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A note on the stability of inviscid zonal jet flows on a rotating sphere

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The linear stability of inviscid zonal jet flows on a rotating sphere is re-examined. A semi-circle theorem for inviscid zonal flows on a rotating sphere is proved. It is also shown that numerically obtained eigenvalues of the linear stability problem do not converge well with a spectral method which was adopted in previous studies, due to an emergence of critical layers near the poles. By using a shooting method where the integral path bypasses the critical layers in the complex plane, the eigenvalues are successfully obtained with $\sim$10 % correction of the critical rotation rates compared to those obtained in Baines (*J. Fluid Mech.*, vol. 73, 1976, pp. 193–213).

Key words: jets, rotating flows, waves in rotating fluids

1. Introduction

Characteristics of two-dimensional barotropic fluid on a rotating sphere, which is one of the simplest models of planetary atmospheres taking into account the effects of planetary rotation and density stratification, have long been investigated (see Hayashi *et al.* 2007; Obuse, Takehiro & Yamada 2010), and on the β-plane (see Kuo 1949; Tung 1981). The stability problem of barotropic zonal flows on a rotating sphere has also been studied in relation to the existence of large-scale zonal flows in the planetary atmospheres, such as on Jupiter and Saturn.

The first aim of this paper is to develop a semi-circle theorem for the inviscid instability of zonal flows on a rotating sphere. The semi-circle theorem was first derived by Howard (1961) for zonal flows in the non-rotating case, and was extended to the β-plane by Pedlosky (1964, 1987). We extend the semi-circle theorem to zonal flows on a rotating sphere, where the radius of the circle depends on the angular velocity of the rotating frame of reference and we minimize the radius by choosing the most convenient frame of reference. A similar method was employed by Thuburn & Haynes (1996) who obtained a semi-circle theorem in which the radius does not coincide with that given in this paper.

The second aim of this paper is to give corrected values of the critical rotation rate of stability. Baines (1976) numerically studied the linear stability of inviscid barotropic zonal flow solutions on a rotating sphere, the streamfunction of which is expressed by the zonal spherical harmonics Y_l^0, as well as inviscid Rossby wave solutions expressed by the spherical harmonics Y_l^m where $m \neq 0$. He solved the eigenvalue problem numerically with a spectral method with the truncation wavenumber up to 20. As suggested by the inflection-point theorem (Rayleigh's criterion), the zonal jet flows

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are stabilized when the rotation rate of the sphere is increased. He obtained for various zonal jet solutions the critical rotation rates at which the stability of zonal jets changes from unstable to stable. He also argued that the values of the critical rotation rates are only slightly above those estimated by the inflection-point theorem. The numerical calculation of eigenvalues by Baines (1976) was significantly challenging at the time prior to the major advance of computational environment, and the obtained values have been frequently employed by many researchers. However, re-examining the numerical calculation, we find that the eigenvalues obtained by the spectral method adopted by Baines (1976) include numerical errors which do not decrease even by increasing the truncation wavenumber as far as practically available in the computation. We should also note Skiba's argument (Skiba 2002) that numerical calculation of some eigenvalues is not stable because of an accumulation of the continuous spectrum.

This paper re-examines the stability of inviscid barotropic zonal flows on a rotating sphere, taking special care with the convergence of the eigenvalues. In § 2, the governing equation and its linearized equation are presented. A semi-circle theorem is derived in § 3. Section 4 elucidates imperfections of the numerical results of the stability eigenvalues obtained by a spectral method, and instead a shooting method is employed to overcome the problems. A conclusion follows in § 5.

2. Governing equations

A two-dimensional incompressible barotropic inviscid flow on a rotating sphere is governed by the equation of vorticity,

$$\frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2 \psi) + 2\Omega \frac{\partial \psi}{\partial \lambda} = 0. \tag{2.1}$$

Here t is the time, λ and ϕ are the longitude and the latitude, and $\mu = \sin\phi$ is the sine latitude; ψ is the streamfunction and $\nabla^2\psi$ is the vorticity, where ∇^2 is the horizontal Laplacian on an unit sphere. The longitudinal and latitudinal components of velocity (u_λ, u_μ) are given by $u_\lambda = -\sqrt{1-\mu^2}(\partial\psi/\partial\mu)$ and $u_\mu = 1/\sqrt{1-\mu^2}(\partial\psi/\partial\lambda)$, respectively. $J(A, B) = (\partial A/\partial\lambda)(\partial B/\partial\mu) - (\partial B/\partial\lambda)(\partial A/\partial\mu)$ is the Jacobian operator, and Ω is the non-dimensional constant rotation rate of the sphere.

A general zonal flow $\psi = \psi_0(\mu)$ is a steady solution of the equation of vorticity (2.1), regardless of the rotation rate. Here we consider steady zonal flow solutions with l jets described by a 4π normalized spherical harmonic function $Y_l^m(\lambda, \mu)$ as

$$\psi_0 = \Psi_0(\mu) = -\frac{1}{l(l+1)} Y_l^0(\mu), \tag{2.2}$$

which we call l-jet flow. Here, the number of jets is defined as the number of extreme points of the longitudinal velocity, which is equal to the number of nodes of the latitudinal distribution of the streamfunction.

