QUASICONVEX SETS

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Introduction. Let I be the closed real number interval: $0 \leqslant \theta \leqslant 1$. Any subset Δ of I containing at least one number interior to I, will be called a *quasiconvexity generating set*. To each quasiconvexity generating set Δ we associate as follows a generalized notion of convexity, here called quasiconvexity or Δ convexity. Two numbers α and β, one of which belongs to Δ, the other being determined by the relation $\alpha + \beta = 1$, are called complementary ratios of Δ. A set Q in a real vector space is said to be Δ convex if for every pair of complementary ratios α and β in Δ and every pair of points a and b lying in Q the point $\alpha a + \beta b$ also lies in Q.

Quasiconvexity generated by the closed unit interval I evidently coincides with ordinary convexity. We are not, however, interested here in this type of quasiconvexity in that for it our theorems become trivial. More illuminating for our purpose is the quasiconvexity generated by the single self-complementary ratio $\frac{1}{2}$. We shall call this type of quasiconvexity midpoint convexity. It is easily verified that the graph of any solution of the functional equation

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

in midpoint convex. Such graphs, particularly the discontinuous ones, have been intensively studied and are known to possess many interesting measure and topological properties.

These known properties and other new properties as well follow from our general results on quasiconvex sets.

Notation. We shall denote by X a real normed vector space of finite dimension ν. The norm of a vector x in X will be written $|x|$. Points or vectors in X and real numbers will be denoted by small letters, sets by capital letters.

Set union will be symbolized by $\cup$, set intersection by $\cap$, and set difference by $-$. The symbols $\supset$ and $\subset$ mean "contains" and "is contained in" respectively. The closure of a set E will be denoted by $\overline{E}$, the interior by $\underline{E}$, the boundary by $\dot{E}$, and the complement $X - E$ of E in X by CE. The null set is represented by 0.

1. Algebra. Let E be an arbitrary subset of X. The set of all points x in X of the form $x = \alpha a + \beta b$ where a and b lie in E and α and β are complementary ratios of Δ is called the Δ divisor set of E and is denoted by ΔE. Since $x = \alpha x + \beta x$, we see that $E \subset \Delta E$: the divisor operation Δ is ascending. The operation Δ is evidently also increasing in the sense that if $A \subset B$ then $\Delta A \subset \Delta B$.

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The Δ divisor iterates of E, $\Delta^n E$ ($n = 0, 1, 2, \ldots$), are defined recursively as follows: $\Delta^0 E = E$ and $\Delta^{n+1} E = \Delta\Delta^n E$ for $n \geqslant 0$. Let $\Delta^\omega E$ be the union of all these iterates $\Delta^n E (n \geqslant 0)$; ω may here be regarded in its usual ordinal sense.

A set Q has been defined to be Δ convex if it contains all its Δ divisors: $Q \supset \Delta Q$. Thus Q is Δ convex if and only if $\Delta Q = Q$.

Since the space X is Δ convex, the intersection of any collection of Δ convex sets in X is easily seen to be Δ convex. The intersection of all Δ convex sets in X containing a given set E is then the minimal Δ convex set containing E. It is called the Δ convex hull of E and is denoted by $\Delta[E]$.

THEOREM 1.1. $\Delta[E] = \Delta^\omega E$.

Proof. We first note by induction that $\Delta[E] \supset \Delta^n E$ for $n < \omega$. This is certainly true for $n = 0$, and if true for $n < \omega$ it is also true for $n + 1$, since the set $\Delta[E]$ being Δ convex,

$$\Delta[E] = \Delta\Delta[E] \supset \Delta\Delta^n E = \Delta^{n+1} E.$$

Therefore $\Delta[E] \supset \Delta^\omega E$. On the other hand $\Delta^\omega E$ is a Δ convex set containing E. For let x be a Δ divisor of some two points a and b of $\Delta^\omega E$. Then, since the sets $\Delta^n E$ are ascending, some integer $n < \omega$ exists such that $a, b \subset \Delta^n E$. Therefore

$$x \subset \Delta(a, b) \subset \Delta\Delta^n E = \Delta^{n+1} E \subset \Delta^\omega E,$$

whence $\Delta^\omega E$ is Δ convex, so that $\Delta[E] \subset \Delta^\omega E$. This completes the proof.

The set $\Delta^* = \Delta[0, 1]$ is evidently a quasiconvexity generating set, and is, moreover, Δ convex. From the linear character of the space X we see that $\Delta^*(a, b) = \Delta[a, b]$.

This set Δ^* plays a special role in the theory of Δ convexity. It is particularly important in the discussion of what we shall call equivalent quasiconvexity generating sets. Let $\{\Delta\}$ denote the class of all Δ convex sets. We say that Δ generates $\{\Delta\}$. Two quasiconvexity generating sets Δ_1 and Δ_2 will be called equivalent, and we write $\Delta_1 \sim \Delta_2$, if they generate the same sets; that is, if $\{\Delta_1\} = \{\Delta_2\}$. Clearly $\sim$ is a true equivalence relation.

THEOREM 1.2. $\Delta^* \sim \Delta$; $\Delta^*[E] = \Delta[E]$; $\Delta^{**} = \Delta^*$.

Proof. Since $\Delta^* \supset \Delta$, every Δ^* convex set is evidently Δ convex. On the other hand every Δ convex set Q is also Δ^* convex. For let x be a Δ^* divisor of some two points a and b of Q; then

$$x \subset \Delta^*(a, b) = \Delta[a, b] \subset \Delta[Q] = Q.$$

Therefore $\Delta^* \sim \Delta$. Since the Δ^* convex set $\Delta^*[E]$ is Δ convex, $\Delta^*[E] \supset \Delta[E]$; and since the Δ convex set $\Delta[E]$ is Δ^* convex, $\Delta[E] \supset \Delta^*[E]$. Consequently $\Delta^*[E] = \Delta[E]$. Finally we have

$$\Delta^{**} = \Delta^*[0, 1] = \Delta[0, 1] = \Delta^*.$$

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THEOREM 1.3. $\{\Delta_1\} \subset \{\Delta_2\}$ *if and only if* $\Delta_1^* \supset \Delta_2^*$.

Proof. If $\Delta_1^* \supset \Delta_2^*$, then every Δ_1^* convex set is plainly Δ_2^* convex, and hence every Δ_1 convex set is Δ_2 convex. On the other hand if every Δ_1 convex set is Δ_2 convex, then, since Δ_1^* is Δ_1 convex and consequently Δ_2 convex, we have

$$\Delta_1^* = \Delta_2[\Delta_1^*] \supset \Delta_2[0, 1] = \Delta_2^*.$$

It follows from this result that $\Delta_1 \sim \Delta_2$ if and only if $\Delta_1^* = \Delta_2^*$. Thus the set Δ^* is the maximal quasiconvexity generating set equivalent to Δ. The equivalence relation $\sim$ divides all the quasiconvexity generating sets, essentially all subsets of I, into pairwise disjoint non-null equivalence classes each of which may be uniquely represented by its maximal element namely by Δ^*, where Δ is any element in the class.

