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Amenability and Fixed Point Properties of Semitopological Semigroups in Modular Vector Spaces

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Abstract. In this paper, we initiate the study of fixed point properties of amenable semitopological semigroups in modular spaces. Takahashi's fixed point theorem for amenable semigroups of nonexpansive mappings, and T. Mitchell's fixed point theorem for reversible semigroups of nonexpansive mappings in Banach spaces are extended to the setting of modular spaces. Among other things, we also generalize another classical result due to Mitchell characterizing the left amenability property of the space of left uniformly continuous functions on semitopological semigroups by introducing the notion of a semi-modular space as a generalization of the concept of a locally convex space.

1 Introduction

Finding conditions ensuring the existence of a fixed point for a nonexpansive mapping or more generally for a semigroups of nonexpansive mappings on a bounded subset of a Banach space have been studied since the early sixties with the work of DeMarr [5] who proved in 1963 that any commutative semigroup of nonexpansive mappings on a nonempty compact convex set in a Banach space possesses a common fixed point. In 1969, Takahashi improved DeMarr's result by showing that it remains true even for left amenable semigroups. Afterwards, Mitchell [16] established in 1970 that the left amenability condition required by Takahashi can be weakened to left reversibility. However, if we consider nonexpansive mappings on weakly compact sets, the situation is totally different, and DeMarr's result is no longer true even for a single map. In fact, Alspach [2], has constructed a fixed point free nonexpansive mapping on a non-empty weakly compact convex subset of $L^1([0,1])$. Therefore, it follows that a commutative (or even amenable) semigroup of nonexpansive mappings on a nonvoid weakly compact convex subset of a Banach space need not have a common fixed point. Hence, when we deal with weak topologies, one has to make either some restrictions on the spaces, or put additional conditions on the mappings or on the given set. In the former case, Browder [4] showed that DeMarr's result is true for weakly compact convex sets if the underlying Banach space is assumed to be uniformly convex. The author [20] extended Browder's result for commuting families to amenable semigroups. Independently of Browder, Kirk [9] in 1965 proved that a nonexpansive mapping on a non-empty weakly compact convex subset (of a Banach space) possessing a normal structure (*i.e.,* contains a non-diametral point) has a fixed point.

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The aim of this paper is to extend the study of fixed point properties of semitopological semigroups in Banach spaces to a more general class of spaces called modular spaces. Among other things, we generalize most of the results cited above to this new framework, and by introducing the concept of a semi-modular space, we extend a classical fixed point theorem in locally convex spaces. The concept of a modular finds its origin in the work of Orlicz [19] published during the early 30s. However the formal definition of a modular is due to Nakano [18] in connection with the theory of ordered linear spaces. In this paper we aim to extend the study of fixed point properties of semigroups to this setting. For references in fixed point theory in modular spaces, refer to [8, 11, 19].

2 Preliminaries and Notation

Let E be a vector space over the field of all real numbers. A modular function on E is a mapping $\rho\colon E \to [0,\infty]$ with the following properties:

(1) $\rho(x) = 0$ if and only if $x = 0$.
(2) $\rho(\alpha.x) = \rho(x)$ for all $\alpha \in \mathbb{R}$ with $|\alpha| = 1$.
(3) $\rho(\alpha.x + (1-\alpha).y) \le \rho(x) + \rho(y)$ for all $\alpha \in [0,1]$ and $x, y \in E$.

A modular ρ is said to be convex if it satisfies the property:

$$\rho\big(\alpha.x + (1-\alpha).y\big) \le \alpha\rho(x) + (1-\alpha)\rho(y)$$

for every pair of points $x, y \in E$ and every $\alpha \in [0,1]$. If property (1) is replaced by the weaker condition $x = 0 \Rightarrow \rho(x) = 0$, then we will call ρ a semi-modular.

To a given modular function ρ on the vector space E, we assign the space

$$E_\rho := \Big\{x \in E\colon \lim_{\alpha\to 0} \rho(\alpha.x) = 0\Big\}$$

called a modular space associated with ρ. As readily checked, E_ρ is a vector subspace of E. Since ρ can take the value $+\infty$, then it does not define a norm on E_ρ in general. In the case where ρ is a convex modular, it is then possible to make E_ρ into a normed space by letting

$$\|x\|_\rho = \inf\Big\{\alpha > 0 : \rho\Big(\frac{x}{\alpha}\Big) \le 1\Big\} \text{ for all } x \in E_\rho.$$

From a modular function one can define a kind of topology on the modular space that resembles to a topology induced by a metric.

(1) $(x_\alpha)_{\alpha\in J}$ in E_ρ is said to converge to x, if we have $\lim_\alpha \rho(x_\alpha - x) = 0$.
(2) A subset C of E_ρ is said to be ρ-closed, if whenever $(x_\alpha)_{\alpha\in J}$ is a net of elements of C converging to some $x \in E_\rho$, then $x \in C$.
(3) C is said to be ρ-compact if each net in C possesses a convergent subnet.
(4) We say that C is ρ-bounded if its diameter $\delta_\rho(C) := \sup_{x,y\in C} \rho(x-y) < \infty$.
(5) ρ is said to be lower semi-continuous if: $\rho(x-y) \le \liminf_\alpha \rho(x - y_\alpha)$ whenever $x \in E_\rho$, and $y_\alpha \to y$ is a convergent net in E_ρ.
(6) Finally we say that ρ satisfies the Δ_2-condition if there is a positive constant $k \ge 2$ such that: $\rho(2x) \le k\rho(x)$ for all $x \in E_\rho$.