In order to examine the linear stability of the inviscid zonal flow $\psi_0(\mu)$, we substitute $\psi = \psi_0(\mu) + \psi'(\lambda, \mu, t)$ into (2.1) and neglect the second-order terms of ψ'. Assuming that $\psi' = \hat{\psi}(\mu)\exp[\mathrm{Im}\, m(\lambda - ct)]$, we finally have a linearized equation of vorticity

$$[U(\mu) - c]\nabla_m^2 \hat{\psi} + \left\{ 2\Omega - \frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\left[(1-\mu^2)U(\mu)\right] \right\} \hat{\psi} = 0. \tag{2.3}$$

Here, $U(\mu) = -\mathrm{d}\psi_0(\mu)/\mathrm{d}\mu$ is the angular velocity of the basic zonal flow and ∇_m^2 is defined as $\nabla_m^2 = (\mathrm{d}/\mathrm{d}\mu)(1-\mu^2)(\mathrm{d}/\mathrm{d}\mu) - m^2/(1-\mu^2)$. The boundary conditions at the north and the south poles are given by

$$\hat{\psi}(\pm 1) = 0. \tag{2.4}$$

Equations (2.3) and (2.4) constitute an eigenvalue problem with the eigenvalue c being the complex angular phase velocity.

3. Semi-circle theorem

We introduce the latitudinal displacement of the perturbation $\eta = \hat{\eta}(\mu)\exp[\mathrm{Im}\, m(\lambda - ct)]$. The material derivative of η is related to the latitudinal component of the perturbation velocity u'_μ as

$$u'_\mu = \frac{\mathrm{D}\eta}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial \lambda}\right)\eta. \tag{3.1}$$

Then, $\hat{\psi}$ can be expressed by $\hat{\eta}$ as $\hat{\psi} = \sqrt{1-\mu^2}(U-c)\hat{\eta}$. Substituting $\hat{\eta}$ into (2.3) and taking the inner product with $\sqrt{1-\mu^2}\hat{\eta}^*$, where * indicates complex conjugate, we obtain

$$\int \mathrm{d}\mu[(U-c_r)^2 - c_i^2]P = 2(\Omega + c_r)\int \mathrm{d}\mu(U-c_r)Q + 2c_i^2\int \mathrm{d}\mu Q, \tag{3.2}$$

$$c_r\int \mathrm{d}\mu(P+2Q) = \int \mathrm{d}\mu U(P+Q) - \Omega\int \mathrm{d}\mu Q. \tag{3.3}$$

Here, c_r and c_i are the real and imaginary parts of c, and $P = P(\mu)$ and $Q = Q(\mu)$ denote $P(\mu) = (1-\mu^2)^2\,|\mathrm{d}\hat{\eta}/\mathrm{d}\mu|^2 + (m^2-1)\,|\hat{\eta}|^2 > 0$, and $Q(\mu) = (1-\mu^2)\,|\hat{\eta}|^2 > 0$. Expansion of $\phi = \sqrt{1-\mu^2}\hat{\eta}$ by the associated Legendre polynomials, $\phi = \sum_{n=m}^{\infty}\phi_n^m P_n^m(\mu)$ gives

$$\int \mathrm{d}\mu(P+2Q) \geqslant m(m+1)\int \mathrm{d}\mu Q. \tag{3.4}$$

When $\Omega \geqslant 0$, (3.3) yields

$$c_r \leqslant \frac{\displaystyle\int \mathrm{d}\mu U_{max}(P+Q)}{\displaystyle\int \mathrm{d}\mu(P+2Q)} \leqslant U_{max}, \tag{3.5a}$$

$$c_r \geqslant \frac{\displaystyle\int \mathrm{d}\mu U_{min}(P+Q)}{\displaystyle\int \mathrm{d}\mu(P+2Q)} - \Omega\frac{\displaystyle\int \mathrm{d}\mu Q}{\displaystyle\int \mathrm{d}\mu(P+2Q)} \geqslant U_{min} - \frac{\Omega}{m(m+1)}, \tag{3.5b}$$

where we have made an assumption for the angular velocity,

$$U_{max} = \max_{-1\leqslant\mu\leqslant 1} U(\mu) > 0, \quad U_{min} = \min_{-1\leqslant\mu\leqslant 1} U(\mu) < 0. \tag{3.6}$$

Thus, we obtain the following condition for the phase velocity:

$$U_{min} - \frac{\Omega}{m(m+1)} \leqslant c_r \leqslant U_{max} \quad (\text{for } \Omega \geqslant 0). \tag{3.7}$$

The assumption (3.6) is temporal, and we will remove it at the end of this proof.

