MARGINAL STANDARDIZATION OF UPPER SEMICONTINUOUS PROCESSES. WITH APPLICATION TO MAX-STABLE PROCESSES

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Abstract

Extreme value theory for random vectors and stochastic processes with continuous trajectories is usually formulated for random objects where the univariate marginal distributions are identical. In the spirit of Sklar's theorem from copula theory, such marginal standardization is carried out by the pointwise probability integral transform. Certain situations, however, call for stochastic models whose trajectories are not continuous but merely upper semicontinuous (USC). Unfortunately, the pointwise application of the probability integral transform to a USC process does not, in general, preserve the upper semicontinuity of the trajectories. In this paper we give sufficient conditions to enable marginal standardization of USC processes and we state a partial extension of Sklar's theorem for USC processes. We specialize the results to max-stable processes whose marginal distributions and normalizing sequences are allowed to vary with the coordinate.

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1. Introduction

A common way to describe multivariate max-stable distributions is as follows: their margins are univariate max-stable distributions; after standardization of the marginal distributions to a common one, the joint distribution has a specific representation describing the dependence structure. The separation into margins and dependence is in line with Sklar's theorem [17], which provides a decomposition of a multivariate distribution into its margins and a copula, that is, a multivariate distribution with standard uniform margins. If the margins of the original distribution are continuous, the copula is unique and can be found by applying the probability integral transform to each variable. Conversely, to recover the original distribution, it suffices to apply the quantile transformation to each copula variable. Although it is more common in extreme value theory to standardize to the Gumbel or the unit-Fréchet distribution rather than to the uniform distribution, the principle is the same. The advantage of breaking up a distribution into its margins and a copula is that both components can be modelled separately.

Applications in spatial statistics have spurred the development of extreme value theory for stochastic processes. If the trajectories of the process are continuous almost surely, then the process can be reduced to a process with standardized margins and continuous trajectories via

the probability integral transform applied to each individual variable. Conversely, the original process can be recovered from the standardized one by applying the quantile transform to each standardized variable. Moreover, these maps, sending one continuous function to another one, are measurable with respect to the sigma-field on the space of continuous functions that is generated by the finite-dimensional cylinders. These results hinge on the following two properties. First, the marginal distributions of a stochastic process with continuous trajectories depend continuously on the index variable. Second, the distribution of the random continuous function associated to the process is determined by the finite-dimensional distributions.

For stochastic processes with upper semicontinuous (USC) trajectories, however, the two properties mentioned above do not hold: the marginal distributions need not depend in a continuous way on the index variable, and the distribution of path functionals such as the supremum of the process is not determined by the finite-dimensional distributions of the process. Still, max-stable processes with USC trajectories have been proposed as models for spatial extremes of environmental variables [2], [5], [16]. As in the continuous case, construction of, and inference on, such models is carried out by a separation of concerns regarding the margins and the dependence structure. However, up to the present time, there is no theoretical foundation for such an approach. Another possible application of max-stable USC processes is random utility maximization when the alternatives range over a compact metric space rather than a finite set [6], [7], [12].

In this paper we aim to fill the gap in theory and develop a framework for marginal standardization for stochastic processes with USC trajectories. For the mathematical framework, we follow [9] and [12] and we work within the space USC($\mathbb{D}$) of USC functions on a locally compact subset $\mathbb{D}$ of some Euclidean space. The space USC($\mathbb{D}$) is equipped with the hypo-topology: a USC function is identified with its hypograph, a closed subset of $\mathbb{D} \times \mathbb{R}$; the hypo-topology on USC($\mathbb{D}$) is then defined as the trace topology inherited from the Fell hit-and-miss topology on the space $\mathcal{F}$ of closed subsets of $\mathbb{D} \times \mathbb{R}$ [15], [18].

The theory is specialized to max-stable USC processes. Our definition of max-stability allows the shape parameter of the marginal distributions to vary with the index variable of the stochastic process. As a consequence, the stabilizing affine transformations in the definition of max-stability may depend on the index variable too. However, the coordinatewise affine transformation of a USC function does not necessarily produce a USC function, so that care is needed in the formulation of the definition and the results. In [3] and [12], in contrast, the marginal distributions were assumed to be Fréchet with unit shape parameter, so that the stabilizing sequences in the definition of max-stability are the same for all margins. It is often taken for granted that the general case may be reduced to this simpler case, but, as argued in the paper, for USC processes this is not guaranteed.

In Section 2 we set up the necessary background concerning USC processes. A general class of measurable transformations on the space of USC functions is introduced in Section 3. Under regularity conditions on the marginal distributions, this class includes the pointwise probability integral transform and its inverse. This property allows us to state a partial generalization of Sklar's theorem for USC processes in general (Section 4) and for max-stable ones in particular (Section 5). Section 6 concludes. Some additional results are deferred to Appendices A–D.

2. Random USC processes

We review some essential definitions and properties of random USC processes, or USC processes for short. The material in this section may, for instance, be found in [1, Chapter 5], [8, Chapter 1.1 and Appendix B], [15], [19], and [21].

Let $\mathbb{D}$ be a nonempty, locally compact subset of some finite-dimensional Euclidean space. A function $x : \mathbb{D} \to [-\infty, \infty]$ is USC if and only $\limsup_{n \to \infty} x(s_n) \leq x(s)$ whenever $s_n \to s \in \mathbb{D}$. An equivalent property is that the set $\{s \in \mathbb{D} : \alpha \leq x(s)\}$ is closed for each $\alpha \in \mathbb{R}$. The definition of *lower semicontinuous* (LSC) functions is similar, with a lim inf rather than a lim sup and with the inequalities being reversed.

For $x : \mathbb{D} \to [-\infty, +\infty]$, the *epigraph* and *hypograph* are commonly defined by

$$\operatorname{epi} x = \{(s, \alpha) \in \mathbb{D} \times \mathbb{R} : x(s) \leq \alpha\}, \qquad \operatorname{hypo} x = \{(s, \alpha) \in \mathbb{D} \times \mathbb{R} : \alpha \leq x(s)\}.$$

Observe that $\operatorname{epi} x$ and $\operatorname{hypo} x$ are subsets of $\mathbb{D} \times \mathbb{R}$, even if the range of x contains $-\infty$ or $+\infty$. A function is upper or lower semicontinuous if and only if its hypograph or epigraph, respectively, is closed.

Let $\operatorname{USC}(\mathbb{D})$ be the collection of all USC functions from $\mathbb{D}$ into $[-\infty, \infty]$. By identifying the function $z \in \operatorname{USC}(\mathbb{D})$ with the set $\operatorname{hypo} z \subset \mathbb{D} \times \mathbb{R}$, any topology on the space $\mathcal{F} = \mathcal{F}(\mathbb{D} \times \mathbb{R})$ of closed subsets of $\mathbb{D} \times \mathbb{R}$ results in a trace topology on the space of USC functions.

The Fell *hit-and-miss* topology on $\mathcal{F}$ is defined as follows. Let $\mathcal{K}$ and $\mathcal{G}$ denote the families of compact and open subsets of $\mathbb{D} \times \mathbb{R}$, respectively. A base for the Fell topology on $\mathcal{F}$ is the family of sets of the form

$$\mathcal{F}^K_{G_1,\dots,G_n} = \{F \in \mathcal{F} : F \cap K = \varnothing, \, F \cap G_1 \neq \varnothing, \dots, F \cap G_n \neq \varnothing\}$$

for $K \in \mathcal{K}$ and $G_1, \dots, G_n \in \mathcal{G}$. A net of closed sets converges to a limit set F if and only if every compact K missed by F is eventually also missed by the net and if every open G hit by F is eventually also hit by the net.

The Fell topology on $\mathcal{F}(\mathbb{D} \times \mathbb{R})$ induces a trace topology on $\operatorname{USC}(\mathbb{D})$, the *hypo-topology*. Subsets of $\operatorname{USC}(\mathbb{D})$ which are open or closed with respect to the hypo-topology will be referred to as *hypo-open* or *hypo-closed*, respectively. A sequence x_n in $\operatorname{USC}(\mathbb{D})$ is said to *hypo-converge* to x in $\operatorname{USC}(\mathbb{D})$ if and only if $\operatorname{hypo} x_n$ converges to $\operatorname{hypo} x$ in the Fell topology. Since the underlying space $\mathbb{D} \times \mathbb{R}$ is locally compact, Hausdorff, and second-countable (LCHS), the space $\operatorname{USC}(\mathbb{D})$ thus becomes a compact, Hausdorff, second-countable space; see, for example, [8, Theorem B.2]. A convenient pointwise criterion for hypo-convergence is the following: a sequence $x_n \in \operatorname{USC}(\mathbb{D})$ hypo-converges to $x \in \operatorname{USC}(\mathbb{D})$ if and only if,

$$\begin{aligned} &\text{for all } s \in \mathbb{D} : \text{for all } s_n \in \mathbb{D}, \, s_n \to s : \limsup_{n \to \infty} x_n(s_n) \leq x(s), \\ &\text{for all } s \in \mathbb{D} : \text{there exists } s_n \in \mathbb{D}, \, s_n \to s : \liminf_{n \to \infty} x_n(s_n) \geq x(s). \end{aligned} \tag{2.1}$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and let $\mathcal{B}(\mathcal{F})$ be the Borel σ-field on $\mathcal{F} = \mathcal{F}(\mathbb{D} \times \mathbb{R})$ generated by the Fell topology. A random closed set of $\mathbb{D} \times \mathbb{R}$ is a Borel measurable map from Ω into $\mathcal{F}$. Similarly, a USC process ξ on $\mathbb{D}$ is a Borel measurable map from Ω into $\operatorname{USC}(\mathbb{D})$. Equivalently, $\operatorname{hypo} \xi$ is a random closed subset of $\mathbb{D} \times \mathbb{R}$ taking values in the collection of all closed hypographs. Some authors use the term 'normal integrand' to refer to such processes, as a special case of stochastic processes with USC realizations; see [15, p. 12]. In this paper, a USC process is always a function $\xi : \mathbb{D} \times \Omega \to [-\infty, \infty]$ such that the map $\Omega \to \mathcal{F} : \omega \mapsto \operatorname{hypo} \xi(\cdot, \omega)$ is Borel measurable.

The *capacity functional* of $\operatorname{hypo} \xi$ is the map $T : \mathcal{K} \to [0, 1]$ defined by

$$T(K) = \mathbb{P}\{(\operatorname{hypo} \xi) \cap K \neq \varnothing\}, \qquad K \in \mathcal{K}.$$

We also call T the capacity functional of ξ rather than of hypo ξ. Since $\mathbb{D} \times \mathbb{R}$ is LCHS, the Borel σ-field $\mathcal{B}(\mathcal{F})$ is generated by the sets

$$\mathcal{F}_K = \{F \in \mathcal{F} : F \cap K \neq \varnothing\}, \qquad K \in \mathcal{K};$$

see [8, p. 2]. The collection of complements of the sets $\mathcal{F}_K$ being a π-system, the capacity functional determines the distribution of a random closed set or a USC process.

We emphasize that the Borel σ-field on $\mathrm{USC}(\mathbb{D})$ is strictly larger than the one generated by the finite-dimensional cylinders. Without additional hypotheses, the finite-dimensional distributions of a USC process do not determine its distribution as a USC process. The evaluation mappings $\mathrm{USC}(\mathbb{D}) \to [-\infty, \infty]: x \mapsto x(s)$ being hypo-measurable, a USC process ξ is also a stochastic process (i.e. a collection of random variables) with USC trajectories. The converse is not true, however: for such stochastic processes, the map $\Omega \to \mathrm{USC}(\mathbb{D}): \omega \mapsto \xi(\cdot, \omega)$ is not necessarily hypo-measurable; see [13, Proposition 14.28] for a counterexample. Measurability of maps to or from $\mathrm{USC}(\mathbb{D})$ with respect to the Borel σ-field induced by the hypo-topology will be referred to as *hypo-measurability*.