For more details about the Δ_2-condition, see [8, 10, 17].

Given a semigroup S (*i.e.*, a set with an associative binary operation), if it is equipped with a separated topology such that for every $a \in S$, the mappings $s \mapsto as$

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and $s \mapsto sa$ are continuous from S into itself, then we say that S is a semitopological semigroup. From now on, we will assume that S is a semitopological semigroup. Let $C_b(S)$ denote the Banach algebra of all bounded continuous functions on S equipped with the sup norm topology given by $\|f\| = \sup_{s\in S} |f(s)|$ for all $f \in C_b(S)$. To each $s \in S$, we assign a bounded operator of $C_b(S)$ defined by $\ell_s f(t) := f(st)$ called left translation operator induced by s. Let $\mathrm{LUC}(S)$ be the subset of $C_b(S)$ consisting of those functions f for which the mapping $s \mapsto \ell_s f : S \to C_b(S)$ is norm continuous. It is a well-known fact that $\mathrm{LUC}(S)$ is a translation invariant closed sub-algebra of $C_b(S)$ containing constant functions (see [3] for more details). A mean on $\mathrm{LUC}(S)$ is a member m of the continuous dual $\mathrm{LUC}(S)^*$ with the properties $m(1) = 1 = \|m\|$. If in addition, we have $m(\ell_s f) = m(f)$ for all $f \in \mathrm{LUC}(S)$, then m is called a left invariant mean, and we say that S is left amenable if $\mathrm{LUC}(S)$ possesses a left invariant mean. Let $\mathrm{AP}(S)$ denote the subspace of $C_b(S)$ of those functions f for which the left orbit $\mathcal{L}(f) := \{\ell_s f; s \in S\}$ is relatively compact (*i.e.,* has a compact closure) with respect to the sup norm topology of $C_b(S)$. It is known that $\mathrm{AP}(S)$ is a translation invariant closed sub-algebra of $C_b(S)$ containing constant functions. S is called left reversible, if for all $s, s' \in S$, we have $\overline{sS} \cap \overline{s'S} \neq \varnothing$. Here, $\overline{sS}$ stands for the closure in S of $sS := \{st; t \in S\}$. Relationships between $\mathrm{AP}(S)$, $\mathrm{LUC}(S)$, and left reversibility can be summarized as follows:

- $\mathrm{AP}(S) \subset \mathrm{LUC}(S) \subset C_b(S)$ for all semitopological semigroup S.
- $\mathrm{AP}(S) = \mathrm{LUC}(S) \subset C_b(S)$ if S is compact.
- $\mathrm{AP}(S) = \mathrm{LUC}(S) = C_b(S)$ if S is a compact topological semigroup (*i.e.,* the operation of S is jointly continuous).
- $\mathrm{AP}(S) \subset \mathrm{LUC}(S) = \ell^\infty(S)$ if S is discrete.

It is known that if S is left reversible, then $\mathrm{AP}(S)$ has a left invariant mean; see [12]. However, the converse is not true. For example, the partially bicyclic semigroup $S_2 = \langle e, a, b, c : ab = ac = e \rangle$ generated by an identity e with three elements a, b, and c is not left reversible, whereas $\mathrm{AP}(S)$ has an invariant mean, which is a result due to T. Mitchell during a 1984 conference on analysis on semigroups in Virginia. When S is discrete, it is straightforward that S is left reversible if it is left amenable, and the converse of this fact is not true; by considering the free group on two generators. Further, the implication does not hold for the semitopological case, and even when the whole $C_b(S)$ has a left invariant mean. Indeed, there is a regular topological space S such that the continuous real-valued functions are precisely given by the constant maps; see [7]. If we let $a.b = a$ for all $a, b \in S$, then we define a semitopological semigroup structure. Furthermore, it is easy to check that given $s \in S$ the point mass $f \mapsto f(s)$ defines a left invariant mean on $C_b(S)$; however, S is not left reversible.

Let S be a semitopological semigroup, and let E be an arbitrary non-trivial real vector space. Given a modular function ρ on E, and a non-empty subset $K \subset E_\rho$, an action of S on K is a mapping $. : S \times K \to K$ subject to the condition $(ss').x = s.(s'.x)$ for all $s, s' \in S$, and $x \in K$. We say that the action is jointly continuous if, it is continuous when $S \times K$ is given the product topology. The action is called k-Lipschitzian if it satisfies the following property:

$$\rho(s.x - s.y) \leq k.\rho(x - y) \text{ for all } s \in S, \text{ and } x, y \in K.$$

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In particular, a 1-Lipschitzian is precisely a nonexpansive action in the modular sense. We will say that a point $x \in K$ is a common fixed for S, if it has the property for all $s.x = x$ for all $s \in S$, and a subset X of K is said to be S-invariant if we have $s.X \subset X$ for all $s \in S$.

3 Main Results

In this section, we establish our main results.

Theorem 3.1 *Let S be a semitopological semigroup. Assume that it satisfies either one of the following conditions:*

(i) *LUC(S) has a left invariant mean.*
(ii) *S is left reversible.*

Then S possesses the following nonlinear fixed point property:

(F1) *Whenever $S \times K \to K$ is a jointly continuous $\frac{2}{k}$-Lipschitzian action on a non-empty ρ-compact convex ρ-bounded subset K of a modular space E_ρ induced by a lower semi-continuous convex modular ρ with the Δ_2-condition with constant k, there exists in K a common fixed point for S. Moreover, the fixed point set of S in K is a singleton when $k > 2$.*

Remark 3.2 It may seem strange to assume a boundedness condition in the presence of compactness. Since ρ is allowed to take the value $+\infty$, a ρ-compact set may fail to be bounded. In fact, let us fix $x \in E_\rho$ such that $\rho(x) = +\infty$ and put $K = \{0, x\}$. Then K is ρ-compact because given a net $(x_j)_{j\in J}$ in K, we have:

Case 1 There is $j_o \in J$ such that $x_j = x$ for all $j \geq j_o$ or $x_j = 0$ for all $j \geq j_o$. In both cases the net $(x_j)_{j\in J}$ is ρ-convergent.