An obvious inequality,

$$0 \geqslant \int \mathrm{d}\mu (U-U_{min})(U-U_{max})P = \int \mathrm{d}\mu [U^2 - (U_{max}+U_{min})U + U_{min}U_{max}]P, \quad (3.8)$$

with (3.2) and (3.3) yields

$$0 \geqslant \int \mathrm{d}\mu (c_r^2 + c_i^2)(P+2Q) - (U_{max}+U_{min})UP + U_{max}U_{min}P + 2\Omega UQ, \quad (3.9)$$

which leads to

$$\left[\left(c_r - \frac{U_{max}+U_{min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{max}-U_{min}}{2}\right)^2\right]\int \mathrm{d}\mu (P+2Q) \leqslant |\Omega|(U_{max}-U_{min})\int \mathrm{d}\mu Q. \quad (3.10)$$

Then, using (3.4), we obtain

$$\left(c_r - \frac{U_{max}+U_{min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{max}-U_{min}}{2}\right)^2 \leqslant \frac{|\Omega|}{m(m+1)}(U_{max}-U_{min}). \quad (3.11)$$

Here we should note that if the angular velocity of the system of coordinates is changed from Ω to $\Omega+\omega$, where $U_{min} \leqslant \omega \leqslant U_{max}$ (see (3.6)), then U and c_r become $U-\omega$ and $c_r-\omega$ with c_i unchanged, i.e. the left-hand side of (3.11) is unchanged. Therefore, by taking ω which minimizes $|\Omega+\omega|$ we obtain more restricted ranges for c_r and c_i as

$$U_{min} - \frac{|\Omega+U|_{min}}{m(m+1)} \leqslant c_r \leqslant U_{max} \quad (\text{for } \Omega > 0), \quad (3.12)$$

and

$$\left(c_r - \frac{U_{max}+U_{min}}{2}\right)^2 + c_i^2 \leqslant \left(\frac{U_{max}-U_{min}}{2}\right)^2 + \frac{|\Omega+U|_{min}}{m(m+1)}(U_{max}-U_{min}), \quad (3.13)$$

which gives the semi-circle theorem. Remarkably this semi-circle theorem is valid even when $U(\mu)$ does not satisfy (3.6), because then we can choose rotating coordinates where $U_{min}-\omega < 0 < U_{max}-\omega$. Therefore the assumption (3.6) is unnecessary for the semi-circle theorem to hold.

A semi-circle theorem has been obtained by Thuburn & Haynes (1996), in which the radius of the circle is different from that obtained here. Our derivation is different in that the present P and Q allow us to utilize a property of Legendre functions. We note that the radius of (3.13) is smaller than or equal to that of Thuburn & Haynes (1996) except when $3-2\sqrt{2} \leqslant |\Omega+U|_{min}/|\Omega+U|_{max} < 1/3$ and both $|\Omega+U|_{min}$ and $|\Omega+U|_{max}$ are sufficiently small. We should note that another semi-circle theorem was stated in the Appendix of Ishioka & Yoden (1992), where $|\Omega+U|_{min}/(m(m+1))$ in (3.13) is replaced by $|\Omega+U|_{max}/(m^2+1+m^2/(2|m|+3))$.

4. Re-examination of the stability of inviscid zonal flow

In this section, we re-examine the linear stability of the inviscid zonal flow (2.2) on a rotating sphere. This problem was previously investigated by Baines (1976), but we show that some numerical corrections are necessary, taking into account singular

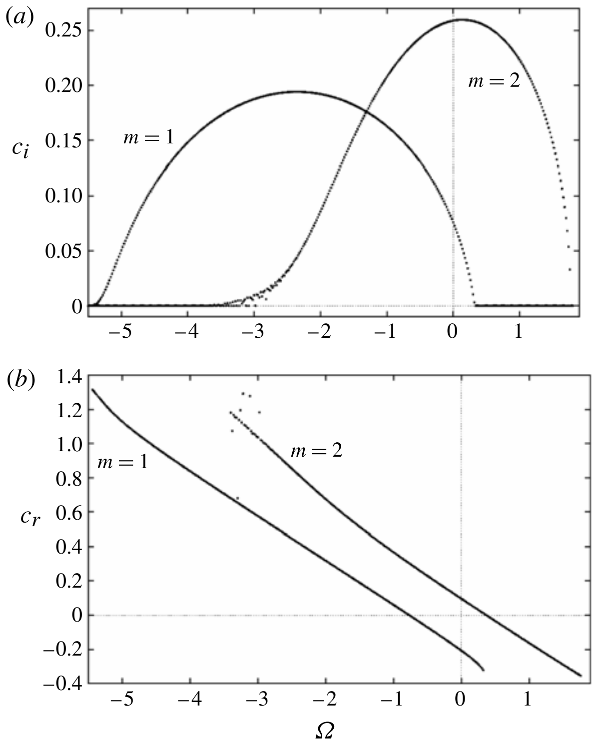

FIGURE 1. The eigenvalues of linear stability of 3-jet zonal flow for $m = 1$ and 2 obtained with the spectrum method with the truncation wavenumber $N = 213$: (*a*) and (*b*) show the imaginary and real parts of the phase angular velocity c_i and c_r, respectively.The horizontal and vertical axes are the rotation rate Ω and the eigenvalues respectively.