3. Transformations of USC functions

Obviously, a USC function remains USC if multiplied by a positive, continuous function. More generally, we will need to prove that under certain assumptions the mappings derived from the probability integral transform or the quantile transform are hypo-measurable mappings from $\mathrm{USC}(\mathbb{D})$ to itself. To do so, we will rely on the function class $\mathcal{U}(\mathbb{D})$ defined and studied next.

Definition 3.1. A function $U : \mathbb{D} \times [-\infty, \infty] \to [-\infty, \infty]$ belongs to the class $\mathcal{U}(\mathbb{D})$ if it has the following two properties:

(i) for every $s \in \mathbb{D}$, the map $x \mapsto U(s, x)$ is nondecreasing and right-continuous;

(ii) for every $x \in \mathbb{R} \cup \{\infty\}$, the map $s \mapsto U(s, x)$ is USC.

See Remark 3.1 for the reason why we did not include $x = -\infty$ in condition (ii).

Proposition 3.1. *Let $U \in \mathcal{U}(\mathbb{D})$.*

(i) *If z_n hypo-converges to z in $\mathrm{USC}(\mathbb{D})$ and $s_n \to s$ in $\mathbb{D}$ as $n \to \infty$, then $\limsup_{n \to \infty} U(s_n, z_n(s_n)) \leq U(s, z(s))$.*

(ii) *For every $z \in \mathrm{USC}(\mathbb{D})$, the function $U_*(z): \mathbb{D} \to [-\infty, \infty]$ defined by $(U_*(z))(s) = U(s, z(s))$ also belongs to $\mathrm{USC}(\mathbb{D})$.*

(iii) *For every compact $K \subset \mathbb{D} \times \mathbb{R}$, the set $\{z \in \mathrm{USC}(\mathbb{D}): \mathrm{hypo}\, U_*(z) \cap K \neq \varnothing\}$ is hypo-closed.*

(iv) *The map $U_*: \mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ is hypo-measurable.*

Proof. (i) Consider two cases: $z(s) = \infty$ and $z(s) < \infty$.
Suppose first that $z(s) = \infty$. By the assumptions on U, we have

$$\limsup_{n \to \infty} U(s_n, z_n(s_n)) \leq \limsup_{n \to \infty} U(s_n, \infty) \leq U(s, \infty) = U(s, z(s)).$$

Suppose next that $z(s) < \infty$. Let $y > z(s)$. By hypo-convergence, we have

$$\limsup_{n \to \infty} z_n(s_n) \leq z(s);$$

see (2.1). As a consequence, there exists a positive integer $n(y)$ such that $z_n(s_n) \leq y$ for all $n \geq n(y)$. By the properties of U, we have

$$\limsup_{n \to \infty} U(s_n, z_n(s_n)) \leq \limsup_{n \to \infty} U(s_n, y) \leq U(s, y).$$

Since $y > z(s)$ was arbitrary and since $x \mapsto U(s, x)$ is nondecreasing and right-continuous, we find that $\limsup_{n \to \infty} U(s_n, z_n(s_n)) \leq \inf_{y > z(s)} U(s, y) = U(s, z(s))$.

(ii) In statement (i), set $z_n = z$ to see that $\limsup_{n \to \infty} U(s_n, z(s_n)) \leq U(s, z(s))$ whenever $s_n \to s$ in $\mathbb{D}$ as $n \to \infty$.

(iii) Let $K \subset \mathbb{D} \times \mathbb{R}$ be compact. Suppose that z_n hypo-converges to z in USC($\mathbb{D}$) as $n \to \infty$ and that hypo $U_*(z_n) \cap K \neq \varnothing$ for all positive integers n. We need to prove that hypo $U_*(z) \cap K \neq \varnothing$ too.

For each positive integer n, there exists $(s_n, x_n) \in K$ such that $U(s_n, z_n(s_n)) \geq x_n$. Since K is compact, we can find a subsequence $n(k)$ such that $(s_{n(k)}, x_{n(k)}) \to (s, x) \in K$ as $k \to \infty$. By (i), we have

$$U(s, z(s)) \geq \limsup_{k \to \infty} U(s_{n(k)}, z_{n(k)}(s_{n(k)})) \geq \limsup_{k \to \infty} x_{n(k)} = x.$$

As a consequence, $(s, x) \in$ hypo $U_*(z) \cap K$.

(iv) The collection of sets of the form $\{z \in \mathrm{USC}(\mathbb{D}) : \mathrm{hypo}(z) \cap K \neq \varnothing\}$, where K ranges over the compact subsets of $\mathbb{D} \times \mathbb{R}$, generates the Borel σ-field on USC($\mathbb{D}$) induced by the hypo-topology. By (iii), the inverse image under U_* of each such set is hypo-closed and, thus, hypo-measurable. We conclude that U_* is hypo-measurable. $\qquad \square$

Remark 3.1. In Definition 3.1(ii), we did not include $x = -\infty$. The reason is that this case is automatically included: by Proposition 3.1(ii) applied to the function $z(s) \equiv -\infty$, the map $s \mapsto U(s, -\infty)$ is necessarily USC too.

Definition 3.1(ii) is also necessary for the conclusion of Proposition 3.1(ii) to hold: given x, define $z(s) \equiv x$.

A convenient property of $\mathcal{U}(\mathbb{D})$ is that it is closed under an appropriate type of composition. This allows the deconstruction of complicated transformations into more elementary ones.

Lemma 3.1. *If U and V belong to $\mathcal{U}(\mathbb{D})$, the function W defined by $(s, x) \mapsto V(s, U(s, x))$ also belongs to $\mathcal{U}(\mathbb{D})$, and $W_* = V_* \circ U_*$.*

Proof. First, fix $s \in \mathbb{D}$. The map $x \mapsto V(s, U(s, x))$ is nondecreasing: for $-\infty \leq x \leq y \leq \infty$, we have $U(s, x) \leq U(s, y)$ and, thus, $V(s, U(s, x)) \leq V(s, U(s, y))$. The map $x \mapsto V(s, U(s, x))$ is also right-continuous: if x_n converges from the right to x as $n \to \infty$, then so does $u_n = U(s, x_n)$ to $u = U(s, x)$ and, thus, $V(s, u_n)$ to $V(s, u)$.

Next, fix $x \in \mathbb{R} \cup \{\infty\}$. We need to show that the function $s \mapsto V(s, U(s, x))$ is USC. But the function z defined by $s \mapsto U(s, x)$ is USC, and, by Proposition 3.1, so is the function $s \mapsto V(s, z(s)) = V(s, U(s, x))$. $\qquad \square$

Example 3.1. Here are some simple examples of functions U in the class $\mathcal{U}(\mathbb{D})$ and the associated mappings $U_* : \mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$.

(i) If $f : [-\infty, \infty] \to [-\infty, \infty]$ is nondecreasing and right-continuous, the function $(s, x) \mapsto f(x)$ belongs to $\mathcal{U}(\mathbb{D})$. The associated map USC($\mathbb{D}$) $\to$ USC($\mathbb{D}$) is $z \mapsto f \circ z$.

(ii) If $y \in \mathrm{USC}(\mathbb{D})$, the functions $(s, x) \mapsto x \vee y(s)$ and $(s, x) \mapsto x \wedge y(s)$ both belong to $\mathcal{U}(\mathbb{D})$. The associated maps $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ are $z \mapsto z \vee y$ and $z \mapsto z \wedge y$, respectively.

(iii) If $a \colon \mathbb{D} \to (0, \infty)$ is continuous then the function $(s, x) \mapsto a(s)\,x$ belongs to $\mathcal{U}(\mathbb{D})$. If $b \colon \mathbb{D} \to \mathbb{R}$ is USC then the map $(s, x) \mapsto x + b(s)$ belongs to $\mathcal{U}(\mathbb{D})$. The associated maps $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ are $z \mapsto az$ and $z \mapsto z + b$, respectively.

A more elaborate example of a function in $\mathcal{U}(\mathbb{D})$ is induced by the collection of right-continuous quantile functions of the marginal distributions of a stochastic process with USC trajectories. See Appendix A for some background on right-continuous quantile functions.

Lemma 3.2. *Let $\xi = (\xi(s) \colon s \in \mathbb{D})$ be a stochastic process indexed by $\mathbb{D}$ and with values in $[-\infty, \infty]$. Let $F_s \colon x \in [-\infty, \infty] \to F_s(x) = \mathbb{P}[\xi(s) \leq x]$ denote the right-continuous marginal distribution function of $\xi(s)$. Define*

$$Q_s(p) = \sup\{y \in \mathbb{R} \colon F_s(y) \leq p\}, \qquad (s, p) \in \mathbb{D} \times [0, 1].$$

If ξ has USC trajectories, the function $(s, x) \mapsto Q_s((x \vee 0) \wedge 1)$ belongs to $\mathcal{U}(\mathbb{D})$.

Proof. First, by Lemma A.1, the map $[0, 1] \ni p \mapsto Q_s(p)$ is nondecreasing and right-continuous for every $s \in \mathbb{D}$, and, hence, the same is true for the map $[-\infty, \infty] \ni x \mapsto Q_s((x \vee 0) \wedge 1)$.

Second, fix $x \in [-\infty, \infty]$ and write $p = (x \vee 0) \wedge 1 \in [0, 1]$. We need to show that the map $s \mapsto Q_s(p)$ is USC. Let $s_n \to s$ in $\mathbb{D}$ as $n \to \infty$; we need to show that $Q_s(p) \geq \limsup_{n \to \infty} Q_{s_n}(p)$. Let Q' be the right-continuous quantile function of the random variable $\limsup_{n \to \infty} \xi(s_n)$. The trajectories of ξ are USC, and, thus, $\xi(s) \geq \limsup_{n \to \infty} \xi(s_n)$, which implies that $Q_s(p) \geq Q'(p)$. By Lemma A.2, we then find $Q_s(p) \geq Q'(p) \geq \limsup_{n \to \infty} Q_{s_n}(p)$, as required. $\qquad \square$

4. Sklar's theorem for USC processes

A d-variate copula is the cumulative distribution function of a d-dimensional random vector with standard uniform margins. Sklar's [17] celebrated theorem states two things.

(I) For every copula C and every vector $F_1, \ldots, F_d$ of univariate distribution functions, the function $(x_1, \ldots, x_d) \mapsto C(F_1(x_1), \ldots, F_d(x_d))$ is a d-variate distribution function with margins $F_1, \ldots, F_d$.

(II) Every d-variate distribution function F can be represented in this way.

Reformulated in terms of random vectors, the two statements read as follows.

(I) For every random vector $(U_1, \ldots, U_d)$ with uniform components and for every vector $F_1, \ldots, F_d$ of univariate distribution functions, the random vector $(Q_1(U_1), \ldots, Q_d(U_d))$ has marginal distributions $F_1, \ldots, F_d$, where Q_j is the (right- or left-continuous) quantile function corresponding to F_j.

(II) Every random vector $(X_1, \ldots, X_d)$ can be represented in this way.