Case 2 For all $j \in J$, there exist $j' \geq j$ and $j^{''} \geq j'$ with $x_{j'} = x$ and $x_{j''} = 0$. So there are subnets converging to x and 0. However, K is ρ-unbounded as it has an infinite ρ-diameter. *Case 2* For all $j \in J$, there exist $j' \geq j$ and $j^{''} \geq j'$ with $x_{j'} = x$ and $x_{j''} = 0$. So there are subnets converging to x and 0. However, K is ρ-unbounded as it has an infinite ρ-diameter. *Cas 2* For all $j \in J$, there exist $j' \geq j$ and $j^{''} \geq j'$ with $x_{j'} = x$ and $x_{j''} = 0$. So there are subnets converging to x and 0. However, K is ρ-unbounded as it has an infinite ρ-diameter.

The following lemma will be crucial in proving this result.

Lemma 3.3 *If the conditions of the theorem are satisfied, then there exist two non-empty sets $\Gamma_* \subset K_* \subset K$ with the following properties:*

(i) *Γ_* and K_* are minimal with respect to being non-empty, ρ-compact, and S-invariant with in addition K_* being convex.*
(ii) *$s.\Gamma_* = \Gamma_*$ for all $s \in S$.*
(iii) *If $\delta_\rho(\Gamma_*) > 0$, then there exists a point $u_* \in co(\Gamma_*)$ (convex hull of Γ^*) such that:*
   (a) *$\rho(x - u_*) < \delta_\rho(\Gamma_*)$ for all $x \in \Gamma_*$.*
   (b) $\sup_{x\in\Gamma_*} \rho(x - u_*) < \frac{k}{2}.\delta_\rho(\Gamma_*)$.

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Proof Put

$$\mathscr{C} := \{C : C \subset K, C \neq \varnothing, \rho\text{-closed, convex, and } S\text{-invariant}\}.$$

ordered by reverse inclusion (*i.e.*, $C \leq C' \Leftrightarrow C' \subset C$). Then C is inductive. In fact, given a simply ordered sub-family $(C_j)_{j\in J}$ of elements of $\mathscr{C}$, then for all $j_1, \dots, j_n \in J$, we have $\mathscr{C} \ni \bigcap_{i=1}^n C_{j_i} = C_{j_q}$ for some q. So $\mathscr{C}$ has the finite intersection property. Now let $\mathcal{F}(J)$ denote the set of all non-void finite subsets of J ordered by inclusions. Using the axiom of choice, consider a family $(x_\sigma)_{\sigma\in\mathcal{F}(J)}$ of elements of K such that $x_\sigma \in \bigcap_{j\in\sigma} C_j$ for all σ. By ρ-compactness, we can assume without loss of generality (by taking a ρ-convergent subnet if necessary) that $x_\sigma \to x$ for some $x \in K$. We assert that $x \in \bigcap_j C_j$. Indeed, if it is not the case, then let $x \notin C_j$ for some j. Since for all $\sigma \geq \{j\}$, we have $x_\sigma \in C_j$ and C_j is ρ-closed, by passing to the limit it follows that $x \in C_j$, which is absurd. Therefore, $\bigcap_j C_j \in \mathscr{C}$ is an upper bound in $\mathscr{C}$ for the C_j's, which shows that our claim is true. Hence by Zorn's lemma, $\mathscr{C}$ has a maximal element which we denote by K_*. For the existence of Γ_* we shall consider the following cases:

• Case 1: LUC(S) has a left invariant mean. Let

$$\tau_\rho := \{F^c \cap K_*; F \subset K \text{ and } F \text{ being } \rho\text{-closed}\}$$

with F^c standing for the complement of F in K. Then it is readily checked that τ_ρ defines a topology on K_*. Note that if $F_1, \dots, F_n$ are non-void ρ-closed subsets of K, then their union is also ρ-closed. In fact, let $F = \bigcup_j F_j$ and $x_t \to x$ be a convergent net of points of F. By contradiction, let us assume that $x \notin F$. Then we have $d(x, F_j) = \inf_{y\in F_j} \rho(x - y) > 0$ for all $j = 1, \dots, n$, because by ρ-compactness of each F_j, we have $d(x, F_j) = \rho(x - a_j) > 0$ (for some $a_j \in F_j$). Let $0 < \epsilon < \min_j d(x, F_j)$ and an index t such that $\rho(x - x_t) < \epsilon$. Then the element x_t does not lie in F which is impossible. Therefore, τ_ρ is closed under finite intersections, and since arbitrary intersections of ρ-closed sets are ρ-closed, τ_ρ is also closed under arbitrary unions. Hence, τ_ρ is a well defined topology on K_*. Furthermore, as readily checked a net in K_* is τ_ρ-convergent whenever it is ρ-convergent. Hence (K_*, τ_ρ) is a compact Hausdorff space (separateness following from the convexity and the Δ_2-condition of ρ). Hence, if we consider the restriction of the action on K_*, together with the joint continuity assumption it follows (see [21, Lemma 2.2]) that for all $f \in C(K_*)$ and $x \in K_*$, the mapping $f_x : s \mapsto f(s.x)$ from S into $\mathbb{R}$ lies in LUC(S). Hence, together with the existence of a left invariant mean on LUC(S), the existence of a non-void τ_ρ-compact and S-invariant subset ω of K_* such that $s.\omega = \omega$ for all $s \in S$ follows from [21, Lemma 2.12]. Then it is obvious to let $\Gamma_* := \omega$.