behaviour of eigenfunctions. The 1-jet and 2-jet zonal flows are linearly stable due to conservation laws of angular momentum, energy and enstrophy (see Baines 1976). However, l-jet zonal flows with $l \geqslant 3$ can be unstable, and we consider the cases of $3 \leqslant l \leqslant 9$, the same range of l as Baines. The unstable modes of l-jet zonal flow do not contain the spherical harmonics $Y_n^m(\lambda, \mu)$ with $|m| \geqslant l$ as proved by Skiba (1989) and Ishioka & Yoden (1992). Also, zonal modes $Y_n^0(\mu)$ are all neutral modes. Therefore, it is sufficient to study disturbances with the azimuthal wavenumber $1 \leqslant |m| \leqslant l - 1$.

4.1. *Stability analysis with a spectral method*

First, we present numerical results of stability obtained by a spectral method, essentially in the same way as Baines (1976). In order to solve the eigenvalue problem of (2.3) and (2.4) for a given azimuthal wavenumber m of the disturbance, we assume the streamfunction $\hat{\psi}(\mu) = \sum_{n=m}^{N} \psi_n^m P_n^m(\mu)$ where ψ_n^m are the expansion coefficients and N is the truncation wavenumber. On evaluating the terms $U(\mu)\nabla_m^2\hat{\psi}(\mu)$ and $(\mathrm{d}^2/\mathrm{d}\mu^2)[(1-\mu^2)U(\mu)]\hat{\psi}(\mu)$ in (2.3), we adopt a transform method, employing in the physical space the numbers of longitudinal and latitudinal grid points I and J satisfying $I \geqslant 3N + 1$ and $J > 3N/2$ in order to eliminate aliasing errors.

Figure 1 shows the numerical eigenvalues for $m = 1, 2$ in the case of the 3-jet zonal flow. Baines (1976) calculated the eigenvalue for the same problem, and concluded that the 3-jet flow is unstable for $\Omega_B^- = -5.35 < \Omega < 1.76 = \Omega_B^+$. We show in

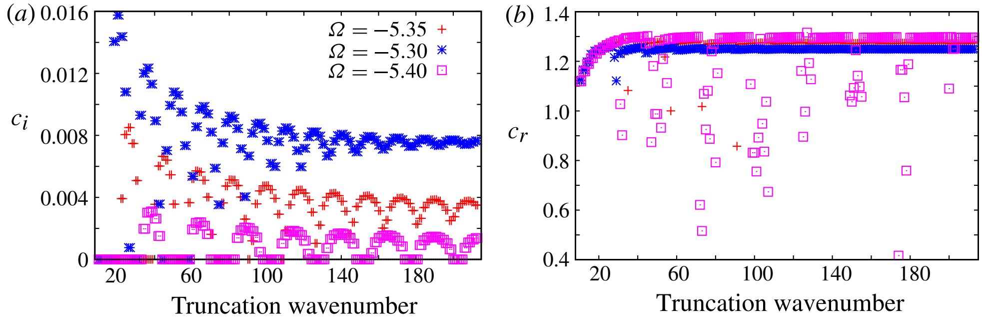

FIGURE 2. (Colour online) The unstable eigenvalues around $\varOmega = \varOmega_B^-$, which is the critical rotation rate obtained by Baines: (*a*) and (*b*) show the imaginary and real parts of phase angular velocity, c_i and c_r, respectively. The horizontal and vertical axes are the truncation wavenumber N and the eigenvalues, respectively.

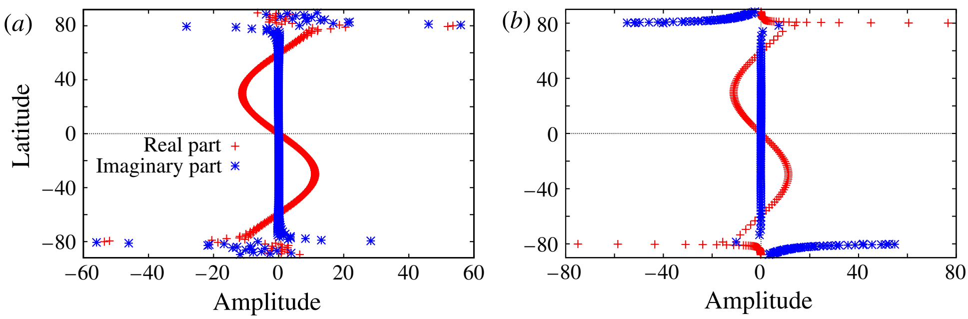

FIGURE 3. (Colour online) The vorticity of unstable eigenfunctions in the case of 3-jet zonal flow at $\varOmega = \varOmega_B^-$ with azimuthal wavenumber $m = 1$: (*a*) and (*b*) are obtained by the spectral method with truncation wavenumber $N = 213$ and by the shooting method, respectively.

figure 2 the eigenvalues obtained in our numerical calculation around Baines' negative critical rotation rate $\varOmega_B^- = -5.35$ as a function of the truncation wavenumber N. The imaginary part of phase angular velocity c_i does not converge even when the truncation wavenumber is increased up to 10 times of that used by Baines, although the real part c_r can be obtained with three-digit accuracy, which is equal to 1.27 at $\varOmega = \varOmega_B^-$.