We also investigate the extent to which these statements hold for USC processes. According to Proposition 4.1, the first statement remains true provided the sections $s \mapsto Q_s(p)$ of the right-continuous quantile functions are USC. According to Proposition 4.2, the Sklar representation is valid for USC processes whose marginal distribution functions have USC sections, and even

then, the equality in distribution is not guaranteed. In Section 5 we will specialize the two propositions to max-stable processes.

Proposition 4.1. (À la Sklar (I) for USC processes.) *Let Z be a USC process having standard uniform margins. Let $(F_s \colon s \in \mathbb{D})$ be a family of (right-continuous) distribution functions and let $Q_s(p) = \sup\{x \in \mathbb{R} \colon F_s(x) \le p\}$ for all $(s, p) \in \mathbb{D} \times [0, 1]$. Define a stochastic process ξ by $\xi(s) = Q_s((Z(s) \vee 0) \wedge 1)$ for $s \in \mathbb{D}$. Then the following two statements are equivalent:*

(i) *ξ is a USC process with marginal distributions given by F_s;*

(ii) *for every $p \in [0, 1]$, the function $s \mapsto Q_s(p)$ is USC.*

Proof. For every $s \in \mathbb{D}$, the random variable $(Z(s) \vee 0) \wedge 1$ is equal almost surely to $Z(s)$, so that its distribution is uniform on $[0, 1]$ too. By Lemma A.1(iv), the distribution function of $\xi(s)$ is given by F_s. As a consequence, Q_s is the right-continuous quantile function of the random variable $\xi(s)$.

If (i) holds then the trajectories $s \mapsto \xi(s)$ are USC. By Lemma 3.2, the function $s \mapsto Q_s(p)$ is USC for every $p \in [0, 1]$.

Conversely, if (ii) holds then the function U defined by $(s, x) \mapsto Q_s((x \vee 0) \wedge 1)$ belongs to $\mathcal{U}(\mathbb{D})$ defined in Definition 3.1. Let U_* be the associated map $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$; see Proposition 3.1. Then $\xi = U_*(Z)$ is a USC process since Z is a USC process and U_* is hypo-measurable by Proposition 3.1(iv). $\qquad\square$

Proposition 4.2. (À la Sklar (II) for USC processes.) *Let ξ be a USC process. Let $F_s(x) = \mathbb{P}[\xi(s) \le x]$ for $x \in [-\infty, \infty]$ and let $Q_s(p) = \sup\{x \in \mathbb{R} \colon F_s(x) \le p\}$ for $p \in [0, 1]$. Suppose that the following two conditions hold:*

(a) *for every $s \in \mathbb{D}$, the distribution of $\xi(s)$ has no atoms in $[-\infty, \infty]$;*

(b) *for every $x \in \mathbb{R} \cup \{+\infty\}$, the function $s \mapsto F_s(x)$ is USC.*

Then the following statements hold:

(i) *the process Z defined by $Z(s) = F_s(\xi(s))$ is a USC process with standard uniform margins;*

(ii) *the process $\tilde{\xi}$ defined by $\tilde{\xi}(s) = Q_s(Z(s)) = Q_s(F_s(\xi(s)))$ is a USC process such that $\mathbb{P}[\tilde{\xi}(s) = \xi(s)] = 1$ for every $s \in \mathbb{D}$. In particular, the finite-dimensional distributions of $\tilde{\xi}$ and ξ are identical.*

Proof. (i) By condition (a), the marginal distributions of Z are standard uniform. By condition (b), the function U defined by $(s, x) \mapsto F_s(x)$ belongs to $\mathcal{U}(\mathbb{D})$ (Definition 3.1) and we have $Z = U_*(\xi)$ with $U_* \colon \mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D})$ as in Proposition 3.1. By Proposition 3.1(iv), the map U_* is hypo-measurable, so that Z is a USC process too.

(ii) By Lemma A.1(v), we have $\mathbb{P}[\tilde{\xi}(s) = \xi(s)] = 1$ for every $s \in \mathbb{D}$. The function Q_s is the right-continuous quantile function of $\xi(s)$ and the stochastic process $(\xi(s) \colon s \in \mathbb{D})$ has USC trajectories. Then Proposition 4.1(ii) is fulfilled by an application of Lemma 3.2. By Proposition 4.1(i), $\tilde{\xi}$ is a USC process. $\qquad\square$

Remark 4.1. Although $\mathbb{P}[\tilde{\xi}(s) = \xi(s)] = 1$ for all $s \in \mathbb{D}$, it is not necessarily true that $\mathbb{P}[\tilde{\xi} = \xi] = 1$, and not even that $\tilde{\xi}$ and ξ have the same distribution as USC processes; see Examples B.2 and B.3. However, if $\mathbb{P}[\text{for all } s \in \mathbb{D} \colon 0 < F_s(\xi(s)) < 1] = 1$ and if $Q_s(F_s(x)) = x$ for every s and every x such that $0 < F_s(x) < 1$, then clearly $\mathbb{P}[\tilde{\xi} = \xi] = 1$.

The following Lemma helps clarifying the meaning of the assumptions of Proposition 4.2.

Lemma 4.1. (*Regularity of the marginal distributions with respect to the space variable.*) *If ξ is a USC process, conditions (a) and (b) in Proposition 4.2 together imply that the map $s \mapsto F_s(x) = \mathbb{P}[\xi(s) \leq x]$ is continuous, for any fixed $x \in \mathbb{R}$.*

Proof. By condition (a), we have $\mathbb{P}[\xi(s) < x] = F_s(x)$ for each fixed (s, x), so that by Lemma A.1(iii), we have

$$\{s \in \mathbb{D} : x \leq Q_s(p)\} = \{s \in \mathbb{D} : F_s(x) \leq p\}$$

for every $x \in [-\infty, \infty]$ and $p \in [0, 1]$. Since ξ is a USC process, the function $s \mapsto Q_s(p)$ is USC for $p \in [0, 1]$ (Lemma 3.2). The set in the display is thus closed. But then the map $s \mapsto F_s(x)$ is *lower* semicontinuous, and, thus, by (b), continuous. $\qquad \square$

Remark 4.2. Condition (a) in Proposition 4.2, continuity of the marginal distributions, can perhaps be avoided by using an additional randomization device, 'smearing out' the probability masses of any atoms, as in [14]. However, even if this may ensure uniform margins, it may destroy upper semicontinuity; see Example 4.1. Since our interest is in extreme value theory, in particular in max-stable distributions (Section 5), which are continuous, we do not pursue this issue further.

Remark 4.3. Without condition (b) in Proposition 4.2, the trajectories of the stochastic process $(Z(s) : s \in \mathbb{D})$ may not be USC; see Example 4.2. However, condition (b) is not necessarily USC either; see Example B.1.

The examples below illustrate certain aspects of Propositions 4.1 and 4.2. In the examples, we do not go into measurability issues, that is, we do not prove that the mappings ξ from the underlying probability space into USC($\mathbb{D}$) are hypo-measurable. To do so, one can, for instance, rely on [8, Example 1.2, p. 3 and Theorem 2.25, p. 37].

Example 4.1. (*What may happen without (a) in Proposition 4.2.*) Consider two independent, uniformly distributed variables X and Y on $[0, 1]$. Take $\mathbb{D} = [-1, 1]$, and define

$$\xi(s) = \begin{cases} |s|X & \text{if } -1 \leq s < 0, \\ 0 & \text{if } s = 0, \\ sY & \text{if } 0 < s \leq 1. \end{cases}$$

Condition (b) in Proposition 4.2 is fulfilled: the trajectories $s \mapsto F_s(x) = \mathbb{P}[\xi(s) \leq x]$ are continuous for every $x \neq 0$ and still USC for $x = 0$. The marginal probability integral transform produces $Z(s) = X$ for $s < 0$ and $Z(s) = Y$ for $s > 0$, while $Z(0) = F_0(0) = 1$. The process Z has USC trajectories but $Z(0)$ is not uniformly distributed. If one wanted to modify the definition of Z at 0 so that the modified version, $\tilde{Z}$, would still have USC trajectories and such that $\tilde{Z}(0)$ would be uniformly distributed, then one would need to have $\tilde{Z}(0) \geq X \vee Y$, contradicting the uniformity assumption. We conclude that it is impossible to write $\xi(s)$ as $Q_s(\tilde{Z}(s))$, where $\tilde{Z}$ is a USC process with standard uniform margins.

Example 4.2. (*What may happen without (b) in Proposition 4.2.*) Consider two independent, uniformly distributed variables X and Y on $[0, 1]$. On $\mathbb{D} = [0, 2]$, define

$$\xi(s) = X \vee (Y\mathbf{1}_{\{1\}}(s)) = \begin{cases} X & \text{if } s \neq 1, \\ X \vee Y & \text{if } s = 1, \end{cases}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Then ξ is a USC process and the distribution function, F_s, of $\xi(s)$ is given by

$$F_s(x) = \mathbb{P}[\xi(s) \le x] = \begin{cases} x & \text{if } s \ne 1, \\ x^2 & \text{if } s = 1, \end{cases}$$

where $x \in [0, 1]$. As a consequence, the function $s \mapsto F_s(x)$ is LSC but not USC if $0 < x < 1$, so that condition (b) in Proposition 4.2 is violated. Reducing to standard uniform margins yields

$$Z(s) = F_s(\xi(s)) = \begin{cases} X & \text{if } s \ne 1, \\ (X \vee Y)^2 & \text{if } s = 1. \end{cases}$$

The event $\{0 < Y^2 < X < 1\}$ has positive probability, and on this event we have $(X \vee Y)^2 < X$, so that the trajectory $s \mapsto Z(s)$ is not USC. Hence, statement (i) in Proposition 4.2 fails.

Some additional examples are postponed to Appendix B, which show that condition (b) in Proposition 4.2 is not necessary, and that the USC processes ξ and $\tilde{\xi}$ in Proposition 4.2 may be different in law as random elements of $\text{USC}(\mathbb{D})$.

5. Max-stable processes

We apply Propositions 4.1 and 4.2 to *max-stable* USC processes. These processes and their standardized variants are introduced in Subsection 5.2, after some preliminaries on generalized extreme value distributions in Subsection 5.1. The main results are given in Subsection 5.3, followed by some examples in Subsection 5.4.

5.1. Generalized extreme value distributions

The distribution function of the generalized extreme value (GEV) distribution with parameter vector $\theta = (\gamma, \mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R} \times (0, \infty)$ is given by

$$F(x; \theta) = \begin{cases} \exp\left[-\left\{1 + \dfrac{\gamma(x - \mu)}{\sigma}\right\}^{-1/\gamma}\right] & \text{if } \gamma \ne 0 \text{ and } \sigma + \gamma(x - \mu) > 0, \\ \exp\left[-\exp\left\{-\dfrac{(x - \mu)}{\sigma}\right\}\right] & \text{if } \gamma = 0 \text{ and } x \in \mathbb{R}. \end{cases} \tag{5.1}$$

The corresponding quantile function is

$$Q(p; \theta) = \begin{cases} \mu + \dfrac{\sigma[\{-1/\log(p)\}^{\gamma} - 1]}{\gamma} & \text{if } \gamma \ne 0, \\ \mu + \sigma \log\left\{-\dfrac{1}{\log(p)}\right\} & \text{if } \gamma = 0, \end{cases} \tag{5.2}$$

for $0 < p < 1$. The support is equal to the interval $\{x \in \mathbb{R}: \sigma + \gamma(x - \mu) > 0\}$. In particular, the lower endpoint is equal to $Q(0; \theta) = -\infty$ if $\gamma \le 0$ and $Q(0; \theta) = \mu - \sigma/\gamma$ if $\gamma > 0$.