• Case 2: S is left reversible. Since $S \times K_* \to K_*$ is a jointly continuous action and K_* is compact, the existence of Γ_* follows by applying [13, Lemma 3.4].

So it remains to prove (iii). Let us assume that the ρ-diameter of Γ^* is positive and fix $k \geq 2$ provided by the Δ_2 condition. We will follow an idea of DeMarr in [5] in the case of Banach spaces. Let us introduce

$$\mathscr{E} := \Big\{E : E \subset \Gamma_*, E \neq \varnothing, \rho(x - y) \in \Big[\frac{2}{k}.\delta_\rho(\Gamma_*), \delta_\rho(\Gamma_*)\Big] \text{ if } x, y \in E \text{ and } x \neq y\Big\}.$$

We claim that $\mathscr{E}$ is non-void. In fact, for all $j \in \mathbb{N}$ let us pick $(x_j, y_j) \in \Gamma_*^2$ such that $\delta_\rho(\Gamma_*) - \frac{1}{j} \leq \rho(x_j - y_j)$. From the ρ-compactness of Γ_*, let $(x_{j_\alpha}, y_{j_\alpha}) \to (x, y)$ be a

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convergent subnet. Then by lower semi-continuity of ρ it follows that

$$\rho(x-y) \le \liminf_{\alpha} \rho(x_{j_\alpha} - y_{j_\alpha}) \le \delta_\rho(\Gamma_*)$$

by definition of $\mathscr{E}$. On the other hand, from the Δ_2 condition, it follows for all α that

$$\begin{aligned}\delta_\rho(\Gamma_*) - \frac{1}{j_\alpha} &\le \rho(x_{j_\alpha} - y_{j_\alpha}) \\ &= \rho\Big(2\frac{x_{j_\alpha} - y_{j_\alpha}}{2}\Big) \\ &= \rho\Big(\frac{2(x_{j_\alpha} - y_{j_\alpha} - x + y)}{2} + \frac{2(x-y)}{2}\Big) \\ &\le \frac{k}{2}\big(\rho(x_{j_\alpha} - y_{j_\alpha} - x + y) + \rho(x-y)\big).\end{aligned}$$

Therefore, by passing to the limit, it follows $\delta_\rho(\Gamma_*) \le \frac{k}{2}.\rho(x-y)$. Hence, the two inequalities yield $E = \{x, y\} \in \mathscr{E}$, which shows that our claim is true. Now let us order $\mathscr{E}$ upwards by inclusions (*i.e., $E \le E' \Leftrightarrow E \subset E'$*). Then it is straightforward that $(\mathscr{E}, \le)$ is inductive. So let E_o be a maximal element of $\mathscr{E}$. We assert that E_o is a finite set. Indeed, E_o is ρ-totally bounded (*i.e.*, covered by finitely many sets with arbitrary small ρ-diameters), because if not, then there would be $\epsilon > 0$ and a sequence $(x_j)_j$ of elements of E_o such that for all $j \ge 2$, we have

$$x_j \notin \bigcup_{i=1}^{j-1} \{x : \rho(x - x_i) \le \epsilon\}.$$

We can assume that $x_j \to x$ (for some x), because if it does not converge, we take a convergent subnet whose existence is guaranteed by the ρ-compactness of Γ_*. Then for all $i, j \in \mathbb{N}$ with $i \ne j$, we have by construction $x_i \ne x_j$, and therefore as before, we have

$$\begin{aligned}\frac{2}{k}.\delta_\rho(\Gamma_*) &\le \rho(x_{2j} - x_{2j+1}) \\ &\le \frac{k}{2}\big(\rho(x_{2j} - x) + \rho(x_{2j+1} - x)\big) \\ &\xrightarrow[j\to\infty]{} 0.\end{aligned}$$

Thus, $\delta_\rho(\Gamma_*) = 0$ which leads to a contradiction. Hence our assertion is right. Now let $r := \frac{1}{k}.\delta_\rho(\Gamma_*)$. By pre-compactness of E_o, let $E_o \subset \bigcup_{i=1}^{n}\{x : \rho(x-u_i) < r\}$ with $u_i \in E_o$ due to the convexity and the Δ_2-condition. Then we must have $E_o = \{u_1, \dots, u_n\}$, because if $x \in E_o$ and $x \ne u_i$ for all i, then for some j we have $\rho(x - u_j) < r$ (since E_o is covered by ρ-balls centred at those points); then it follows that

$$\frac{2}{k}.\delta_\rho(\Gamma_*) \le \rho(x - u_j) < \frac{1}{k}.\delta_\rho(\Gamma_*)$$

which is absurd. Hence, $E_o = \{u_1, \dots, u_n\}$. Put $u_* = \frac{1}{n}\sum_{i=1}^{n} u_i \in co(\Gamma_*)$. Then given $x \in \Gamma_*$ we have two possibilities:

- First possibility: $x = u_j$ for some j. Then by convexity we have

$$\rho(x - u_*) \le, \frac{1}{n}\sum_{i=1}^{n}\rho(u_i - u_j) = \frac{1}{n}\sum_{i=1, i\ne j}^{n}\rho(u_i - u_j) \le \frac{n-1}{n}.\delta_\rho(\Gamma_*) < \delta_\rho(\Gamma_*).$$

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• Second possibility: $x \neq u_j$ for all j. Then for some j, we have $\rho(x-u_j) < \frac{2}{k}.\delta_\rho(\Gamma_*)$ because if not, then we would have $E_o < E'_o = E_o \cup \{x\} \in \mathscr{E}$ contradicting the maximality of E_o. Thus,

$$\rho(x-u_*) \leq \frac{1}{n}\sum_{i=1}^{n}\rho(x-u_j) = \frac{1}{n}\sum_{i=1, i\neq j}^{n}\rho(x-u_i) + \frac{\rho(x-u_j)}{n} < \delta_\rho(\Gamma_*).$$

Hence, (a) holds. For (b), let us fix a sequence $(x_j)_j$ of elements of Γ_* such that $\sup_{x\in\Gamma_*}\rho(x-u_*) \leq \rho(x_j-u_*) + \frac{1}{j}$ for all j. Due to the compactness of Γ_*, we can assume (by taking a convergent subnet if necessary) that $x_j \to x$ (for some $x \in \Gamma_*$). Then it follows that $\sup_{x\in\Gamma_*}\rho(x-u_*) \leq \frac{k}{2}\big(\rho(x-x_j)+\rho(x_j-u_*)\big)+\frac{1}{j}$ for all j. Thus, taking the limit yields $\sup_{x\in\Gamma_*}\rho(x-u_*) \leq \frac{k}{2}.\rho(x-u_*)$, which shows that (b) holds as well. Hence, the proof is completed. ∎

We are now ready to proceed to the proof of the theorem.

Proof Let K_* and let Γ_* be as in the lemma, and k be the Δ_2-condition constant. We have the following cases:

• Case 1: $k > 2$. Let us fix $s \in S$. Then the Lipschitz condition together with (ii) of the lemma yield:

$$\begin{aligned}\delta_\rho(\Gamma_*) = \delta_\rho(s.\Gamma_*) &= \sup_{x,y\in\Gamma_*}\rho(s.x-s.y)\\ &\leq \frac{2}{k}.\sup_{x,y\in\Gamma_*}\rho(x-y)\\ &= \frac{2}{k}.\delta_\rho(\Gamma_*).\end{aligned}$$

Consequently, $\delta_\rho(\Gamma_*) = 0$, and it follows that $\Gamma_* = \{x_*\}$ for some $x_* \in K$ that is for sure a common fixed point for S, since Γ_* is S-invariant. Note that, when $k > 2$, due to the Lipschitz condition there is a unique common fixed point.

• Case 2: $k = 2$. If $\delta_\rho(\Gamma_*) = 0$, then we are done. Otherwise, by (b), we have

$$\sup_{x\in\Gamma_*}\rho(x-u_*) < \delta_\rho(\Gamma_*) \text{ for some } u_* \in co(\Gamma_*). \tag{3.1}$$

Let $\rho := \sup_{x\in\Gamma_*}\rho(x-u_*)$. Then as readily checked

$$K_*^* := \bigcap_{y\in\Gamma_*}\{x \in K_* : \rho(x-y) \leq \rho\}$$

sits in $\mathcal{C}$ (see beginning of proof Lemma 3.3). Thus, by minimality, we have $K_*^* = K_*$. But by (3.1), there are some points $y, y' \in \Gamma_*$ such that $\rho(y-y') > \rho$, which implies that $y' \notin \{x \in K_* : \rho(x-y) \leq \rho\} \supset \Gamma_*$ which is absurd. Hence, if Case 2 happens, then we must have $\delta_\rho(\Gamma_*) = 0$ and therefore Γ_* contains only a single point, which is certainly a common fixed point for S. ∎

From the proof of Lemma 3.3, we derive the following useful proposition.

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Proposition 3.4 *Let E_ρ be a modular space induced by a lower semi-continuous convex modular function ρ satisfying the Δ_2-condition with constant k. Then for all ρ-compact subset K of E containing at least two points, there exists a point u_* in the convex hull of K such that:*

(i) $\rho(x - u_*) < \delta_\rho(K)$ *for all* $x \in K$*;*
(ii) $\sup_{x\in K} \rho(x - u_*) < \frac{k}{2}.\delta_\rho(K)$.

Remark 3.5 Proposition 3.4 generalizes [5, Lemma 1] to the setting of modular spaces.

Our main theorem yields the following known classical results.

Corollary 3.6 ((DeMarr's theorem) [5, Main theorem]) *Let B be a Banach space and let X be a nonempty compact convex subset of B. If F is a non-empty commutative family of nonexpansive mappings of X into itself, then the family F has a common fixed point in X.*

Proof In fact, since a commutative family generates a commutative semigroup that is well known to be left amenable, DeMarr's result follows from Theorem 3.1 upon letting $E = B$, $\rho = \|\cdot\|$, and $k = 2$. ∎

In connection, Takahashi has proved that DeMarr's theorem remains true if F is replaced with a left amenable (discrete) semigroup (*i.e.*, a semigroup S such that $\ell^\infty(S)$ has a left invariant mean). Then Theorem 3.1 yields.

Corollary 3.7 (Takahashi's theorem [22, Main theorem]) *Let K be a non-empty compact convex subset of a Banach space B and S be an amenable semigroup of nonexpansive mappings of K into K. Then there exists an element z in K such that s.z = z for each s in S.*

Mitchell improved Takahashi's result by weakening the condition "F be a commuting family" to "F being a left reversible semigroup", and established the following corollary.