Let Q be a given Δ convex set and α and β be positive complementary ratios of Δ^*. We shall in the sequel make frequent use of the following three types of projection mappings, which since the ratios α and β are chosen from Δ^* will be called Δ^* projections.

The projection f defined by the equation $f(x) = \alpha s + \beta x$ is a contraction toward the point s. If $s \subset Q$, then $f(Q) \subset Q$; for if $s \subset Q$ and $x \subset Q$, then $f(x) \subset \Delta^* Q = Q$.

The projection f defined by the equation $x = \alpha s + \beta f(x)$ is an expansion away from the point s. If $s \subset Q$, then $f(CQ) \subset CQ$; for if $s \subset Q$ and $f(x) \subset Q$, then $x \subset \Delta^* Q = Q$.

The projection f defined by the equation $s = \alpha x + \beta f(x)$ is a reflection through the point s. If $s \subset CQ$, then $f(Q) \subset CQ$; for if $x \subset Q$ and $f(x) \subset Q$, then $s \subset \Delta^* Q = Q$.

In each of the projections f defined above the point s is the centre of projection and any image set is similar to its original. The projection f is a topological mapping, and its inverse f^{-1} is also a projection: if f is a Δ^* contraction, expansion, or reflection with centre s, then f^{-1} is a Δ^* expansion, contraction, or reflection respectively also with centre s.

We may summarize the above results on Δ^* projections as follows: a Δ^* contraction of Q toward a point of Q lies in Q; a Δ^* expansion of CQ away from a point of Q lies in CQ; a Δ^* reflection of Q through a point of CQ lies in CQ.

2. Density. In this section we investigate some elementary topological properties of quasiconvex sets. All the results here stem from the following density property of Δ^*.

THEOREM 2.1. Δ^* *is dense in* I.

Proof. Suppose to the contrary that the open set $I - \overline{\Delta^*}$ is non-null. Let J be an open interval component of this set with end points a and b, which lie in $\overline{\Delta^*}$. Thus there exist point sequences a_κ and b_κ of Δ^* with $a_\kappa \rightarrow a$ and $b_\kappa \rightarrow b$. Let α and β be positive complementary ratios of Δ. Then the point $x = \alpha a + \beta b$

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lies in J, and the points $x_\kappa = \alpha a_\kappa + \beta b_\kappa$ lie in $\Delta\Delta^* = \Delta^*$. Now $x_\kappa \to x$; whence x lies in $\overline{\Delta^*}$ and hence not in J. This contradiction proves the theorem.

THEOREM 2.2. *A quasiconvex set is dense in its convex hull.*

Proof. Every point in the convex hull of a set lies in the convex hull of some finite subset of that set. Thus it suffices to prove that $\Delta[A]$ is dense in $I[A]$ for every finite set A. This is clearly true for the null set and hence by induction is true for every finite set if it is true for a finite set A containing a point a whenever it is true for the set $A - a$. To demonstrate this let x be a point of $I[A]$; then x may be expressed in the form $x = \alpha a + \beta b$ where α and β are complementary ratios of I and $b \subset I[A - a]$. Since Δ^* is dense in I there exist complementary ratios α_κ and β_κ of Δ^* with $\alpha_\kappa \to \alpha$ and $\beta_\kappa \to \beta$. Furthermore by the induction hypothesis points b_κ of $\Delta[A - a]$ exist with $b_\kappa \to b$. Thus the points $x_\kappa = \alpha_\kappa a + \beta_\kappa b_\kappa$ lie in $\Delta^*[A] = \Delta[A]$. But clearly $x_\kappa \to x$, so $\Delta[A]$ is dense in $I[A]$.

The interior of the convex hull of a set E will be called the near interior of E and the complement of E in its near interior the near complement of E. Thus $\underline{I(E)}$ is the near interior of E and $\underline{I(E)} - E$ the near complement of E. Note that E is non-planar if and only if its near interior is non-null. We shall say of a non-planar set E that it is nearly convex if it contains its near interior: $E \supset \underline{I(E)}$, that is, if its near complement is null.

Nearly convex sets play an important role in the theory of quasiconvexity. Thus many of our theorems read: A quasiconvex set having such and such a property is nearly convex. We assume that any quasiconvex set forming the subject of a theorem is non-planar, so that the notion of near convexity is applicable to it. The hypotheses of the theorem usually ensure this.

LEMMA 2.3. *Let Q be a Δ convex set with near interior G; let S_q be an open sphere about $q \subset Q$; and let p be a point of G different from q. Then an open sphere $S_p \subset G$ about p and positive complementary ratios α, $\beta \subset \Delta^*$ exist such that for every non-null open subset V of S_p a point $a \subset Q$ can be found for which the expansion f away from a defined by the equation $x = \alpha a + \beta f(x)$ has the property that $q \subset f(V) \subset S_q$.*

Proof. Let p be the origin and let the radius of S_q be $2\epsilon\rho$ where $\rho = |q| > 0$. We may assume $\epsilon \leqslant 1$. Since $p \subset G$, some sphere S, say of radius $2\lambda\rho$, about p lies in G. Let α and β be positive complementary ratios of Δ^* chosen so that $\beta < \lambda/(1 + \lambda)$ whence $\beta/\alpha < \lambda$. Let S_p be the open sphere about p of radius $\epsilon\beta\rho < \lambda\rho$. Thus $S_p \subset S \subset G$. Consider the expansion g away from q defined by the equation $x = \alpha g(x) + \beta q$. Evidently for $x \subset S_p$ we have

$$| g(x) | = \frac{1}{\alpha} | x - q | < \frac{1}{\alpha} (\epsilon\beta\rho + \beta\rho) < 2\lambda\rho,$$

whence $g(V) \subset g(S_p) \subset S \subset G$ for any open subset V of S_p. Since Q is dense in S and hence in the open set $g(V)$, some point $a \subset Q \cap g(V)$ can be selected.

Let $v = g^{-1}(a)$; then $v \subset V$ and $g(v) = a \subset Q$. Thus $v = \alpha a + \beta q$. Now consider the expansion f away from a defined by the equation $x = \alpha a + \beta f(x)$. We observe that $q = f(v)$, so for $x \subset S_p$

$$| f(x) - q | = \frac{1}{\beta} | x - v | < \frac{1}{\beta} (\epsilon\beta\rho + \epsilon\beta\rho) = 2\epsilon\rho$$

whence $f(S_p) \subset S_q$. Thus we conclude that $q = f(v) \subset f(V) \subset f(S_p) \subset S_q$.