For every n, there exist unique scalars $a_{n,\theta} \in (0, \infty)$ and $b_{n,\theta} \in \mathbb{R}$ such that the following max-stability relation holds:

$$F^n(a_{n,\theta} x + b_{n,\theta}; \theta) = F(x; \theta), \qquad x \in \mathbb{R}. \tag{5.3}$$

In fact, a nondegenerate distribution is max-stable if and only if it is GEV. This property motivates the use of such distributions for modeling maxima over many variables. The location

and scale sequences are given by

$$a_{n,\theta} = n^\gamma, \qquad b_{n,\theta} = \begin{cases} \dfrac{(\sigma - \gamma\mu)(n^\gamma - 1)}{\gamma} & \text{if } \gamma \neq 0, \\ \sigma \, \log n & \text{if } \gamma = 0. \end{cases} \tag{5.4}$$

For quantile functions, max-stability means that

$$Q(p^{1/n}; \theta) = a_{n,\theta} Q(p; \theta) + b_{n,\theta}, \qquad p \in [0, 1]. \tag{5.5}$$

Note that $Q(\mathrm{e}^{-1}; \theta) = \mu$ and, thus, $Q(\mathrm{e}^{-1/n}; \theta) = a_n \mu + b_n$.

In Sklar's theorem, the uniform distribution on $[0, 1]$ plays the role of a pivot. Here, it is natural to standardize to a member of the GEV family. Multiple choices are possible. We opt for the unit-Fréchet distribution, given by

$$\Phi(x) = F(x; \, 1, 1, 1) = \begin{cases} 0 & \text{if } x \leq 0, \\ \mathrm{e}^{-1/x} & \text{if } x > 0. \end{cases} \tag{5.6}$$

The unit-Fréchet quantile function is given by $Q(p; \, 1, 1, 1) = -1/\log(p)$ for $0 < p < 1$. If the law of X is unit-Fréchet then the law of $Q(\Phi(X); \theta)$ is GEV(θ).

Consider now a stochastic process $\{\xi(s), \, s \in \mathbb{D}\}$, such that $\xi(s)$ follows a GEV distribution with parameter θ depending on s, so that $\mathbb{P}[\xi(s) \leq x] = F(x; \, \theta(s))$, with F as in (5.1). In view of the conditions in Propositions 4.1 and 4.2, the following lemma is relevant.

Lemma 5.1. *Let $\theta \colon \mathbb{D} \to \Theta$. The functions $s \mapsto F(x; \, \theta(s))$ and $s \mapsto Q(p; \, \theta(s))$ are USC for every $x \in \mathbb{R}$ and $p \in [0, 1]$ if and only if θ is continuous.*

Proof. If θ is continuous then the maps $s \mapsto F(x; \, \theta(s))$ and $s \mapsto Q(p; \, \theta(s))$ are continuous and, thus, USC.

Conversely, suppose that the functions $s \mapsto F(x; \, \theta(s))$ and $s \mapsto Q(p; \, \theta(s))$ are USC for every $x \in \mathbb{R}$ and $p \in [0, 1]$. The argument is similar to the proof of Lemma 4.1. By Lemma A.1(iii), we have

$$\{s \in \mathbb{D} \colon x \leq Q(p; \, \theta(s))\} = \{s \in \mathbb{D} \colon F(x; \, \theta(s)) \leq p\}$$

for every $x \in [-\infty, \infty]$ and $p \in [0, 1]$; note that GEV distribution functions are continuous functions of x. Since the map $s \mapsto Q(p; \, \theta(s))$ is USC, the above sets are closed. But then the map $s \mapsto F(x; \, \theta(s))$ must be LSC, and, thus, continuous. Finally, the map sending a GEV distribution to its parameter is continuous with respect to the weak topology; see Lemma D.1. Hence, θ is continuous. $\qquad\square$

Remark 5.1. If $\theta \colon \mathbb{D} \to \Theta$ is continuous then the normalizing functions $s \mapsto a_{n,\theta(s)}$ and $s \mapsto b_{n,\theta(s)}$ are continuous as well. Consider the map $U_n(s, x) = \{x - b_{n,\theta(s)}\}/a_{n,\theta(s)}$ for $(s, x) \in \mathbb{D} \times [-\infty, \infty]$. Clearly, $U_n \in \mathcal{U}(\mathbb{D})$. By Proposition 3.1, the transformation USC($\mathbb{D}$) $\to$ USC($\mathbb{D}$): $z \mapsto (z - b_{n,\theta})/a_{n,\theta}$ is then well defined and hypo-measurable.

5.2. Max-stable USC processes

Max-stable USC processes have been defined in the literature [8], [18], [20] as USC processes whose distribution is invariant under the componentwise maximum operation, up to an affine rescaling involving *constants* which are not allowed to depend upon the index variable $s \in \mathbb{D}$.

Likewise, the processes in [3] and [12] have marginal distributions which are Fréchet with shape parameter $\gamma \equiv 1$ and lower endpoint $\mu - \gamma/\sigma \equiv 0$, so that the scaling sequences are $a_n \equiv n$ and $b_n \equiv 0$. Moreover, max-stability is defined in [3] in terms of finite-dimensional distributions only. Definition 5.1 below extends these previous approaches by allowing for an index-dependent rescaling, through scaling *functions* a_n and b_n, and by viewing the random objects as random elements in USC($\mathbb{D}$).

Definition 5.1. A USC process ξ is *max-stable* if, for all integer $n \geq 1$, there exist functions $a_n \colon \mathbb{D} \to (0, \infty)$ and $b_n \colon \mathbb{D} \to \mathbb{R}$ such that, for each vector of n independent and identically distributed (i.i.d.) USC processes $\xi_1, \ldots, \xi_n$ with the same law as ξ, we have

$$\bigvee_{i=1}^{n} \xi_i \stackrel{\mathrm{D}}{=} a_n \xi + b_n \quad \text{in USC}(\mathbb{D}), \tag{5.7}$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. A max-stable USC process ξ^* is said to be *simple* if, in addition, its marginal distributions are unit-Fréchet as in (5.6). In that case, the norming functions are given by $a_n(s) = n$ and $b_n(s) = 0$ for all $n \geq 1$ and $s \in \mathbb{D}$, i.e. for i.i.d. USC processes $\xi_1^*, \ldots, \xi_n^*$ with the same law as ξ^*, we have

$$\bigvee_{i=1}^{n} \xi_i^* \stackrel{\mathrm{D}}{=} n\xi^* \quad \text{in USC}(\mathbb{D}). \tag{5.8}$$

In (5.7) and (5.8), the meaning is that the induced probability distributions on the space USC($\mathbb{D}$) equipped with the sigma-field of hypo-measurable sets are equal. In Definition 5.1, it is implicitly understood that the functions a_n and b_n are such that the right-hand side of (5.7) still defines a USC process. If a_n is continuous and b_n is USC then this is automatically the case; see Lemma 3.1 and Example 3.1(iii).

Equation (5.7) is not necessarily the same as saying that $(\bigvee_{i=1}^{n} \xi_i - b_n)/a_n$ is equal in distribution to ξ. The reason is that it is not clear that $(\bigvee_{i=1}^{n} \xi_i - b_n)/a_n$ is a USC process. Whether this is the case or not remains an open question.

The evaluation map USC($\mathbb{D}$) $\to [-\infty, \infty] \colon z \mapsto z(s)$ is hypo-measurable for all $s \in \mathbb{D}$. Equation (5.7) then implies the following distributional equality between random variables:

$$\bigvee_{i=1}^{n} \xi_i(s) \stackrel{\mathrm{D}}{=} a_n(s)\xi(s) + b_n(s), \qquad s \in \mathbb{D}.$$

As a consequence, the marginal distribution of $\xi(s)$ is max-stable and, therefore, GEV with some parameter vector $\theta(s) \in \Theta$. The normalizing functions a_n and b_n must then be of the form $a_n(s) = a_{n,\theta(s)}$ and $b_n(s) = b_{n,\theta(s)}$ as in (5.4). If $\theta \colon \mathbb{D} \to \Theta$ is continuous then the normalizing functions are continuous too, and, by Remark 5.1, the USC process $(\bigvee_{i=1}^{n} \xi_i - b_n)/a_n$ is well defined and equal in distribution to ξ.

5.3. Sklar's theorem for max-stable USC processes

We investigate the relation between general and simple max-stable USC processes via the pointwise probability integral transform and its inverse. Max-stability of USC processes is defined in (5.7) via an equality of distributions on USC($\mathbb{D}$) rather than of finite-dimensional distributions. It is therefore not clear from the outset that max-stability is preserved by pointwise transformations.

In Proposition 5.1 we state a necessary and sufficient condition on the GEV margins in order to construct a general max-stable USC process starting from a simple one. In Propositions 5.2

and 5.3 we treat the converse question, that is, when can a max-stable USC process be first reduced to a simple one and then be reconstructed from it.

Proposition 5.1. (À la Sklar (I) for max-stable USC processes.) *Let ξ^* be a simple max-stable USC process. Let $\theta \colon \mathbb{D} \to \Theta$. Define a stochastic process ξ by $\xi(s) = Q(\Phi(\xi^*(s)); \theta(s))$ for $s \in \mathbb{D}$. Then the following two statements are equivalent:*

(i) *ξ is a USC process with marginal distributions* GEV$(\theta(s))$;

(ii) *for every $p \in [0, 1]$, the function $s \mapsto Q(p; \theta(s))$ is USC.*

If these conditions hold then ξ is a max-stable USC process with normalizing functions $a_n(s) = a_{n,\theta(s)}$ and $b_n(s) = b_{n,\theta(s)}$.

Proof. By Proposition 4.2(i) applied to ξ^*, the stochastic process $Z(s) = \Phi(\xi^*(s))$ induces a USC process whose margins are uniform on $[0, 1]$. The equivalence of statements (i) and (ii) then follows from Proposition 4.1.

Assume that (i) and (ii) are fulfilled. We need to show that the right-hand side of (5.7) defines a USC process and that the stated equality in distribution holds.

For positive integer n, define $U_n(s, x) = Q(\Phi(nx); \theta(s))$ for $(s, x) \in \mathbb{D} \times [-\infty, \infty]$. In view of Lemma 3.1 and Example 3.1, the map U_n belongs to $\mathcal{U}(\mathbb{D})$. Moreover, max-stability (5.5) implies that

$$U_n(s, x) = Q(\Phi(x)^{1/n}; \theta(s)) = a_{n,\theta(s)} Q(\Phi(x); \theta(s)) + b_{n,\theta(s)}.$$

It follows that

$$a_{n,\theta(s)} \xi(s) + b_{n,\theta(s)} = U_n(s, \xi^*(s)).$$

By Proposition 3.1, the function $s \mapsto U_n(s, z(s))$ belongs to USC$(\mathbb{D})$ for every $z \in$ USC$(\mathbb{D})$, and the map $U_{n,*}$ from USC$(\mathbb{D})$ to itself sending $z \in$ USC$(\mathbb{D})$ to this function is hypo-measurable. We conclude that $a_n \xi + b_n$ is a USC process.