Corollary 3.8 (Mitchell's theorem [16, Main theorem]) *Let S be a left reversible semigroup of nonexpansive mappings on a non-empty compact convex subset K of a Banach space B. Then there exists an element z in K such that s.z = z for all s ∈ S.*

If we let FPT stand for an abbreviation of "Fixed Point Theorem", then in summary the following implications hold:

$$\text{Theorem 3.1} \Longrightarrow \text{Mitchell FPT} \Longrightarrow \text{Takahashi FPT} \Longrightarrow \text{DeMarr FPT}.$$

Next we will show that Theorem 3.1 remains true if the action is assumed to be separately continuous, and the amenability condition is imposed on some (suitable) smaller space sitting inside $\mathrm{LUC}(S)$. We first let $C(\Gamma)$ denote the Banach algebra of all τ_ρ-continuous real-valued functions on Γ. Or equivalently, the collection of those $f : \Gamma \to \mathbb{R}$ such that: whenever $\rho(x_\alpha - x) \to 0$, $f(x_\alpha) \to f(x)$. The topology τ_ρ is as in the proof of Lemma 3.3. We have the following theorem.

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Theorem 3.9 *Let S be a semitopological semigroup. If $AP(S)$ has a left invariant mean, then for all $k \geq 2$, the semigroup S possesses the following fixed point property: $F(k)$: $S \times K \to K$ is a separately continuous $\frac{2}{k}$-Lipschitzian action on a non-empty ρ-compact convex ρ-bounded subset K of a modular space E_ρ, induced by a lower semi-continuous convex modular ρ satisfying the Δ_2-condition with constant k; then there exists in K a common fixed point for S.*

Conversely, if S has fixed point property $F(2)$, then $AP(S)$ possesses a left invariant mean.

Remark 3.10 In the theorem we have dropped the left reversibility condition assumed in Theorem 3.1, because as shown in [12], if a semitopological semigroup S is left reversible, then $\mathrm{AP}(S)$ has a left invariant mean.

The following lemma will be needed for the proof of this result.

Lemma 3.11 *For any non-empty S-invariant ρ-closed subset Γ of K, the mapping $\phi_x : S \to \mathbb{R}$ given by $\phi_x(s) = \phi(s.x)$ sits inside $AP(S)$ for all $x \in \Gamma$, and $\phi \in C(\Gamma)$.*

Proof Let $\Gamma \subset K$ be a non-empty S-invariant ρ-closed subset. Fix $x \in \Gamma$ and $\phi \in C(\Gamma)$. Consider $\Psi : \Gamma \to C_b(S)$ given by $\Psi(y) = \phi_y$. We assert that Ψ is continuous when $C_b(S)$ is given the sup norm topology. Let $x_\alpha \to x$ be a convergent net in Γ. First we will need to show that ϕ is ρ-uniformly continuous. By contradiction, let us assume that it is not. Then let $\delta > 0$ and $(a_n)_n, (b_n)_n$ such that

$$\rho(a_n - b_n) \leq \frac{1}{n} \quad \text{and} \quad |\phi(a_n) - \phi(b_n)| \geq \delta \text{ for all } n \in \mathbb{N}.$$

By ρ-compactness of Γ, there is a subnet $(n_\lambda)_\lambda$ such that $a_{n_\lambda} \to a$ and $b_{n_\lambda} \to b$ for some $a, b \in \Gamma$. On the one hand, by using the Δ_2-condition, we also have $a_{n_\lambda} - b_{n_\lambda} \to a - b$ as well, and on the other the difference converges to 0; thus, $a = b$. By passing to the limit, it follows $|\phi(a) - \phi(b)| = 0 \geq \delta$, which is impossible. So ϕ must be ρ-uniformly continuous. Now let us fix $\epsilon > 0$ and $\delta_\epsilon > 0$ such that $\rho(x - y) \leq \delta_\epsilon \Rightarrow |\phi(x) - \phi(y)| \leq \epsilon$. Fix α_ϵ with the property that $\alpha \geq \alpha_\epsilon$ implies $\rho(x_\alpha - x) \leq \frac{k}{2}\delta_\epsilon$. Then given $s \in S$, we have

$$\rho(s.x_\alpha - s.x) \leq \frac{2}{k}\rho(x_\alpha - x) \leq \delta_\epsilon \Longrightarrow |\phi(s.x_\alpha) - \phi(s.x)| \leq \epsilon.$$

Thus, it follows, $\|\Psi(x_\alpha) - \Psi(x)\| = \sup_{s \in S} |\phi(s.x_\alpha) - \phi(s.x)| \leq \epsilon$ whenever $\alpha \geq \alpha_\epsilon$, which shows that Ψ is continuous. Next, consider the compact set $\overline{S.x}^{\tau_\rho}$ the closure of $S.x := \{s.x; s \in S\}$. So by continuity, $\Psi(\overline{S.x}^{\tau_\rho})$ is a norm compact subset of $C_b(S)$. On the other hand, given $s \in S$ it is easy to check that we have $r_s \phi_x = \Psi(s.x)$; where r_s denotes the right translate by s defined by $r_s f(t) = f(ts)$. It is a known fact (see [3] for details) that $\phi_x \in \mathrm{AP}(S) \Leftrightarrow \mathcal{R}(\phi_x) = \{r_s \phi_x; s \in S\}$ is relatively compact in the sup norm topology of $C_b(S)$. As $\mathcal{R}(\phi_x) \subset \Psi(\overline{S.x}^{\tau_\rho})$ it then follows that $\mathcal{R}(\phi_x)$ is relatively norm compact that means that ϕ_x lies in $\mathrm{AP}(S)$. ■

Now we can proceed to the proof of the theorem.