THEOREM 2.4. *A quasiconvex set with non-null interior is nearly convex.*

Proof. Let Q be a Δ convex set containing an open sphere S_q with center q. We are to show that Q contains its near interior G. Since $q \subset Q$, we consider to this end any point p in G different from q. According to the preceding lemma there exists a point $a \subset Q$ and a Δ^* expansion f away from a with the property that $f(p) \subset S_q \subset Q$ whence $p \subset f^{-1}(Q)$. Now f^{-1} is a Δ^* contraction toward the point $a \subset Q$, so $p \subset f^{-1}(Q) \subset Q$. Therefore $G \subset Q$.

3. Measure. Let μ^* be an outer measure function and μ_* the corresponding inner measure function defined on subsets of X. If E is a measurable set then $\mu^*(E) = \mu_*(E)$ and we write $\mu(E)$ for this common value. We assume that μ^* and μ_* are homogeneous measures in the following sense: if f is a projection with ratio of similarity θ, then for every set E we have $\mu^*(f(E)) = \theta^\nu \mu^*(E)$ and $\mu_*(f(E)) = \theta^\nu \mu_*(E)$.

THEOREM 3.1. *The near complement of a quasiconvex set of positive outer measure has zero inner measure.*

Proof. Let Q be a Δ convex set of positive outer measure. Suppose, contrary to the theorem, that the near complement P of Q has positive inner measure. Let p be a point of inner density of P and let $\eta = \frac{1}{2}$. Then an open sphere about p of radius ρ exists such that for every smaller concentric open sphere S_p we have

$$\mu_*(S_p \cap P) > \eta\mu(S_p).$$

Now let q be a point of outer density of Q. Then an open sphere S_q of radius $\rho_q < \rho$ exists with

$$\mu^*(S_q \cap Q) > (1 - \eta^{\nu+1})\, \mu(S_q),$$

whence

$$\mu_*(S_q - Q) < \eta^{\nu+1}\mu(S_q).$$

According to Lemma 2.3 a point of Q and a Δ^* expansion f away from this point can be found with the property that $|b - q| < \eta\rho_q$, where $b = f(p)$. Let S_b be the sphere about b of radius $\rho_b = \eta\rho_q$. Then $\mu(S_b) = \eta^\nu \mu(S_q)$, and, since $\eta = \frac{1}{2}$, $S_b \subset S_q$. Furthermore, the inverse set $S_p = f^{-1}(S_b)$, being a contraction of S_b, is an open sphere about $p = f^{-1}(b)$ with radius $\rho_p < \rho_b < \rho_q < \rho$. Hence $\mu_*(S_p \cap P) > \eta\mu(S_p)$ and consequently

$$\mu_*(f(S_p \cap P)) > \eta\mu(f(S_p)) = \eta\mu(S_b) = \eta^{\nu+1}\mu(S_p).$$

Since f is a Δ^* expansion away from a point of Q, we have $f(P) \subset f(CQ) \subset CQ$. Therefore

$$f(S_p \cap P) = f(S_p) \cap f(P) \subset S_b \cap CQ \subset S_q - Q$$

so

$$\mu_*(f(P \cap S_p)) \leqslant \mu_*(S_q - Q) < \eta^{\nu+1}\mu(S_q)$$

in contradiction to a preceding inequality. This contradiction proves that P has zero inner measure.

Let P be the near complement of a quasiconvex set Q. Under the assumption that Q has positive outer measure we have shown that P has zero inner measure. Under the stronger hypothesis that Q has positive inner measure we now prove the stronger conclusion that P is the null set, that is, Q is nearly convex.

THEOREM 3.2. *A quasiconvex set of positive inner measure is nearly convex.*

Proof. Let Q be a Δ convex set of positive inner measure. Then Q contains a measurable set F of positive measure. Let q be a point of density of F and let $\eta = \frac{1}{2}$. Then there exists an open sphere S about q of radius ρ such that

$$\mu(F \cap S) > (1 - \alpha_\nu \eta^\nu)\mu(S)$$

where $\alpha_\nu = \alpha^\nu/(\alpha^\nu + \beta^\nu)$, α and β being positive complementary ratios of Δ^* with $\alpha \leqslant \beta$. Let S_q be the sphere about q of radius $\eta\rho$. We contend that $S_q \subset Q$. Suppose, to the contrary, that some point p, which we may assume to be the origin of S_q, does not lie in Q. Let S_p be the sphere about p of radius $\eta\rho$; then, since $\eta = \frac{1}{2}$, $S_p \subset S$. Let $F_p = F \cap S_p$; then

$$\mu(F_p) = \mu(F \cap S_p) = \mu(F \cap S) - \mu(F \cap S - S_p).$$

Consequently

$$\mu(F \cap S - S_p) \leqslant \mu(S - S_p) = \mu(S) - \mu(S_p) = (1 - \eta^\nu)\mu(S),$$

wherefore

$$\mu(F_p) > [(1 - \alpha_\nu\eta^\nu) - (1 - \eta^\nu)]\mu(S) = \beta_\nu\mu(S_p),$$

the ratio β_ν being complementary to α_ν. Let f be the Δ^* reflection through the point p of CQ defined by the equation $p = \alpha x + \beta f(x)$. Therefore $f(F_p) \subset f(Q) \subset CQ \subset CF$. Moreover, since $\alpha \leqslant \beta$, we have

$$|f(x)| = \frac{\alpha}{\beta}|x| \leqslant |x|$$

whence $f(F_p) \subset S_p$. Consequently the set F_p and its reflection $f(F_p)$ are disjoint measurable subsets of S_p, so that

$$\mu(S_p) \geqslant \mu(F_p) + \mu(f(F_p)) = \left(1 + \frac{\alpha^\nu}{\beta^\nu}\right)\mu(F_p) = \frac{1}{\beta_\nu}\mu(F_p)$$

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in contradiction to a preceding inequality. This contradiction proves that the quasiconvex set Q contains the sphere S_p and hence is nearly convex.

THEOREM 3.3. *Quasiconvexity generated by a set of positive inner measure is equivalent to convexity.*

Proof. Let Δ be a quasiconvexity generating set of positive inner measure. Consider the Δ convex set $\Delta^* \supset \Delta$, and note that $\Delta \sim \Delta^* = I$.

We remark that quasiconvexity generated by a set of measure zero may be equivalent to convexity. The Cantor ternary set is an example of such a set.