Next, we prove that ξ is max-stable. Let $\xi_1, \ldots, \xi_n$ be i.i.d. USC processes with the same law as ξ. Furthermore, let $\xi_1^*, \ldots, \xi_n^*$ be i.i.d. USC processes with the same law as ξ^*. For every $i \in \{1, \ldots, n\}$, we have

$$\xi_i \overset{\mathrm{D}}{=} \xi = U_{1,*}(\xi^*) \overset{\mathrm{D}}{=} U_{1,*}(\xi_i^*).$$

The last equality in distribution comes from the hypo-measurability of $U_{1,*}$. By independence, it follows that

$$(\xi_1, \ldots, \xi_n) \overset{\mathrm{D}}{=} (U_{1,*}(\xi_1^*), \ldots, U_{1,*}(\xi_n^*)).$$

Write $(\tilde{\xi}_1, \ldots, \tilde{\xi}_n) = (U_{1,*}(\xi_1^*), \ldots, U_{1,*}(\xi_n^*))$. By monotonicity, we have, for $s \in \mathbb{D}$,

$$\bigvee_{i=1}^n \tilde{\xi}_i(s) = \bigvee_{i=1}^n Q(\Phi(\xi_i^*(s)); \theta(s)) = Q\left(\Phi\left(\bigvee_{i=1}^n \xi_i^*(s)\right); \theta(s)\right).$$

Since ξ^* is a simple max-stable USC process, we have $\bigvee_{i=1}^n \xi_i^* \overset{\mathrm{D}}{=} n\xi^*$ in USC$(\mathbb{D})$. But then also

$$\bigvee_{i=1}^n \xi_i \overset{\mathrm{D}}{=} \bigvee_{i=1}^n \tilde{\xi}_i$$

$$\overset{\mathrm{D}}{=} Q(\Phi(n\xi^*); \theta)$$

$$= Q(\Phi(\xi^*)^{1/n}; \theta)$$

$$= a_{n,\theta} Q(\Phi(\xi^*); \theta) + b_{n,\theta}$$
$$= a_{n,\theta}\xi + b_{n,\theta}.$$

We conclude that ξ is max-stable, as required. $\qquad\square$

Proposition 5.2. (À la Sklar (II) for USC processes with GEV margins.) *Let ξ be a USC process with $\mathrm{GEV}(\theta(s))$ margins for $s \in \mathbb{D}$. If $\theta\colon \mathbb{D} \to \Theta$ is continuous then the following statements hold:*

(i) *the process ξ^* defined by $\xi^*(s) = -1/\log F(\xi(s); \theta(s))$ is a USC process with unit-Fréchet margins;*

(ii) *the process $\tilde{\xi}$ defined by $\tilde{\xi}(s) = Q(\Phi(\xi^*(s)); \theta(s))$ is a USC process and, with probability 1, for all $s \in \mathbb{D}$,*

$$\tilde{\xi}(s) = \begin{cases} \xi(s) & \text{if } \xi^*(s) < \infty, \\ \infty & \text{if } \xi^*(s) = \infty. \end{cases}$$

Proof. The marginal distributions of ξ are GEV and depend continuously on s. Conditions (a) and (b) in Proposition 4.2 are therefore satisfied.

By Proposition 4.2(i), the process Z defined by $Z(s) = F(\xi(s); \theta(s))$ is a USC process with standard uniform margins. It then follows that ξ^* defined by $\xi^*(s) = -1/\log Z(s)$ is a USC process too, its margins being unit-Fréchet.

Since $0 \leq Z(s) \leq 1$ by construction, we have $0 \leq \xi^*(s) \leq \infty$ and, thus, $\Phi(\xi^*(s)) = Z(s)$. We find that $\tilde{\xi}(s) = Q(Z(s); \theta(s)) = Q(F(\xi(s); \theta(s)); \theta(s))$. By Proposition 4.2(ii), the process $\tilde{\xi}$ is a USC process.

Recall that GEV distribution functions are continuous and strictly increasing on their domains. As a consequence, for all x such that $x \geq Q(0; \theta(s))$, we have

$$Q(F(x; \theta(s)); \theta(s)) = \begin{cases} x & \text{if } F(x; \theta(s)) < 1, \\ \infty & \text{if } F(x; \theta(s)) = 1. \end{cases}$$

Moreover, Lemma C.1 implies that $\xi \geq Q(0; \theta)$ almost surely. Since $\xi^*(s) < \infty$ if and only if $F(\xi(s); \theta(s)) < 1$, we arrive at the stated formula for $\tilde{\xi}$. $\qquad\square$

Proposition 5.3. (À la Sklar (II) for max-stable processes.) *Let ξ be a USC process with $\mathrm{GEV}(\theta(s))$ margins for $s \in \mathbb{D}$. Assume that, for every compact $K \subset \mathbb{D}$, we have $\sup_{s \in K} F(\xi(s); \theta(s)) < 1$ with probability 1. As in Proposition 5.2, define two USC processes ξ^* and $\tilde{\xi}$ by $\xi^*(s) = -1/\log F(\xi(s); \theta(s))$ and $\tilde{\xi}(s) = Q(\Phi(\xi^*(s)); \theta(s))$ for $s \in \mathbb{D}$. Then, almost surely, $\xi = \tilde{\xi}$. Furthermore, the following two statements are equivalent:*

(i) *the USC process ξ is max-stable;*

(ii) *the USC process ξ^* is simple max-stable.*

Proof. The fact that ξ^* is a USC process with unit-Fréchet margins is a consequence of the continuity of θ and Proposition 5.2(i). Let $(K_n)_{n \in \mathbb{N}}$ be a compact cover of $\mathbb{D}$; the existence of a compact cover is guaranteed by the fact that $\mathbb{D}$ is locally compact and second-countable, whence Lindelöf. We have $\mathbb{P}[\text{for all } n \in \mathbb{N}: \sup_{s \in K_n} F(\xi(s); \theta(s)) < 1] = 1$ by assumption and, thus, also $\mathbb{P}[\text{for all } s \in \mathbb{D}: F(\xi(s); \theta(s)) < 1] = 1$. Proposition 5.2(ii) implies that $\xi = \tilde{\xi}$ almost surely.

The functions $s \mapsto a_{n,\theta(s)}$ and $s \mapsto b_{n,\theta(s)}$ are continuous. The map $\mathbb{D} \times [-\infty, \infty] \to [-\infty, \infty]$ defined by $(s, x) \mapsto a_{n,\theta(s)} x + b_{n,\theta(s)}$ belongs to $\mathcal{U}(\mathbb{D})$. By Proposition 3.1, the map from USC($\mathbb{D}$) to itself sending z to the function $s \mapsto a_{n,\theta(s)} z(s) + b_{n,\theta(s)}$ is well defined and hypo-measurable. It follows that $a_{n,\theta} \xi + b_{n,\theta}$ is a USC process.

Suppose first that (ii) holds. By Proposition 5.1, the USC process $\tilde{\xi}$ is max-stable with norming functions $s \mapsto a_{n,\theta(s)}$ and $s \mapsto b_{n,\theta(s)}$. Since ξ and $\tilde{\xi}$ are equal almost surely in USC($\mathbb{D}$), they are also in equal in law. Statement (i) follows.

Conversely, suppose that (i) holds. Let $(\xi_i^*)_{i=1}^n$ and $(\xi_i)_{i=1}^n$ be vectors of i.i.d. USC processes with common laws equal to the ones of ξ^* and ξ, respectively. For $z \in \text{USC}(\mathbb{D})$, write $-1/\log F(z, \theta) = (-1/\log F(z(s), \theta(s)))_{s \in \mathbb{D}}$. The mapping $z \mapsto -1/\log F(z, \theta)$ from USC($\mathbb{D}$) to itself is hypo-measurable, by an argument as in the proof of Proposition 5.2. For $i \in \{1, \ldots, n\}$, we have, in USC($\mathbb{D}$),

$$\xi_i^* \stackrel{\mathrm{D}}{=} \xi^* = -\frac{1}{\log F(\xi, \theta)} \stackrel{\mathrm{D}}{=} -\frac{1}{\log F(\xi_i, \theta)}.$$

By independence, we thus have $(\xi_i^*)_{i=1}^n \stackrel{\mathrm{D}}{=} (-1/\log F(\xi_i, \theta))_{i=1}^n$. Property (i) and max-stability (5.3) now yield

$$\bigvee_{i=1}^n \xi_i^* \stackrel{\mathrm{D}}{=} -\frac{1}{\log F(\bigvee_{i=1}^n \xi_i; \theta)}$$

$$\stackrel{\mathrm{D}}{=} -\frac{1}{\log F(a_{n,\theta} \xi + b_{n,\theta}; \theta)}$$

$$= -\frac{1}{\log\{F(\xi; \theta)^{1/n}\}}$$

$$= n \left[-\frac{1}{\log F(\xi; \theta)} \right]$$

$$= n \xi^*.$$

We conclude that ξ^* is simple max-stable. $\square$

Remark 5.2. (*Regarding the finiteness of ξ^* in Proposition 5.2(ii).*) Recall that USC functions reach their suprema on compacta. Let $(K_n)_{n \in \mathbb{N}}$ be a compact cover of $\mathbb{D}$. For every $z \in \text{USC}(\mathbb{D})$, we have $z(s) < \infty$ for all $s \in \mathbb{D}$ if and only if $\sup_{s \in K_n} z(s) < \infty$ for all $n \in \mathbb{N}$. The event $\bigcap_{n \in \mathbb{N}} \{\sup_{s \in K_n} \xi^*(s) < \infty\}$ is thus the same as the event {for all $s \in \mathbb{D}$: $\xi^*(s) < \infty$}.

Remark 5.3. (*Regarding the continuity assumption on θ in Proposition 5.2.*) According to Lemmas 3.2 and 5.1, imposing the continuity of the GEV parameter vector $\theta(s)$ as a function of $s \in \mathbb{D}$ is equivalent to imposing the upper semicontinuity of the function $s \mapsto F(x, \theta(s))$ for each fixed $x \in \mathbb{R}$.

5.4. Examples

In comparison to Proposition 4.2, we have added to Proposition 5.2 the assumption that the margins be GEV. Although their distribution functions are continuous and strictly increasing on their support, this does not resolve the issues arising when the marginal distributions are not continuous in space, as in Example 4.2. In Example 5.1, which parallels Example 4.2, the GEV parameter function $s \mapsto \theta(s)$ is not continuous and pointwise standardization to unit-Fréchet margins produces a stochastic process whose trajectories are no longer USC almost surely.

Example 5.1. (*What may happen without the continuity of θ.*) Consider two independent, unit-Fréchet distributed variables X and Y. As in Example 4.2, take $\mathbb{D} = [0, 2]$, and define

$$\xi(s) = X \vee (Y\mathbf{1}_{\{1\}}(s)) = \begin{cases} X & \text{if } s \neq 1, \\ X \vee Y & \text{if } s = 1. \end{cases}$$

Then again, ξ is a USC process. It is even a max-stable one with normalizing functions $a_n \equiv n$ and $b_n \equiv 0$. The marginal distribution functions are

$$F(x; \theta(s)) = \mathbb{P}[\xi(s) \leq x] = \begin{cases} \mathrm{e}^{-1/x} & \text{if } s \neq 1, \\ \mathrm{e}^{-2/x} & \text{if } s = 1, \end{cases}$$

where $x \geq 0$. Then the function $s \mapsto F_s(x)$ is lower rather than USC, and the marginal GEV parameter vector is

$$\theta(s) = (\gamma(s), \mu(s), \sigma(s)) = \begin{cases} (1, 1, 1) & \text{if } s \neq 1, \\ (1, 2, 2) & \text{if } s = 1, \end{cases}$$

which is not continuous as a function of s. Standardizing to Fréchet margins yields

$$\xi^*(s) = -\frac{1}{\log F(\xi(s); \theta(s))} = \begin{cases} X & \text{if } s \neq 1, \\ \dfrac{(X \vee Y)}{2} & \text{if } s = 1. \end{cases}$$

The event $\{0 < X < Y < 2X\}$ has positive probability, and on this event, we have $(X \vee Y)/2 = Y/2 < X$, so that the trajectory $s \mapsto \xi^*(s)$ is not USC.