The eigenfunction at $\varOmega = \varOmega_B^-$ is shown in figure 3(*a*). It is observed that the vorticity diverges near $\mu = \pm 1$, indicating singularities near the north and south poles. The critical points, where $U(\mu) - c_r = 0$, appear around $\pm 79.7^\circ$ in latitude. The divergence behaviour of c_i is caused by lack of resolution around the critical layers of the eigenfunction emerging near the poles.

On the other hand, when $\varOmega > 0$, the eigenvalues converge fairly well. We find that the positive critical rotation rate $\varOmega_c^+$ is 1.77194, and the critical azimuthal wavenumber $m_c = 2$. Figure 4 shows the eigenvalues around $\varOmega = \varOmega_B^+$. The eigenvalues can be obtained with 0.1 % accuracy when the truncation wavenumber is increased up to 63. The slightly unstable eigenfunction at $\varOmega = 1.7719$ is shown in figure 5. Obviously, no critical point is found, in contrast to the cases near the negative critical rotation rate.

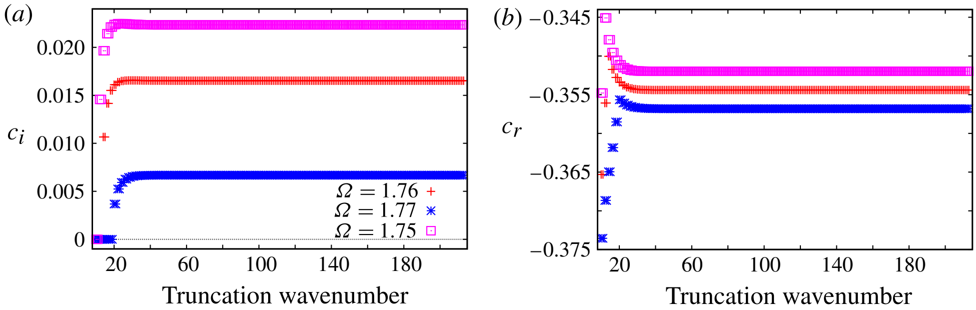

FIGURE 4. (Colour online) Same as figure 2 but for the unstable eigenvalues around $\Omega = \Omega_B^+$: (*a*) imaginary part, (*b*) real part.

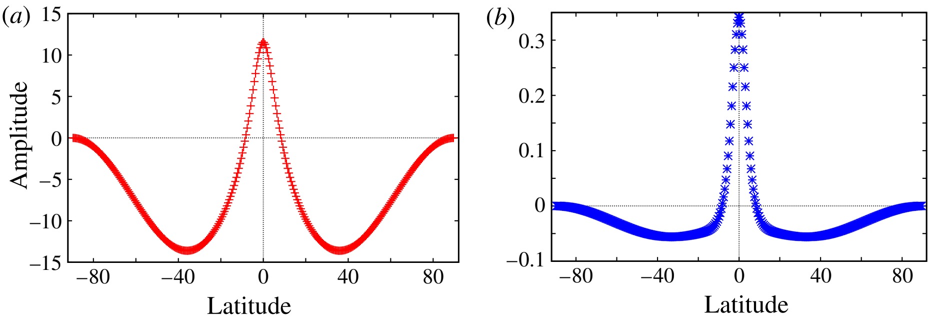

FIGURE 5. (Colour online) Same as figure 3 but for the vorticity of unstable eigenfunctions of 3-jet zonal flow Y_3^0 at $\Omega = 1.7719$ with azimuthal wavenumber $m = 2$ with the truncation wavenumber $N = 213$: (*a*) imaginary part, (*b*) real part.

4.2. *Stability analysis with a shooting method*

In the previous subsection, it is shown that in the spectral method c_i does not converge even when the truncation wavenumber is increased, because the critical layers appear in the eigenfunctions. In this subsection, instead of the spectral method, we make use of a shooting method to overcome the difficulty.

Equation (2.3) is expressed in the normal form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\begin{pmatrix}\hat{\psi}\\ \hat{\phi}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{l(l+1)U(\mu)+2\Omega}{\{U(\mu)-c\}(1-\mu^2)} + \dfrac{m^2}{(1-\mu^2)^2} & \dfrac{2\mu}{1-\mu^2}\end{pmatrix}\begin{pmatrix}\hat{\psi}\\ \hat{\phi}\end{pmatrix}, \tag{4.1}$$

where $\hat{\phi} = \mathrm{d}\hat{\psi}/\mathrm{d}\mu$.

For a given value of Ω, we obtain the solution $\hat{\psi}^-$ and $\hat{\phi}^-$ by integrating the normal form of (4.1) from the edge point $\mu = -1$ to a certain point $\mu_0 \in (-1, 1)$ with the boundary conditions (2.4). Then we obtain the other solution $\hat{\psi}^+$ and $\hat{\phi}^+$ by integrating it from $\mu = 1$ to $\mu = \mu_0$.