Our theorems on measure of quasiconvex sets may be stated as follows: Every quasiconvex set Q is either extremely measurable or extremely non-measurable. By this we mean that if Q is measurable its measure is as small or as large as possible, and if Q is non-measurable its inner measure is as small as possible and its outer measure as large as possible: zero being as small a measure as possible and the measure of the convex hull of Q being as large a measure as possible.

4. Subcontinua. In this section we investigate what happens when a quasiconvex set or its near complement contains a certain type of continuum. We show that a quasiconvex set containing a non-planar continuum is nearly convex and that a quasiconvex set whose near complement contains a certain type of non-planar continuum is zero dimensional in the topological sense of dimension.

THEOREM 4.1. *A quasiconvex set containing a non-planar continuum is nearly convex.*

Let Q be a Δ convex set containing a non-planar continuum K. We may suppose that K contains the origin and ν linearly independent vectors k_λ ($\lambda = 1, \ldots, \nu$). Let α and β be positive complementary ratios of Δ. If $x_1, \ldots x_\nu$ are points of Q then it is easily verified by taking successive α, β linear combinations that the point

$$x = \theta_1 x_1 + \ldots + \theta_\nu x_\nu$$

also lies in Q where $\theta_1 = \alpha^{\nu-1}$ and $\theta_\lambda = \alpha^{\nu-\lambda}\beta$ for $\lambda = 2, \ldots, \nu$. Let $H_\lambda (\lambda = 1, \ldots \nu)$ be the θ_λ contraction of the continuum K toward the origin. Thus H consists of all points of the form $\theta_\lambda x$ for $x \subset K$. We have just indicated that the vector sum H of the ν continua H_λ lies in Q. However, since the vectors $h_\lambda = \theta_\lambda k_\lambda$ form a basis for X, this vector sum set H has a non-null interior. (For the proof of this see the paper which follows entitled, *On the vector sum of continua.*) Therefore Q has a non-null interior and hence is nearly convex.

We now digress into some lemmas concerning convex sets.

Let K be a compact convex set in X. We shall call a point a an apex of K if for every neighbourhood N of a there exists an open space H containing a whose intersection with K lies in $N : H \cap K \subset N$. An apex of K is evidently

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a boundary point of K, but not necessarily conversely. Let $A(K)$ be the set of apices of K.

LEMMA 4.2. *$A(T \cap K) = T \cap A(K)$ for every supporting plane T of K; if $I[E] = K$, then $E \supset A(K)$; $I[A(K)] = K$.*

Proof. It is evident that $T \cap A(K) \subset A(T \cap K)$. Suppose then that $a \subset A(T \cap K)$. Thus for every neighbourhood N of a there exists a half plane V of dimension $\nu - 1$ open in T such that $a \subset V \cap (T \cap K) = V \cap K \subset N$. Let L be that linear subspace of dimension $\nu - 2$, a plane in T, which bounds V; and let T' be a variable plane of dimension $\nu - 1$ which contains L and is different from T. Furthermore let V' be that half plane open in T which is bounded by $L = T' \cap T$ and lies on the same side of the supporting plane T of K as does K; and let H' be that open half space of dimension ν bounded by T' which contains V. Then for some H' we have $H' \cap K \subset N$; else $H' \cap K - N \neq 0$ for all H', so that by choosing T' approaching T in such a way that V' approaches V it would follow from the compactness of $K - N$ that $V \cap K - N \neq 0$. This completes the proof of the first part of the lemma.

To prove the second part, suppose that $I[E] = K$ but that $a \subset A(K) - E$. Since $a \subset A(K) \subset K = I[E]$ there exists a finite set $F \subset E \subset K$ whose convex hull contains a although the set F itself does not contain a. Therefore the open set CF is a neighbourhood of the apex a of K, so an open half space H containing a exists such that $H \cap K \subset CF$, whence $H \cap F = H \cap F \cap K = 0$. Therefore $F \subset CH$; that is, the closed half space CH is a convex set containing F but not a, in contradiction to $a \subset I[F]$.

The proof of the third part of the lemma proceeds by induction on the dimension of K. It is clearly true for dimension 1. Assume it true for dimension $\nu - 1 \geqslant 1$, and let K be of dimension ν. Furthermore let t be any boundary point of K and let T be a supporting plane to K at t. From the induction hypothesis and the first part of the lemma we see that

$$t \subset T \cap K = I[A(T \cap K)] = I[T \cap A(K)] \subset I[A(K)].$$

Thus every boundary point of K belongs to the convex set $I[A(K)]$ so that $K \subset I[A(K)]$. It is obvious that $K \supset I[A(K)]$. This concludes the proof of the lemma. We note that $A(K)$ is the minimal set whose convex hull is K.

We shall say that a set E is indented if E together with some plane T bounds a non-null bounded open set $W : \dot{W} \subset T \cup E$. A point $p \subset E$ will be called an indentation point of E provided every neighbourhood of p contains an indented subset of E.

LEMMA 4.3. *Every indented set contains an indentation point.*

Proof. Let E be an indented set; and let T be a plane and W a non-null bounded open set such that $\dot{W} \subset T \cup E$. Consider the compact set $K = I[W]$. Since $K = I[A(K)]$ is non-planar, the set $A(K)$ is also non-planar. Therefore $A(K)$ contains some point, say p, not in the plane T. Now $A(K)$ is the minimal

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set whose convex hull is K, so $p \subset \overline{W}$. Moreover, the apex p is not interior to $\overline{W}$; hence $p \subset {}^{\cdot}W \subset T \cup E$. But $p \not\subset T$, so $p \subset E$. Consider any neighbourhood N_p of p. Since $p \not\subset T$, a neighbourhood N of p can be found such that $\overline{N} \cap T = 0$ and $\overline{N} \subset N_p$. Now p is an apex of K, so there exists an open half space H containing p such that $H \cap K \subset N$. Since $p \subset H \cap {}^{\cdot}W$, the set $V = H \cap W$ is a non-null open set containing p on its boundary ${}^{\cdot}V$. Evidently $V \subset N$, so ${}^{\cdot}V \cap T \subset \overline{N} \cap T = 0$. Consequently we see from the inclusion ${}^{\cdot}V \subset {}^{\cdot}H \cup {}^{\cdot}W \subset {}^{\cdot}H \cup T \cup E$ that ${}^{\cdot}V \subset {}^{\cdot}H \cup E$, ${}^{\cdot}H$ being the bounding plane of H. Thus ${}^{\cdot}V$ is an indented subset of E and ${}^{\cdot}V \subset \overline{N} \subset N_p$. Since this is so for any neighbourhood N_p of p, we conclude that the point $p \subset E$ is an indentation point of E.