To conclude, we present a construction principle for simple max-stable USC processes. In combination with Proposition 5.1, this provides a device for the construction of max-stable USC processes with arbitrary GEV margins. The method is similar to the one proposed in Theorem 2 of [16]. Proving max-stability of the USC process that we construct requires special care, since max-stability in $\mathrm{USC}(\mathbb{D})$ does not follow from max-stability of the finite-dimensional distributions.

Example 5.2. Let $Y_1 > Y_2 > Y_3 > \cdots$ denote the points of a Poisson point process on $(0, \infty)$ with intensity measure $y^{-2}\,\mathrm{d}y$. Let $V, V_1, V_2, \ldots$ be i.i.d. USC processes, independent of the point process $(Y_i)_i$. Assume that V satisfies the following properties:

- $\mathbb{P}[\inf_{s \in \mathbb{D}} V(s) \geq 0] = 1$;

- $\mathbb{E}[\sup_{s \in \mathbb{D}} V(s)] < \infty$;

- the mean function $f(s) = \mathbb{E}[V(s)]$ is strictly positive and continuous on $\mathbb{D}$.

By Lemma C.3 below, $\inf_{\mathbb{D}} V$ is indeed a random variable. Note that we do *not* impose that $\inf_{\mathbb{D}} V > 0$ almost surely.

Define a stochastic process ξ on $\mathbb{D}$ by

$$\xi(s) = \sup_{i \geq 1} Y_i \frac{1}{f(s)} V_i(s), \qquad s \in \mathbb{D}.$$

We will show that ξ is 'almost surely' a simple max-stable USC process, in the following sense:

(a) with probability 1, the trajectories of ξ are USC;

(b) for some set $\Omega_1 \subset \Omega$ of probability 1 on which the trajectories $\xi(\cdot, \omega)$ are USC, the map $\xi \colon \Omega_1 \to \mathrm{USC}(\mathbb{D})$ is hypo-measurable.

Consider the space of nonnegative USC functions

$$\mathrm{USC}(\mathbb{D})_+ = \{z \in \mathrm{USC}(\mathbb{D}) \colon \text{for all } s \in \mathbb{D}, z(s) \ge 0\} = \bigcap_{s \in \mathbb{D}} \{z \in \mathrm{USC}(\mathbb{D}) \colon z(s) \ge 0\}.$$

For each fixed $s \in \mathbb{D}$, the set $\{z \in \mathrm{USC}(\mathbb{D}) \colon z(s) \ge 0\}$ is hypo-closed, since it is the complement of the hypo-open set $\{z \in \mathrm{USC}(\mathbb{D}) \colon \mathrm{hypo}\, z \cap \{(s, 0)\} = \varnothing\}$. The set $\mathrm{USC}(\mathbb{D})_+$ is, thus, hypo-closed, as an intersection of hypo-closed sets.

Set $E = \mathrm{USC}(\mathbb{D})_+ \setminus \{0_{\mathbb{D}}\}$, where $0_{\mathbb{D}}$ denotes the null function on $\mathbb{D}$. Since $\mathrm{USC}(\mathbb{D})$ is a compact space with a countable basis [8, Theorem B.2, p. 399], the space E is locally compact with a countable basis. Classical theory of point processes applies and it is possible to define Poisson processes on E by augmentation and/or continuous mappings.

Set $W_i = V_i/f$. Then, for $i \in \mathbb{N}$, W_i is a random element of $\mathrm{USC}(\mathbb{D})$ such that $W_i \in E$ with probability 1. The point process Γ defined by

$$\Gamma = \sum_{i \ge 1} \delta_{(Y_i, W_i)}$$

is, thus, a Poisson process on $(0, \infty) \times E$ with mean measure $\mathrm{d}\Lambda(y, w) = y^{-2}\,\mathrm{d}y \otimes \mathrm{d}P_W(w)$, where P_W is the law of W_1; see [11, Proposition 3.8].

The 'product' mapping $T \colon (0, \infty) \times E \to E$ defined by $T(y, w) = yw$, for $y > 0$ and $w \in E$, is hypo-measurable. Provided that the image measure $\mu = \Lambda \circ T^{-1}$ is finite on compact sets of E, we find that the point process

$$\Pi = \sum_{i \ge 1} \delta_{(Y_i W_i)}$$

is a Poisson process with mean measure μ on E; see [11, Proposition 3.7]. To check the finiteness of μ on compact sets of E, we must check that, for $K \subset \mathbb{D}$ and $x > 0$, writing

$$\mathcal{F}_{K \times \{x\}} = \left\{ z \in \mathrm{USC}(\mathbb{D})_+ \colon \sup_{s \in K} z(s) \ge x \right\},$$

we have $\mu(\mathcal{F}_{K \times \{x\}}) < \infty$. Indeed, the set $\mathcal{F}_{K \times \{x\}}$ is hypo-closed in $\mathrm{USC}(\mathbb{D})$, and since $\mathrm{USC}(\mathbb{D})$ is compact, the compact sets in E are the hypo-closed sets F in $\mathrm{USC}(\mathbb{D})$ such that $F \subset E$. Thus, $\mathcal{F}_{K \times \{x\}}$ is compact. Also, any compact set in E must be contained in such a $\mathcal{F}_{K \times \{x\}}$. Now,

$$\mu(\mathcal{F}_{K \times \{x\}}) = -\Lambda\left\{ (r, w) \in (0, \infty) \times \mathrm{USC}(\mathbb{D}) \colon r \max_{s \in K} w(s) \ge x \right\}$$

$$= \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{r \ge \inf_{s \in K}(x/W(s))\}} \frac{\mathrm{d}r}{r^2} \right]$$

$$= \frac{1}{x} \mathbb{E}\left[\sup_{s \in K_j} W(s) \right]$$

$$\le \frac{1}{x} \frac{\mathbb{E}[\sup_{\mathbb{D}} V]}{\inf_{\mathbb{D}} f},$$

which is finite by assumption on V.

To prove (a), we adapt an argument of [4, Proof of Theorem 2.1]. For fixed $K \subset \mathbb{D}$ compact and $x > 0$, we have $\mu(\mathcal{F}_{K \times \{x\}}) < \infty$ and, thus, $\Pi(\mathcal{F}_{K \times \{x\}}) < \infty$ almost surely. Thus, there exists a set Ω_1 of probability 1, such that the following two statements hold for all $\omega \in \Omega_1$:

(i) $W_i(\cdot, \omega) \in E$ for all $i \in \mathbb{N}$;

(ii) $\Pi(\mathcal{F}_{K \times \{x\}})(\omega) < \infty$ for every rational $x > 0$ and every compact rectangle $K \subset \mathbb{D}$ with rational vertices.

Take $\omega \in \Omega_1$. We need to show that for $s \in \mathbb{D}$ and $x \in \mathbb{Q}$ such that $\xi(s, \omega) < x$, we have $\limsup_{t \to s} \xi(t, \omega) \leq x$. Fix s and x as above, which implies that $x > 0$. Let $K_n \searrow \{s\}$ be a collection of compact rational rectangles as above such that s is in the interior of K_n. Then $\mathcal{F}_{\{(s,x)\}} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_{K_n \times \{x\}}$ (the inclusion '$\subset$' is immediate; the inclusion '$\supset$' is obtained by choosing, for $z \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_{K_n \times \{x\}}$ and for $n \in \mathbb{N}$, a point $s_n \in K_n$ such that $z(s_n) \geq x$ and then observing that $s_n \to s$ and, thus, $z(s) \geq x$ by upper semicontinuity). From our choice of Ω_1, we have $\Pi(\mathcal{F}_{K_1 \times \{x\}}, \omega) < \infty$, so that the downward continuity property of the measure $\Pi(\cdot, \omega)$ applies and

$$\Pi(\mathcal{F}_{\{(s,x)\}}, \omega) = \lim_{n \to \infty} \Pi(\mathcal{F}_{K_n \times \{x\}}, \omega).$$

By our choice of s, x, and ω, the left-hand side in the display is 0. Since the sequence on the right-hand side is integer valued, there exists n_0 such that, for all $n \geq n_0$, we have $\Pi(\mathcal{F}_{K_n \times \{x\}}, \omega) = 0$. This implies that, for $n \geq n_0$, we have

$$\sup_{t \in K_n} Y_i(\omega) W_i(t, \omega) < x, \qquad i \in \mathbb{N}.$$

We complete the proof of (a) by noting that

$$\limsup_{t \to s} \xi(t, \omega) \leq \sup_{t \in K_{n_0}} \xi(t, \omega) = \sup_{i \in \mathbb{N}} \sup_{t \in K_{n_0}} Y_i(\omega) W_i(t, \omega) \leq x.$$

To prove (b), we need to show that, for any compact $K \subset \mathbb{D} \times \mathbb{R}$, the set

$$A = \{\omega \in \Omega_1 : \text{hypo}\, \xi(\cdot, \omega) \cap K \neq \varnothing\}$$

is a measurable subset of Ω. Note first that $\xi(\cdot, \omega) \in \text{USC}(\mathbb{D})_+$ for $\omega \in \Omega_1$; use property (i) above of Ω_1. It follows that $A = \Omega_1$ as soon as K is not a subset of $\mathbb{D} \times (0, \infty)$. Assume that $K \subset \mathbb{D} \times (0, \infty)$.

On Ω_1, we have $\text{hypo}\, \xi \cap K \neq \varnothing$ if and only if $\Pi(\{z \in E : \text{hypo}\, z \cap K \neq \varnothing\}) \geq 1$. Now $\mathcal{F}_K = \{z \in E : \text{hypo}\, z \cap K \neq \varnothing\}$ is a hypo-measurable subset of E, so that $X = \Pi(\mathcal{F}_K)$ is a random variable. It follows that $A = X^{-1}([1, \infty)) \cap \Omega_1$ is measurable, yielding (b).