Proof The necessary condition follows using method of Theorem 3.1 with in Lemma 3.3 the space $AP(S)$ substituted for $LUC(S)$, which can be done by virtue of Lemma 3.11. For the sufficiency condition, let us assume that S has fixed point property $F(2)$. By virtue of [6], it is enough to show that for all $f \in AP(S)$ we have $\overline{co}^p(\mathcal{R}(f))$ (closure of the convex envelope of the right orbit $\mathcal{R}(f) = \{r_s f; s \in S\}$ of f in the topology of pointwise convergence) contains a constant function. So let $f \in AP(S)$ be fixed. Then $\mathcal{R}(f)$ is relatively norm compact in $C_b(S)$; see [3]. Therefore, $K := \overline{co}^{\|\cdot\|}(\mathcal{R}(f))$ is norm compact too. Let S act on K through $s.g = r_s g$ for all $s \in S$, and $g \in K$. By letting $E = C_b(S), \rho = \|\cdot\|$, and $k = 2$, then all the conditions in $F(2)$ are met. Therefore, by assumption there is $g \in K$ such that $s.g = g$ for all $s \in S$. Without loss of generality we may assume that S has a unit say e, because if it is not the case, we adjoin a unit as in the proof of [21, Theorem 2.15]. Therefore, we have $g(s) = r_s g(e) = g(e)$ for all $s \in S$. Hence the constant $c = g(e) \equiv g$ lies in $K \subset \overline{co}^p(\mathcal{R}(f))$. Hence by arbitrariness of f it follows that $AP(S)$ has a left invariant mean. ∎

Now our next purpose is to provide a linear version of Theorem 3.1 characterizing left amenability property of the space $LUC(S)$ on a semitopological semigroup S extending [15, Theorem 1] to more general spaces introduced here. We first need to introduce the following definitions.

Let E be a vector space and let $\rho = \{\rho_i; i \in I\}$ be a collection of semi-modular functions on E. Let E_{ρ_i} be the semi-modular space generated by ρ_i. The semi-modular space E_ρ associated with ρ is defined by:

$$E_\rho := \bigcap_{i \in I} E_{\rho_i}.$$

E_ρ is said to be separated, if it has the property: for all $x \in E_\rho$

$$\rho_i(x) = 0 \text{ for all } i \in I \text{ implies } x = 0.$$

We will say that a property (P) holds for ρ, if it does for each ρ_i, *e.g.*, a net $(x_\alpha)_\alpha$ is said to be ρ-convergent, if and only if $\rho_i(x_\alpha - x) \to 0$ for all $i \in I$. If $K \subset E_\rho$ is a non-empty convex subset of E_ρ, then a mapping $f\colon K \to (\mathbb{R} \text{ or } K)$ is said to be affine, if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $x, y \in K$ and $t \in [0,1]$. An action $S \times K \to K$ is said to be an affine action if for all $s \in S$, $x \mapsto s.x$ from K into itself is an affine mapping. Put

$$A(K) := \{f\colon K \to \mathbb{R} \text{ such that } f \text{ is affine and } \rho\text{-continuous}\}.$$

If K is ρ-compact and convex, then we will say that K is ρ-admissible if $A(K)$ separates points of K (*i.e.*, whenever $x, y \in K$ and $f(x) = f(y)$ for all $f \in A(K)$, then $x = y$).

Semi-modular spaces include trivially modular spaces, and the class of all locally convex spaces.

We are now ready to state our next result.

Theorem 3.12 *Let S be a semitopological semigroup. Then $LUC(S)$ has a left invariant mean if and only if S has the following fixed point property:*

(F3) *Whenever $.\colon S \times K \to K, (s,x) \mapsto s.x$ is a jointly continuous affine action on a non-empty ρ-admissible ρ-compact convex subset K of a semi-modular space E_ρ, then there exists in K a common fixed point for S.*

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Proof Let us assume that LUC(S) has a left invariant mean m. Let τ_ρ be as in the proof of Theorem 3.1. As we have already shown, τ_ρ defines a compact Hausdorff topology on K with the property that a net $(x_\alpha)_\alpha$ is ρ-convergent if and only if it is with respect to τ_ρ. It is known (see [21]) that if LUC(S) has a left invariant mean, then there exists $\Psi \in \overline{co(\beta K)}^{\tau_{w*}}$ such that $\Psi(f \circ s) = \Psi(f)$ for all $f \in C(K)$ and $s \in S$; where $co(\beta K)$ denotes the convex hull of the Stone-Čech compactification βK of K, and the closure is taken in the weak* topology of $C(K)^*$. Let $(\Psi_\alpha)_{\alpha \in J}$ be a a net in $co(\beta K \simeq K)$ converging weak* to Ψ. Set

$$\Psi_\alpha = \sum_{i=1}^{n_\alpha} t_i^\alpha \delta_{x_i^\alpha},\ x_i^\alpha \in K,\ t_i^\alpha \geq 0 \text{ with } \sum_i t_i^\alpha = 1.$$