LEMMA 4.4. *Every neighbourhood of an indentation point of the near complement P of a quasiconvex set contains a non-null open subset with boundary in P.*

Proof. Let Q be a Δ convex set with near interior G and near complement P; and let N be a neighbourhood of an indentation point p of P. Thus the set $N \cap G$ is a neighbourhood of p and hence contains some open sphere S_p about p. Let the radius of S_p be 5ρ and let S be the open sphere about p of radius ρ. Since p is an indentation point of P, a plane T and a non-null bounded open set W exist such that ${}^{\cdot}W \subset T \cup P$ and $W \subset S$. We shall for convenience assume that the plane T contains the origin. Let φ be a linear functional vanishing on T such that W intersects the open half space $\varphi > 0$. Define 4μ to be the upper bound of $\varphi(w)$ as w ranges over the non-null bounded set W; then $\mu > 0$. Let H be the open half space $\varphi > 8\mu$; and let K be the closed half space $\varphi \leqslant 0$. Consider the expansion f_q away from q defined by the equation $x = \alpha_q + \beta f_q(x)$ where α and β are fixed positive complementary ratios of Δ^* so chosen that $\frac{1}{3} < \alpha < \frac{3}{8}$. We assert that the open expansion sets $f_q(W)$ cover $K \cap \overline{W}$ as q ranges over $Q \cap H$. To prove this let k be a given point of $K \cap \overline{W}$; and let g be the expansion away from k defined by the equation $x = \alpha g(x) + \beta k$. By definition of μ some point $w \subset W$ exists such that $\varphi(w) > 3\mu$. Since $\varphi(k) \leqslant 0$ and $\alpha < \frac{3}{8}$ we have

$$\varphi(g(w)) = \frac{1}{\alpha}\varphi(w) - \frac{\beta}{\alpha}\varphi(k) > 8\mu,$$

so that $H \cap g(W) \neq 0$. Furthermore, for any point $w \subset W \subset S$ we have

$$g(w) - p = \frac{1}{\alpha}(w - p) - \frac{\beta}{\alpha}(k - p),$$

so that

$$|g(w) - p| \leqslant \frac{\rho}{\alpha} + \frac{\beta\rho}{\alpha} < 5\rho$$

since $\alpha > \frac{1}{3}$ and $k \subset \overline{W} \subset \overline{S}$. Therefore $g(W) \subset S_p \subset G$. The set $H \cap g(W)$ is then a non-null open subset of G. Since Q is dense in G, some point $q \subset Q \cap$

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$H \cap g(W)$ exists. Let $w = g^{-1}(q)$. Then $w = \alpha q + \beta k \subset W$, so that $k = f_q(w) \subset f_q(W)$.

This proves that the open sets $f_q(W)$ cover the compact set $K \cap \overline{W}$ as q ranges over $Q \cap H$. Consequently a finite subset Y of $Q \cap H$ exists such that the sets $f_y(W)$ cover $K \cap \overline{W}$ as y ranges over Y. Define $U = \bigcup_y f_y(W)$. Thus U is an open set and $K \cap \overline{W} \subset U$. Now $f_y(^{\cdot}W) \subset K \cup CQ$. For since f_y is a Δ^* expansion away from a point of Q we have $f_y(CQ) \subset CQ$. And since $\varphi(y) > 0$ and $\varphi(k) \leqslant 0$, we have

$$\varphi(f_y(k)) = \beta^{-1}[\varphi(k) - \alpha\varphi(y)] < 0,$$

whence $f_y(K) \subset K$. But $^{\cdot}W \subset K \cup CQ$, so

$$f_y(^{\cdot}W) \subset f_y(K \cup CQ) = f_y(K) \cup f_y(CQ) \subset K \cup CQ.$$

Therefore $^{\cdot}U \subset \bigcup_y f_y(^{\cdot}W) \subset K \cup CQ$, the union being finite.

Consider the following open subset of W: $V = W - \overline{U}$. We shall show that V is a non-null open subset of N whose boundary lies in P. Evidently V is an open subset of $W \subset N$. We note that $^{\cdot}V \subset CV = CW \cup \overline{U}$. Since U and V are disjoint open sets, U and $\overline{V}$ are also disjoint, so $^{\cdot}V \subset \overline{V} \subset CU$. It is clear that $^{\cdot}V \subset \overline{V} \subset \overline{W}$. Combining these inclusions with the inclusions $^{\cdot}W \subset K \cup CQ$ and $^{\cdot}U \subset K \cup CQ$, we obtain the result that $^{\cdot}V \subset {^{\cdot}W} \cup {^{\cdot}U} \subset K \cup CQ$. But since $^{\cdot}V \cap K \subset \overline{W} \cap K \subset U$ and $^{\cdot}V \subset CU$ and $^{\cdot}V \subset G$, we conclude that $^{\cdot}V \subset G \cap CQ = P$.

To complete the demonstration we must show that V is non-null. To do this we prove that each of the open sets $f_y(W)$ composing U lies in the open half space $\varphi < 2\mu$. For let $y \subset Y$ and $w \subset W$; then $\varphi(y) > 8\mu$ and $\varphi(w) < 4\mu$, so that

$$\varphi(f_y(w)) = \frac{1}{\beta}\varphi(w) - \frac{\alpha}{\beta}\varphi(y) < \frac{4\mu}{\beta} - \frac{8\alpha\mu}{\beta} < 2\mu$$

since $\alpha > \frac{1}{3}$. The upper bound of $\varphi(w)$ for $w \subset W$ is 4μ, so we see that $V = W - \overline{U} \neq 0$. This completes the proof of the lemma.

THEOREM 4.5. *A quasiconvex set whose near complement is indented has topological dimension zero.*

Proof. Let Q be a Δ convex set whose near complement P is indented; and let p be a fixed indentation point of P. Consider any point $q \subset Q$ and any open sphere S_q about q. Since p lies in the near complement G of Q there exists according to Lemma 2.3 a sphere $S_p \subset G$ about p and positive complementary ratios $\alpha, \beta \subset \Delta^*$ such that for any non-null open subset V of G a point $a \subset Q$ can be found for which the Δ^* expansion away from a defined by the equation $x = \alpha a + \beta f(x)$ has the property that $q \subset f(V) \subset S_q$. Let V be a non-null open subset of S_p, such as constructed in the preceding lemma, with boundary in CQ. Therefore the neighbourhood $f(V)$ of Q lies in S_q and its boundary, being a Δ^* expansion away from a point of Q of the set $^{\cdot}V \subset CQ$. This is so for every point q of Q and every sphere S_q about q. Thus we conclude that Q has topological dimension zero.

For $\nu = 2$ this theorem takes the following form.

THEOREM 4.6. *In a two dimensional vector space any quasiconvex set whose near complement contains a non-linear closed connected set has topological dimension zero.*

Proof. The proof consists in showing that any non-linear closed connected set F is indented. Now either F is convex and the result is obvious, or else there exists a line L whose intersection with F is not connected. Therefore, according to a theorem of Janiszewski, one of the components of $F \cup L$ must be bounded; so F is indented.