A standard argument yields that the margins of ξ are unit-Fréchet. To show that ξ is simple max-stable, we need to show that, for independent random copies $\xi_1, \ldots, \xi_n$ of ξ, the capacity functionals of $\bigvee_{i=1}^n \xi_i$ and $n\xi$ are the same. That is, we need to show that, for every compact set in $\mathbb{D} \times \mathbb{R}$, we have

$$\mathbb{P}\left[\text{hypo}\left(\bigvee_{i=1}^n \xi\right) \cap K = \varnothing\right] = \mathbb{P}[\text{hypo}(n\xi) \cap K = \varnothing]. \tag{5.9}$$

The left-hand side is equal to

$$\mathbb{P}\left[\left(\bigcup_{i=1}^n \text{hypo}\, \xi_i\right) \cap K = \varnothing\right] = \mathbb{P}[\text{for all } i = 1, \ldots, n : \text{hypo}\, \xi_i \cap K = \varnothing]$$

$$= (\mathbb{P}[\text{hypo}\, \xi \cap K = \varnothing])^n. \tag{5.10}$$

Without loss of generality, we may assume that

$$K = \bigcup_{j=1}^{p} (K_j \times \{x_j\}),$$

where $p \in \mathbb{N}$, $K_j \subset \mathbb{D}$ is compact, and x_j is real for $j \in \{1, \ldots, p\}$. Indeed, the capacity functional of a USC process is entirely determined by its values on such compacta; see [8, p. 340]. We may also choose $x_j > 0$, since otherwise both sides of (5.9) vanish. For such a set K, we have

$$
\begin{aligned}
&\mathbb{P}[\text{hypo}\, \xi \cap K = \varnothing] \\
&\quad = \mathbb{P}\Big[\Pi\Big\{z \in E : \text{there exists } 1 \leq j \leq p, \max_{K_j} z \geq x_j\Big\} = 0\Big] \\
&\quad = \exp\Big(-\Lambda\Big\{(r, w) \in (0, \infty) \times E : \text{there exists } j \leq p, r \max_{s \in K_j} w(s) \geq x_j\Big\}\Big) \\
&\quad = \exp\Big(-\mathbb{E}\Big[\int_0^\infty \mathbf{1}_{\{\text{there exists } j : r \geq \min_{s \in K_j}(x_j/W(s))\}} \frac{\mathrm{d}r}{r^2}\Big]\Big) \\
&\quad = \exp\Big(-\mathbb{E}\Big[\int_0^\infty \mathbf{1}_{\{r \geq \min_j \min_{s \in K_j}(x_j/W(s))\}} \frac{\mathrm{d}r}{r^2}\Big]\Big) \\
&\quad = \exp\Big(-\mathbb{E}\Big[\max_j \frac{\max_{s \in K_j} W(s)}{x_j}\Big]\Big).
\end{aligned}
$$

For ξ replaced by $n\xi$, we obtain the same result, but with x_j replaced by x_j/n. In view of (5.10), the desired equality (5.9) follows.

6. Conclusion

The aim of this paper has been to extend Sklar's theorem from random vectors to USC processes. We have stated necessary and sufficient conditions to be able to construct a USC process with general margins by applying the pointwise quantile transformation to a USC process with standard uniform margins (Propositions 4.1 and 5.1). Furthermore, we have stated sufficient conditions for the pointwise probability integral transform to be possible for USC processes (Propositions 4.2, 5.2, and 5.3). These conditions imply, in particular, that the marginal distribution functions are continuous with respect to the space variable (Lemma 4.1). We have also provided several examples of things that can go wrong when these conditions are not satisfied. However, finding *necessary* and sufficient conditions remains an open problem.

The motivation has been to extend the margins versus dependence paradigm used in multivariate extreme value theory to max-stable USC processes. The next step is to show that marginal standardization is possible in max-domains of attraction too. One question, for instance, is whether the standardized weak limit of the pointwise maxima of a sequence of USC processes is equal to the weak limit of the pointwise maxima of the sequence of standardized USC processes; see [11, Proposition 5.10]. Interesting difficulties arise: weak convergence of finite-dimensional distributions does not imply, and is not implied by, weak hypoconvergence; Khinchin's convergence-of-types lemma does not apply in its full generality to unions of random closed sets; see [8, p. 254, 'Affine normalization']. This topic will be the subject of further work.

Appendix A. Right-continuous quantile functions

The *right-continuous* quantile function, Q, of a random variable X taking values in $[-\infty, \infty]$ and with distribution function $F(x) = \mathbb{P}[X \leq x]$, $x \in [-\infty, \infty]$, is defined as

$$Q(p) = \sup\{x \in \mathbb{R} \colon F(x) \leq p\}, \qquad p \in [0, 1]. \tag{A.1}$$

By convention, $\sup \varnothing = -\infty$ and $\sup \mathbb{R} = \infty$. The fact that Q is right continuous is stated in Lemma A.1(ii). The function Q is denoted by F_+^{-1} in [10, p. 749], where properties (ii) and (iv) of Lemma A.1 are mentioned. The corresponding statements for the right-continuous inverse, $p \mapsto \inf\{x \in \mathbb{R} \colon F(x) \geq p\}$, are well known; see, for instance, Section 0.2 of [11].

Lemma A.1. *Let X be a random variable taking values in $[-\infty, \infty]$. Define $Q \colon [0, 1] \to [-\infty, \infty]$ as in (A.1).*

 (i) *For all $p \in [0, 1]$, we have $Q(p) = \sup\{x \in \mathbb{R} \colon \mathbb{P}(X < x) \leq p\}$.*

 (ii) *The function Q is nondecreasing and right continuous.*

 (iii) *For every $x \in [-\infty, \infty]$ and every $p \in [0, 1]$, we have $x \leq Q(p)$ if and only if $\mathbb{P}[X < x] \leq p$.*

 (iv) *If V is uniformly distributed on $[0, 1]$ then the distribution function of $Q(V)$ is F, i.e. $Q(V)$ and X are identically distributed.*

 (v) *If the law of X has no atoms in $[-\infty, \infty]$ then $\mathbb{P}[X = Q(F(X))] = 1$.*

Proof. (i) Fix $p \in [0, 1]$. Since $\mathbb{P}[X < x] \leq \mathbb{P}[X \leq x] = F(x)$, we have

$$Q(p) := \sup\{x \in \mathbb{R} \colon F(x) \leq p\} \leq \sup\{x \in \mathbb{R} \colon \mathbb{P}[X < x] \leq p\} := x_0.$$

Conversely, if $Q(p) = +\infty$, there is nothing to prove. Assume then that $Q(p) < +\infty$, so that $p < 1$. Let $y > Q(p)$. We need to show that $y > x_0$ too, that is, $\mathbb{P}[X < y] > p$. But $\mathbb{P}[X < y] = \sup\{F(z) \colon z < y\}$, and this supremum must be larger than p, since, for all $z > Q(p)$, we have $F(z) > p$.

(ii) The sets $\{x \in \mathbb{R} \colon F(x) \leq p\}$ become larger with p, the function Q is nondecreasing. Next, we show that Q is right continuous at any $p \in [0, 1]$. If $Q(p) = \infty$, there is nothing to show, so suppose that $Q(p) < \infty$ (in particular, $p < 1$). Let $\varepsilon > 0$. Then $Q(p) + \varepsilon > Q(p)$ and, thus, $F(Q(p) + \varepsilon) = p + \delta > p$ for some $\delta > 0$. For $r < p + \delta$, we have $F(Q(p) + \varepsilon) > r$ too, and, thus, $Q(r) < Q(p) + \varepsilon$, as required.

(iii) First suppose that $x < Q(p)$; we show that $\mathbb{P}[X < x] \leq p$. The $x = \infty$ case is impossible, and if $x = -\infty$ then $\mathbb{P}[X < x] = 0 \leq p$. So suppose that $x \in \mathbb{R}$. Using statement (i), there exists $y \in \mathbb{R}$ with $x \leq y \leq Q(p)$ such that $\mathbb{P}[X < y] \leq p$. But then also $\mathbb{P}[X < x] \leq p$.

 Second, suppose that $x > Q(p)$; we show that $\mathbb{P}[X < x] > p$. Clearly, we must have $Q(p) < \infty$, and so we can without loss of generality assume that x is real. But then $\mathbb{P}[X < x] > p$ by statement (i).

 Finally, consider $x = Q(p)$; we show that $\mathbb{P}[X < Q(p)] \leq p$. If $Q(p) = -\infty$ then $\mathbb{P}[X < Q(p)] = 0 \leq p$. If $Q(p) > -\infty$ then $\mathbb{P}[X < y] \leq p$ for all $y < Q(p)$, and, thus, $\mathbb{P}[X < Q(p)] = \sup_{y < Q(p)} \mathbb{P}[X < y] \leq p$ too.

(iv) Without loss of generality, assume that $0 \leq V \leq 1$ (if not, then replace V by $(V \vee 0) \wedge 1$, which is almost surely equal to V). Let $x \in [-\infty, \infty]$. By statement (iii), we have $x \leq Q(V)$

if and only if $\mathbb{P}[X < x] \leq V$. As a consequence, $\mathbb{P}[x \leq Q(V)] = \mathbb{P}[\mathbb{P}[X < x] \leq V] = 1 - \mathbb{P}[X < x] = \mathbb{P}[x \leq X]$. We conclude that $Q(V)$ and X are identically distributed and, thus, that $Q(V)$ has distribution function F too.

(v) By definition, $x \leq Q(F(x))$ for every $x \in \mathbb{R}$. For $x = -\infty$ and $x = +\infty$, the same inequality is trivially fulfilled too (recall that $F(\infty) = 1$ and $Q(1) = \infty$). As a consequence, $X \leq Q(F(X))$.

Conversely, let $\mathcal{P}$ be the collection of $p \in [0, 1]$ such that the set $\{x \in \mathbb{R} \colon F(x) = p\}$ has positive Lebesgue measure. These sets being disjoint for distinct p, the set $\mathcal{P}$ is at most countably infinite. If $x < Q(F(x))$ then there exists $y > x$ such that $F(x) = F(y)$ and, thus, $F(x) \in \mathcal{P}$. However, the law of $F(X)$ is standard uniform, so that $\mathbb{P}[F(X) \in \mathcal{P}] = 0$. Hence, $X = Q(F(X))$ almost surely. □

Lemma A.2. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables defined on the same probability space and taking values in $[-\infty, \infty]$. Let Q_n and Q be the right-continuous quantile functions (A.1) of X_n and $\limsup_{n \to \infty} X_n$, respectively. Then*

$$Q(p) \geq \limsup_{n \to \infty} Q_n(p), \qquad p \in [0, 1].$$

Proof. If $Q(p) = \infty$, there is nothing to show, so suppose that $Q(p) < \infty$. Let $y > Q(p)$; we will show that $y \geq \limsup_{n \to \infty} Q_n(p)$. This being true for all $y > Q(p)$, we will have proved the proposition.

By Lemma A.1(i), we have $\mathbb{P}[\limsup_{n \to \infty} X_n < y] > p$. By Fatou's lemma, there exists a positive integer $n(y)$ such that $\mathbb{P}[X_n < y] > p$ for all integer $n \geq n(y)$. But for such n, we have $y > Q_n(p)$ too. Hence, $y \geq \limsup_{n \to \infty} Q_n(p)$, as required. □

Appendix B. Additional examples related to Proposition 4.2

Example B.1. (*Condition (b) in Proposition 4.2 is not necessary.*) Let $\mathbb{D} = [-1, 1]$ and let X and V be independent random variables, X standard normal and V uniform on $[0, 1]$. Define

$$\xi(s) = \begin{cases} X & \text{if } -1 \leq s \leq 0, \\ X - 1 & \text{if } 0 < s < V, \\ X & \text{if } V \leq s \leq 1. \end{cases}$$

Let Φ be the standard normal cumulative distribution function and choose $x \in \mathbb{R}$. Then $F_s(x) = \mathbb{P}[\xi(s) \leq x] = \Phi(x)$ if $s \in [-1, 0]$, while, for $s \in (0, 1]$, we have

$$F_s(x) = \mathbb{P}[s < V]\mathbb{P}[X - 1 \leq x] + \mathbb{P}[V \leq s]\mathbb{P}[X \leq x] = (1 - s)\Phi(x + 1) + s\Phi(x).$$

The function $s \mapsto F_s(x)$ is constant on $s \in [-1, 0]$, while it decreases linearly from $\Phi(x + 1)$ to $\Phi(x)$ for s from 0 to 1, the right-hand side limit at 0 being equal to $\Phi(x + 1)$, which is greater than $\Phi(x)$, the value at $s = 0$ itself. Hence, the function $s \mapsto F_s(x)$ is lower but not USC, and condition (b) in Proposition 4.2 does not hold. Nevertheless, the random variables $Z(s) = F_s(\xi(s))$, $s \in \mathbb{D}$, are given as

$$Z(s) = F_s(\xi(s)) = \begin{cases} \Phi(X) & \text{if } -1 \leq s \leq 0, \\ (1 - s)\Phi(X) + s\Phi(X - 1) & \text{if } 0 < s < V, \\ (1 - s)\Phi(X + 1) + s\Phi(X) & \text{if } V \leq s \leq 1. \end{cases}$$

The trajectory of Z is continuous at $s \in [-1, 1] \setminus \{V\}$ and USC at $s = V$, hence, USC overall.