The matching condition consists of continuity of streamfunction $\hat{\psi}$ and its derivative $\hat{\phi}$, which is expressed by

$$f(\Omega, c_r, c_i) = \begin{vmatrix} \hat{\psi}^+(\mu_0) & \hat{\psi}^-(\mu_0) \\ \hat{\phi}^+(\mu_0) & \hat{\phi}^-(\mu_0) \end{vmatrix} = 0. \tag{4.2}$$

The point μ_0 can be chosen at any point on the integral path. However, in this problem, the critical points are expected to exist around both the poles $\mu = \pm 1$. We then select μ_0 as the end point of each integration and take $\mu_0 = 0.1$ which is far from both the poles.

In the above integrations, we should consider the singular points $\mu = \pm 1$ and critical points μ_c such that $U(\mu_c) - c = 0$.

First, in order to avoid the difficulty arising from the singular points, we change the starting point of the numerical integration from $\mu = \pm 1$ to certain nearby points. The values of $\hat{\psi}$ and $\hat{\phi}$ are obtained by using a power series expansion of the solution: at the south pole $\mu = -1$, $\hat{\psi}$ is expanded into a power series of $z = \mu + 1$ as

$$\hat{\psi} = z^{m/2} \left(1 + \sum_{j=1}^{J_t} a_j z^j \right), \tag{4.3}$$

where J_t is a sufficiently large number and is taken up to 20. The coefficients a_j of the series are successively determined by expanding (2.3) around $\mu = -1$. The starting point should be close to the south pole to keep the accuracy of the power series expansion. Moreover, near marginal stability, the critical layer approaches the pole, which means that the convergence radius becomes small, and therefore we have to pay attention to the choice of the starting point. The same scenario holds also for the north pole. Second, we have to solve the singular behaviour of the solution around the critical point $\mu = \mu_c$. Near marginal stability, the critical point approaches the interval $[-1, 1]$, and the numerical integration along the μ-axis rapidly becomes difficult. Then, in order to find the marginal stability eigenvalue as the limit of unstable eigenvalues, we deform the integral path in the complex μ-plane to bypass the critical points in such a way that $\pi \leqslant \arg \mu \leqslant 2\pi$ or $0 \leqslant \arg \mu \leqslant \pi$ if $U'(\mu_c) > 0$ or $U'(\mu_c) < 0$, respectively.

Specifically, we employ a piecewise linear path as shown in figure 6. On integrating the normal form (4.1) from the south pole (A) $\mu = -1$ to (E) μ_0, we divide the integration path into four sections: (A)→(B)→(C)→(D)→(E), where the power series expansion is employed for the section (A)→(B), and the integrals for the other sections are performed by the fourth-order Runge–Kutta method with the number of grid points being $\sim 3 \times 10^4$. From the north pole (I) $\mu = 1$ to (E), we perform the calculation in a similar way to the above in the order of (I)→(H)→(G)→(F)→(E).

We perform this shooting method to determine $c = c_r + \mathrm{i}c_i$ for given values of Ω by use of the Newton method. The stopping condition of the Newton method is that the rate of the correction of the eigenvalues is less than 10^{-8}. The Jacobi matrix

$$\begin{pmatrix} \dfrac{\partial \mathrm{Re}\,[f]}{\partial c_r} & \dfrac{\partial \mathrm{Re}\,[f]}{\partial c_i} \\ \dfrac{\partial \mathrm{Im}\,[f]}{\partial c_r} & \dfrac{\partial \mathrm{Im}\,[f]}{\partial c_i} \end{pmatrix} \tag{4.4}$$

is evaluated by the central finite difference method with $\Delta c_r,\ \Delta c_i = 10^{-6}$.

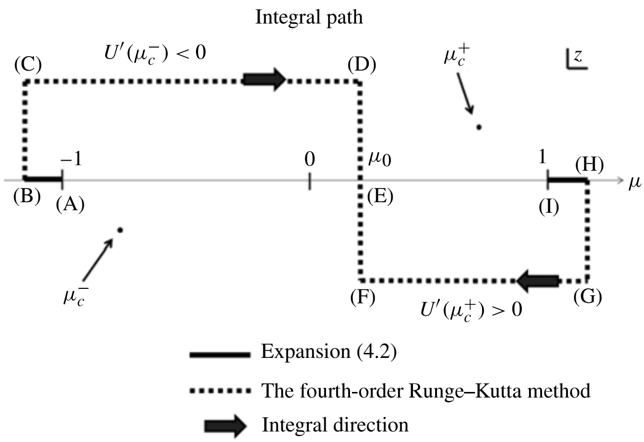

FIGURE 6. Schematic of the integral path on the complex plane $\mu \in \mathbb{C}$. The solid and the dashed lines indicate the integrations by the expansion method and the Runge–Kutta method, respectively. The critical points μ_c are given by the roots of $U(\mu_c) - c = 0$.