For all $\alpha \in J$ define $x_\alpha := \sum_{i=1}^{n_\alpha} t_i^\alpha x_i^\alpha \in K$. By taking a convergent subnet if necessary, we may assume that $x_\alpha \to x$ for some $x \in K$ by compactness of K. Let $f \in A(K)$ and $s \in S$ be fixed. Let σ_s denote the mapping $x \mapsto s.x$. Then $f \circ \sigma_s \in A(K)$ and

$$\begin{aligned} f(s.x) &= \lim_\alpha (f \circ \sigma_s)\Big(\sum_{i=1}^{n_\alpha} t_i^\alpha x_i^\alpha\Big) = \lim_\alpha \sum_{i=1}^{n_\alpha} t_i^\alpha (f \circ \sigma_s)(x_i^\alpha) \\ &= \lim_\alpha \Psi_\alpha(f \circ \sigma_s) = \Psi(f \circ \sigma_s) = \Psi(f) = \lim_\alpha \Psi_\alpha(f) \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha} t_i^\alpha f(x_i^\alpha) = \lim_\alpha f\Big(\sum_{i=1}^{n_\alpha} t_i^\alpha x_i^\alpha\Big) \\ &= \lim_\alpha f(x_\alpha) = f(x). \end{aligned}$$

Thus, $f(s.x) = f(x)$ for all $f \in A(K)$ and $s \in S$. So by separateness, it follows that $s.x = x$ for all $s \in S$; which means that x is a common fixed point for S.

For the converse, let us assume that S possesses the fixed point property (F3). Let E=LUC$(S)^*$. For all $f \in$ LUC(S), let us assign the modular function (that is actually a semi-norm) $\rho_f\colon E \to [0,\infty)$ defined by $\rho_f(\phi) := |\phi(f)|$; and put

$$\rho = \{\rho_f; f \in \text{LUC}(S)\}.$$

Put K= collection of all means on LUC(S), and define a jointly continuous action on K by letting $s.m(f) = m(\ell_s f)$ for all $s \in S$, and $m \in K$. The ρ-admissibility condition of K is automatic, since $\{ev_f; f \in \text{LUC}(S)\} \subset A(K)$ separates points of $E_\rho = E$, because $E_{\rho_f} = E$ for all $f \in$ LUC(S). Hence, by assumption, there exists $m \in K$ such that $s.m = m$ for all $s \in S$, which implies that m is a left invariant mean. ■

Corollary 3.13 ([15, Theorem 1]) *Let S be a semitopological semigroup. Then the following properties are equivalent:*

(P2) *LUC(S) has a left invariant mean.*

(F2) *Whenever S acts affinely on a compact convex subset Y of a locally convex linear topological space, where the map $S \times Y \to Y$ is jointly continuous, then Y contains a common fixed point of S.*

Proof Indeed, if E is a separated locally convex space with family Q of semi-norms defining its topology, then $E_Q = E$. The corollary follows from Theorem 3.12 by

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taking $\rho = Q$. As the continuous dual E^* separates points, each non-empty compact convex set in $E_\rho = E$ is ρ-admissible. ∎

In the previous theorem if we let (F4) denote the fixed point property obtained from (F3) with the phrase "separately continuous and equicontinuous" substituted for "jointly continuous", then while proving the necessary condition, one can only assume the existence of a left invariant mean on $\mathrm{AP}(S)$ instead of on the whole of $\mathrm{LUC}(S)$, and in fact, (F4) turns out to imply left amenability of $\mathrm{AP}(S)$. We recall that an action $S \times K \to K$ is termed ρ-equicontinuous if, for all $\epsilon > 0$, there corresponds $\delta_\epsilon > 0$ such that for every pair x, y of points of K, we have

$$\rho(x-y) \le \delta_\epsilon \Longrightarrow \sup_{s\in S} \rho(s.x - s.y) \le \epsilon.$$

More generally, if K is a subset of a semi-modular space E_ρ induced by a family $\rho = \{\rho_i; i \in J\}$ of modular functions on E, an action of S on K is termed ρ-equicontinuous if, it is ρ_i-equicontinuous for all $i \in J$.

Then we have the following result.

Theorem 3.14 *Let S be a semitopological semigroup. Then $AP(S)$ has a left invariant mean if and only if S has the following property.*

(F4) *Whenever $.: S \times K \to K, (s, x) \mapsto s.x$ is a separately continuous ρ-equicontinuous affine action on a non-empty ρ-admissible ρ-compact convex subset K of a semi-modular space E_ρ, there is a common fixed point for S in K.*

Remark 3.15 We point out that Lemma 3.11 is still valid if the Lipschitz condition of the action is replaced by a ρ-equicontinuity.

Proof of Theorem 3.14 The necessary condition follows from a similar argument as in the proof of Theorem 3.12 by taking into account of the previous remark. ∎

Open Problems

The Δ_2-condition has played a crucial role while proving Theorem 3.1 and 3.9. So it is natural to raise.

Problem 1 Is the Δ_2-condition removable in either one of the theorems?

It would be interesting to have fixed point properties dealing with weak modular topologies, but it is very difficult to determine the conjugate space associated with a given modular space. We do not even know if such a space exists.

Problem 2 Does any modular vector space have a modular conjugate (*i.e.*, a subspace of the algebraic dual consisted by linear forms ϕ such that $\phi(x_\alpha) \to 0$ whenever $\rho(x_\alpha) \to 0$)?

If the answer to this question is positive, then we will let E_ρ^* denote the continuous dual of a modular space E_ρ. Then we can ask the following questions.

Problem 3 Is Theorem 3.1 still valid if K is compact with respect to the weak topology $\tau = \sigma(E_\rho, E_\rho^*)$?

Problem 4 Does Theorem 3.12 still hold if we remove the ρ-admissibility condition? In other words, is this theorem still true for any non-empty compact convex set?

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