5. Connectedness. Many examples of pathological connected sets may be constructed in a more or less systematic fashion as graphs of solutions of the functional equation $\varphi(x + y) = \varphi(x) + \varphi(y)$ [17]. It is thus of interest to investigate the connectedness of quasiconvex sets and their near complements.

LEMMA 5.1. *If E is connected, then ΔE is connected.*

Proof. Let p be any point of ΔE. Then p may be expressed in the form $p = \alpha a + \beta b$ where α, β are complementary ratios of Δ and a, $b \subset E$. Consider the contraction f defined by the vector formula $f(x) = \alpha a + \beta x$. The set $f(E)$, being a Δ contraction of E toward a point of E, lies in ΔE. Furthermore, $f(E)$ is similar to E and hence is connected. Now $p = f(b)$ lies in $f(E)$ and $a = f(a)$ also lies in $f(E)$. Therefore $f(E)$ is a connected set containing p and intersecting the connected set E. This proves ΔE connected.

THEOREM 5.2. *A quasiconvex set containing a non-planar connected set is connected.*

Proof. Let Q be a Δ convex set containing a non-planar connected set E and let Q' be that component of Q which contains E. Then Q' is closed in Q and $\Delta Q' \subset \Delta Q = Q$. According to the preceding lemma $\Delta Q'$ is a connected superset of Q'. Consequently $\Delta Q' = Q'$, so Q' is a non-planar connected Δ convex set. It is therefore dense in some open set namely its near interior G'. We assert that $Q' = Q$. For suppose to the contrary that Q' is a proper subset of Q. Then since Q' is closed in Q there exists a point q of Q at a positive distance from Q. Evidently a slight Δ^* contraction f toward q can be obtained such that the convex open sets G' and $f(G')$ interesct. Now Q' is a connected subset of Q dense in G', and $f(Q')$ is a connected subset of Q, dense in $f(G')$; the union $Q' \cup f(Q')$ is then connected. But Q' is a component of Q, so $f(Q') \subset Q'$. On the other hand the contraction set $f(Q')$ lies slightly closer to q than does Q'. This contradiction proves $Q' = Q$, wherefore Q is connected.

Let A be a plane or planar portion. We shall say that a set E is semiconnected parallel to A if no plane parallel to A separates E. The set E will be called semiconnected if it is semiconnected to every plane.

THEOREM 5.3. *A quasiconvex set containing a planar portion and semiconnected parallel to this planar portion is connected.*

Proof. Let Q be a Δ convex set containing a planar portion A and semiconnected parallel to A. Suppose, contrary to the theorem, that Q is not connected. Then Q is separated by some closed set F cutting X. Thus F also cuts the near interior G of Q. Let V be a component of $G - F$, and W a component of $G - \overline{V}$. The space X is locally connected, so the sets W and V, being components of open sets, are open subsets of G. Since any point in $G - F$ lies in some open component of $G - F$, we see that $G \cap^{\cdot} V \subset F$. Similarly $G \cap^{\cdot} W \subset \overline{V}$. Now $W \cap \overline{V} = 0$ so $\overline{V} \cap W = 0$; whence $\overline{V} \subset CW$ and $\overline{W} \subset CV$. It is clear from this that the boundary $B = G \cap^{\cdot} W$ of W in G is given by

$$B = G \cap \overline{W} \cap \overline{V} = G \cap^{\cdot} W \cap^{\cdot} V \subset F \subset CQ.$$

Furthermore, since G is connected and V and W are non-null, this set B also is non-null.

Now consider a point $p \subset B$. Evidently a point $a \subset Q$ and an open sphere S about p and lying in G can be found such that any Δ^* contraction of the planar portion $A \subset Q$ toward the point $a \subset Q$ which intersects S also cuts S. We shall call the intersection sets with S of these Δ^* contractions Q-discs. Thus the Q-discs are parallel planar portions dense in S which cut S and lie in Q. Hence they do not intersect B, so any Q-disc which intersects W or V lies wholly in W or V. Since p is a limit point of both W and V, it is also a limit point of discs lying in W and of discs lying in V. Let D_p be the disc formed by intersection with S of the plane through p parallel to A. Therefore $D_p \subset \overline{W}$ and $D_p \subset \overline{V}$. Also $D_p \subset S \subset G$. Consequently $D_p \subset G \cap \overline{W} \cap \overline{V} = B$. Thus every point $p \subset B$ lies in some disc D_p open in the plane parallel to A through p.

Let D be the intersection with G of a plane parallel to A which intersects the non-null closed set B. Then D is convex and hence connected. Moreover, $B \cap D$ is a non-null set closed in D, which, as we have just shown, is also open in D. Thus $B \cap D = D$, so that $D \subset B \subset F$. Therefore $\overline{D} \subset F \subset CQ$, so the plane extension T of D does not intersect Q. This plane T, parallel to A, then separates Q in contradiction to the semiconnectedness of Q parallel to A. This contradiction proves Q connected.

The following theorem is similar to the theorem just proved and can be proved in a similar fashion: A quasiconvex set containing a linear portion A is connected if it cannot be separated by a cylinder parallel to A.

THEOREM 5.4. *A bounded quasiconvex set containing a planar portion and semiconnected parallel to this planar portion is nearly convex.*

Proof. Let Q be a bounded Δ convex set containing a planar portion A and semiconnected parallel to A. We may assume that A lies in the near interior G of Q, for we could otherwise replace A by a suitable Δ contraction of A which does lie in G. Moreover, we shall for convenience suppose that A contains the origin. If T is the plane containing A, then a radius $\rho > 0$ exists such that every point of T in the ρ sphere about the origin lies in A. Let φ be a linear functional of norm 1 vanishing on T. Since Q is semiconnected parallel to T, the set $\varphi(Q)$ of real numbers contains an open λ neighbourhood of 0 for some $\lambda > 0$. Let $\epsilon = \min(\lambda, \rho)$ and let $\mu\epsilon$ be a bound of the bounded set Q. Choose complementary ratios α and β of Δ^* such that $0 < \alpha < (1 + \mu)^{-1}$; and define $\eta = \min[\alpha, 1 - \alpha(1 + \mu)]$. We contend that Q contains the open sphere S of radius $\eta\epsilon$ about the origin. To prove this consider a point $x \subset S$; thus $|x| < \eta\epsilon$. Now

$$\alpha^{-1}\varphi(x) \leqslant \alpha^{-1}|x| < \alpha^{-1}\eta\epsilon \leqslant \epsilon \leqslant \lambda;$$

so there exists a point $q \subset Q$ such that $\varphi(x) = \alpha\varphi(q)$. Consider the point t defined by the equation $x = \alpha q + \beta t$. We see that

$$\varphi(t) = \beta^{-1}[\varphi(x) - \alpha\varphi(q)] = 0;$$

so $t \subset T$. Also by choice of η we have

$$|t| \leqslant \beta^{-1}(|x| + \alpha|q|) < \beta^{-1}(\eta\epsilon + \alpha\mu\epsilon) \leqslant \epsilon \leqslant \rho,$$

whence $t \subset Q$. Thus $x = \alpha q + \beta t \subset \Delta^* Q = Q$. This proves that Q contains the open sphere S and hence is nearly convex.