We also perform this shooting method to determine Ω_c and c_r for $c_i = 0$. Skiba (2002) argued that numerical calculation of some eigenvalues is not stable because of the accumulation of the continuous spectrum. In our calculation we checked the numerical convergence of the eigenvalue by changing the number of grid points to 6×10^3 and 6×10^5 and confirmed that the relative errors of c_r and c_i (or Ω_c and c_r) are less than 0.1 %. Also, we have changed the increments of Δc_r and Δc_i for the evaluation of the Jacobi matrix from 10^{-6} to 10^{-5} and found that the relative errors of the critical rotation rates and the eigenvalues remain less than 0.1 %. Further, we have checked that the obtained eigenvalues and eigenfunctions are consistent with the inflection-point theorem and the semi-circle theorem, and that the ratio of the energy and the enstrophy of the eigenfunction is $l(l+1)$ as derived by Skiba (2009) for the zonal flow Y_l^0.

Figure 7 shows the stability eigenvalues obtained for the 3-jet zonal flow. For the sake of comparison, the eigenvalues obtained by the spectral method are also shown. It is seen that the eigenvalues obtained by the shooting method converge better than those obtained by the spectral method. We find the negative critical rotation rate $\Omega_c^- = -5.45685$ and the critical azimuthal wavenumber $m_c = 1$. The unstable eigenfunction at $\Omega = \Omega_B^-$ is shown in figure 3(*b*). The vorticity around the critical layers is more accurately shown in the shooting method solution, compared with the spectral method solution in figure 3(*a*).

We also show the critical rotation rates of other zonal jet solutions (2.2) in table 1. There are ∼10 % differences between the critical rotation rates of Baines, $\Omega_B^\pm$, and of the present study, $\Omega_c^\pm$. When the number of zonal jets l is odd and $\Omega < 0$, the critical layers emerge around both the north and the south poles. When l is even, the critical layer arises around the south (north) pole for $\Omega > 0 (<0)$. When l is odd and $\Omega > 0$, the critical eigenfunction does not have a singularity.

The inflection-point theorem states that when the basic flow is unstable, there is at least one zero point of $\tilde{\beta} = 2\Omega + \mathrm{d}Y_l^0(\mu)/\mathrm{d}\mu$ in the interval $\mu \in [-1, 1]$. This condition gives the possible range of the critical rotation rates, the upper and the lower bounds of which are given in table 1 as $\Omega_I^\pm$. However, $\Omega_I^\pm$ do not coincide with $\Omega_c^\pm$, with

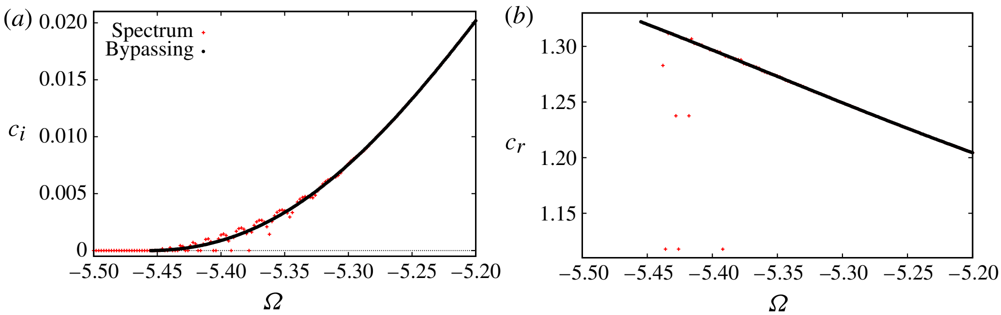

FIGURE 7. (Colour online) The stability eigenvalues for the 3-jet zonal flow $m = 1$ around the negative critical rotation rate: (*a*) and (*b*) show the imaginary and real parts of phase angular velocity c_i and c_r, respectively. The horizontal and vertical axes indicate the rotation rate Ω and the eigenvalues, respectively. The black dots are the results by the shooting method, while the crosses (red online) are those by spectral method with $N = 213$.

l-jet	$\Omega_c^{\pm}$	$m_c^{\pm}$	$\mu_c^{\pm}$	Baines $\Omega_B^{\pm}$	$m_B^{\pm}$	Relative error (%)	$\Omega_I^{\pm}$	Relative difference (%)
3	−5.4568	1	±1	−5.35	1	1.95	$-3\sqrt{7}$	45.4
	1.7719	2	—	1.76	2	0.673	$3\sqrt{7}/4$	11.9
4	−9.7700	1	1	8.78	1	9.49	−15	53.5
	9.7700	1	−1	8.78	1	9.49	15	53.5
5	−19.22	1	±1	−18.2	1	5.21	$-15\sqrt{11}/2$	29.4
	4.022	3	—	3.90	3	3.03	$-15\sqrt{11}/16$	177
6	−28.389	1	1	−25.0	1	11.9	$-21\sqrt{13}/2$	33.3
	28.389	1	−1	25.0	1	11.9	$21\sqrt{13}/2$	33.3
7	−44.445	1	±1	−40.0	1	10.0	$-14\sqrt{15}$	24.2
	7.8929	3	—	7.226	3	8.44	$35\sqrt{15}/32$	86.3
8	−59.618	1	1	−48.4	1	18.8	$-18\sqrt{17}$	24.4
	59.618	1	−1	48.4	1	18.8	$18\sqrt{17}$	24.4
9	−83.340	1	±1	−69.3	1	16.8	$-45\sqrt{19}/2$	17.6
	13.665	3	—	11.5	1	15.8	$-315\sqrt{19}/256$	139

