Shapiro, Part II, Israel Math. Conf. Proc. 3, Weizmann, Jerusalem, 1990, pp. 279–289.

[Sh7] ———, *Local coefficients as Artin factors for real groups.* Duke Math. J. **52**(1985), 973–1007.

[Ta] M. Tadic, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case.* Ann. Sci. Ec. Norm. Sup. **19**(1986), 335–382.

[V] D. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules.* Invent. Math. **48**(1978), 75–98.

[Ze] A.V. Zelevinsky, *Induced representations of reductive p-adic groups II. On irreducible representations of GL(n).* Ann. Sci. École Norm. Sup. **13**(1980), 165–210.

[Zh] Y. Zhang, *The holomorphy and nonvanishing of normalized local intertwining operators.* Pacific J. Math. **180**(1997), 385–398.

Deptartment of Mathematics
University of Toronto
Toronto, ON M5S 3G3
e-mail: henrykim@math.toronto.edu