A result similar to the theorem just proved can be similarly proved, namely: A bounded quasiconvex set Q is nearly convex if it contains a linear portion A and if every line parallel to this linear portion intersecting the near interior of Q also intersects Q, that is, if Q is opaque parallel to A.

THEOREM 5.5. *If the near complement of a quasiconvex set is semiconnected it is also connected.*

Proof. Let Q be a quasiconvex set with near interior G whose near complement P is semiconnected. Suppose, contrary to the theorem, that P is not connected. Then some set F in G cuts G and separates P. Let V be a component of $G - F$ and W a component of $G - \overline{V}$. Then as in 5.3 the boundary B of W in G is non-null and $B \subset F \subset Q$. Let $G_\kappa (\kappa = 1, 2, \ldots)$ be a sequence of bounded non-null convex open sets intersecting B, the sequence strictly increasing to G in the sense that $\overline{G}_\kappa \subset G_{\kappa+1}$ and $G = \cup G_\kappa$. We define sets W_κ $(\kappa = 0, 1, 2, \ldots)$ recursively as follows. Let $G_0 = 0$ and $W_0 = 0$, and suppose W_κ to be a connected open subset of $G_\kappa - B$. Since B cuts G and intersects $G_{\kappa+1}$ we see that W and V also intersect $G_{\kappa+1}$ so that B cuts $G_{\kappa+1}$. Therefore the set $G_{\kappa+1} - B$ contains the connected open set W_κ and possesses at least two components. Let $V_{\kappa+1}$ be a component such that $V_{\kappa+1} \cap W_\kappa = 0$ and let $W_{\kappa+1}$ be the component of $G_{\kappa+1} - \overline{V}_{\kappa+1}$ containing W_κ. Thus $W_\kappa \subset W_{\kappa+1}$. Since $G_\kappa (\kappa = 1, 2, \ldots)$ is an open ν-cell, the boundary $B_\kappa = G_\kappa \cap \cdot W_\kappa$ of W_κ in G_κ is according to the Phragmen-Brouwer theorem connected. Clearly $\overline{B}_\kappa$ lies in the compact set $\overline{G}_\kappa$ and hence is a continuum. Furthermore, $\overline{B}_\kappa \subset B \subset Q$. Now $\overline{B}_\kappa$ must be a planar set, else Q by containing a non-planar continuum would be nearly convex and its near complement null. Therefore B_κ is a planar portion crosscutting the convex set G_κ. We note that $B_\kappa \subset B_{\kappa+1}$. For any point $p \subset B_\kappa$ lies in $G_\kappa \cap \overline{W}_\kappa$ and hence in $G_{\kappa+1} \cap \overline{W}_{\kappa+1}$. If p were not in $B_{\kappa+1}$ it would lie in $W_{\kappa+1}$ and hence not in B. But this is impossible since $B_\kappa \subset B$. The union $B_\omega \subset B$ of the sets B_κ is then a planar portion crosscutting the union G of the sets G_κ. Consequently P is separated by the planar portion B_ω and hence is not semiconnected. This contradiction proves P connected.

This theorem suggests the question: Is a semiconnected quasiconvex set connected? The answer is no, as shown by the following example.

By using a procedure similar to that of Jones [17] a midpoint convex set Q dense in the Cartesian plane can be constructed having the property that Q intersects every perfect set not lying in a countable union of horizontal and vertical lines and having the further property that every horizontal line and every vertical line intersects Q in precisely one point. Clearly Q is semiconnected. However, the horizontal and vertical lines through any point not in Q form four complementary open quadrants one of which evidently contains no point of Q on its boundary. Thus Q can be separated by a right angle and hence is not connected; in fact, according to 4.6, Q has topological dimension zero.

From the theorem that a semiconnected near complement of a quasiconvex set is connected, we deduce the following three results.

THEOREM 5.6. *The near complement of a bounded semiconnected quasiconvex set is connected.*

Proof. The near complement P of a bounded semiconnected quasiconvex set Q is semiconnected and hence connected. For if, to the contrary, P is not semiconnected, it is a non-null set separated by a planar portion lying in Q. But then Q, being semiconnected, is, according to Theorem 5.3, nearly convex, whence P is null—a contradiction.

A set whose convex hull is the entire space X will be called totally unbounded.

THEOREM 5.7. *The near complement of a totally unbounded semiconnected quasiconvex set is connected.*

Proof. The near complement $P = CQ$ of a totally unbounded convex set Q is semiconnected and hence connected. For if, to the contrary, P is not semiconnected, it is a non-null set separated by some plane T, which, since Q is unbounded, lies in Q. Let f be a Δ reflection through a point of the non-null set P. Thus $f(T)$ is a plane lying in P and hence separating Q. This, however, is a contradiction, for Q is semiconnected.

We have shown that the near complement of a bounded or of a totally unbounded semiconnected quasiconvex set is connected. However, if the set is neither bounded nor totally unbounded its near complement may not be connected. For let Q be the intersection with the upper half plane $y > 0$ of the midpoint convex set consisting of all points (x, y) in the Cartesian plane such that $y > \varphi(x)$ where φ is a discontinuous solution of the functional equation

$\varphi(x + y) = \varphi(x) + \varphi(y)$. Now the set A of real numbers x for which $\varphi(x) \leqslant 0$ is everywhere dense; so for every $x \subset A$ the vertical line $y > 0$ with abscissa x lies in Q. Therefore Q is a semiconnected midpoint convex set whose near interior, the upper half plane $y > 0$, possesses only linear components. We see from 5.3 that Q is connected.

THEOREM 5.8. *If the near complement of a totally unbounded quasiconvex set contains a non-planar connected subset, it is connected.*

Proof. If the near complement $P = CQ$ of a totally unbounded Δ convex set Q contains a non-planar connected subset E, then P is semiconnected and hence connected. For if, to the contrary, P is not semiconnected, it is a non-null set separated by some plane T, which since Q is totally unbounded, lies in Q. Let G be the near interior of the non-planar set E. Then G is a non-null open set. Evidently a Δ^* contraction f toward a point of Q can be found such that the plane $f(T)$ intersects G. Thus $f(T)$ lies in Q and hence separates E. This, however, is impossible, for E is connected.