TABLE 1. The critical rotation rates of inviscid zonal flows $\Psi_0 = -Y_l^0(\mu)/l(l+1)$. Column 1 shows the number of jets l of the basic flows. Columns 2, 3, and 4 indicate the results of the present study: the critical rotation rate $\Omega_c^{\pm}$, the critical azimuthal wavenumber $m_c^{\pm}$, and the sine latitude of critical layers $\mu_c^{\pm}$. Columns 5, 6, and 7 are the results of Baines (1976) for the sake of comparison: the critical rotation rate $\Omega_B^{\pm}$, the critical azimuthal wavenumber $m_B^{\pm}$, and the relative errors of critical rotation rates between Baines (1976) and the present study. Columns 8 and 9 indicate the critical rotation rates $\Omega_I^{\pm}$ estimated by the inflection-point theorem and their relative difference from the critical rotation rate $\Omega_c^{\pm}$ obtained by the present study.

relative differences up to ∼170 %. This suggests that stability characteristics of zonal flows on a rotating sphere are rather different from those of parallel flows on a plane where the inflection point is often related to the emergence of instability.

We remark that in the case of the 3-jet flow, the critical rotation rate Ω_c^- can be obtained analytically. In the 3-jet case, under the assumption that the critical point exists at each pole, we find $c = c_r = \sqrt{7}/2$ and $U - c_r = 5\sqrt{7}(\mu - 1)(\mu + 1)/8$. Substituting these into (2.3) with $\{(1-\mu^2)U\}'' = -3(3+1)U$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{(1-\mu^2)\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}\mu}\right\} - \frac{\alpha}{1-\mu^2}\hat{\psi} = -3(3+1)\hat{\psi}, \tag{4.5}$$

where $\alpha = m^2 + 16\Omega/5\sqrt{7} + 48/5$. If α is a square of an integer number, $\alpha = \tilde{m}^2$, the linear operator of the left-hand side of (4.5) becomes $\nabla^2_{\tilde{m}}$ and the eigenfunction is the associated Legendre function $P_3^{\tilde{m}}(\mu)$. We find the similarity of $\hat{\psi}$ to $P_3^2(\mu)$, and we choose $\tilde{m} = 2$, which yields $\Omega_c = -33\sqrt{7}/16 = -5.456862\ldots$, in agreement with the numerical result. The case of $m = 2, \tilde{m} = 3$ corresponds to $\Omega_c = -3.807$ where an unstable mode arises from the neutral mode. According to the numerical results, for all other combinations of $m, \tilde{m}$, unstable modes do not arise from the neutral modes.

5. Conclusion and discussion

In this paper we re-examine the linear stability of inviscid barotropic zonal flows on a rotating sphere. A semi-circle theorem for zonal flows on a rotating sphere is derived. The critical rotation rates for stability of zonal flows are obtained more accurately than the previous study by Baines (1976).

By the spectral method, the critical eigenvalues could not be obtained accurately for an even number of jets, because of the emergence of the critical layers near the north and the south poles when the zonal flow approaches the marginal stability state. A similar difficulty also arises for an odd number of jets with a negative rotation rate. To obtain the critical eigenvalues and critical rotation rates with sufficient accuracy, we make use of the shooting method and the power series expansion method, taking into account the singular points. As a result, we find that the critical rotation rates of Baines (1976) should be corrected by $\sim$10 %. On the other hand, in the cases of an odd number of jets, the positive critical rotation rates are obtained without difficulty by the spectral method, because of the absence of the critical layers.

So far in this paper, we have discussed the stability problem of Rossby waves each streamfunction of which is expressed by a single spherical harmonic of Y_l^0, i.e. zonal flows. In the aforementioned paper, Baines also studied the stability of non-zonal Rossby waves, i.e. ψ_0 proportional to $Y_l^m(\lambda, \mu)$, $(m \neq 0)$. However, we have found in high-resolution computations, that some of the stability results for these flows in Baines (1976) also suffer from inaccuracy due to the singular point where the coefficient of the highest-order derivative of the eigenfunction vanishes. The traditional technique of bypassing the singular point in the complex plane, which we have employed in this paper, is applicable only to the problems of space dimension one. Accurate results of the stability eigenvalues for the two-dimensional problem is therefore still open to further study.

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we use a spectral transformation library 'ISPACK' (http://www.gfd-dennou.org/library/ispack/) and a FORTRAN90 wrapper library 'SPMODEL' (http://www.gfd-dennou.org/library/spmodel/).

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