Example B.2. (*Processes ξ and $\tilde{\xi}$ in Proposition 4.2 may be different in law (1).*) Let X and Y be independent, uniform random variables on $[0, 1]$. For $s \in \mathbb{D} = [0, 1]$, define

$$\xi(s) = X + \mathbf{1}_{\{Y=s\}} = \begin{cases} X & \text{if } s \neq Y, \\ X+1 & \text{if } s = Y. \end{cases}$$

Since $\mathbb{P}[Y = s] = 0$ for every $s \in [0, 1]$, the law of $\xi(s)$ is standard uniform. Conditions (a) and (b) in Proposition 4.2 are trivially fulfilled with $F_s(x) = (x \vee 0) \wedge 1$ for $x \in [-\infty, \infty]$ and $Q_s(p) = p$ for $p \in [0, 1)$, while $Q_s(1) = \infty$. We obtain

$$Z(s) = F_s(\xi(s)) = \begin{cases} X & \text{if } s \neq Y, \\ 1 & \text{if } s = Y, \end{cases}$$

and, thus,

$$\tilde{\xi}(s) = Q_s(Z(s)) = \begin{cases} X & \text{if } s \neq Y, \\ \infty & \text{if } s = Y. \end{cases}$$

The processes $\tilde{\xi}$ and ξ have different capacity functionals and, thus, a different distribution as random elements in $\mathrm{USC}(\mathbb{D})$: for $K = [0, 1] \times \{x\}$ with $x > 2$, we have $\mathbb{P}[\text{hypo}\, \xi \cap K \neq \varnothing] = 0$, while $\mathbb{P}[\text{hypo}\, \tilde{\xi} \cap K \neq \varnothing] = 1$.

Example B.3. (*Processes ξ and $\tilde{\xi}$ in Proposition 4.2 may be different in law (2).*) Let X and Y be independent random variables, with X uniformly distributed on $[0, 1] \cup [2, 3]$ and Y uniformly distributed on $[0, 1]$. For $s \in \mathbb{D} = [0, 1]$, define

$$\xi(s) = X \vee (1.5 \times \mathbf{1}_{\{Y=s\}}) = \begin{cases} X & \text{if } s \neq Y \text{ or } X > 1, \\ 1.5 & \text{if } s = Y \text{ and } X \leq 1. \end{cases}$$

Then $\mathbb{P}[\xi(s) = X] = 1$ for all $s \in [0, 1]$, so that the marginal distribution and quantile functions F_s and Q_s do not depend on s and are equal to those of X, denoted by F_X and Q_X. The random variable $U = F_X(X)$ is uniformly distributed on $[0, 1]$. Since $F_X(1.5) = F_X(1) = \frac{1}{2}$, we have

$$Z(s) = F_X(\xi(s)) = \begin{cases} U & \text{if } s \neq Y \text{ or } X > 1, \\ \frac{1}{2} & \text{if } s = Y \text{ and } X \leq 1. \end{cases}$$

However, as $Q_X(\frac{1}{2}) = 2$, we obtain

$$\tilde{\xi}(s) = Q_X(Z(s)) = \begin{cases} X & \text{if } s \neq Y \text{ or } X > 1, \\ 2 & \text{if } s = Y \text{ and } X \leq 1. \end{cases}$$

On the event $\{X \leq 1\}$, which occurs with probability $\frac{1}{2}$, the hypographs of ξ and $\tilde{\xi}$ are different.

Appendix C. Lower bounds of USC processes

Lemma C.1. *If ξ is a USC process then the function $\ell(s) = \sup\{x \in \mathbb{R} : \mathbb{P}[\xi(s) < x] = 0\}$ for $s \in \mathbb{D}$ is USC. Moreover, $\xi \vee \ell$ is also a USC process, and we have $\mathbb{P}[\xi \geq \ell] = \mathbb{P}[\xi = \xi \vee \ell] = 1$.*

Proof. The function ℓ is equal to the function $s \mapsto Q_s(0)$, with Q_s the right-continuous quantile function (A.1) of ξ_s. By Lemma 3.2, the function ℓ is USC, and by Example 3.1, the map $\mathrm{USC}(\mathbb{D}) \to \mathrm{USC}(\mathbb{D}) : z \mapsto z \vee \ell$ is hypo-measurable, so that $\xi \vee \ell$ is a USC process too.

By Lemma C.2, there exists a countable subset, $\mathbb{Q}_\ell$, of $\mathbb{D}$ with the following property: for all $x \in \mathrm{USC}(\mathbb{D})$, we have $x(s) \geq \ell(s)$ for all $s \in \mathbb{D}$ if and only if $x(t) \geq \ell(t)$ for all $t \in \mathbb{Q}_\ell$. Since $\ell(t) = Q_t(0)$, we have $\mathbb{P}[\xi(t) \geq \ell(t)] = 1$ for all $t \in \mathbb{Q}_\ell$; see Lemma A.1(iii). Since $\mathbb{Q}_\ell$ is countable, also $\mathbb{P}[\text{for all } t \in \mathbb{Q}_\ell : \xi(t) \geq \ell(t)] = 1$. By the property of $\mathbb{Q}_\ell$ mentioned earlier, the event $\{\text{for all } t \in \mathbb{Q}_\ell : \xi(t) \geq \ell(t)\}$ is equal to both $\{\xi \geq \ell\}$ and $\{\xi = \xi \vee \ell\}$. $\qquad\square$

Lemma C.2. *For every $z \in \mathrm{USC}(\mathbb{D})$, there exists a countable set $\mathbb{Q}_z \subset \mathbb{D}$ such that*

$$z(s) = \inf_{\varepsilon > 0} \sup_{t \in \mathbb{Q}_z : \, d(s,t) \leq \varepsilon} z(t), \qquad s \in \mathbb{D}. \tag{C.1}$$

In particular, for every $x \in \mathrm{USC}(\mathbb{D})$, we have $x(s) \geq z(s)$ for all $s \in \mathbb{D}$ if and only if $x(t) \geq z(t)$ for all $t \in \mathbb{Q}_z$.

Proof. The set hypo z is a subset of the metrizable separable space $\mathbb{D} \times \mathbb{R}$. Hence, it is separable too. Let $\mathbb{Q}$ be a countable, dense subset of hypo z. Let $\mathbb{Q}_z$ be the set of $t \in \mathbb{D}$ such that $(t, x) \in \mathbb{Q}$ for some $x \in \mathbb{R}$. Then $\mathbb{Q}_z$ is a countable subset of $\mathbb{D}$.

Let $y(s)$ denote the right-hand side of (C.1). Since z is USC, we have $z(s) \geq y(s)$ for all $s \in \mathbb{D}$.

Conversely, let $s \in \mathbb{D}$. If $z(s) = -\infty$ then trivially $y(s) \geq z(s)$. Suppose that $z(s) > -\infty$. Let $-\infty < \alpha < z(s)$, so that $(s, \alpha) \in$ hypo z. Find a sequence $(t_n, \alpha_n) \in \mathbb{Q}$ such that $(t_n, \alpha_n) \to (s, \alpha)$ as $n \to \infty$. Then $(t_n, \alpha_n) \in$ hypo z and, thus, $z(t_n) \geq \alpha_n$ for all n. Moreover, $t_n \to s$ and $\alpha_n \to \alpha$ as $n \to \infty$. It follows that $y(s) \geq \alpha$. Since this is true for all $\alpha < z(s)$, we find $y(s) \geq z(s)$.

We prove the last statement. Let $x \in \mathrm{USC}(\mathbb{D})$ and suppose that $x(t) \geq z(t)$ for all $t \in \mathbb{Q}_z$. The function x is equal to its own USC hull, i.e.

$$x(s) = \inf_{\varepsilon > 0} \sup_{t \in \mathbb{D} : \, d(s,t) \leq \varepsilon} x(t), \qquad s \in \mathbb{D}.$$

Combine this formula together with (C.1) to see that $x(s) \geq z(s)$ for all $s \in \mathbb{D}$. $\qquad\square$

Lemma C.3. *If ξ is a USC process then $\inf_{s \in F} \xi(s)$ is a random variable for any closed set $F \subset \mathbb{D}$.*

Proof. Let $\mathbb{Q}$ be a countable, dense subset of F. Since every $\xi(s)$ is a random variable, it suffices to show that $\inf_{s \in F} \xi(s) = \inf_{s \in \mathbb{Q}} \xi(s)$. The inequality '$\leq$' is trivial. To see the other inequality, suppose that x is such that $\xi(s) \geq x$ for all $s \in \mathbb{Q}$. Then $(s, x) \in$ hypo ξ for all $s \in \mathbb{Q}$, and, thus, $(s, x) \in$ hypo ξ for all $s \in F$, since hypo ξ is closed. It follows that $\xi(s) \geq x$ for all $s \in F$. $\qquad\square$

Appendix D. Continuity of the GEV parameter

Recall the GEV distributions from Subsection 5.1.

Lemma D.1. *Let $F_n = F(\cdot, \theta_n)$, $n \geq 0$, be GEV distribution functions with associated GEV parameters $\theta_n = (\mu_n, \sigma_n, \gamma_n)$. If $(F_n)_n$ converges weakly to F_0 then also $\lim_{n \to \infty} \theta_n = \theta_0$ in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$.*

Proof. By the continuity of the GEV distribution, weak convergence is the same as pointwise convergence, and, thus, $\lim_{n \to \infty} F_n(x) = F_0(x)$ for all $x \in \mathbb{R}$.

Recall the expression (5.2) of the quantile function $Q(\cdot; \theta)$. Pointwise convergence of monotone functions implies pointwise convergence of their inverses at continuity points of the limit [11, Chapter 0]. Setting $p = e^{-1}$, we obtain

$$\mu_n = Q(e^{-1}; \theta_n) \to Q(e^{-1}; \theta_0) = \mu_0, \qquad n \to \infty.$$

As a consequence, for $p \in (0, 1)$,

$$Q(p; \gamma_n, 0, \sigma_n) = Q(p; \theta_n) - \mu_n \to Q(p; \theta_0) - \mu_0 = Q(p; \gamma_0, 0, \sigma_0), \qquad n \to \infty. \quad \text{(D.1)}$$

This implies that, for $x, y > 0$ such that $y \neq 1$,

$$\lim_{n \to \infty} \frac{x^{\gamma_n} - 1}{y^{\gamma_n} - 1} = \frac{x^{\gamma_0} - 1}{y^{\gamma_0} - 1}.$$

For $\gamma_n = 0$, the above expressions are to be understood as $\log(x) / \log(y)$. A subsequence argument then yields that $(\gamma_n)_n$ must be bounded, and a second subsequence argument confirms that $\gamma_n \to \gamma_0$ as $n \to \infty$. Combine this convergence relation with (D.1) and use the identity $Q_n(p; \gamma_n, 0, \sigma_n) = \sigma_n Q(p; \gamma_n, 0, 1)$ to conclude that $\sigma_n \to \sigma_0$ as $n \to \infty$. □

Acknowledgements

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