The following example shows that the near complement of a bounded quasiconvex set may possess exactly two non-planar components.

Let a and b be rationally independent real numbers. Then any rational linear combination x of a and b is uniquely expressible in the form $x = \alpha + \beta$ where α represents a rational multiple of a and β a rational multiple of b. Let Q be the set of all points (x, y) in the Cartesian plane such that x is expressible in the form $x = \alpha + \beta$ with $|\alpha| < 1$, $|\beta| < 1$, $|x| < 1$, and such that $-1 + |\beta| < y < 1 - |\alpha|$. It is easily verified that Q is a midpoint convex set whose near interior G is the square $|x| < 1$, $|y| < 1$. Moreover, the y-axis separates the near complement $G - Q$ of Q, but $G - Q$ is otherwise semiconnected. Thus $G - Q$ is not connected, but that part of it to either side of the y-axis is a non-linear semiconnected and hence connected set.

History. A function φ defined for all real numbers satisfying the functional equation

$$\varphi(x + y) = \varphi(x) + \varphi(y) \tag{1}$$

will be called additive. In 1821 Cauchy [9] showed that any additive function φ is also rationally homogeneous, that is,

$$\varphi(\xi x) = \xi\varphi(x) \tag{2}$$

for all rational numbers ξ; whence he deduced that a continuous additive function φ is real homogeneous, that is, satisfies (2) for all real ξ and hence is of the form

$$\varphi(x) = x\varphi(1). \tag{3}$$

From (1) and (2) it follows that for any additive function φ we have

$$\varphi(\sum \xi_\kappa x_\kappa) = \sum \xi_\kappa \varphi(x_\kappa), \tag{4}$$

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the sum being finite and the ξ_κ being rational. In 1905 Hamel [14], using the then newly discovered well-ordering theorem of Zermelo, constructed a set H, now called a Hamel basis, with the property that every real number x can be represented uniquely (with the exception of zero coefficients) in the form

$$x = \sum \xi_\kappa x_\kappa, \tag{5}$$

where the sum is finite, the ξ_κ rational, and the x_κ belong to H. Thus we see from (4) that an additive function φ is exactly determined by its values on a Hamel basis H. If the functional values of φ are arbitrarily selected for x in H and determined for the remaining real numbers x by (4), then the resulting function φ is additive. It is continuous if and only if the ratio $\varphi(x)/x$ is constant as x ranges over the basis H. In this way Hamel completely solved the problem of the existence of discontinuous additive functions. Other interesting properties of Hamel bases and their application to discontinuous additive functions have been studied by Burstin [8], Sierpinski [30], and Jones [17, 18].

A function φ defined on an open interval of real numbers satisfying the functional inequality

$$\varphi(\tfrac{1}{2}\, x + \tfrac{1}{2}\, y) \leqslant \tfrac{1}{2}\, \varphi(x) + \tfrac{1}{2}\, \varphi(y) \tag{6}$$

will be called midpoint convex. Such functions were introduced in 1905 by Jensen [15, 16] who showed that any midpoint convex function φ is rationally convex, that is,

$$\varphi(\alpha x + \beta y) \leqslant \alpha\varphi(x) + \beta\varphi(y) \tag{7}$$

for all rational complementary ratios α and β; whence it follows that a continuous midpoint convex function is convex, that is, satisfies (7) for all real complementary ratios α and β.

Now it is easily seen from (4) that an additive function φ satisfies (7) with equality holding and hence is midpoint convex. Thus from the point of view of attaining generality it is desirable to consider the midpoint convex functions rather than the additive functions. Historically, however, results were first discovered for additive functions and then later extended to midpoint convex functions.

Generally speaking the problem was this: Find constraints, in themselves very weak, which, when placed on an additive or midpoint convex function, are sufficiently strong to force that function to be continuous; that is: What pathological properties do the discontinuous additive and midpoint convex functions possess?

Some density properties of additive functions were noted in 1875 by Darboux [10, 11] who showed that an additive function is continuous if it is bounded above or below on some interval. In his paper on the generation of discontinuous additive functions Hamel [14] pointed out that the graph of such a function is everywhere dense in the plane. In 1915, Bernstein and Doetsch [6] showed that the graph of a discontinuous midpoint convex function is dense above some convex function ($-\infty$ being allowed).

Measure properties of additive and midpoint convex functions have been extensively investigated. The first result in this direction, namely, that a measurable additive function is continuous, was discovered in 1913 by Fréchet [12]. This same theorem has since been proved many times: in 1920 by Sierpinski [31] and by Banach [2], in 1936 by Kac [19], in 1945 by Alexiewicz and Orlicz [1], and in 1947 by Kestleman [20]. It was somewhat generalized in 1924 by Sierpinski [34], who observed that an additive function majorized by a measurable function is continuous. That a measurable midpoint convex function is continuous was shown by Blumberg [7] in 1919 and in 1920 by Sierpinksi [32]. It should be mentioned that these measure results are closely connected with the work of Steinhaus [35] on the distances between points of a set.

These researches into measure and density properties culminated in 1929 in two papers by Ostrowski [22, 23] which include practically all the previous results. In one paper [22] Ostrowski showed that a midpoint convex function bounded above on a set of positive measure is continuous; and in the other paper [23] that the x-projection of the set of those graph points of a discontinuous midpoint convex function which lie in any plane neighbourhood above the lower bounding curve of the function has positive outer measure and zero inner measure.

The connectivity properties of graphs of discontinuous additive functions were first studied in 1942 by Jones [17, 18], who showed that every such graph is either connected or totally disconnected, and that it is connected if and only if it intersects every non-vertical continuum. Jones also pointed out how many pathological properties that connected sets may possess can be exhibited by the graphs of discontinuous additive functions or by sets closely related with such graphs, thus unifying and simplifying a large collection of examples scattered throughout the literature.

Other papers not mentioned in this historical survey which appear in our bibliography are: [4, 5, 13, 21, 24, 25, 26, 27, 28, 33]. An excellent account of the development of convex functions and sets (including midpoint convexity) and their generalizations may be found in a recent article by Beckenbach [3].

Conclusion. Our point of view throughout this paper has been on sets rather than on functions. Theorems concerning quasiconvex sets are applicable to the study of functions; for the graph of an additive function is, as we have already mentioned, midpoint convex, and the set of points (x, y) such that $y \geqslant \varphi(x)$, where φ is a midpoint convex function, is also a midpoint convex set. With few exceptions the theorems concerning additive and midpoint convex functions can be deduced from results on quasiconvex sets; though it is not generally conversely true that the set results can be made to follow from the function theorems.

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